Report 2962

GEOMETRICAL CHARACTERISTICS OF STREAMLINED SHAPES

NAVAL SHIP RESEARCH AND DEVELOPMENT CENTER

Washington, D.C. 20007

GEOMETRICAL CHARACTERISTICS OF STREAMLINED SHAPES

HYDROMECHANICS LABORATORY

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95

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TABLE OF CONTENTS

LIST OF FIGURES

Page

NOMENCLATURE

a	subscript indicating afterbody
a_i	polynomial coefficients
C_p	prismatic coefficient of semibody
$\bar{C}_p$	prismatic coefficient of whole body
d	general independent parameter at end
D	diameter of offset of body
D(x)	polynomial corresponding to d
D_f	diameter of flat face
e	ratio given in Eq. [252]
f	"quadratic" polynomial
f	subscript indicating forebody
g	"square root" polynomial
h	"cubic" polynomial
k	curvature
k_o	curvature at $x = 0$
$K_o(x)$	polynomial corresponding to k_o
k_1	curvature at $x = 1$
$K_1(x)$	polynomial corresponding to k_1
L	length of body
m	relative axial position of maximum section $m = X_m/L$
P_n	polynomial
Q(x)	polynomial for restraining conditions
r	end radius
R(x)	polynomial corresponding to r
s	end slope

$S(x)$	polynomial corresponding to s
x	normalized axial coordinate
X	axial coordinate
X_m	axial length to maximum section
X_n	axial length of nose of parallel middle body
X_t	axial length of tail of parallel middle body
y	normalized radius or offset
z	general function of offset of least-squares fitted body
z_1	general function of offset of actual body
α	unspecified constant
α_i	independent conditions
β	unspecified constant
β_i	conditions of restraint
γ	unspecified constant

ABSTRACT

Streamlined shapes, both two-dimensional and axisymmetric, are analytically expressed by appropriate polynomials in terms of independent parameters. Permissible ranges of the independent parameters are examined with respect to selected geometrical constraints.

ADMINISTRATIVE INFORMATION

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INTRODUCTION

Streamlined bodies may be defined as those bodies with negligible drag due to separation of flow on the tail. Since separation is a viscous flow phenomenon, its occurrence is governed by the Reynolds number of the flow past the body. In fact at extremely low Reynolds numbers all bodies are in effect streamlined since they suffer no separation. However, for the higher range of Reynolds numbers of practical interest, it has been experimentally recognized that elongated bodies, that is, bodies with rounded noses and tapered tails, act as streamlined shapes. In this respect the terms elongated bodies and streamlined bodies have often been used synonymously even though some elongated bodies have proved to be poor streamlined bodies.

A general system is developed for analytically defining and determining the suitability of shapes for use as streamlined bodies, both two-dimensional and axisymmetric. Rounded, pointed, cusped, and flat faced ends, with and without parallel middle bodies, in any combination of forebody and afterbody are included. Known streamlined shapes may be fitted into the system by least-squares fittings. Polynomial expressions are used which lend t emselves most readily to automatic computation.

The shapes of elongated bodies have been devised by potential flow methods such as source-sink distributions [1] in order to obtain knowledge of the flow field such as the pressure distribution on the body. However, modern methods of numerical calculation [2] by high-speed computers provide pressure distributions for bodies of arbitrary shapes quite readily so this aspect is no longer a prime consideration.

The devising of streamlined shapes has had its greatest impetus from the development of aircraft wings for two-dimensional figures and of dirigible bodies for axisymmetric figures. Analytic expressions were first used for defining shapes from which pressure distributions were determined. It was then decided to start with what wei considered suitable pressure distributons and determine the shapes afterwards. The results are tables of offsets as given by the NACA airfoil series [3] for two-dimensional figures and by You · ana Young [4] for bodies of revolution with rounded noses and pointed t ls.

Since both the pressure distribution and the axial distribution of the offset of the body control the boundary-layer development leading to

[1] References are listed on page 83.

separation, the specification of pressure distribution alone does not ensure the best streamlined body per se. The approach of this paper is to specify the geometrical shape of families of bodies whose hydrodynamic suitability can then be determined by further analysis such as that of the boundary-layer development.

A notable family of streamlined bodies of revolution was developed by Landweber and Gertler [5] which have rounded noses and rounded tails. "Quadratic" polynomials are used in which the square of the radial offset is a polynomial function of axial distance. What is significant is that independent polynomials are related to the independent parameters determining the shape. This technique was presented earlier by Admiral David Taylor [6] in his studies of the mathematical description of ship hulls. The method of factorial analysis is also used by Landweber and Gertler for determining the coefficients of the independent polynomials. In addition the limiting ranges of suitability of the independent parameters are analyzed by an envelope technique which was also previously described by Admiral Taylor. [6] The "quadratic" polynomial developed from delineations of dirigible shapes and is presented in a primitive form by Lyon. [7]

The "square root" polynomial consists of an ordinary polynomial with the addition of a square root term to accomodate rounded noses. It was used in the specification of the 4 & 5-digit series of airfoil sections by NACA [8, 9] and by Kwik [10, 11] for rudder sections.

The method of this report follows in part the technique of Landweber and Gertler [5] with the initial difference of dividing the body into a fore body and after body and then normalizing the coordinate system. The result is that there are only two independent parameters remaining to consider for the same degree of generality. Also both the "quadratic" polynomial and "square root" polynomial are subjected to the analysis by division into independent polynomials; the factorial analysis, where allowable, for determining the coefficients of the independent polynomials; and the envelope analysis for establishing the range of values of the independent parameters for suitability as desirable shapes. The results are analytical expressions for the cases of rounded, pointed, and cusped ends, with and without parallel middle bodies, in any combination of forebody and afterbody. The suitability requirements are presented on charts giving permissible ranges of the two independent parameters. If there were more than two remaining independent parameters, such a simple display would not be possible.

The special case of the flat faced nose with hydrodynamic continuity is finally analyzed by a "cubic" polynomial.

GENERAL ANALYSIS

The shapes of families of bodies of revolution and of two-dimensional symmetrical bodies may be stated functionally as

$$Y = f(X:\alpha_i;\beta_j) \qquad \begin{array}{l} i = 1,\ 2,\ \dots \\ j = 1,\ 2,\ \dots \end{array} \qquad [1]$$

where Y is the radius of the body of revolution or the offset of the two dimensional body,

X is the axial distance of the body measured from the nose,

α_i are the parameters to be varied which specify the family such as length L, maximum diameter or thickness D, etc., and

β_j are the boundary conditions or restraints which give desirable contours such as closed body conditions $Y = 0$ at $X = 0$ and at $X = L$, etc.

The complexity of the analytical analysis is greatly reduced by considering the nose and tail portions separately in a normalized coordinate system (x,y): $y = 0$ at $x = 0$ and $y = 1$ at $x = 1$. For the curved body the split into forward and after bodies is made at the position of maximum radius or thickness. For the parallel middle body the cuts are made at both ends of the parallel portion of the body. For the flat faced body an additional cut is made at the edge of the flat face.

For the completely curved body, if the axial distance to the maximum section is X_m, then the normalized coordinates become

$$y = \frac{2Y}{D} \qquad [2]$$

$$x_f = \frac{X}{X_m} \quad \text{for the forward portion} \qquad [3]$$

and

$$x_a = \frac{L - X}{L - X_m} \quad \text{for the after portion} \qquad [4]$$

For the parallel middle body, if X_n and X_t are the X-coordinates of the end of the nose and the beginning of the tail respectively, then the normalized coordinates become

$$y = \frac{2Y}{D} \qquad [5]$$

$$x = \frac{X}{X_n} \quad \text{for the nose} \qquad [6]$$

and

$$x = \frac{L - X}{L - X_t} \quad \text{for the tail} \qquad [7]$$

For the flat faced body the normalized coordinate y becomes

$$y = \frac{2Y - D_f}{D - D_f} \qquad [8]$$

where D_f is the diameter or width of flat face.

To achieve "hydrodynamic continuity" as contrasted with mathematical continuity it is only necessary that the position, slope,and curvature match at the junction of the forward and after bodies. Since at the junction $y = 1$ and $\frac{dy}{dx} = 0$, it is evident that the position and slope requirements are always met. The curvature condition however remains to be satisfied. At the junction the curvature is given by $\frac{d^2Y}{dX^2}$ and in normalized coordinates

$$k_f = \left(\frac{X_m}{L - X_m} \right)^2 k_a \qquad [9]$$

or

$$k_f = \left(\frac{m}{1 - m} \right)^2 k_a \qquad [10]$$

where k_f = curvature of forebody at $x = 1$,

k_a = curvature of afterbody at $x = 1$, and

$m = \frac{X_m}{L}$, relative axial position of maximum section.

For parallel middle bodies it is obvious that $k_f = k_a = 0$.

In normalized coordinates the contour is given by

$$y = f(x;\alpha_i;\beta_j) \qquad \begin{matrix} i = 1, 2, \ldots \\ j = 1, 2, \ldots \end{matrix} \qquad [11]$$

where α_i and β_j are now defined in normalized coordinates. In this study only two independent parameters α_1 and α_2 are to be considered for simplicity of analysis.

If a functional form like that of a polynomial is selected as

$$y = \sum_{n=0}^{n=n} a_n x^n = P_n(x) \qquad [12]$$

a resolution into linearly independent polynomials may be obtained like that of linearly independent vectors such as

$$y = \sum_i \alpha_i P_{n,i}(x) + \sum_j \beta_j P_{n,j}(x) \qquad [13]$$

or

$$y = \sum_i \alpha_i P_{n,i}(x) + Q(x) \qquad [14]$$

where

$$Q(x) = \sum_j \beta_j P_{n,j}(x) \qquad [15]$$

This permits the effect of the controllable parameters α_i to be obtained independently of each other.

The independent polynomials may be determined by substituting conditions α_i and β_j into the general polynomials and evaluating the polynomial coefficients by a solution of the resulting simultaneous algebraic equations and then by a gathering of terms. Another method is to use the factorial properties of polynomials which is illustrated in the specific cases to follow.

Not all variations of the α_i produce desirable shapes. Conditions for zero values, values of one, maxima or minima, and inflection points may be investigated. For example the condition for zero values of y

$$y(x;\alpha_1,\alpha_2) = 0 \ , \ 0 \le x \le 1 \qquad [16]$$

may be studied as follows.

If the α_1 and α_2 are now considered as variables and x as an adjustable parameter, a line may be defined for each x. An envelope to these lines may be developed which represents the boundary of regions for values of α_1 and α_2 with and without an additional zero value of y that at $x = 0$. The envelope condition is given by

$$\frac{\partial}{\partial x} y(x;\alpha_1,\alpha_2) = 0 \qquad [17]$$

From Equations [16] and [17]

$$\alpha_1 = f_1(x) \qquad [18]$$

$$\alpha_2 = f_2(x) \qquad [19]$$

A plot of α_1 against α_2 for the range of values of x, $0 \le x \le 1$, gives the envelope curve.

"QUADRATIC" POLYNOMIAL REPRESENTATION

GENERAL

The functional relation

$$y^2 = \sum_{n=0}^{n=n} a_n x^n \qquad [20]$$

is to be called the "quadratic" polynomial for want of a better name. It is very suitable for describing bodies of revolution for which volume is an important consideration since it represents the axial distribution of the cross-sectional area. It has the additional advantage of providing a means of accommodating the analytical description of bodies with rounded ends; something the ordinary polynomial cannot do.

In addition to bodies with rounded ends the "quadratic" polynomial may be applied to bodies with pointed ends and cusped ends. Of course since the representations apply equally well to noses and tails, any combination can be formed such as bodies with rounded noses and pointed tails, etc.

Although any number of adjustable parameters α_i may be used, the analysis is to be limited to two as being sufficient for describing suitable figures. Two adjustable parameters for each partial body in the normalized coordinate system is equivalent to six adjustable parameters for the whole body in the natural coordinate system.

ROUNDED ENDS

The adjustable parameters α_i are

α_1) r = radius of curvature at $x = 0$, $r > 0$

$$r = + \frac{1}{\left(\frac{d^2x}{dy^2}\right)_{x=0}} \qquad [21]$$

α_2) k_1 = curvature at $x = 1$, $k_1 \geq 0$

$$k_1 = - \left(\frac{d^2y}{dx^2}\right)_{x=1} \qquad [22]$$

The signs of the equations for r and k_1 are chosen so as to require r and k_1 to be positive for desirable shapes.

The boundary conditions β_j are

$$\begin{aligned} &\beta_1) \quad x = 0, \; y = 0 \\ &\beta_2) \quad x = 1, \; y = 1 \qquad [23] \\ &\beta_3) \quad x = 1, \; \frac{dy}{dx} = 0 \end{aligned}$$

Since there are five conditions in all, $n = 4$.

The α_i and β_j are substituted into the polynomial. Differentiating Equation [20] successively with respect to y gives

$$2y = (a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3)\,\frac{dx}{dy} \qquad [24]$$

and

$$2 = (a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3)\,\frac{d^2x}{dy^2} + (2a_2 + 6a_3x + 12a_4x^2)\left(\frac{dx}{dy}\right)^2 \qquad [25]$$

Since $a_1 \neq 0$, $\frac{dx}{dy} = 0$ at $x = 0$. This automatically provides a rounded end. Then

$$\alpha_1)\quad a_1 = 2r$$

The other substitutions yield

$$\begin{aligned} &\alpha_2)\quad 2a_2 + 6a_3 + 12a_4 = -2k_1 \\ &\beta_1)\quad a_o = 0 \\ &\beta_2)\quad a_o + a_1 + a_3 + a_4 = 1 \\ &\beta_3)\quad a_1 + 2a_2 + 3a_3 + 4a_4 = 0 \end{aligned} \qquad [26]$$

The solution of Equations [26] by determinants shows that the a's are linear functions of r and k_1 . Hence y^2 is also a linear function of r and k_1 and may be written as

$$y^2 = r\,R(x) + k_1K_1(x) + Q(x) \qquad [27]$$

where $R(x)$, $K_1(x)$, and $Q(x)$ are also polynomials of the fourth degree in x . It is possible to determine $R(x)$, $K_1(x)$, and $Q(x)$ by first solving for the a's from the simultaneous equations and then regrouping terms applicable to $R(x)$, $K_1(x)$, and $Q(x)$. Another method is developed in Reference 5 by utilizing the factorial properties of polynomials as follows.

It is evident that the relations for α_i and β_j correspond to

$$
\begin{aligned}
&\alpha_1) \quad \frac{d}{dx}\, y^2(0) = 2r \\
&\alpha_2) \quad \frac{d^2}{dx^2}\, y^2(1) = -\,2k_1 \\
&\beta_1) \quad y^2(0) = 0 \\
&\beta_2) \quad y^2(1) = 1 \\
&\beta_3) \quad \frac{d}{dx}\, y^2(1) = 0
\end{aligned}
\qquad [28]
$$

Since the foregoing apply identically to r and k_1 , it is further evident that

$$
\begin{aligned}
&\alpha_1) \quad R'(0) = 2\ ,\ K_1'(0) = Q'(0) = 0 \\
&\alpha_2) \quad K_1''(1) = -\,2\ ,\ R''(1) = Q''(1) = 0 \\
&\beta_1) \quad R(0) = K_1(0) = Q(0) = 0 \\
&\beta_2) \quad Q(1) = 1\ ,\ R(1) = K_1(1) = 0 \\
&\beta_3) \quad R'(1) = K_1'(1) = Q'(1) = 0
\end{aligned}
\qquad [29]
$$

where $R' = \frac{dR}{dx}$, $R'' = \frac{d^2R}{dx^2}$, etc.

Evaluation of $R(x)$

Since $R(0) = R(1) = R'(1) = R''(1) = 0$ and $R(x)$ is a polynomial of the fourth degree, $R(x)$ may be written as

$$R(x) = \alpha x \ (x - 1)^3 \qquad [30]$$

Since $R'(0) = 2$, $\alpha = -2$.

Then

$$R(x) = -2x \ (x - 1)^3 \qquad [31]$$

Evaluation of $K_1(x)$

Since $K_1(0) = K_1'(0) = K_1(1) = K_1'(1) = 0$, $K_1(x)$ may be written factorially as

$$K_1(x) = \beta x^2 \ (x - 1)^2 \qquad [32]$$

Since $K_1''(1) = -2$, $\beta = -1$.

Then

$$K_1(x) = - x^2 \ (x - 1)^2 \qquad [33]$$

Evaluation of $Q(x)$

Since $Q'(0) = Q'(1) = Q''(1) = 0$, $Q'(x)$ may be written factorially as

$$Q'(x) = \gamma x \ (x - 1)^2 \qquad [34]$$

Then integrating

$$Q(x) = \gamma \left(\frac{x^4}{4} - \frac{2x^3}{3} + \frac{x^2}{2} \right) + C \qquad [35]$$

With $Q(0) = 0$ and $Q(1) = 1$, $C = 0$ and $\gamma = 12$. Then

$$Q(x) = x^2(3x^2 - 8x + 6) \qquad [36]$$

For rounded ends in summary

$$y^2 = r\ R(x) + k_1 K_1(x) + Q(x) \qquad [27]$$

with

$$R(x) = -\ 2x\ (x - 1)^3 \qquad [31]$$

$$K_1(x) = -\ x^2\ (x - 1)^2 \qquad [33]$$

$$Q(x) = x^2(3x^2 - 8x + 6) \qquad [36]$$

As a check let $r = k_1 = 1$ which are the conditions for a sphere. Then

$$y^2 = -\ x^2 + 2x \qquad [37]$$

which is the contour of a sphere with center at $x = 1$.

Graphs of $R(x)$, $K_1(x)$, and $Q(x)$ are given in Figure 1.

Permissible Ranges of Parameters r and k_1

Not all combinations of r and k_1 give desirable shapes. Although the fourth degree polynomial does not lend itself very easily to peculiar shapes for positive r and k_1, it is interesting to analyze the possible limitations in terms of simple criteria:

1. Zero condition. $y^2 = 0$ for $0 \le x \le 1$.

   Negative values of y^2 would be meaningless.
2. Unity condition. $y^2 = 1$ for $0 \le x \le 1$.

   Bulges above $y = 1$ are undesirable.
3. Maximum or minimum condition. $\frac{dy}{dx} = 0$ for $0 \le x \le 1$.

   No other maximum or minimum is to be permitted than at $x = 1$.
4. Inflection point condition. $\frac{d^2y}{dx^2} = 0$ for $0 \le x \le 1$.

   Inflection points are undesirable on noses.

Zero Condition.

$$y^2 = f(x;r,k_1) = 0 \ , \quad 0 \le x \le 1 \qquad [38]$$

The envelope in r and k_1 with x as the variable parameter is given by

$$f' = \frac{\partial f}{\partial x} = 0 \qquad [39]$$

The two envelope conditions, Equations [38] and [39], provide two simultaneous equations in r and k_1 which are solved by Cramer's rule to give $r(x)$ and $k_1(x)$:

$$r = \frac{x^2(x - 2)}{(x - 1)^3} \qquad [40]$$

$$k_1 = \frac{x^2 - 4x + 6}{(x - 1)^2} \qquad [41]$$

The envelope curve is shown in Figure 2. Desirable values of r and k_1 are on the "inside curved" side of the envelope curve.

Unity Condition. The unity condition is that

$$y^2 = f(x;r,k_1) = 1 \qquad 0 \leq x \leq 1 \qquad [42]$$

The envelope in r and k_1 with x as the variable parameter is given by

$$\frac{\partial}{\partial x}(f - 1) = 0 \qquad [43]$$

The two envelope conditions, Equations [42] and [43], provide two simultaneous equations in r and k_1 which are solved by Cramer's rule to give

$$r = 1 + \frac{1}{x} \qquad [44]$$

$$k_1 = \left(1 - \frac{1}{x}\right)^2 \qquad [45]$$

Eliminating x gives simply

$$k_1 = (2 - r)^2 \qquad [46]$$

The envelope curve is shown in Figure 2. Desirable values of r and k_1, that is, without bulges, are on the "inside curved" side of the envelope curve.

Maximum or Minimum Condition. The maximum or minimum condition is given by

$$\frac{dy}{dx} = f' = 0 \qquad [47]$$

The envelope curve in r and k_1 with x as the variable parameter is given by

$$f'' = 0 \qquad [48]$$

The envelope curve is shown in Figure 2. A better understanding of the envelope curve is developed in Figure 3. Each point on the envelope curve represents a tangent giving the locus of values of r and k_1 which provide a maximum or minimum at each value of x other than the maximum at $x = 1$ which prevails at all times. Two such loci are represented. Their point of intersection provides a value of r and k_1 representing maxima or minima at two values of x. Evidently from any point in the region outside the envelope curve, two tangents may be drawn to the envelope curve. Thus the region outside the envelope curve represents values of r and k_1 giving two maxima or minima. The region inside the envelope curve provides no maximum or minimum. Finally there is only one maximum or minimum specified by the envelope curve itself.

The two envelope conditions, Equations [47] and [48], provide two simultaneous equations in r and k_1 which are solved by Cramer's rule to give $r(x)$ and $k_1(x)$ as

$$r = \frac{6x^2}{6x^2 - 4x + 1} \qquad [49]$$

$$k_1 = \frac{6(x - 1)^2}{6x^2 - 4x + 1} \qquad [50]$$

Inflection Point Condition. The inflection point condition is given by

$$\frac{d^2y}{dx^2} = 0$$

For $y^2 = f(x)$

$$2ff'' - f'^2 = 0 \qquad [51]$$

and the envelope condition

$$f''' = 0 \qquad [52]$$

The two conditions provide two simultaneous equations in r and k_1 in terms of x. Since the boundary condition leads to a quadratic relation, Cramer's rule does not apply. For specified values of x the two simultaneous equations may be solved by direct substitution of one equation into the other. The results are shown in Figures 2 and 4.

POINTED ENDS

The adjustable parameters α_i are

α_1) s = slope at $x = 0$, $s > 0$

$$s = \left(\frac{dy}{dx} \right)_{x=0} \qquad [53]$$

α_2) k_1 = curvature at $x = 1$, $k_1 > 0$

$$k_1 = -\left(\frac{d^2y}{dx^2} \right)_{x=1} \qquad [54]$$

The sign of the equation for k_1 is chosen so as to require k_1 to be positive for desirable shapes.

The boundary conditions β_j are

$$\begin{aligned} &\beta_1) \quad x = 0 \ , \ y = 0 \\ &\beta_2) \quad x = 1 \ , \ y = 1 \\ &\beta_3) \quad x = 1 \ , \ \frac{dy}{dx} = 0 \end{aligned} \qquad [55]$$

Since the "quadratic" polynomial automatically gives infinite slope at $x = 0$, an additional condition is necessary to give controlled slopes at $x = 0$. Hence the degree of the polynomial becomes five.

For α_1)

$$\frac{dy}{dx} = \frac{a_1 + 2a_2x + \dots + na_n x^{n-1}}{2y} \qquad [56]$$

Since $y = 0$ at $x = 0$, $\frac{dy}{dx} \to \infty$ unless $a_1 = 0$. For $a_1 = 0$, $\frac{dy}{dx}$ is indeterminate at $x = 0$. Then by L'Hôpital's Rule

$$\underset{x \to 0}{L} \frac{dy}{dx} = \left(\frac{dy}{dx} \right)_{x=0} = s = \frac{a_2}{s} \qquad [57]$$

or

$$s^2 = a_2$$

α_1 then requires that

$$a_1 = 0$$
$$a_2 = s^2 \qquad [58]$$

The other substitutions yield

$$\alpha_2)\quad 2a_2 + 6a_3 + 12a_4 + 20a_5 = -2k_1$$
$$\beta_1)\quad a_o = 0$$
$$\beta_2)\quad a_o + a_1 + a_2 + a_3 + a_4 + a_5 = 1 \qquad [59]$$
$$\beta_3)\quad a_1 + 2a_2 + 3a_3 + 4a_4 + 5a_5 = 0$$

y^2 is then a linear function of s^2 and k_1 or

$$y^2 = s^2\, S(x) + k_1 K_1(x) + Q(x) \qquad [60]$$

It is evident that the relations for α_i and β_j correspond to

$$\begin{aligned}
&\alpha_1) \quad \frac{d}{dx} y^2 \ (0) = 0 \\
&\qquad \frac{d^2}{dx^2} y^2 \ (1) = 2 s^2 \\
&\alpha_2) \quad \frac{d^2}{dx^2} y^2 (1) = - 2k_1 \\
&\beta_1) \quad y^2 \ (0) = 0 \\
&\beta_2) \quad y^2 \ (1) = 1 \\
&\beta_3) \quad \frac{d}{dx} y^2 \ (1) = 0
\end{aligned} \qquad [61]$$

Since the foregoing apply identically to s and k_1, it is further evident that

$$\begin{aligned}
&\alpha_1) \quad S'(0) = K_1'(0) = Q'(0) = 0 \\
&\qquad S''(0) = 2 \ , \ K_1''(0) = Q''(0) = 0 \\
&\alpha_2) \quad K_1''(1) = -2 \ , \ S''(1) = Q''(1) = 0 \\
&\beta_1) \quad S(0) = K_1(0) = Q(0) = 0 \\
&\beta_2) \quad Q(1) = 1 \ , \ S(1) = K_1(1) = 0 \\
&\beta_3) \quad S'(1) = K_1'(1) = Q'(1) = 0
\end{aligned} \qquad [62]$$

Evaluation of $S(x)$

Since $S(0) = S'(0) = S(1) = S'(1) = S''(1) = 0$ and $S(x)$ is a polynomial of the fifth degree, $S(x)$ may be written factorially as

$$S(x) = \alpha x^2 (x - 1)^3 \qquad [63]$$

Since $S''(0) = 2$, $\alpha = -1$.

Then

$$S(x) = -x^2 (x - 1)^3 \qquad [64]$$

Evaluation of $K_1(x)$

Since $K_1(0) = K_1'(0) = K_1''(0) = K_1(1) = K_1'(1) = 0$, $K_1(x)$ may be written factorially as

$$K_1(x) = \beta x^3 (x - 1)^2 \qquad [65]$$

Since $K_1''(1) = -2$, $\beta = -1$.

Then

$$K_1(x) = -x^3 (x - 1)^2 \qquad [66]$$

Evaluation of $Q(x)$

Since $Q'(0) = Q''(0) = Q'(1) = Q''(1) = 0$, $Q'(x)$ may be written factorially as

$$Q'(x) = \gamma x^2 (x - 1)^2 \qquad [67]$$

Then integrating

$$Q(x) = \gamma\left(\frac{x^5}{5} - \frac{x^4}{2} + \frac{x^3}{3}\right) + C \qquad [68]$$

With $Q(0) = 0$ and $Q(1) = 1$, $C = 0$ and $\gamma = 30$.

Then

$$Q(x) = x^3(6x^2 - 15x + 10) \qquad [69]$$

For pointed ends in summary

$$y^2 = s^2\ S(x) + k_1K_1(x) + Q(x) \qquad [60]$$

with

$$S(x) = -\ x^2\ (x - 1)^3 \qquad [64]$$

$$K_1(x) = -\ x^3\ (x - 1)^2 \qquad [66]$$

$$Q(x) = x^3\ (6x^2 - 15x + 10) \qquad [69]$$

Graphs of $S(x)$, $K_1(x)$, and $Q(x)$ are given in Figure 5 .

Permissible Ranges of Parameters s^2 and k_1

Four conditions are to be applied: the zero condition, the unity condition, the maximum condition, and the inflection point condition.

Zero Condition. The envelope curve is specified by

$$y^2 = f(x;s^2,k_1) = 0 \qquad 0 \le x \le 1 \qquad [70]$$

and

$$f' = 0 \qquad [71]$$

The results are

$$s^2 = \frac{x^2(3x - 5)}{(x - 1)^3} \qquad [72]$$

and

$$k_1 = \frac{3x^2 - 10x + 10}{(x - 1)^2} \qquad [73]$$

The envelope curve is plotted in Figure 6.

Unity Condition. The envelope curve is specified by

$$f - 1 = 0 \qquad [74]$$

and

$$f' = 0 \qquad [75]$$

The relations are then

$$s^2 = \frac{3x^2 + 4x + 3}{x^2} \qquad [76]$$

$$k_1 = \frac{3x^3 - 4x^2 - x + 2}{x^2} \qquad [77]$$

The envelope curve is plotted in Figure 6.

Maximum or Minimum Condition. The envelope curve is specified by

$$f' = 0 \qquad [78]$$

and

$$f'' = 0 \qquad [79]$$

The results are

$$s^2 = \frac{30x^2}{10x^2 - 10x + 3} \qquad [80]$$

and

$$k_1 = \frac{30(x-1)^2}{10x^2 - 10x + 3} \qquad [81]$$

The envelope curve is plotted in Figure 6.

Inflection Point Condition. The enevelope curve is specified by

$$2ff'' - f'^2 = 0 \qquad [82]$$

and

$$f''' = 0 \; . \qquad [83]$$

The variation of s^2 with k_1 is obtained numerically by direct substitution in solving the nonlinear simultaneous equations of the envelope curve.

For $x = 0$ an indeterminate condition exists. By L'Hopital's Rule

$$\left(\frac{d^2y}{dx^2}\right)_{x=0} = \frac{-3s^2 - k_1 + 10}{s} \qquad [84]$$

and

$$\left(\frac{d^3y}{dx^3}\right)_{x=0} = \frac{3(15s^2 + 9k_1 - 70)}{s} \qquad [85]$$

The boundary curve at $x = 0$ is then

$$\left(\frac{d^2y}{dx^2} \right)_{x=0} = 0 \qquad [86]$$

or

$$- 3s^2 - k_1 + 10 = 0 \qquad [87]$$

The point of tangency at $x = 0$ is given by Equation [87] and

$$\left(\frac{d^3x}{dx^3} \right)_{x=0} = 0 \quad \text{or} \quad 15s^2 + 9k_1 - 70 = 0 \ . \qquad [88]$$

Then

$$s^2 = \frac{5}{3} \quad \text{and} \quad k_1 = 5 \ . \qquad [89]$$

The envelope curve is shown in Figures 6 and 7.

CUSPED ENDS

The adjustable parameters α_i are

α_1) k_o = curvature at $x = 0$, $k_o > 0$

$$k_o = +\left(\frac{d^2y}{dx^2} \right)_{x=0} \qquad [90]$$

α_2) k_1 = curvature at $x = 1$, $k_1 > 0$

$$k_1 = -\left(\frac{d^2y}{dx^2} \right)_{x=1} \qquad [91]$$

The boundary conditions β_j are

$$\begin{aligned} &\beta_1) \quad x = 0 \ , \ y = 0 \\ &\beta_2) \quad x = 0 \ , \ \frac{dy}{dx} = 0 \\ &\beta_3) \quad x = 1 \ , \ y = 1 \\ &\beta_4) \quad x = 1 \ , \ \frac{dy}{dx} = 0 \end{aligned} \qquad [92]$$

As will be shown the cusped end requires two additional conditions which makes $n = 7$.

For β_1: $a_o = 0$

For β_2:

$$\frac{dy}{dx} = \frac{a_1 + 2a_2x + 3a_3x^2 + \ldots + na_nx^{n-1}}{2\ (a_1x + a_2x^2 + \ldots + a_nx^n)^{\frac{1}{2}}} \qquad [93]$$

For $\frac{dy}{dx} = 0$ at $x = 0$, $a_1 = 0$.

$$\text{Then } \frac{dy}{dx} = \frac{2a_2 + 3a_3x + \dots + na_nx^{n-2}}{2\,(a_2 + a_3x + \dots + a_nx^{n-2})^{\frac{1}{2}}} \qquad [94]$$

$$\left(\frac{dy}{dx}\right)_{x=0} = a_2^{\frac{1}{2}} \qquad [95]$$

$$\text{For } \left(\frac{dy}{dx}\right)_{x=0} = 0 \quad , a_2 = 0 \;. \qquad [96]$$

For α_1 :

$$\frac{d^2y}{dx^2} = \frac{6a_3x + 12a_4x^2 + \dots + n(n-1)a_nx^{n-2} - 2\left(\frac{dy}{dx}\right)^2}{2y} \qquad [97]$$

$$= \frac{6a_3x + 12a_4x^2 + \dots + n(n-1)a_nx^{n-2}}{2(a_3x^3 + a_4x^4 + \dots + a_nx^n)^{\frac{1}{2}}} - \frac{(3a_3x^2 + 4a_4x^2 + 4a_4x^3 + \dots + na_nx^{n-1})^2}{4(a_3x^3 + a_4x^4 + \dots + a_nx^n)^{3/2}} \qquad [98]$$

$$= \frac{2[6a_3x + 12a_4x^2 + \dots + n(n-1)a_nx^{n-2}](a_3x^3 + a_4x^4 + \dots + a_nx^n) - (3a_3x^2 + 4a_4x^3 + \dots + na_nx^{n-1})^2}{4\,(a_3x^3 + a_4x^4 + \dots + a_nx^n)^{3/2}} \qquad [99]$$

$$= \frac{2[6a_3 + 12a_4x + \dots + n(n-1)a_nx^{n-3}](a_3 + a_4x + \dots + a_nx^{n-3}) - (3a_3 + 4a_4x + \dots + na_nx^{n-2})^2}{4\,(a_3x^{1/3} + a_4x^{4/3} + \dots + a_nx^{n-8/3})^{3/2}} \qquad [100]$$

$$\text{At } x = 0 \,,\; \frac{d^2y}{dx^2} \to \infty$$

Let $a_3 = 0$

$$\frac{d^2y}{dx^2} = \frac{2[12a_4 + 20a_5x + \dots + n(n-1)a_nx^{n-4}](a_4 + a_5x + \dots + a_nx^{n-4}) - (4a_4 + 5a_5x + \dots + na_nx^{n-3})^2}{4(a_4 + a_5x + \dots + a_nx^{n-4})^{3/2}} \qquad [101]$$

At $x = 0$

$$\frac{d^2y}{dx^2} = 2a_4^{\frac{1}{2}} \qquad [102]$$

$$4a_4 = k_o^{\,2} \qquad [103]$$

The other substitutions yield

$$\alpha_2: \quad 12a_4 + 20a_5 + 30a_6 = -\,2k_1$$

$$\beta_3: \quad a_4 + a_5 + a_6 = 1 \qquad [104]$$

$$\beta_4: \quad 4a_4 + 5a_5 + 6a_6 = 0$$

y^2 is then a linear function of $k_o^{\,2}$ and k_1

$$y^2 = k_o^{\,2}\, K_o(x) + k_1\, K_1(x) + Q(x) \qquad [105]$$

It is evident that the relations for α_i and β_j correspond to

$$\alpha_1) \quad \frac{d^3}{dx^3}\, y^2(0) = 0$$

$$\frac{d^4}{dx^4}\, y^2(0) = 6k_o^{\,2}$$

$$\alpha_2) \quad \frac{d^2}{dx^2}\, y^2(1) = -\,2k_1$$

$$[106]$$

$$\beta_1) \quad y^2(0) = 0$$

$$\beta_2) \quad \frac{d}{dx}\, y^2(0) = 0$$

$$\frac{d^2}{dx^2}\, y^2(0) = 0$$

β_3) $y^2(1) = 1$

β_4) $\frac{dy^2}{dx}(1) = 0$

Since the foregoing apply identically to k_o and k_1, it is further evident that

$$
\begin{aligned}
&\alpha_1)\quad K_o'''(0) = K_1'''(0) = Q'''(0) = 0\\
&\qquad K_o^{iv}(0) = 6\ ;\ K_1^{iv}(0) = Q^{iv}(0) = 0\\
&\alpha_2)\quad K_1''(1) = -2\ ;\ K_o''(1) = Q''(1) = 0\\
&\beta_1)\quad K_o(0) = K_1(0) = Q(0) = 0\\
&\beta_2)\quad K_o'(0) = K_1'(0) = Q'(0) = 0\\
&\qquad K_o''(0) = K_1''(0) = Q''(0) = 0\\
&\beta_3)\quad Q(1) = 1\ ;\ K_o(1) = K_1(1) = 0\\
&\beta_4)\quad K_o'(1) = K_1'(1) = Q(1) = 0
\end{aligned}
\qquad [107]
$$

Evaluation of $K_o(x)$

Since $K_o(0) = K_o'(0) = K_o''(0) = K_o'''(0) = K_o(1) = K_o'(1) = K_o''(1) = 0$ and $K_o(x)$ is a polynomial of the seventh degree, $K_o(x)$ may be written factorially as

$$K_o(x) = \alpha x^4 (x - 1)^3 \qquad [108]$$

Since $K_o^{iv}(0) = 6$, $\alpha = -1/4$.

Then

$$K_o(x) = -\tfrac{1}{4}x^4 \ (x - 1)^3 \qquad [109]$$

Evaluation of $K_1(x)$

Since $K_1(0) = K_1'(0) = K_1''(0) = K_1'''(0) = K_1^{iv}(0) = K_1(1) = K_1'(1) = 0$, $K_1(x)$ may be written factorially as

$$K_1(x) = \beta x^5 \ (x - 1)^2 \qquad [110]$$

Since $K_1''(1) = -2$, $\beta = -1$.

Then

$$K_1(x) = -x^5 \ (x - 1)^2 \qquad [111]$$

Evaluation of $Q(x)$

Since $Q'(0) = Q''(0) = Q'''(0) = Q^{iv}(0) = Q'(1) = Q''(1) = 0$, $Q'(x)$ may be written factorially as

$$Q'(x) = \gamma x^4 \ (x - 1)^2 \qquad [112]$$

Then integrating

$$Q(x) = \gamma\left(\frac{x^7}{7} - \frac{x^6}{3} + \frac{x^5}{5} \right) + C \qquad [113]$$

With $Q(0) = 0$ and $Q(1) = 1$, $C = 0$ and $\gamma = 105$.

Then

$$Q(x) = x^5(15x^2 - 35x + 21) \qquad [114]$$

For cusped ends in summary

$$y^2 = k_o^2\ K_o(x) + k_1\ K_1(x) + Q(x) \qquad [105]$$

with

$$K_o(x) = -\tfrac{1}{4}x^4\ (x - 1)^3 \qquad [109]$$

$$K_1(x) = -\ x^5\ (x - 1)^2 \qquad [111]$$

$$Q(x) = x^5(15x^2 - 35x + 21) \qquad [114]$$

Graphs of $K_o(x)$, $K_1(x)$, and $Q(x)$ are given in Figure 8.

Permissible Ranges of Parameters k_o and k_1

As before four conditions are to be applied: the zero condition, the unity condition, the maximum and minimum condition, and the inflection point condition.

Zero Condition. The envelope curve is specified by

$$f = 0 \qquad [115]$$

and

$$f' = 0 \qquad [116]$$

The results are

$$k_o^{\,2} = \frac{4x^2(5x - 7)}{(x - 1)^3} \qquad [117]$$

and

$$k_1 = \frac{10x^2 - 28x + 21}{(x - 1)^2} \qquad [118]$$

The envelope curve is shown in Figure 9.

Unity Condition. The envelope curve is specified by

$$f - 1 = 0 \qquad [119]$$

and

$$f' = 0 \qquad [120]$$

The results are

$$k_o^{\,2} = \frac{4(5x^4 + 8x^3 + 9x^2 + 8x + 5)}{x^4} \qquad [121]$$

and

$$k_1 = \frac{10x^5 - 8x^4 - 5x^3 - 2x^2 + 4}{x^5} \qquad [122]$$

The envelope curve is shown in Figure 9.

Maximum or Minimum Condition. The envelope curve is specified by

$$f' = 0 \qquad [123]$$

and

$$f'' = 0 \qquad [124]$$

The results are

$$k_o^2 = \frac{420x^2}{21x^2 - 28x + 10} \qquad [125]$$

and

$$k_1 = \frac{210(x - 1)^2}{21x^2 - 28x + 10} \qquad [126]$$

The envelope curve is plotted in Figure 9.

Inflection Point Condition. The envelope curve is specified by

$$2ff'' - f'^2 = 0 \qquad [127]$$

and

$$f''' = 0 \qquad [128]$$

The variation of k_o^2 with k_1 is obtained numerically by direct substitution in solving the nonlinear simultaneous equations of the envelope curve.

The results are shown in Figure 10. The various regions giving the number of inflection points are delineated in the same figure.

"SQUARE ROOT" POLYNOMIAL REPRESENTATION

GENERAL

The functional relation

$$y = a_{\frac{1}{2}} x^{\frac{1}{2}} + \sum_{n=0}^{n=n} a_n x^n \qquad [129]$$

is to be called the "square root" polynomial for want of a better name. It is suitable for describing two-dimensional shapes with rounded ends.

Of course without the square root term an ordinary polynomial remains.

The same analysis procedure used for the "quadratic" polynomial is to be applied where possible to the "square root" polynomial for the same cases: rounded ends, pointed ends and cusped ends.

ROUNDED ENDS

The adjustable parameters α_i are

α_1) r = radius of curvature at $x = 0$, $r \geq 0$

$$r = + \frac{1}{\left(\frac{d^2x}{dy^2} \right)_{x=0}} \qquad [130]$$

α_2) k_1 = curvature at $x = 1$, $k_1 \geq 0$

$$k_1 = - \left(\frac{d^2y}{dx^2} \right)_{x=1} \qquad [131]$$

The signs of the equations for r and k_1 are chosen so as to require r and k_1 to be positive for desirable shapes.

The boundary conditions β_j are

$$\begin{aligned} &\beta_1)\quad x = 0\ ,\ y = 0 \\ &\beta_2)\quad x = 1\ ,\ y = 1 \\ &\beta_3)\quad x = 1\ ,\ \frac{dy}{dx} = 0 \end{aligned} \qquad [132]$$

Since there are five conditions in all, $n = 3$.

The α_i and β_j are substituted into the polynomial. For α_1 : Differentiating Equation [129] with respect to y gives

$$1 = (\tfrac{1}{2}a_{\frac{1}{2}}x^{-\frac{1}{2}} + a_1 + 2a_2x + 3a_3x^2)\frac{dx}{dy} \qquad [133]$$

or

$$\frac{dx}{dy} = \frac{x^{\frac{1}{2}}}{\tfrac{1}{2}a_{\frac{1}{2}} + a_1x^{\frac{1}{2}} + 2a_2x^{3/2} + 3a_3x^{5/2}} \qquad [134]$$

At $x = 0$, $\frac{dx}{dy} = 0$ which ensures a rounded end. Differentiating Equation [133] with respect to y gives

$$0 = (-\tfrac{1}{4}a_{\frac{1}{2}}x^{-3/2} + 2a_2 + 6a_3x)(\frac{dx}{dy})^2 + (\tfrac{1}{2}a_{\frac{1}{2}}x^{-\frac{1}{2}} + a_1 + 2a_2x + 3a_3x^2)\frac{d^2x}{dy^2} \qquad [135]$$

or

$$\frac{1}{\frac{d^2x}{dy^2}} = \frac{(\tfrac{1}{2}a_{\frac{1}{2}} + a_1x^{\frac{1}{2}} + 2a_2x^{3/2} + 3a_3x^{5/2})(\tfrac{1}{2}a_{\frac{1}{2}} + a_1x^{\frac{1}{2}} + 2a_2x^{3/2} + 3a_3x^{5/2})^2}{\tfrac{1}{4}a_{\frac{1}{2}} - 2a_2x^{3/2} - 6a_3x^{5/2}} \qquad [136]$$

At $x = 0$

$$r = \tfrac{1}{2} a_{\frac{1}{2}}^{\,2}$$

or

$$a_{\frac{1}{2}} = \sqrt{2r}$$

$$\begin{aligned}
&\alpha_2)\quad -\tfrac{1}{4} a_{\frac{1}{2}} + 2a_2 + 6a_3 = -k_1 \\
&\beta_1)\quad a_o = 0 \\
&\beta_2)\quad a_{\frac{1}{2}} + a_1 + a_2 + a_3 = 1 \\
&\beta_4)\quad \tfrac{1}{2} a_{\frac{1}{2}} + a_1 + 2a_2 + 3a_3 = 0
\end{aligned} \qquad [137]$$

The presence of the square root term prevents the use of the factorial analysis.

The solution as simultaneous equations in the a's produces

$$y = \sqrt{2r}\, R(x) + k_1\, K_1(x) + Q(x) \qquad [138]$$

with

$$R(x) = \sqrt{x} - \frac{x}{8}\,(3x^2 - 10x + 15) \qquad [139]$$

$$K_1(x) = -\tfrac{1}{2} x\,(x-1)^2 \qquad [140]$$

$$Q(x) = x\,(x^2 - 3x + 3) \qquad [141]$$

Graphs of $R(x)$, $K_1(x)$, and $Q(x)$ are shown in Figure 11.

Permissible Ranges of Parameters r and k_1

For the same reasons as the "quadratic" polynomial the conditions to be considered are the zero condition, the unity condition, the maximum or minimum condition, and the inflection point condition.

Zero Condition. The envelope conditions are

$$y = g(x;r,k_1) = 0 \qquad 0 \leq x \leq 1 \tag{142}$$

and

$$g' = 0 \tag{143}$$

The results are

$$\sqrt{2r} = \frac{2(x - 3)}{5x^{-1/2} - x^{-3/2} + x - 5} \tag{144}$$

and

$$k_1 = \frac{-36x^{-1/2} + 20x^{1/2} + 12x^{-3/2} + x^2 - 12x + 15}{2(x - 1)(5x^{-1/2} - x^{-3/2} + x - 5)} \tag{145}$$

The results are plotted in Figure 12.

Unity Condition. The envelope curve is specified by

$$g - 1 = 0 \qquad [146]$$

and

$$g' = 0 \qquad [147]$$

The results are

$$\sqrt{2r} = \frac{2(x - 1)^3}{-x^{1/2} + 5x^{3/2} + x^3 - 5x^2} \qquad [148]$$

and

$$k_1 = \frac{(x - 1)(4x^{-1/2} + 20x^{1/2} + x^2 - 10x - 15)}{2(-x^{1/2} + 5x^{3/2} + x^3 - 5x^2)} \qquad [149]$$

The envelope curve is plotted in Figure 12.

Maximum or Minimum Condition. The envelope curve is specified by

$$g' = 0 \qquad [150]$$

and

$$g'' = 0 \qquad [151]$$

Then

$$\sqrt{2r} = \frac{24(x - 1)^2 x^{3/2}}{1 - 12x + 15x^2 + 12x^{7/2} - 36x^{5/2} + 20x^{3/2}} \qquad [152]$$

and

$$k_1 = \frac{6(x - 1)(-1 + 5x + x^{5/2} - 5x^{3/2})}{1 - 12x + 15x^2 + 12x^{7/2} - 36x^{5/2} + 20x^{3/2}} \qquad [153]$$

The envelope curve is plotted in Figure 12.

Inflection Point Condition. The envelope conditions are

$$g'' = 0 \qquad [154]$$

and

$$g''' = 0 \qquad [155]$$

The results are

$$\sqrt{2r} = \frac{16x^{5/2}}{8x^{5/2} - 5x + 2} \qquad [156]$$

and

$$k_1 = \frac{2(2x^{5/2} - 5x + 3)}{8x^{5/2} - 5x + 2} . \qquad [157]$$

The envelope curves are plotted in Figures 12 and 13.

POINTED ENDS

The ordinary polynomial is utilized

$$y = \sum_{n=0}^{n=n} a_n x^n \qquad [158]$$

The adjustable parameters α_i are

α_1) s = slope at $x = 0$, $s > 0$

$$s = \left(\frac{dy}{dx}\right)_{x=0} \qquad [159]$$

α_2) k_1 = curvature at $x = 1$, $k_1 \geq 0$

$$k_1 = -\left(\frac{d^2y}{dx^2}\right)_{x=1} \qquad [160]$$

The sign of the equation for k_1 is chosen so as to require k_1 to be positive for desirable shapes.

The boundary conditions β_j are

$$\begin{aligned} &\beta_1) \quad x = 0 \ , \ y = 0 \\ &\beta_2) \quad x = 1 \ , \ y = 1 \\ &\beta_3) \quad x = 1 \ , \ \frac{dy}{dx} = 0 \end{aligned} \qquad [161]$$

Since there are five conditions in all, $n = 4$.

Substitution of α_i and β_j into the polynomial produces

$$\begin{aligned} &\alpha_1) \quad a_1 = s \\ &\alpha_2) \quad 2a_2 + 6a_3 + 12a_4 = -k_1 \\ &\beta_1) \quad a_o = 0 \end{aligned} \qquad [162]$$

$\beta_2)\quad a_o + a_1 + a_2 + a_3 + a_4 = 1$

$\beta_3)\quad a_1 + 2a_2 + 3a_3 + 4a_4 = 0$

y is a linear function of s and k_1 or

$$y(x) = s\ S(x) + k_1\ K_1(x) + Q(x) \qquad [163]$$

The relations for α_i and β_j correspond to

$$\begin{aligned}
&\alpha_1)\quad y'(0) = s\\
&\alpha_2)\quad y''(1) = -k_1\\
&\beta_1)\quad y(0) = 0\\
&\beta_2)\quad y(1) = 1\\
&\beta_3)\quad y'(1) = 0
\end{aligned} \qquad [164]$$

It is evident that

$$\begin{aligned}
&\alpha_1)\quad S'(0) = 1\ ;\ K_1'(0) = Q'(0) = 0\\
&\alpha_2)\quad K_1''(1) = -1\ ;\ S''(1) = Q''(1) = 0\\
&\beta_1)\quad S(0) = K(0) = Q(0) = 0\\
&\beta_2)\quad Q(1) = 1\ ;\ S(1) = K_1(1) = 0\\
&\beta_3)\quad S'(1) = K_1'(1) = Q'(1) = 0
\end{aligned} \qquad [165]$$

Since $S(0) = S(1) = S'(1) = S''(1) = 0$, $S(x) = \alpha x(x-1)^3$. Since $S'(0) = 1$, $\alpha = -1$ and

$$S(x) = -x(x-1)^3 \qquad [166]$$

Since $K_1(0) = K_1'(0) = K_1(1) = K_1'(1) = 0$, $K_1(x) = \beta x^2(x - 1)^2$. Since $K_1''(1) = -1$, $\beta = \frac{1}{2}$ and

$$K_1(x) = -\tfrac{1}{2}x^2(x - 1)^2 \qquad [167]$$

Since $Q'(0) = Q'(1) = Q''(1) = 0$, $Q'(x) = \gamma x(x - 1)^2$. Since $Q(0) = 0$ and $Q(1) = 1$, $\gamma = 7$ and then

$$Q(x) = x^2(3x^2 - 8x + 6) \qquad [168]$$

In summary for pointed ends

$$y = s\ S(x) + k_1\ K_1(x) + Q(x) \qquad [163]$$

$$S(x) = -\ x(x - 1)^3 \qquad [166]$$

$$K_1(x) = -\tfrac{1}{2}x^2(x - 1)^2 \qquad [167]$$

$$Q(x) = x^2(3x^2 - 8x + 6) \qquad [168]$$

The polynomials are plotted in Figure 14.

Permissible Ranges of Parameters s and k_1

Four conditions are to be considered: the zero condition, the unity condition, the maximum or minimum condition, and the inflection point condition.

Zero Condition. The envelope curve is specified by

$$g = 0 \qquad [169]$$

and

$$g' = 0 \qquad [170]$$

The results are

$$s = \frac{2x^2(x - 2)}{(x - 1)^3} \qquad [171]$$

and

$$k_1 = \frac{2(x^2 - 4x + 6)}{(x - 1)^2} \qquad [172]$$

The envelope curve is plotted in Figure 15.

Unity Condition. The envelope curve is specified by

$$g - 1 = 0 \qquad [173]$$

and

$$g' = 0 \qquad [174]$$

The results are

$$s = 2(1 + \frac{1}{x}) \qquad [175]$$

and

$$k_1 = 2(1 - \frac{1}{x})^2 \qquad [176]$$

The envelope curve is plotted in Figure 15.

Maximum or Minimum Condition. The envelope curve is specified by

$$g' = 0 \qquad [177]$$

and

$$g'' = 0 \qquad [178]$$

The results are

$$s = \frac{12x^2}{6x^2 - 4x + 1} \qquad [179]$$

and

$$k_1 = \frac{12(x - 1)^2}{6x^2 - 4x + 1} \qquad [180]$$

The envelope curve is plotted in Figure 15.

Inflection Point Condition. The envelope curve is specified by

$$g'' = 0 \tag{181}$$

and

$$g''' = 0 \tag{182}$$

The results are

$$s = \frac{4(3x^2 - 3x + 1)}{6x^2 - 8x + 3} \tag{183}$$

and

$$k_1 = \frac{12(x - 1)^2}{6x^2 - 8x + 3} \tag{184}$$

For $x = 0$ the boundary line given by $g'' = 0$ is

$$6s + k_1 - 12 = 0 \ . \tag{185}$$

The envelope curves are shown in Figures 15 and 16.

CUSPED ENDS

The ordinary polynomial is utilized

$$y = \sum_{n=0}^{n=n} a_n x^n \qquad [186]$$

The adjustable parameters α_i are

α_1) k_o = curvature at $x = 0$, $k_o > 0$

$$k_o = \left(\frac{d^2 y}{dx^2} \right)_{x=0} \qquad [187]$$

α_2) k_1 = curvature at $x = 1$, $k_1 > 0$

$$k_1 = - \left(\frac{d^2 y}{dx^2} \right)_{x=1} \qquad [188]$$

The boundary conditions β_j are

$$\begin{aligned} &\beta_1) \quad x = 0 \ , \ y = 0 \\ &\beta_2) \quad x = 0 \ , \ \frac{dy}{dx} = 0 \\ &\beta_3) \quad x = 1 \ , \ y = 1 \\ &\beta_4) \quad x = 1 \ , \ \frac{dy}{dx} = 0 \end{aligned} \qquad [189]$$

Since there are six conditions in all, $n = 5$.

Substitution of α_i and β_j into the polynomial produces

$$\begin{aligned} &\alpha_1) \quad 2a_2 = k_o \\ &\alpha_2) \quad 2a_2 + 6a_3 + 12a_4 + 20a_5 = - k_1 \\ &\beta_1) \quad a_o = 0 \end{aligned} \qquad [190]$$

β_2) $a_1 = 0$

β_3) $a_o + a_1 + a_2 + a_3 + a_4 + a_5 = 1$

β_4) $a_1 + 2a_2 + 3a_3 + 4a_4 + 5a_5 = 0$

y is a linear function of k_o and k_1 in

$$y(x) = k_o K_o(x) + k_1 K_1(x) + Q(x) \qquad [191]$$

The relations for α_i and β_j correspond to

$$\begin{aligned}
&\alpha_1) \quad y''(0) = k_o \\
&\alpha_2) \quad y''(1) = - k_1 \\
&\beta_1) \quad y(0) = 0 \\
&\beta_2) \quad y'(0) = 0 \\
&\beta_3) \quad y(1) = 1 \\
&\beta_4) \quad y'(1) = 0
\end{aligned} \qquad [192]$$

It is evident that

$$\begin{aligned}
&\alpha_1) \quad K_o''(0) = 1 \; ; \; K_1''(0) = Q''(0) = 0 \\
&\alpha_2) \quad K_1''(1) = - 1 \; ; \; K_o''(1) = Q''(1) = 0 \\
&\beta_1) \quad K_o(0) = K_1(0) = Q(0) = 0 \\
&\beta_2) \quad K_o'(0) = K_1'(0) = Q'(0) = 0 \\
&\beta_3) \quad Q(1) = 1 \; ; \; K_o(1) = K_1(1) = 0 \\
&\beta_4) \quad K_o'(1) = K_1'(1) = Q'(1) = 0
\end{aligned} \qquad [193]$$

Since $K_o(0) = K_o'(0) = K_o(1) = K_o'(1) = K_o''(1) = 0$,

$$K_o(x) = \alpha x^2 (x - 1)^3 \qquad [194]$$

Since $K_o''(0) = 1$, $\alpha = \frac{1}{2}$ and

$$K_o(x) = \tfrac{1}{2} x^2 (x - 1)^3 \qquad [195]$$

Since $K_1(0) = K_1'(0) = K_1''(0) = K_1(1) = K_1'(1) = 0$,

$$K_1(x) = \beta x^3 (x - 1)^2 \qquad [196]$$

Since $K_1''(1) = - 1$, $\beta = - \frac{1}{2}$ and

$$K_1(x) = - \tfrac{1}{2} x^3 (x - 1)^2 \qquad [197]$$

Since $Q'(0) = Q''(0) = Q'(1) = Q''(1) = 0$,

$$Q'(x) = \gamma x^2 (x - 1)^2 \qquad [198]$$

Since $Q(0) = 0$ and $Q(1) = 1$, $\gamma = 30$ and

$$Q(x) = x^3 (6x^2 - 15x + 10) \ . \qquad [199]$$

In summary

$$y = k_o K_o(x) + k_1 K_1(x) + Q(x) \qquad [191]$$

$$K_o(x) = \tfrac{1}{2}x^2(x - 1)^3 \qquad [195]$$

$$K_1(x) = -\tfrac{1}{2}x^3(x - 1)^2 \qquad [197]$$

$$Q(x) = x^3(6x^2 - 15x + 10) \qquad [199]$$

These are plotted in Figure 17.

Permissible Ranges of Parameters k_o and k_1

Four conditions are to be investigated: the zero condition, the unity condition, the maximum or minimum condition, and the inflection point condition.

Zero Condition. The envelope curve is specified by

$$g = 0 \qquad [200]$$

and

$$g' = 0 \qquad [201]$$

The results are

$$k_o = -\frac{2x^2(3x - 5)}{(x - 1)^3} \qquad [202]$$

and

$$k_1 = \frac{2(3x^2 - 10x + 10)}{(x - 1)^2} \quad [203]$$

The envelope curve is plotted in Figure 18.

Unity Condition. The envelope curve is specified by

$$g - 1 = 0 \quad [204]$$

and

$$g' = 0 \quad [205]$$

The results are

$$k_o = \frac{2(3x^2 + 4x + 3)}{x^2} \quad [206]$$

and

$$k_1 = \frac{2(x - 1)^2(3x + 2)}{x^2} \quad [207]$$

The envelope curve is shown in Figure 18.

Maximum or Minimum Condition. The envelope curve is specified by

$$g' = 0 \qquad [208]$$

and

$$g'' = 0 \qquad [209]$$

The results are

$$k_c = \frac{60x^2}{10x^2 - 10x + 3} \qquad [210]$$

and

$$k_1 = \frac{60(x - 1)^2}{10x^2 - 10x + 3} \qquad [211]$$

The envelope curve is plotted in Figure 18.

Inflection Point Condition. The envelope curve is specified by

$$g'' = 0 \qquad [212]$$

and

$$g''' = 0 \qquad [213]$$

The results are

$$k_o = \frac{20x^2(6x^2 - 8x + 3)}{-20x^4 + 40x^3 - 28x^2 + 8x - 1} \qquad [214]$$

and

$$k_1 = \frac{20(x - 1)(-6x^3 + 10x^2 - 5x + 1)}{-20x^4 + 40x^3 - 28x^2 + 8x - 1} \qquad [215]$$

The envelope curve is shown in Figure 19.

"CUBIC" POLYNOMIAL REPRESENTATION FOR FLAT FACED NOSES

To achieve "hydrodynamic continuity" at the edge of the flat faced nose, zero curvature is required in addition to infinite slope. A "cubic" polynomial achieves this, namely,

$$y^3 = F(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots \qquad [216]$$

Since

$$\frac{dx}{dy} = \frac{3y^2}{a_1 + 2a_2 x + 3a_3 x^2 + \ldots} \qquad [217]$$

and $y = 0$ at $x = 0$

$$\left(\frac{dx}{dy}\right)_{x=0} = 0 \qquad \text{(infinite slope)} \qquad [218]$$

Also since

$$\frac{d^2x}{dy^2} = \frac{6y - (2a_2 + 6a_3 x + \ldots)\left(\frac{dx}{dy}\right)^2}{a_1 + 2a_2 x + 3a_3 x^2 + \ldots} \qquad [219]$$

and $a_1 \neq 0$

$$\left(\frac{d^2x}{dy^2}\right)_{x=0} = 0 \qquad \text{(zero curvature)} \qquad [220]$$

There is one adjustable parameter a_1

$\alpha_1)$ k_1 = curvature at $x = 1$, $k_1 \quad 0$

$$k_1 = -\left(\frac{d^2y}{dx^2}\right)_{x=1} \qquad [221]$$

The boundary conditions β_j are

$$\begin{aligned} &\beta_1) \quad x = 0 \, , \; y = 0 \\ &\beta_2) \quad x = 1 \, , \; y = 1 \\ &\beta_3) \quad x = 1 \, , \; \frac{dy}{dx} = 0 \end{aligned} \qquad [222]$$

Since there are four conditions in all, the polynomial in x is cubic.

The α_1 and β_j are substituted into the polynomial to give

$$\begin{aligned} &\alpha_1) \quad 2a_2 + 6a_3 = -3k_1 \\ &\beta_1) \quad a_o = 0 \\ &\beta_2) \quad a_o + a_1 + a_2 + a_3 = 1 \\ &\beta_3) \quad a_1 + 2a_2 + 3a_3 = 0 \end{aligned} \qquad [223]$$

The form of the "cubic" polynomial is then

$$y^3 = k_1 K_1(x) + Q(x) \qquad [224]$$

The relations for α_1 and β_j correspond to

$$\alpha_1) \quad \frac{d^2y^3}{dx^2}(1) = -3k_1$$

$$\beta_1)\quad y^3(0) = 0$$
$$\beta_2)\quad y^3(1) = 1 \qquad [225]$$
$$\beta_3)\quad \frac{d}{dx}\, y^3(1) = 0$$

Also it is evident that

$$\alpha_1)\quad K_1''(1) = -3\ ,\ Q''(1) = 0$$
$$\beta_1)\quad K_1(0) = Q(0) = 0 \qquad [226]$$
$$\beta_2)\quad Q(1) = 1\ ,\ K_1(1) = 0$$
$$\beta_3)\quad K_1'(1) = Q'(1) = 0$$

Evaluation of $K_1(x)$

Since $K_1(0) = K_1'(1) = K_1(1) = 0$

$$K_1(x) = \alpha x(x-1)^2 \qquad [227]$$

Since $K_1''(1) = -3$, $\alpha = -\frac{3}{2}$. Then

$$K_1(x) = -\frac{3}{2}\, x(x-1)^2 \qquad [228]$$

Evaluation of $Q(x)$

Since $Q''(1) = Q'(1) = 0$

$$Q' = \beta(x-1)^2 \qquad [229]$$

and

$$Q = \frac{\beta(x - 1)^3}{3} + C \qquad [230]$$

Since $Q(0) = 0$ and $Q(1) = 1$, $\beta = 3$ and $C = 1$

Then

$$Q = (x - 1)^3 + 1 \qquad [231]$$

The independent polynomials are plotted in Figure 20.

Permissible Ranges of k_1

Zero Condition.

$$y^3 = h(x,k) \geq 0 \quad , \quad 0 \leq x \leq 1 \qquad [232]$$

The boundary condition $h = 0$ gives

$$k_1 = \frac{2[(x - 1)^3 + 1]}{3x(x - 1)^2} \qquad [233]$$

For $x = 0$, k_1 is indeterminate and by L'Hôpital's Rule

$$k_1 = 2 \qquad [234]$$

From the plot of k_1 against x in Figure 21 it is evident that the permissible range is

$$k_1 \leq 2 \qquad [235]$$

Unity Condition.

$$h(x,k) \leq 1 \quad , \quad 0 \leq x \leq 1 \qquad [236]$$

The boundary condition is then

$$h = 1 \qquad [237]$$

Then

$$k_1 = \frac{2}{3}\left(1 - \frac{1}{x} \right) \qquad [238]$$

From the plot of k_1 against x in Figure 21 it is evident that the permissible range is

$$k_1 \geq 0 \qquad [239]$$

Maximum or Minimum Condition.

$$\frac{dy}{dx} > 0 \quad , \quad 0 \leq x < 1 \qquad [240]$$

except at $x = 1$. Then

$$\frac{h'}{x - 1} < 0 \qquad [241]$$

The boundary condition is then

$$\frac{h'}{x - 1} = 0 \qquad [242]$$

which produces

$$k_1 = \frac{2(x - 1)}{3x - 1} \qquad [243]$$

From the plot of k_1 against x in Figure 21 it is evident that the permissible range is

$$0 < k_1 < 2 \qquad [244]$$

Inflection Point Condition.

$$\frac{d^2y}{dx^2} < 0 \quad , \quad 0 < x < 1 \qquad [245]$$

or

$$\frac{3hh'' - 2h'^2}{9h^{5/3}} < 0 \qquad [246]$$

For $h > 0$ (positive condition)

$$3hh'' - 2h'^2 = 0 \qquad [247]$$

which reduces to

$$(x - 1)^2 k_1^2 + 2(3x - 2)k_1 = 4(x - 1) \qquad [248]$$

or

$$k_1 = \frac{2 - 3x + x\sqrt{4x - 3}}{(x - 1)^2} \qquad [249]$$

The permissible range is $k_1 \geq 0$ and with the zero condition of $k_1 \leq 2$ the combined permissible range is

$$0 \leq k_1 \leq 2 \qquad [250]$$

as shown in Figure 21.

LEAST SQUARES FIT

GENERAL

Known shapes may be approximately expressed in "quadratic" and "square root" polynomials by a least squares fit. If

$z = y^2$ for "quadratic" polynomials

or

$z = y$ for "square root" polynomials

then in general

$$z = d\ D(x) + k_1\ K_1(x) + Q(x) \qquad [251]$$

where d and $D(x)$ refer to the appropriate type of figure. For example for a rounded nose shape $d = r$ and $D(x) = R(x)$. Since the whole body is to be fitted, both fore and after bodies are to be considered which would mean determining four coefficients d_f, k_{1f}, d_a, and k_{1a} by a least squares fit. The subscript f refers to the forebody and a to the afterbody. However since k_{1f} and k_{1a} are related by

$$k_{1f} = \left(\frac{m}{1 - m}\right)^2 k_{1a} = \frac{1}{e}\ k_{1a} \qquad [252]$$

only three independent coefficients have to be determined.

In general a least squares fit requires that

$$\int (z - z_1)^2 dx \qquad \text{be minimized} \qquad [253]$$

where z_1 represents the body shape to be fitted. Differentiating with respect to the coefficients to be determined produces three simultaneous algebraic equations which are easily evaluated:

$$\left[\int_o^1 D_f{}^2 dx_f\right] d_f + \left[\int_o^1 D_f K_{1f} dx_f\right] k_1 = \int_o^1 D_f z_{1f} dx_f - \int_o^1 D_f Q_f dx_f \qquad [254]$$

$$\left[\int_o^1 D_a{}^2 dx_a\right] d_a + \left[e\int_o^1 D_a K_{1a} dx_a\right] k_1 = \int_o^1 D_a z_{1a} dx_a - \int_o^1 D_a Q_a dx_a \qquad [255]$$

$$\left[\int_o^1 D_f K_{1f} dx_f\right] d_f + \left[e\int_o^1 D_a K_{1a} dx_a\right] d_a + \left[\int_o^1 K_{1f}{}^2 dx_f + e^2\int_o^1 K_{1a}{}^2 dx_a\right] k_1$$
$$= \int_o^1 K_{1f} z_{1f} dx_f + e\int_o^1 K_{1a} z_{1a} dx_a - \int_o^1 K_{1f} Q_f dx_f - e\int_o^1 K_{1a} Q_a dx_a \qquad [256]$$

BODIES OF REVOLUTION

In the normalized coordinates of this paper all ellipsoids are transformed into spheres so that $r = k_1 = 1$ which is plotted in Figure 4 for bodies with rounded ends.

An ellipsoid-like body was developed by Munzner and Reichardt[12] which has an almost constant pressure distribution. The Reichardt body is expressed as

$$\left(\frac{X - a}{a}\right)^2 + \left(\frac{Y}{b}\right)^{2.4} = 1 \qquad [257]$$

where a and b are the semiaxes and in normalized coordinates as

$$(x - 1)^2 + y^{2.4} = 1 \qquad [258]$$

For a least squares fit

$$z_1 = [x(2 - x)]^{\frac{1}{1.2}} \qquad [259]$$

A least squares fit by rounded end "quadratic" polynomials gives

$$r = 1.344$$

and

$$k_1 = 1.085$$

which is plotted in Figure 4.

Other known bodies of revolution may be fitted and plotted in Figure 4 for comparison.

TWO-DIMENSIONAL SHAPES

In normalized coordinates all ellipses have been transformed into a circle

$$y^2 = 2x - x^2 \qquad [260]$$

and for least square fitting

$$z_1 = (2x - x^2)^{\frac{1}{2}} \qquad [261]$$

A least squares fit in "square root" polynomials for rounded ends gives

$$\sqrt{2r} = 1.425$$

and

$$k_1 = 0.926$$

which is plotted in Figure 13.

A long time favorite streamlined shape for propeller struts was the Navy Standard Strut[13] (NSS) which has a rounded nose and a pointed tail. A least squares fit gives for the nose

$$\sqrt{2r} = 1.610$$

and

$$k_1 = 0.713$$

and for the tail

$$s = 2.082$$

and

$$k_1 = 2.457$$

which are plotted in Figures 13 and 16.

The Navy Standard Strut was superseded by the EPH shape[13] which has a rounded tail. A least squares fit yields for the nose portion

$$\sqrt{2r} = 1.414$$

and

$$k_1 = 0.752$$

and for the tail portion

$$\sqrt{2r} = 0.389$$

and

$$k_1 = 1.124$$

which are plotted in Figure 13.

PRISMATIC COEFFICIENT

The fullness of a shape is given by the prismatic coefficient which is the ratio of the volume of the body to the volume of a prism having the maximum cross-section area and the length of the body. For a body of revolution the prism is represented by a cylinder of maximum diameter and length of the body.

The prismatic coefficient of the semibodies of this report is given by

$$C_p = \int_o^1 y^2 dx \qquad [262]$$

The prismatic coefficient of the whole body is given by

$$\bar{C}_p = m\, C_{p_f} + (1 - m)\, C_{f_a} \qquad [263]$$

where the subscripts f and a refer to the forebody and afterbody respectively.

The utility of the quadratic polynomial in computing the prismatic coefficient is evident from

$$C_p = d\int_o^1 D(x)\,dx + k_1\int_o^1 K_1(x)\,dx + \int_o^1 Q dx \qquad [264]$$

For the rounded end

$$C_p = \frac{r}{10} - \frac{k_1}{30} + \frac{3}{5} \qquad [265]$$

and the pointed end

$$C_p = \frac{s^2}{60} - \frac{k_1}{60} + \frac{1}{2} \qquad [266]$$

and the cusped end

$$C_p = \frac{k_o^2}{1120} - \frac{k_1}{168} + \frac{3}{8} \qquad [267]$$

Lines of constant C_p are plotted in Figures 4, 7, and 10.

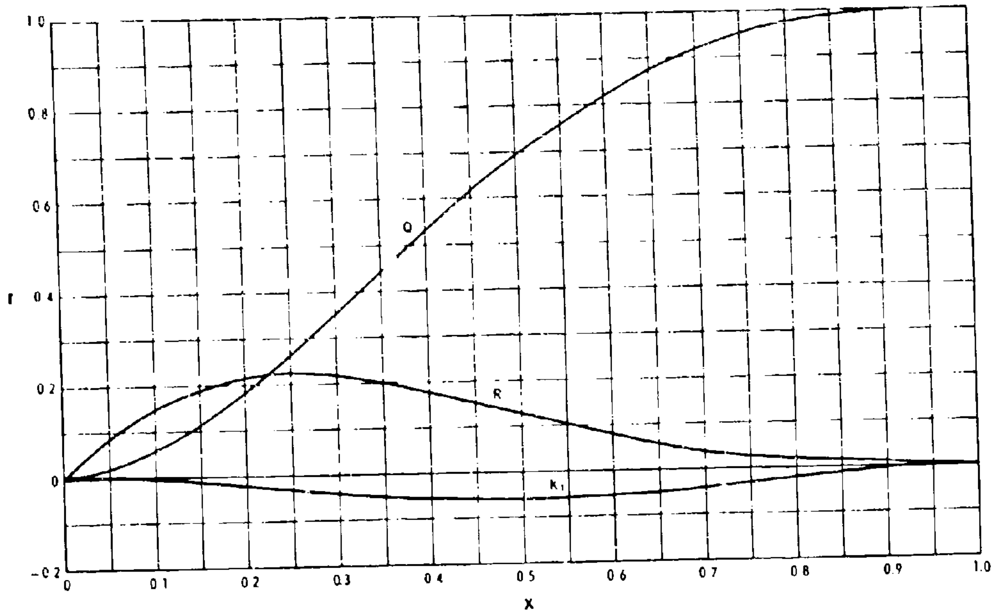

Figure 1 - "Quadratic" Polynomial: Rounded End - Variation of Independent Polynomials

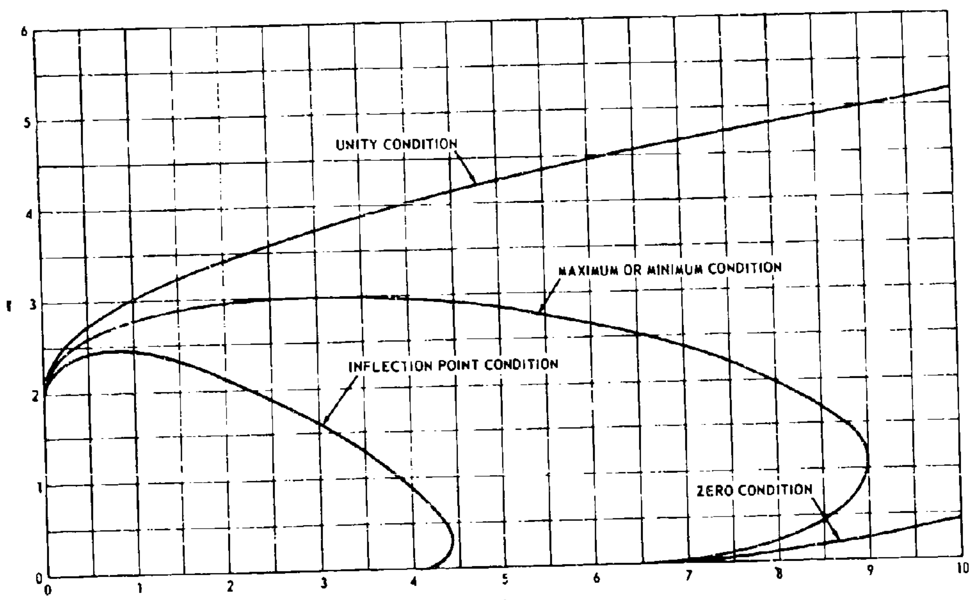

Figure 2 - "Quadratic" Polynomial: Rounded End - Permissible Range of Parameters r and k_1

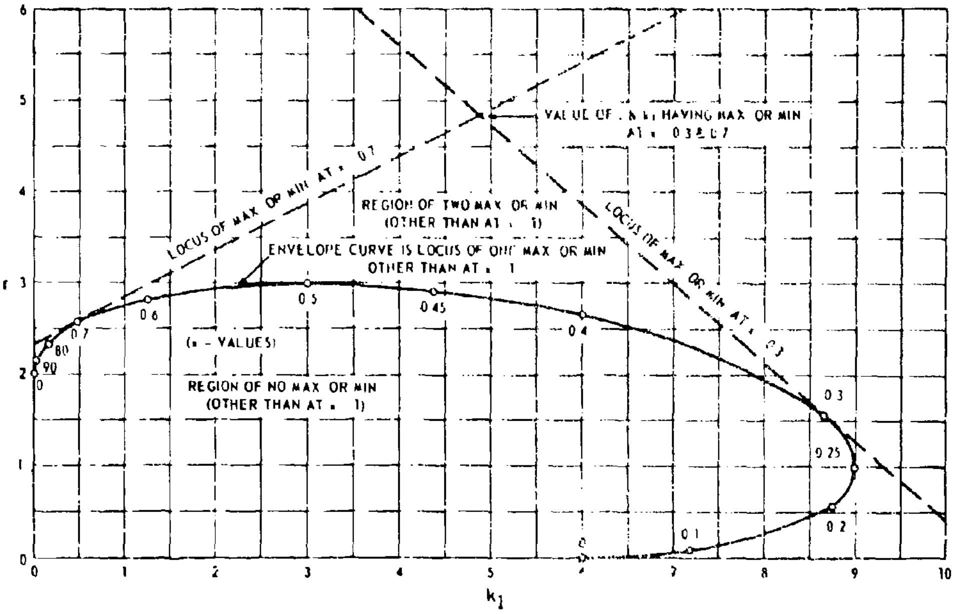

Figure 3 - "Quadratic" Polynomial: Rounded End - Delineation of Regions by Envelope Curve for Maximum and Minimum Condition

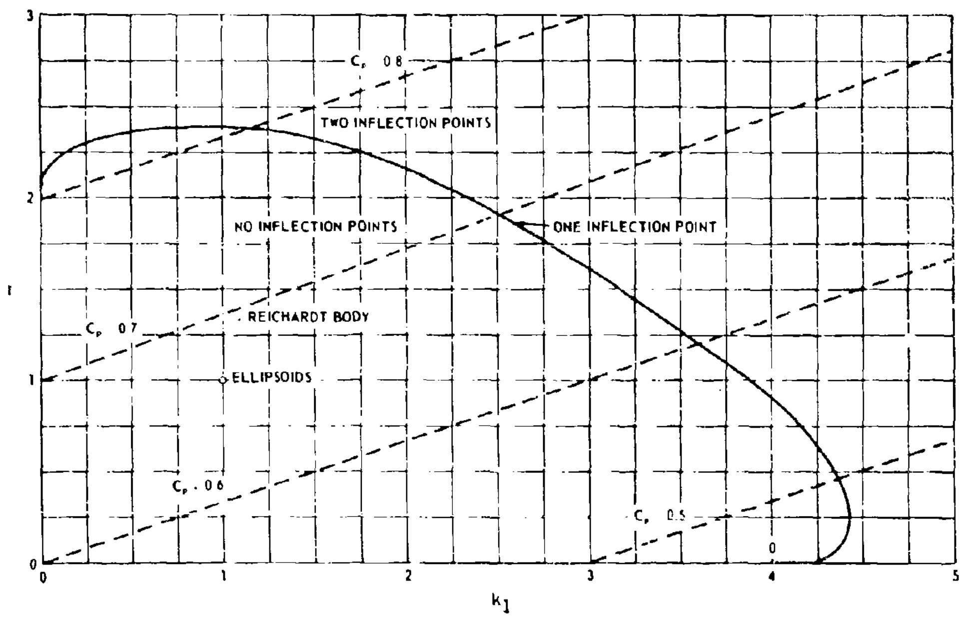

Figure 4 - "Quadratic" Polynomial: Rounded End - Inflection Point Condition

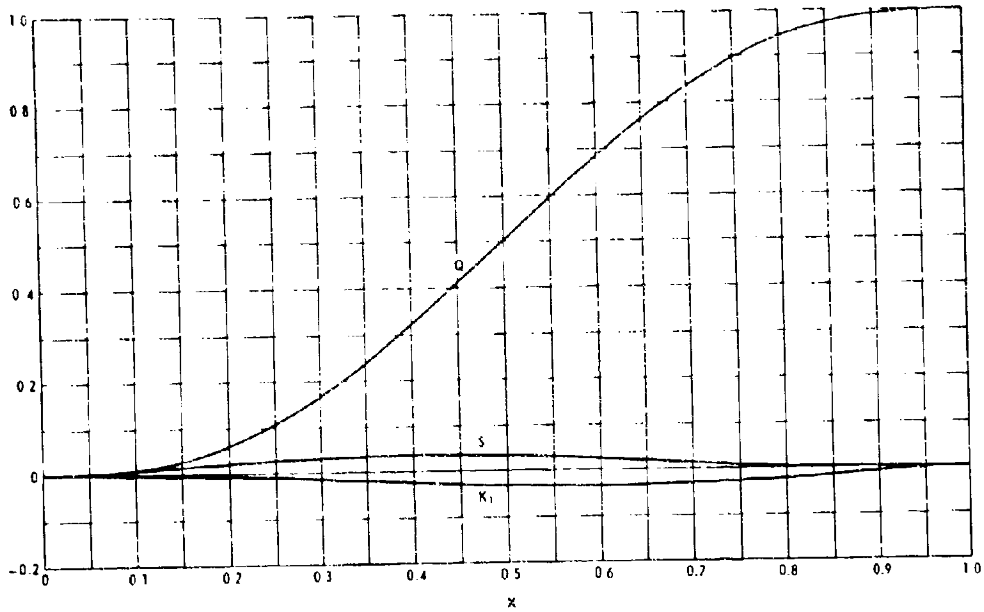

Figure 5 - "Quadratic" Polynomial: Pointed End - Variation of Independent Polynomials

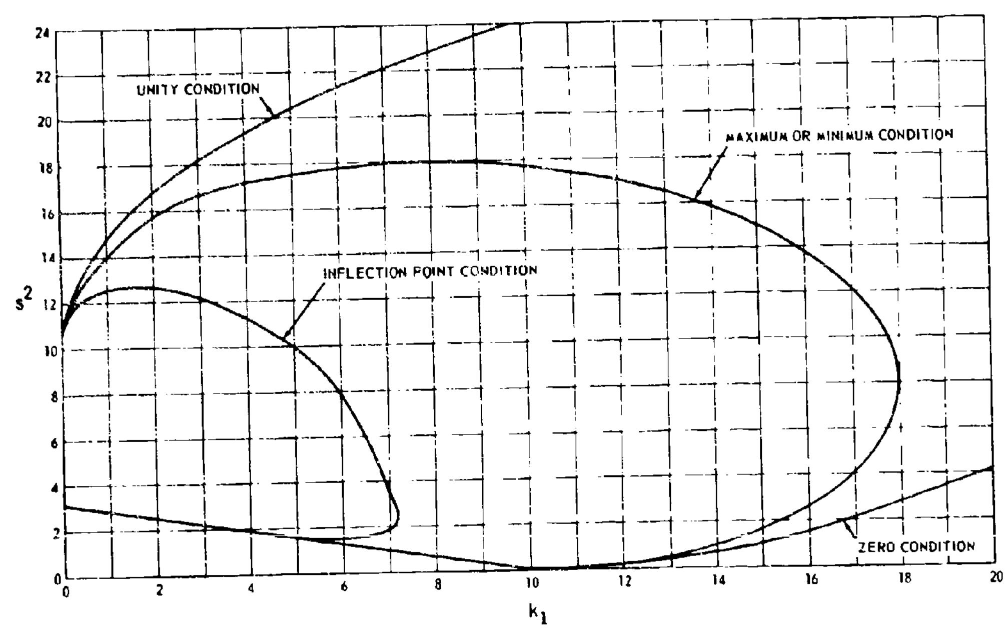

Figure 6 - "Quadratic" Polynomial: Pointed End - Permissible Range of Parameters s^2 and k_1

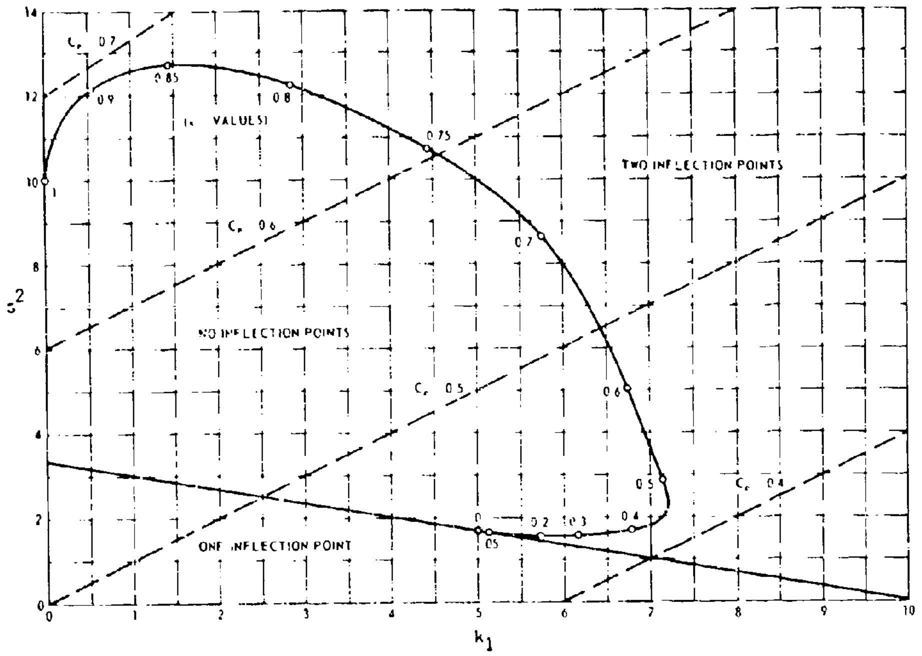

Figure 7 - "Quadratic" Polynomial: Pointed End - Inflection Point Condition

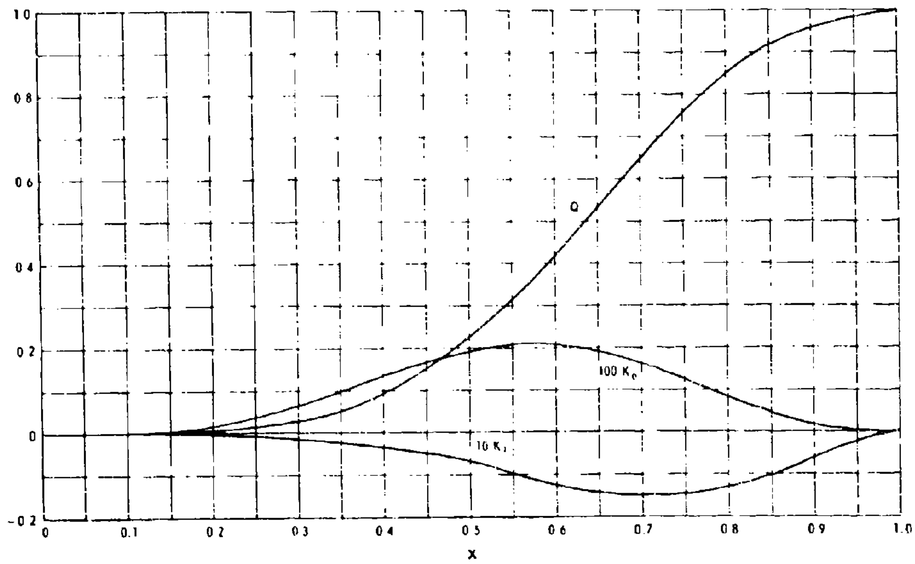

Figure 8 - "Quadratic" Polynomial: Cusped End - Variation of Independent Polynomials

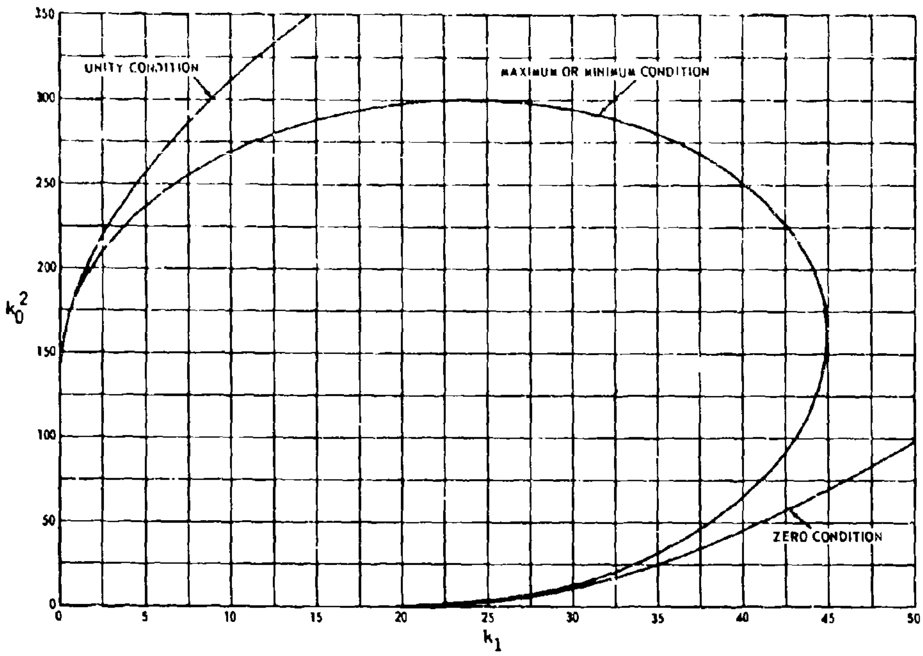

Figure 9 - "Quadratic" Polynomial: Cusped End - Permissible Range of Parameters k_o^2 and k_1

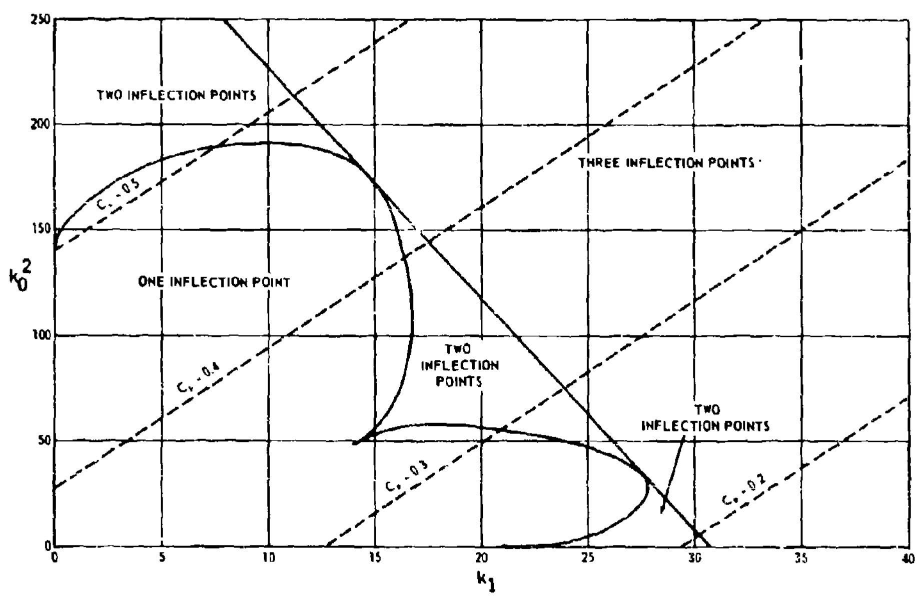

Figure 10 - "Quadratic" Polynomial: Cusped End - Inflection Point Condition

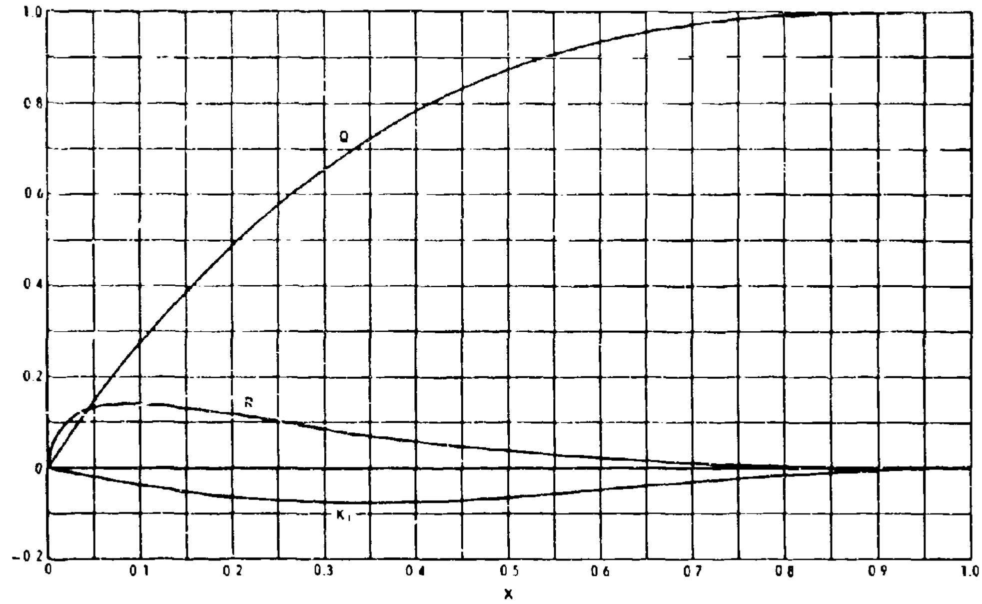

Figure 11 - "Square Root" Polynomial: Rounded End - Variation of Independent Polynomials

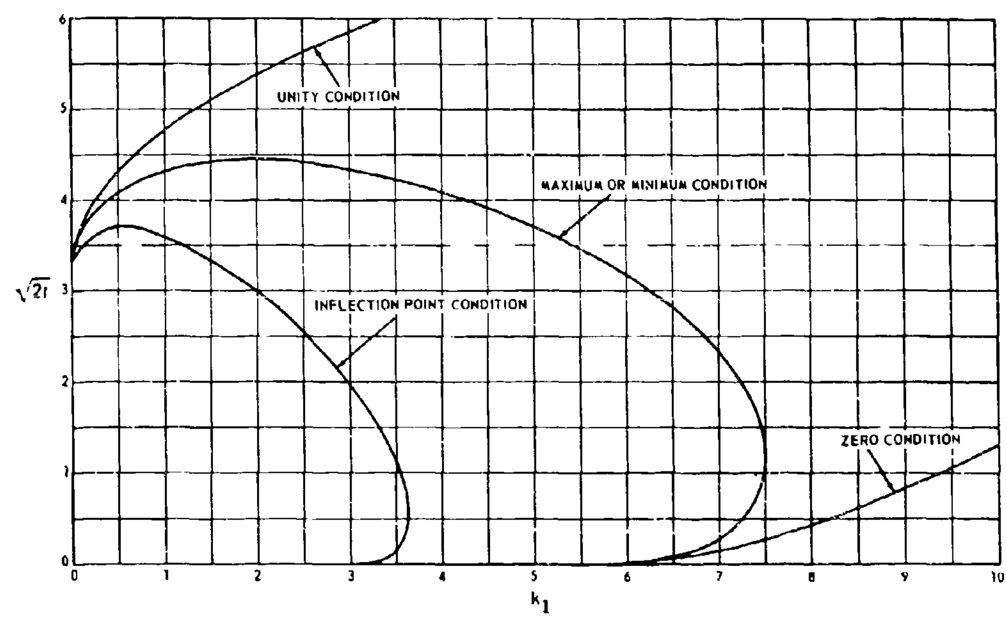

Figure 12 - "Square Root" Polynomial: Rounded End - Permissible Range of Parameters $2r$ and k_1

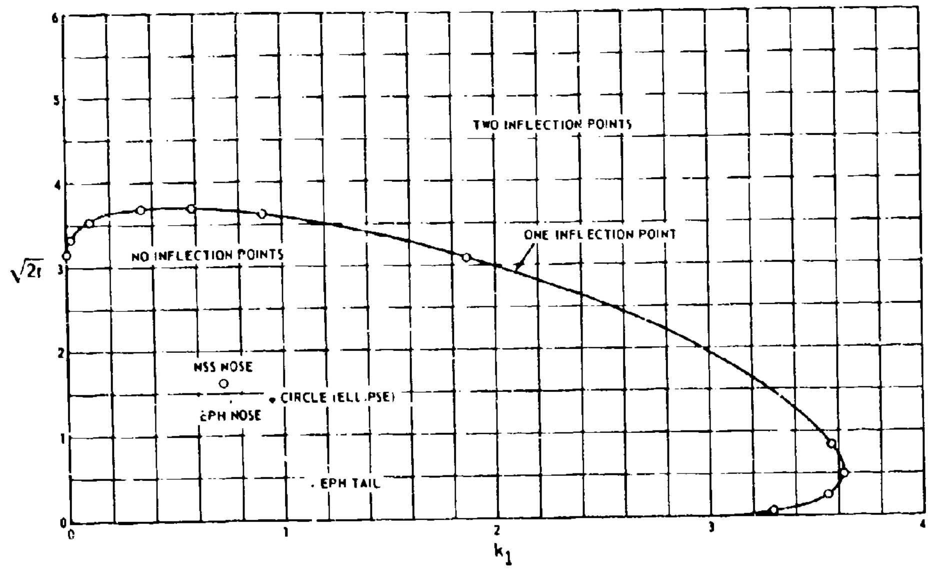

Figure 13 - "Square Root" Polynomial: Rounded End - Inflection Point Condition

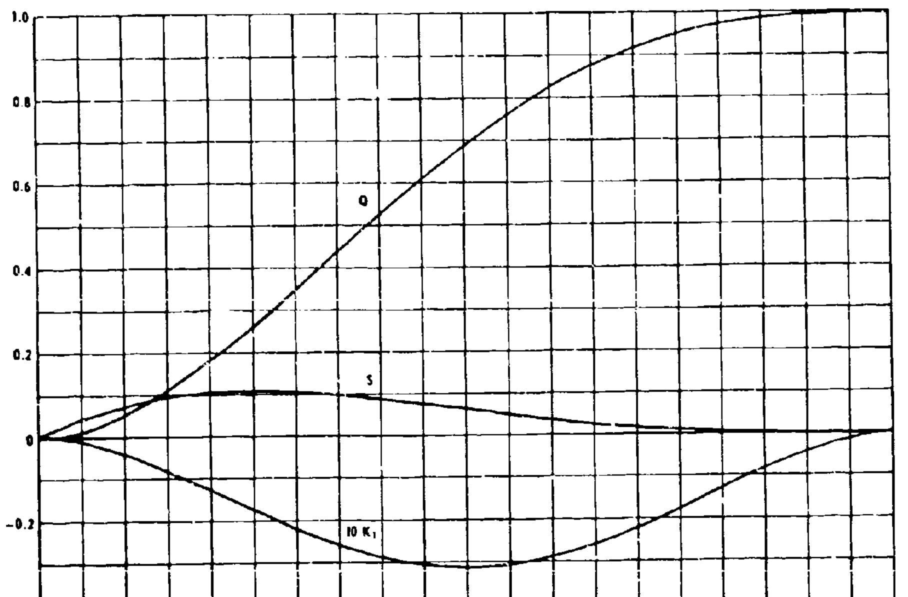

Figure 14 - Ordinary Polynomial: Pointed End - Variation of Independent Polynomials

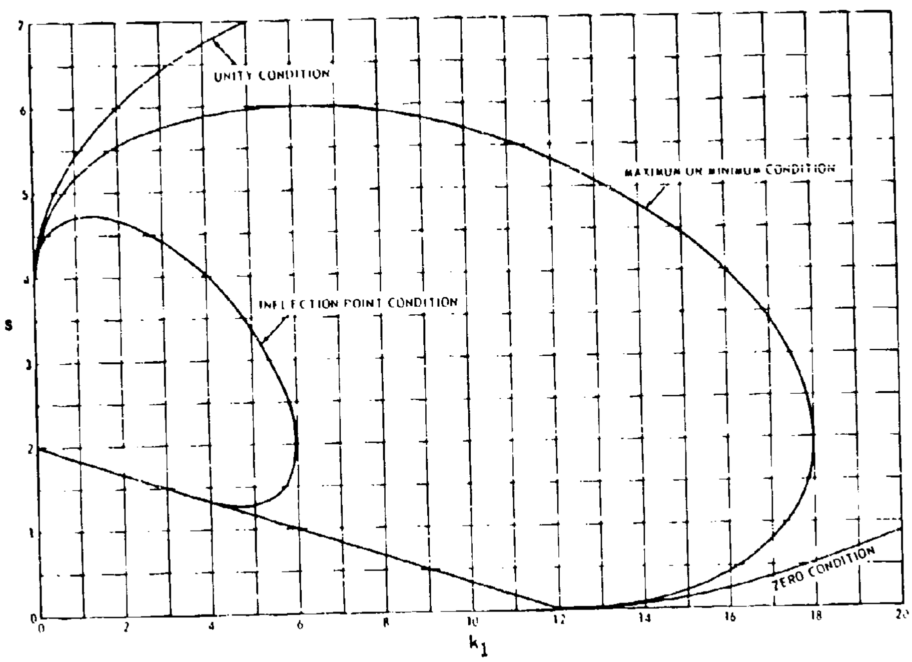

Figure 15 - Ordinary Polynomial: Pointed End - Permissible Range of Parameters s and k_1

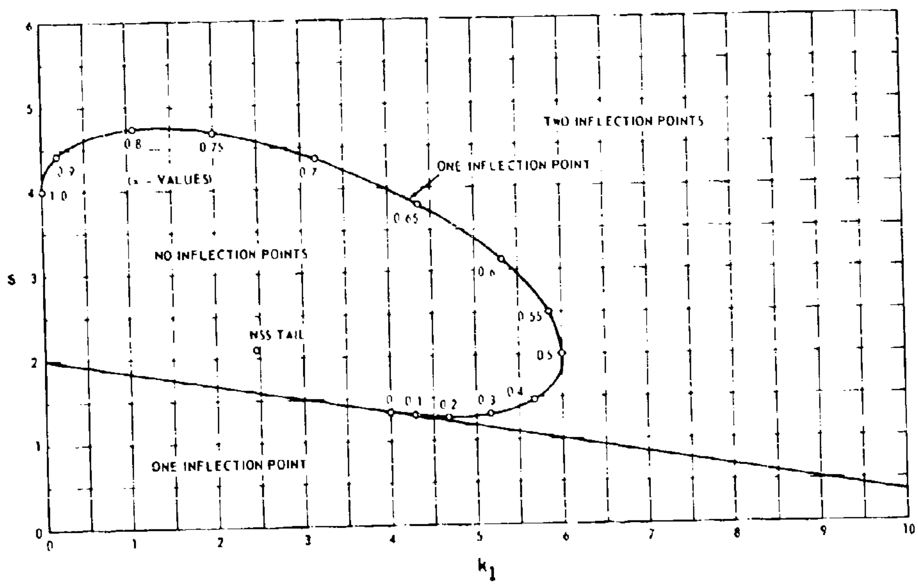

Figure 16 - Ordinary Polynomial: Pointed End - Inflection Point Condition

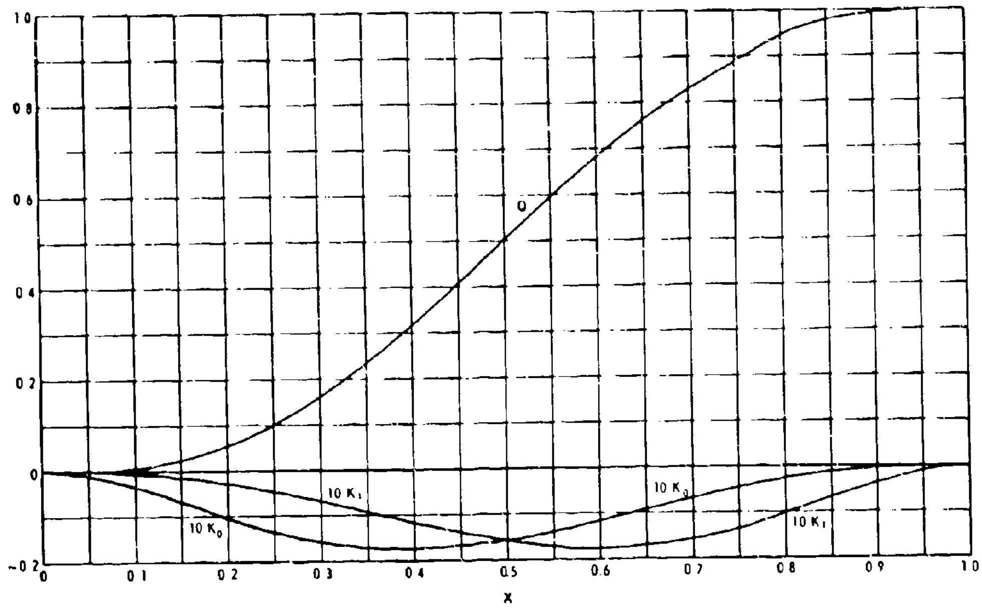

Figure 17 - Ordinary Polynomial: Cusped End - Variation of Independent Polynomials

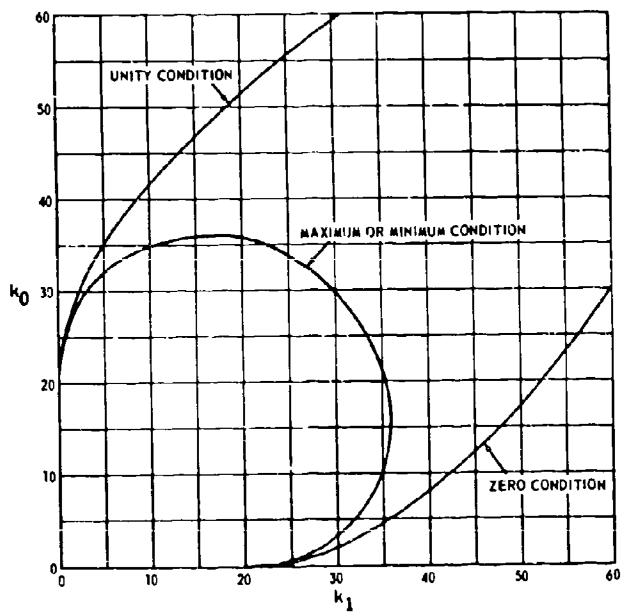

Figure 18 - Ordinary Polynomial: Cusped End - Permissible Range of Parameters k_o and k_1

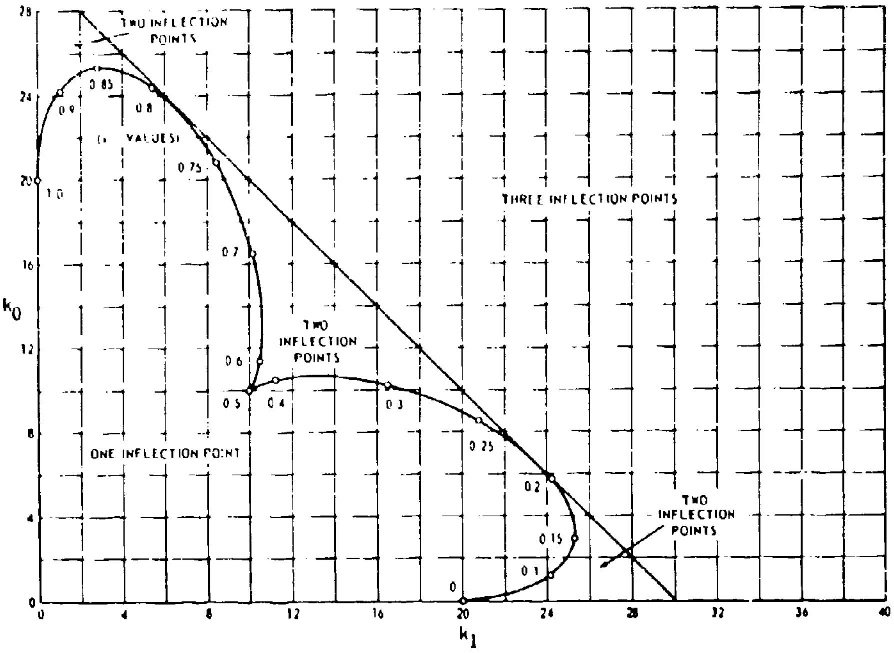

Figure 19 - Ordinary Polynomial: Cusped End - Inflection Point Condition

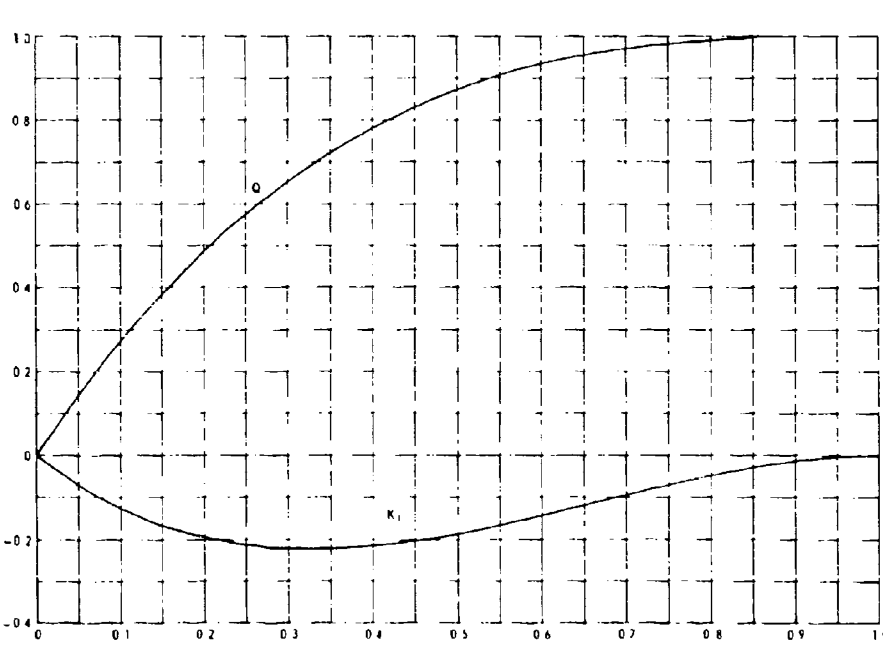

Figure 20 - "Cubic" Polynomial: Flat Face - Variation of Independent Parameters

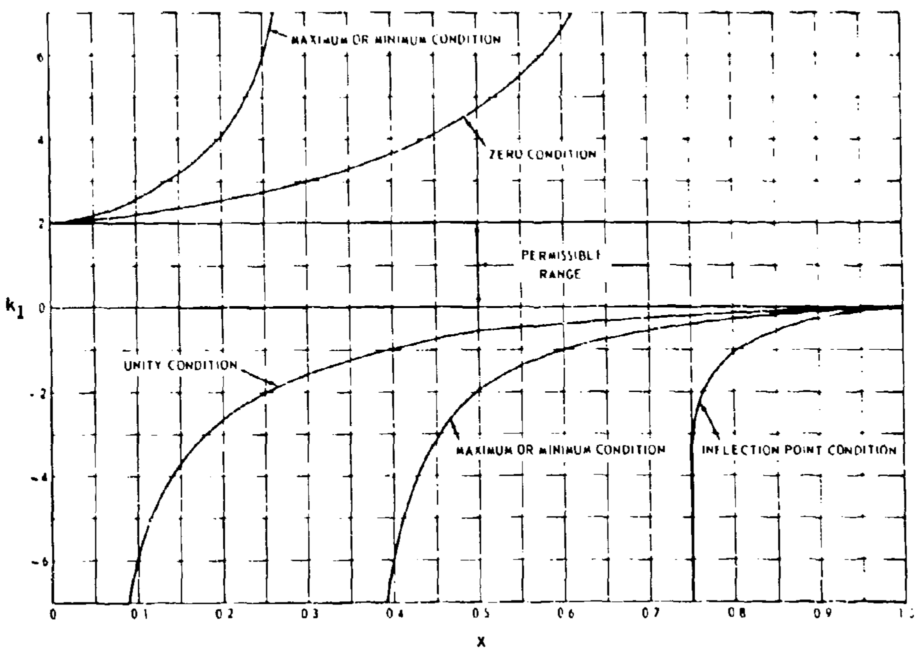

Figure 21 - "Cubic" Polynomial: Flat Face - Permissible Range of Parameter k_1

REFERENCES

1. Robertson, J.M., "Hydrodynamics in Theory and Application," Prentice-Hall Inc., Englewood Cliffs, N.J., 1965.

2. Smith, A.M.O. and Pierce, J., "Exact Solutions of the Neumann Problem. Calculation of Non-Circulatory Plane and Axially Symmetric Flows About or Within Arbitrary Boundaries," Douglas Aircraft Co. Report ES 26988, Apr 1958.

3. Abbott, I.H. et al., "Summary of Airfoil Data," NACA Report 824, 1945.

4. Young, A.D. and Young, E., "A Family of Streamline Bodies of Revolution Suitable for High-Speed and Low-Drag Requirements," Aeronautical Research Council (Great Britain) R & M 2204, Aug 1945.

5. Landweber, L. and Gertler, M., "Mathematical Formulation of Bodies of Revolution," David Taylor Model Basin Report 719, 1950.

6. Taylor, D.W., "Calculation for Ships' Forms and the Light Thrown by Model Experiments upon Resistance, Propulsion and Rolling of Ships," Transactions of the International Engineering Congress, 1915 at San Francisco.

7. Lyon, H.M., "The Effect of Turbulence on the Drag of Airship Models," Aeronautical Research Committee (Great Britain) R & M 1511, Aug 1932.

8. Stack, J. and von Doenhoff, A.E., "Tests of 16 Related Airfoils at High Speeds," NACA Report 492, 1934.

9. Jacobs, E.N. et al., "The Characteristics of 78 Related Airfoil Sections from Tests in the Variable-Density Wind Tunnel," NACA Report 460, 1933.

10. Kwik, K.H., "Grundlagen zur Darstellang der Profilform von Schiffsrudern," Schiff und Hafen, Vol. 14, No. 10, Oct 1962.

11. Kwik, K.H., "Anwendungsbeispiele zur Darstellung der Profilform von Schiffsrudern," Schiff und Hafen, Vol. 16, No. 10, Dec 1964.

12. Munzner, H. and Reichardt, H., "Axially Symmetric Source-Sink Bodies with Predominantly Constant Pressure Distributions," U.S. Naval Ordnance Test Station (Inyokern, Calif.), Tech. Memo. 808-43, Oct 1950 (translated from German Report ZWB/UM/6616).

13. Mandel, P., "Some Hydrodynamic Aspects of Appendage Design," Transactions of Society of Naval Architects and Marine Engineers, Vol. 61, 1953.

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13. ABSTRACT

Streamlined shapes, both two-dimensional and axisymmetric, are analytically expressed by appropriate polynomials in terms of independent parameters. Permissible ranges of the independent parameters are examined with respect to selected geometrical constraints.

14. KEY WORDS	LINK A		LINK B		LINK C	
	ROLE	WT	ROLE	WT	ROLE	WT
Body geometry						
Bodies of revolution						
Airfoil sections						