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PART I: STRONG CONSISTENCY

by
RICHARD E. BARLOW
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ABSTRACT

Assume F and G are distributions on $[0,\infty)$ with densities f and g, respectively. If $G^{-1}F$ is convex on the support of F (an interval), then

$$r(x) = \frac{d}{dx} G^{-1}F(x) = \frac{f(x)}{g[G^{-1}F(x)]}$$

(the generalized failure rate function) is nondecreasing in $x \in [0,\infty)$. We assume G known, r nondecreasing and consider the problem of estimating r. A general class of isotonized fixed and random "window" estimators are proposed and studied. These include the maximum likelihood estimators (MLE's) studied by Grenander (1956), Marshall and Proschan (1965), and Prakasa Rao (1966). By appropriate choice of the window size, we improve, asymptotically, on the MLE. Strong consistency is proved for isotonic window estimators generalizing and simplifying previous proofs. Strong consistency is proved for a class of isotonic estimators when the basic estimator is strongly consistent.

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1. ISOTONIC ESTIMATION IN THE CASE OF CONVEX ORDERING

Assume F and G are distributions on $[0,\infty)$ with densities f and g, respectively. We consider the following convex partial ordering (c-ordering) on the space of distribution functions on $[0,\infty)$. We say that F is c-ordered with respect to G $[F \underset{c}{<} G]$ if and only if $G^{-1}F$ is convex on the support of F (an interval) [van Zwet (1964)]. We say that $F \underset{c}{=} G$ if $G^{-1}F(x) = ax + b$ for some $a > 0$.

If $F \underset{c}{<} G$, then

$$(1.1) \qquad r(x) = \frac{d}{dx} G^{-1}F(x) = \frac{f(x)}{g[G^{-1}F(x)]}$$

is nondecreasing in $x \in [0,\infty)$. We assume G known, $F \underset{c}{<} G$ and consider the problem of estimating r. Maximum likelihood estimators (MLE) for r in the case when G is the exponential or the uniform distribution have been investigated by Grenander (1956), Marshall and Proschan (1965), B.L.S.P. Rao (1966), and Robertson (1967).

Let r_n be an initial or *basic* estimator for r. Let $0 \equiv w_{0,n} < w_{1,n} < \ldots < w_{i,n} < \ldots$ be a subdivision of $[0,\infty)$ and $\mu_n\{w_{j,n}\}$ a sequence of weights on $\left\{w_{j,n}\right\}_{j=0}^{\infty}$. Assume $w_{i,n} \le x < w_{i+1,n}$. We call the interval $[w_{i,n}, w_{i+1,n}]$ a "window." We will be especially interested in the effect on the estimators with respect to the rapidity with which the window goes to zero.

Following Brunk, we call

$$\hat{r}_n(x) = \inf_{t \geq i+1} \sup_{s \leq i} \left[\frac{\sum_{j=s}^{t-1} r_n(w_{j,n})\mu_n\{w_{j,n}\}}{\sum_{j=s}^{t-1} \mu_n\{w_{j,n}\}} \right] \tag{1.2}$$

the *monotonic regression* or more generally the *isotonic regression* of r_n with respect to the discrete measure μ_n. Note that $\hat{r}_n$ is a nondecreasing step function. Let F_n be the empirical distribution function corresponding to an ordered sample $0 \equiv X_0 \leq X_1 \leq X_2 \leq \ldots \leq X_n$ from F. Let $\xi_i = (w_{i+1,n} - w_{i,n})/2$. We call

$$r_n(\xi_i) = \frac{f_n(\xi_i)}{gG^{-1}F_n(\xi_i)} = \frac{F_n(w_{i+1,n}) - F_n(w_{i,n})}{(w_{i+1,n} - w_{i,n})gG^{-1}F_n(\xi_i)}$$

the "naive" estimator for $r(\xi_i)$. This has been extensively studied by Parzen (1962) and others (in the case $G(x) = x \quad (0 \leq x \leq 1)$) and by Watson and Leadbetter (1964) and others (in the case $G(x) = 1 - e^{-x}$ for $x \geq 0$).

If r_n is the naive estimator and $\mu_n\{\xi_i\} = gG^{-1}F_n(\xi_i)(w_{i+1,n} - w_{i,n})$, then

$$\hat{r}_n(x) = \inf_{t \geq i+1} \sup_{s \leq i} \frac{F_n(w_{t,n}) - F_n(w_{s,n})}{\sum_{j=s}^{t-1} gG^{-1}F_n(\xi_j)(w_{j+1,n} - w_{j,n})} . \tag{1.3}$$

If the grid $\{w_{i,n}\}$ is fixed and not random, then (1.3) is the MLE (in the case $G(x) = 1 - e^{-x}$) for a discrete IFR distribution. [Cf. Marshall and Proschan (1965), Section 7.] If $w_{i,n} = X_i$, then

$$\hat{r}_n(x) = \inf_{t \geq i+1} \sup_{s \leq i} \frac{t - s}{n \sum_{j=s}^{t-1} gG^{-1}F_n(\xi_j)(X_{j+1} - X_j)} \tag{1.4}$$

is the MLE studied by Grenander (1956), Marshall and Proschan (1965), and

B.L.S.P. Rao (1966) $\left(\text{in the case } G(x) = 1 - e^{-x} \text{ for } x \geq 0\right)$ and by Robertson (1967) (in the case $G(x) = x$, $0 \leq x \leq 1$) . If $G(x) = 1 - e^{-x}$, then the weight $\mu_n\{\xi_i\} = [1/n](n - i)(X_{i+1} - X_i)$ is the "total time on test" (divided by n) between ordered observations X_i and X_{i+1} . We show in Section 2 that in this case

$$(1.5)\qquad \begin{aligned}\sum_{j=1}^{i} \mu_n\{\xi_j\} &= \int_0^{F_n^{-1}(i/n)} gG^{-1}F_n(u)du \\ &= \phi_{F_n}\left(\frac{i}{n}\right) \overset{a.s.}{\rightarrow} \phi_F(y) = \int_0^{F^{-1}(y)} gG^{-1}F(u)du\end{aligned}$$

as $i/n \to y$ and $n \to \infty$. We call $\phi_F(y)$ the *total time on test distribution* or *measure* (not necessarily a probability distribution).

If r_n is the naive estimator and $\mu_n\{\xi_i\} = w_{i+1,n} - w_{i,n}$, then

$$(1.6)\qquad r_n^*(x) = \inf_{t \geq i+1} \sup_{s \leq i} \frac{1}{w_{t,n} - w_{s,n}} \sum_{j=s}^{t-1} \frac{[F_n(w_{j+1,n}) - F_n(w_{j,n})]}{gG^{-1}F_n(\xi_j)}$$

is the isotonized "naive" estimator. Note that $r_n^*(x)$ is a special kind of average with respect to "discrete Lebesgue measure," while $\hat{r}_n(x)$ is a special kind of average with respect to "total time on test" weights. It might be conjectured that $\hat{r}_n(x)$ will perform better than $r_n^*(x)$ for small samples since the "total time on test" for an interval is a measure of our information over that interval and $\hat{r}_n(x)$ takes advantage of that fact. However, as we shall show in the companion paper†, they are asymptotically equivalent (when the windows of the grid are not too wide).

Another estimator for r can be obtained using the "graphical" estimator

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†Asymptotic Properties of Isotonic Estimators for the Generalized Failure Rate Function - Part II: Asymptotic Distributions).

$$(1.7)\qquad r_n(\xi_i) = \frac{G^{-1}F_n(w_{i+1,n}) - G^{-1}F_n(w_{i,n})}{w_{i+1,n} - w_{i,n}} .$$

This is motivated by the identity

$$G^{-1}F(w_{i+1,n}) - G^{-1}F(w_{i,n}) = \int_{w_{i,n}}^{w_{i+1,n}} r(u)du .$$

If we use this for our basic estimator and let $\mu_n\{\xi_i\} = w_{i+1,n} - w_{i,n}$, we obtain

$$(1.8)\qquad \tilde{r}_n(x) = \inf_{t \geq i+1} \sup_{s \leq i} \frac{G^{-1}F_n(w_{t,n}) - G^{-1}F_n(w_{s,n})}{w_{t,n} - w_{s,n}} .$$

Although all three estimators, $\hat{r}_n(x)$, $r_n^*(x)$, $\tilde{r}_n(x)$ will be shown to be asymptotically equivalent (for certain grids), we can see that $r_n^*(x)$ and $\tilde{r}_n(x)$ are perhaps more alike since they use the same weighting function. The similarity is also apparent if we expand G^{-1} in a Taylor's series about $F_n(x)$. Notice that when $G(x) = x$ $(0 \leq x \leq 1)$, $\hat{r}_n(x) = r_n^*(x) = \tilde{r}_n(x)$.

2. CONVERGENCE OF TOTAL TIME ON TEST MEASURES

The total time on test measure, Φ_{F_n}, plays a crucial role in proving strong consistency of $\hat{r}_n(x)$. It also is important in life test theory for the exponential distribution. Note that when $G(x) = 1 - e^{-x}$ for $x \geq 0$,

$$\Phi_{F_n}\left(\frac{i}{n}\right) = \int_0^{F_n^{-1}(i/n)} gG^{-1}F_n(u)du$$

$$= \sum_{j=1}^{i} gG^{-1}F_n(X_{j-1})(X_j - X_{j-1})$$

$$= \frac{1}{n} \sum_{j=1}^{i} (n-j+1)(X_j - X_{j-1}) .$$

This transform was introduced in Marshall and Proschan (1965) and exploited by B.L.S.P. Rao (1966) $\left(\text{when } G(x) = 1 - e^{-x}\right)$ to obtain the maximum likelihood estimate for the failure rate function of monotone failure rate distributions. Assume $\mu_1 = \int_0^\infty xdF(x) < \infty$. Strong uniform convergence of $\Phi_{F_n}(y)$ to $\Phi_F(y)$, in this case, is an easy consequence of the Glivenko-Cantelli theorem and the strong law since

$$\Phi_{F_n}(1) = \frac{1}{n} \sum_{j=1}^{n} (n-j-1)(X_j - X_{j-1}) = \frac{1}{n} \sum_{j=1}^{n} X_i \xrightarrow{a.s.} \mu_1 .$$

However, strong uniform convergence is not trivial for general distributions, G, when $F^{-1}(1) = \infty$.

The proof of the following theorem is due to H. D. Brunk. Note that in the following theorem we do *not* assume $F \underset{c}{<} G$.

Theorem 2.1:

Suppose that either

(2.1) $$F^{-1}(1) < \infty$$

or

(2.2) $$F^{-1}(1) = \infty \ , \quad \int_0^\infty x dF(x) < \infty \ , \quad \int_0^\infty gG^{-1}F(x)dx < \infty$$

and

(2.3) $$\Psi = gG^{-1}$$

has a continuous derivative ψ on $[0,1]$. Then

$$P\left[\sup_{0\le y\le 1} |\Phi_{F_n}(y) - \Phi_F(y)| \to 0\right] = 1 \ .$$

(Note that $F \underset{c}{<} G$ implies $\int_0^\infty gG^{-1}F(x)dx < \infty$.)

To prove Theorem 2.1, we need the following lemma:

<u>Lemma 2.2</u>:

Let $\psi : [0,1] \to R$ be continuous and set $\Psi(u) = \int_0^u \psi(t)dt$, $u \in [0,1]$. For positive integers i , n , $i \le n$, set

$$\Delta_{n,i} = \Psi\left(\frac{i-1}{n}\right) - \Psi\left(\frac{i}{n}\right) .$$

Let $Z_1, Z_2, \ldots, Z_n$ be independent random variables, each with continuous distribution F . Let $|Z_1|$ and $|Z_1\psi[F(Z_1)]|$ each have finite mean. For fixed positive integer n , let $X_1, X_2, \ldots, X_n$ be the order statistics of $Z_1, \ldots, Z_n$. Then

$$\lim_{n\to\infty} \sum_{i=1}^n \Delta_{n,i} X_i \overset{a.s.}{=} - \int_0^\infty x\psi[F(x)]dF(x) \ .$$

Proof:

We show first that

$$\sum_{i=1}^{n} \left| X_i \left(\Delta_{n,i} + \frac{1}{n} \psi[F(X_i)] \right) \right| \overset{a.s.}{\to} 0 .$$

Set $U_i = F(X_i)$. Then $U_1, \ldots, U_n$ are order statistics of a random sample from the uniform distribution over $[0,1]$. We have

$$-\Delta_{n,i} - \frac{1}{n} \psi[F(X_i)] = \int_{(i-1)/n}^{i/n} [\psi(u) - \psi(U_i)]du .$$

By the Glivenko-Cantelli theorem,

$$\lim_{n} \sup_{1 \le i \le n} \sup_{u \epsilon [(i-1)/n, i/n]} |u - U_i| \overset{a.s.}{=} 0 .$$

But ψ is uniformly continuous on $[0,1]$. Hence with probability 1, given $\varepsilon > 0$, there exists $N_\varepsilon(\omega)$ such that $n \ge N_\varepsilon(\omega)$ implies $|\psi(u) - \psi(U_i)| < \varepsilon$ for all $u \in \left(\frac{i-1}{n} , \frac{i}{n} \right)$, $i = 1,2, \ldots, n$ where ω denotes a sample point. Then $n \ge N_\varepsilon(\omega)$ implies

$$\sum_{i=1}^{n} \left| X_i \left(\Delta_{n,i} + \frac{1}{n} \psi[F(X_i)] \right) \right| \le \frac{\varepsilon}{n} \sum_{i=1}^{n} X_i .$$

But with probability 1, $\frac{1}{n} \sum_{i=1}^{n} |X_i| \to E|Z_1|$, hence with probability 1

$$\lim_{n} \sup \sum_{i=1}^{n} \left| X_i \left(\Delta_{n,i} + \frac{1}{n} \psi[F(X_i)] \right) \right| \le \varepsilon E|Z_1| ,$$

and

$$\lim_n \sum_{i=1}^{n} \left| X_i\left(\Delta_{n,i} + \frac{1}{n}\,\psi[F(X_i)]\right)\right| \overset{a.s.}{=} 0 \ .$$

On the other hand, it follows from the strong law that

$$\frac{1}{n} \sum_{i=1}^{n} X_i\psi[F(X_i)] \overset{a.s.}{\to} \int_0^{\infty} x\psi[F(x)]dF(x) \ .$$

The conclusion of the lemma follows. ||

Proof of Theorem 2.1:

Let $0 \equiv X_0 \leq X_1 \leq X_2 \leq \ldots \leq X_n$ be an ordered sample from F so that

$$\phi_{F_n}\left(\frac{i}{n}\right) = \sum_{j=1}^{i} gG^{-1}\left(\frac{j-1}{n}\right)(X_j - X_{j-1}) \ .$$

Case 1:

Suppose that (2.1) holds. Then for $y \in [0,1]$ we have

$$|\phi_{F_n}(y) - \phi_F(y)| \leq \left| \int_0^{F_n^{-1}(y)} [\Psi(F_n(x)) - \Psi(F(x))]dx \right|$$

$$+ \left| \int_0^{F_n^{-1}(y)} \Psi(F(x))dx - \int_0^{F^{-1}(y)} \Psi(F(x))dx \right| .$$

The first term on the right converges to 0 with probability 1 by the Glivenko-Cantelli theorem, and the second converges to 0 with probability 1 by the strong law.

Case 2:

Suppose that (2.2) and (2.3) hold. Then $F^{-1}(1) = \infty$, which implies

$\Psi(1) = g[G^{-1}(1)] = 0$. For $y < 1$, the argument of Case 1 is valid.

Integrating by parts, we have

$$\Phi_F(y) = \int_0^{F^{-1}(y)} gG^{-1}F(x)dx = \int_0^y gG^{-1}(u)dF^{-1}(u)$$

$$= \int_0^y \Psi(u)dF^{-1}(u) = \Psi(y)F^{-1}(y) - \int_0^y F^{-1}(u)\psi(u)du \ .$$

$$\Phi_F(y) = \Psi(y)F^{-1}(y) - \int_0^{F^{-1}(y)} x\psi[F(x)]dF(x) \ .$$

Since by (2.2), $\int_0^\infty gG^{-1}F(x)dx < \infty$, we have

$$\Phi_F(1) = -\int_0^\infty x\psi[F(x)]dF(x) < \infty \ .$$

Set $g_k = gG^{-1}(k/n)$, $k = 0,1, \ldots, n$

$$\psi(u) = \Psi'(u) \ , \ u \in [0,1] \ ,$$

and

$$\Delta_{n,i} = \Psi\left(\frac{i-1}{n}\right) - \Psi\left(\frac{i}{n}\right) \ .$$

Then

$$\Phi_{F_n}\left(\frac{j}{n}\right) = \sum_{i=1}^{j} \Delta_{n,i}X_i + g_jX_j \qquad j = 1,2, \ldots, n$$

and

$$\Phi_{F_n}(1) = \sum_{i=1}^{n} \Delta_{n,i} X_i$$

since $gG^{-1}(1) = 0$. The conclusion $\Phi_{F_n}(1) \xrightarrow{a.s.} \Phi_F(1)$ now follows from Lemma 2.2. ||

In applying Theorem 2.1 to prove consistency in Section 3, we assume $F = G$ so that Conditions (2.2) and (2.3) become

(i) $\int_0^\infty x dG(x) < \infty$ and

(ii) gG^{-1} has a continuous derivative on $[0,1]$.

An alternative condition on G is $\exists$ a number $0 < \eta < 1 \ni$ for $y \geq \eta$, $gG^{-1}(y)$ is nonincreasing and $\frac{gG^{-1}(y)}{1 - y}$ is nondecreasing. To prove that these conditions are sufficient, we state some additional results.

Lemma 2.3:

Let $Z_{i,n}$, $i = 1,2, \ldots, a_n$, $n = 1,2, \ldots$ have exponential distributions with mean 1 and assume that for every fixed n , $Z_{1,n}, Z_{2,n}, \ldots, Z_{a_n,n}$ are independent. Then, if $\frac{a_n}{\log n} \to \infty$ for $n \to \infty$,

$$\lim_{n\to\infty} \frac{1}{a_n} \sum_{i=1}^{a_n} Z_{i,n} = 1$$

with probability one.

The proof is a straightforward application of the Borel-Cantelli lemma, and we omit it.

Lemma 2.4:

Let $X_1, X_2, \ldots$ be independent and identically distributed with continuous distribution function F and let F_n denote the empirical distribution function based on $X_1, \ldots, X_n$. Then

$$\lim_{n\to\infty} \frac{n^{\frac{1}{2}}}{\log n} \sup_x |F_n(x) - F(x)| = 0$$

with probability 1.

Proof:

From the exact distribution of the one-sided Kolmogorov-Smirnov test statistic, we infer

$$P(\sup |F_n(x) - F(x)| > z) \leq 2P(\sup (F_n(x) - F(x)) > z) =$$

$$= 2z \sum_{i=0}^{[n(1-z)]} \binom{n}{i}\left(1 - z - \frac{i}{n}\right)^{n-i}\left(z + \frac{i}{n}\right)^{i-1} \qquad 0 \leq z < 1 .$$

Consider the function

$$f(z) = \log \binom{n}{i} + (n - i) \log \left(1 - z - \frac{i}{n}\right) + i \log \left(z + \frac{i}{n}\right) + 2nz^2 ,$$

for $0 \leq i \leq n(1 - z)$, i.e., for $z \leq 1 - i/n$, $i = 0,1, \ldots, n$. For $z = 0$, the value of f is the logarithm of a binomial probability, hence $f(0) \leq 0$. Furthermore, for $0 \leq z \leq 1 - i/n$,

$$f'(z) = nz\left(4 - \frac{1}{\left(1 - z - \frac{i}{n}\right)\left(z + \frac{i}{n}\right)}\right) \leq 0 ,$$

and as a result $f(z) \leq 0$ for all z under consideration. Hence

$$\binom{n}{i}\left(1 - z - \frac{i}{n}\right)^{n-i}\left(z + \frac{i}{n}\right)^{i} \leq e^{-2nz^2} ,$$

and

$$P(\sup |F_n(x) - F(x)| > z) \leq 2e^{-2nz^2} \sum_{i=0}^{n} \frac{z}{z + \frac{i}{n}} \leq 2ne^{-2nz^2} .$$

It follows that for every $\varepsilon > 0$

$$\sum_{n=1}^{\infty} P\left(\frac{n^{\frac{1}{2}}}{\log n} \sup |F_n(x) - F(x)| > \varepsilon\right) \leq 2 \sum_{n=1}^{\infty} n^{1-2\varepsilon^2} \log n < \infty ,$$

and the lemma is proved by applying the Borel-Cantelli lemma. ||

<u>Lemma 2.5</u>:

Let $Y_1, Y_2, \ldots$ be independent and identically distributed according to an exponential distribution with mean 1 and let $(0 = Y_{0:n} <)$ $Y_{1:n} < Y_{2:n} < \ldots < Y_{n:n}$ denote the order statistics corresponding to $Y_1, Y_2, \ldots, Y_n$. Then for any $0 \leq c < 1$,

$$\lim_{n\to\infty} \sum_{[cn]+1}^{n} (Y_{j:n} - Y_{j-1:n}) \max\left(1 - \frac{j-1}{n} , e^{-Y_{j-1:n}}\right) = 1 - c$$

with probability 1.

<u>Proof</u>:

Define $Z_{j,n} = (n-j+1)(Y_{j:n} - Y_{j-1:n})$. Since $Z_{1,n}, Z_{2,n}, \ldots, Z_{n,n}$ have independent exponential distributions with mean 1 ,

$$\sum_{[cn]+1}^{n} (Y_{j:n} - Y_{j-1:n})\left(1 - \frac{j-1}{n}\right) = \frac{1}{n} \sum_{[cn]+1}^{n} Z_{j,n}$$

converges almost surely to $(1 - c)$ for $n \to \infty$ by Lemma 2.3. Define $U_{j:n} = 1 - e^{-Y_{j:n}}$. Then $(0 = U_{0:n} <)$ $U_{1:n} < \ldots < U_{n:n}$ are distributed as order statistics from a uniform distribution on $(0,1)$. To prove the lemma, it suffices to show that

$$\sum_{j=1}^{n} (Y_{j:n} - Y_{j-1:n}) \cdot \left|U_{j-1:n} - \frac{j-1}{n}\right| = \frac{1}{n} \sum_{j=1}^{n} Z_{j,n} \cdot \frac{\left|U_{j-1:n} - \frac{j-1}{n}\right|}{1 - \frac{j-1}{n}}$$

converges to zero a.s. Let $a_n = n - n^{2/3}$ and $b_n = n - n^{1/3}$, then $1 - \frac{a_n - 1}{n} \geq n^{-1/3}$ and $1 - \frac{b_n - 1}{n} \geq n^{-2/3}$, hence

$$\frac{1}{n} \sum_{j=1}^{n} Z_{j,n} \cdot \frac{\left|U_{j-1:n} - \frac{j-1}{n}\right|}{1 - \frac{j-1}{n}} \leq \frac{1}{a_n} \sum_{j=1}^{a_n} Z_{j,n} \cdot \sup_{1 \leq j \leq a_n} n^{1/3}\left|U_{j-1:n} - \frac{j-1}{n}\right| +$$

$$+ \frac{1}{b_n - a_n} \sum_{j=a_n+1}^{b_n} Z_{j,n} \cdot \sup_{a_n+1 \leq j \leq b_n} n^{1/3}\left|U_{j-1:n} - \frac{j-1}{n}\right| +$$

$$+ \frac{1}{n - b_n} \sum_{j=b_n+1}^{n} Z_{j,n} \cdot \sup_{b_n+1 \leq j \leq n} n^{1/3}\left|U_{j-1:n} - \frac{j-1}{n}\right| .$$

By Lemmata 2.3 and 2.4, the right-hand side of this inequality tends to zero almost surely, which completes the proof. ||

<u>Theorem 2.6</u>:

Let G be a continuous distribution function on $[0,\infty)$ with density g and assume that there exists a number $0 < \eta < 1$ such that for $y \geq \eta$, $gG^{-1}(y)$ is nonincreasing and $\frac{gG^{-1}(y)}{1 - y}$ is nondecreasing. Then for every $y < c < 1$,

$$\overline{\lim_{n\to\infty}} \left(\Phi_{G_n}(1) - \Phi_{G_n}\left(\frac{[cn]}{n}\right)\right) \leq 1 - c$$

with probability 1. Hence,

$$\lim_{n\to\infty} \left(\Phi_{G_n}(1) - \Phi_G(1)\right) = 0$$

with probability 1.

Proof:

$$\Phi_{G_n}(1) - \Phi_{G_n}\left(\frac{[cn]}{n}\right) = \sum_{[cn]+1}^{n} gG^{-1}\left(\frac{j-1}{n}\right)(X_{j:n} - X_{j-1:n})$$

where $X_{j:n}$ is an order statistic from G. If we define $Y_{j:n}$ by $G^{-1}\left(1 - e^{-Y_{j:n}}\right) = X_{j:n}$, then $Y_{j:n}$ is an order statistic from an exponential distribution with mean 1. With probability 1, there exists a number N such that for $n \geq N$ and $j \geq [cn] + 1$ we have $1 - e^{-Y_{j-1:n}} > \eta$. Since $\frac{gG^{-1}(y)}{1-y}$ is nondecreasing for $y \geq \eta$, we have for $n \geq N$ and $j \geq [cn] + 1$

$$X_{j:n} - X_{j-1:n} = (Y_{j:n} - Y_{j-1:n}) \frac{e^{-Y^*_{j:n}}}{gG^{-1}\left(1 - e^{-Y^*_{j:n}}\right)} \leq$$

$$\leq (Y_{j:n} - Y_{j-1:n}) \frac{e^{-Y_{j-1:n}}}{gG^{-1}\left(1 - e^{-Y_{j-1:n}}\right)}$$

where $Y_{j-1:n} \leq Y^*_{j:n} \leq Y_{j:n}$. If $1 - e^{-Y_{j-1:n}} \geq \frac{j-1}{n}$, we have for $n \geq N$ and $j \geq [cn] + 1$

$$gG^{-1}\left(\frac{j-1}{n}\right)(X_{j:n} - X_{j-1:n}) \leq$$

$$\leq (Y_{j:n} - Y_{j-1:n})\left(1 - \frac{j-1}{n}\right) \cdot \frac{e^{-Y_{j-1:n}} \cdot gG^{-1}\left(\frac{j-1}{n}\right)}{\left(1 - \frac{j-1}{n}\right) \cdot gG^{-1}\left(1 - e^{-Y_{j-1:n}}\right)} \leq$$

$$\leq (Y_{j:n} - Y_{j-1:n})\left(1 - \frac{j-1}{n}\right)$$

again since $\frac{gG^{-1}(y)}{1-y}$ is nondecreasing for $y \geq \eta$. If $1 - e^{-Y_{j-1:n}} < \frac{j-1}{n}$, we use the fact that $gG^{-1}(y)$ is nonincreasing for $y \geq \eta$ to obtain

$$gG^{-1}\left(\frac{j-1}{n}\right)(X_{j:n} - X_{j-1:n}) \leq (Y_{j:n} - Y_{j-1:n})e^{-Y_{j-1:n}}.$$

Hence for $n \geq N$,

$$\Phi_{G_n}(1) - \Phi_{G_n}\left(\frac{[cn]}{n}\right) \leq \sum_{[cn]+1}^{n} (Y_{j:n} - Y_{j-1:n}) \max\left(1 - \frac{j-1}{n}, e^{-Y_{j-1:n}}\right) .$$

Application of Lemma 2.5 completes the proof. ||

We will be mainly interested in isotonic estimators based on grids with wider "windows" than those provided by order statistics. We, therefore, define an analogue to the empirical distribution for more general grids. Let $\left\{w_{i,N}\right\}_{i=1}^{\infty}$ be a subdivision of $[0,\infty)$ and define

$$F^*_{n,N}(x) = F_n(w_{i,N}) \qquad \text{for } w_{i,N} \leq x < w_{i+1,N} .$$

Suppose $\left\{w_{i,N}\right\}_{i=1}^{\infty}$ becomes dense in $[0,\infty)$ as $N \to \infty$ in the sense that given $\varepsilon > 0$, $\exists$ $N^* \ni$ for arbitrary x and all $N > N^*$ $\exists$ $w_{i,N} \ni w_{i,N} \in (x - \varepsilon , x + \varepsilon)$. Clearly, $F^*_{n,N}(x) \leq F_n(x)$ for all x and

$$\lim_{N\to\infty} F^*_{n,N}(x) = F_n(x)$$

since $F^*_{n,N}(x) = F_n(x)$ when the grid includes the order statistics. Also, $F^*_{n,n}(x) = F^*_n(x) \underset{a.s.}{\to} F(x)$ uniformly in $x \in [0,b]$ for every $b < \infty$. The proof is similar to the proof of the Glivenko-Cantelli theorem.

<u>Theorem 2.7</u>:

Suppose the grid $\left\{w_{i,N}\right\}_{i=1}^{\infty}$ becomes dense in $[0,\infty)$ in such a way that $N' > N \Rightarrow \left\{w_{i,N}\right\}_{i=1}^{\infty} \subset \left\{w_{i,N'}\right\}_{i=1}^{\infty}$. Suppose that either Condition (2.1) or (2.2) and (2.3) of Theorem 2.1 hold. Then

$$P\left[\sup_{0\leq y\leq 1} \left|\Phi_{F^*_n}(y) - \Phi_F(y)\right| \to 0\right] = 1 .$$

Proof:

The proof is easy if (2.1) holds. Hence, suppose (2.2) and (2.3) hold. Since by assumption, $\int_0^\infty gG^{-1}F(x)dx < \infty$, given $\varepsilon > 0$ $\exists$ N^* $\ni$ for $N > N^*$

$$\left| \sum_{i=0}^{\infty} gG^{-1}F(w_{i,N})(w_{i+1} - w_i) - \Phi_F(1) \right| < \frac{\varepsilon}{2} .$$

Consider the discretization of F ; i.e.,

$$F_{d,N}(x) = F(w_i) \qquad \text{for } w_i \leq x < w_{i+1}$$

and apply Theorem 2.1 to $F_{d,N}$. Hence,

$$P\Big[\exists\ n_\varepsilon(N,\omega) \ni |\Phi_{F^*_{n,N}}(1) - \Phi_F(1)| < \varepsilon$$

and

$$|\Phi_{F_n}(1) - \Phi_F(1)| < \varepsilon \text{ for all } n > n_\varepsilon(N,\omega) \text{ and all } \varepsilon > 0\Big] = 1$$

where ω denotes a sample point. Since $N' > N$ implies $\{w_{i,N}\} \subset \{w_{i,N'}\}$, it follows that $\Phi_{F^*_{n,N}}(1)$ is closer to Φ_{F_n} than $\Phi_{F^*_{n,N}}$ is. It follows that the convergence is uniform in both n and N so that $\Phi_{F^*_n}(1) \underset{a.s.}{\rightarrow} \Phi_F(1)$. ||

3. STRONG CONSISTENCY OF ISOTONIC ESTIMATORS BASED ON TOTAL TIME ON TEST MEASURE

We assume $F \underset{c}{<} G$ and show that $\hat{r}_n(x) \to r(x)$ with probability one at continuity points of r. We use the fact that, under regularity conditions, the *total time on test* distribution Φ_{F_n} is uniformly strongly consistent. This generalizes and simplifies consistency proofs of Marshall and Proschan (1965) for the case $G(x) = 1 - e^{-x}$ and Robertson (1967) for the case $G(x) = x$ $(0 \le x \le 1)$. The first part of the proof is similar to that of Marshall and Proschan (1965) while the use of the uniform consistency of Φ_{F_n} was suggested by the proof of Robertson (1967).

Theorem 3.1:

Suppose that

(i) $r(x) = f(x)/gG^{-1}F(x)$ is nondecreasing in $x \ge 0$;

(ii) either $\{w_{i,n}\}$ is the grid determined by the order statistics or $\{w_{i,n}\}$ is a grid which becomes dense on the support of F and such that $N' > N$ implies $\{w_{i,N}\} \subset \{w_{i,N'}\}$;

(iii) either $G^{-1}(1) < \infty$ or $G^{-1}(1) = \infty$, $\int_0^\infty x dG(x) < \infty$ and gG^{-1} has a continuous derivative on $[0,1]$.

Then

$$r\left(x_o^-\right) \le \lim\inf \hat{r}_n(x_o) \le \lim\sup \hat{r}_n(x_o) \le r\left(x_o^+\right)$$

with probability one.

Proof:

To show $\lim\sup \hat{r}_n(x_o) \le r\left(x_o^+\right)$. The right hand inequality is trivial if $r(x_o) = 0$ or $r\left(x_o^+\right) = \infty$; otherwise, let $x_1 > x_o$ satisfy $r(x_1) < \infty$ and let

$a_1(n) = a_1$ be the index of the largest grid point $w_j = w_{j,n} \le x_1$. Let $N_1(n) = N_1$ and $N_2(n) = N_2$ be defined by

$$\hat{r}_n(x_o) = \frac{F_n\left(w_{N_2}\right) - F_n\left(w_{N_1}\right)}{\sum_{j=N_1}^{N_2-1} gG^{-1}F_n(w_j)(w_{j+1} - w_j)}$$

$$\le \frac{F_n\left(w_{a_1}\right) - F_n\left(w_{N_1}\right)}{\sum_{j=N_1}^{a_1-1} gG^{-1}F_n(w_j)(w_{j+1} - w_j)} .$$

Let $w_j^* = \dfrac{G^{-1}F(w_j)}{r(x_1)}$ and note that

$$w_{j+1}^* - w_j^* = \int_{w_j}^{w_{j+1}} \frac{r(u)}{r(x_1)}\, du \le w_{j+1} - w_j$$

since $j + 1 \le a_1$ implies $w_{j+1} \le x_1$ and $r(x) \le r(x_1)$ for $x \le x_1$. Hence,

$$\hat{r}_n(x) \le \frac{F_n\left(w_{a_1}\right) - F_n\left(w_{N_1}\right)}{\sum_{j=N_1}^{a_1-1} gG^{-1}F_n(w_j)\left(w_{j+1}^* - w_j^*\right)} .$$

Let $Y_i^* = \dfrac{G^{-1}F(X_i)}{r(x_1)}$ where $X_1, X_2, \ldots, X_n$ is an ordered sample from F and $G^*(x) = P\{Y^* \le x\} = G(r(x_1)y)$. Let $G_n^*(x) = G_n\left(w_i^*\right) = F_n(w_i)$ for $w_i^* \le x < w_{i+1}^*$ where G_n is the empirical distribution corresponding to $Y_1^* \le Y_2^* \le \ldots \le Y_n^*$.

Hence,

$$\hat{r}_n(x) \le \frac{G_n^*\left(w_{a_1}^*\right) - G_n^*\left(w_{N_1}^*\right)}{\sum_{j=N_1}^{a_1-1} gG^{-1}G_n^*\left(w_j^*\right)\left(w_{j+1}^* - w_j^*\right)}$$

$$= \frac{G_n^*\left(w_{a_1}^*\right) - G_n^*\left(w_{N_1}^*\right)}{\Phi_{G_n^*}\left(G_n^*\left(w_{a_1}^*\right)\right) - \Phi_{G_n^*}\left(G_n^*\left(w_{N_1}^*\right)\right)} .$$

Since $w_{N_1}^* \le x_o^*$ while $w_{a_1}^* \uparrow x_1^*$ with probability 1,

$$\liminf_{n\to\infty} \left|G_n^*\left(w_{a_1}^*\right) - G_n^*\left(w_{N_1}^*\right)\right| > 0$$

with probability 1. (Note $G^*(x_o) < G^*(x_1)$ since G is increasing on its support; an interval.) It follows from Theorem 2.1 (letting $F = G^*$), that

$$\limsup_{n\to\infty} \hat{r}_n(x_o) \le \limsup_{n\to\infty} \frac{G_n^*\left(w_{a_1}^*\right) - G_n^*\left(w_{N_1}^*\right)}{\Phi_{G_n^*}\left(G_n^*\left(w_{a_1}^*\right)\right) - \Phi_{G_n^*}\left(G_n^*\left(w_{N_1}^*\right)\right)} \to r(x_1)$$

with probability 1, since $\Phi_{G_n^*}(y) \overset{a.s.}{\to} \Phi_{G^*}(y) = \frac{y}{r(x_1)}$ uniformly for $0 \le y \le 1$.

Letting $x_1 \downarrow x_o$, we see that

$$\limsup_{n\to\infty} \hat{r}_n(x_o) \le r\left(x_o^+\right)$$

with probability 1.

By a similar argument, we have

$$r\left(x_o^-\right) \le \liminf \hat{r}_n(x_o)$$

with probability 1. ||

Remark 1:

Note that the key to the proof was the uniform convergence of $\Phi_{G_n^*}(y) \xrightarrow{a.s.} \Phi_{G^*}(y)$ which does not depend upon F. Hence, a similar consistency proof can be given for the case $F \underset{c}{>} G$. In this circumstance, the estimator becomes

$$\hat{r}_n(x) = \sup_{t \geq i+1} \inf_{s \leq i} \frac{F_n(w_{t,n}) - F_n(w_{s,n})}{\sum_{j=s}^{t-1} gG^{-1}F_n(\xi_j)(w_{j+1,n} - w_{j,n})} .$$

Remark 2:

For the grid based on order statistics, the estimator $r_n(x)$ is *not* consistent. However, as Rao (1966) has shown, isotonizing effectively widens the "windows" so that $\hat{r}_n$ is strongly consistent.

Remark 3: Random Window Estimators

There are at least two ways of specifying window estimators. One can specify the length of the windows and let the number of observations in a window be random or one can specify the number of observations in a window and thus allow the interval length to be random. In practice, it may sometimes be more convenient to specify the number of observations in a window rather than the window length. The MLE for $r(t)$ assuming r nondecreasing is the isotonic regression of a random window estimator which allows precisely one observation per window. Suppose we specify random window intervals to be of the form $\left(X_{[in^\beta]}, X_{[(i+1)n^\beta]}\right)$ for $i = 0,1, \ldots$ and some $\beta(0 \leq \beta < 1)$. Strong consistency of random window estimators can be verified either by modifying the proof of Theorem 2.1 for this case or by the method to be discussed in the next section, since now the basic estimator is consistent.

4. CONSISTENCY OF ISOTONIC ESTIMATORS WHEN THE BASIC ESTIMATOR IS CONSISTENT

Brunk (1965) introduced a general method for proving consistency of isotonic estimators when the basic estimator is consistent. We generalize his results somewhat in order to treat the generalized failure rate function. A fundamental difference between Brunk estimators and our estimators, $\hat{r}_n$, defined on Ω_n ($\Omega_n \subset \Omega$) is that we extend the domain to Ω and define a version of $\hat{r}_n$ on Ω. We then wish to prove that this version converges strongly on Ω (assuming $\Omega_n \to \Omega$).

Let $(\Omega, \mathcal{A}, \mu)$ be a measure space and let Λ be a set of $\mathcal{A}$-measurable functions on Ω with the following properties:

(a) $\lambda_1, \lambda_2 \in \Lambda$ implies $\max(\lambda_1, \lambda_2)$ and $\min(\lambda_1, \lambda_2) \in \Lambda$,

(b) $\Lambda \cap L_2(\mu)$ is convex and closed in L_2,

(c) Λ is a complete inner product space with inner product

$$(\lambda_1, \lambda_2) = \int_\Omega \lambda_1(w)\lambda_2(w)d\mu(w) .$$

For any $\ell \in L_2$, there exists $\hat{\ell} \in \Lambda \cap L_2$ such that

$$\int_\Omega (\ell - \lambda)^2 d\mu \geq \int_\Omega (\ell - \hat{\ell})^2 d\mu + \int_\Omega (\hat{\ell} - \lambda)^2 d\mu \tag{4.1}$$

for all $\lambda \in \Lambda \cap L_2$. This $\hat{\ell} = P(\ell \mid \Lambda \cap L_2)$ is unique $[\mu]$. (Cf. Brunk (1965), Theorem 2.1 and Corollary 2.1.) $P(\ell \mid \Lambda \cap L_2)$ is the projection of ℓ on $\Lambda \cap L_2$.

<u>Lemma 4.1</u>:

If for some $\lambda_2 \in \Lambda$, $\ell \leq \lambda_2$ a.e. $[\mu]$, then $\hat{\ell} \leq \lambda_2$ a.e. $[\mu]$. If for some $\lambda_1 \in \Lambda$, $\ell \geq \lambda_1$ a.e. $[\mu]$, then $\hat{\ell} \geq \lambda_1$ a.e. $[\mu]$.

Proof:

Let $\ell \leq \lambda_2$ a.e. $[\mu]$. If $\hat{\ell} > \lambda_2$ on a set of positive μ measure, consider $\lambda = \min(\hat{\ell},\lambda_2)$. Since λ is A-measurable and $\min(\hat{\ell},\ell) \leq \lambda \leq \hat{\ell}$, λ is square integrable. Because of Condition (a), $\lambda \in \Lambda$, so $\lambda \in \Lambda \cap L_2$. On the set where $\hat{\ell} \leq \lambda_2$, $\lambda = \hat{\ell}$. However on the set where $\hat{\ell} > \lambda_2$, which has positive μ-measure, $\ell \leq \lambda_2 = \lambda < \hat{\ell}$. Hence,

$$(\ell - \lambda)^2 \leq (\ell - \hat{\ell})^2$$

on Ω with strict inequality on a set of positive measure and therefore

$$\int_\Omega (\ell - \lambda)^2 d\mu < \int_\Omega (\ell - \hat{\ell})^2 d\mu$$

which contradicts (3.1). The proof of the second part of the lemma is similar. ||

Note that Brunk's (1965) conditions $\lambda_1 \leq 0$ and $\lambda_2 \geq 0$ or $\mu(\Omega) < \infty$ in his Corollary 3.3 are unnecessary.

Lemma 4.2:

If for some λ_1 , $\lambda_2 \in \Lambda$, $\lambda_1 \leq \ell \leq \lambda_2$ a.e. $[\mu]$, then there exists a version of $\hat{\ell}$ for which

$$\lambda_1 \leq \hat{\ell} \leq \lambda_2$$

on Ω . [Note that $\hat{\ell}$ is defined only on sets of positive μ-measure.]

Proof:

Let $\tilde{\ell}$ be any version of $P(\ell \mid \Lambda \cap L_2)$ on Ω and let

$$\hat{\ell} = \min(\lambda_2 , \max(\tilde{\ell},\lambda_1)) .$$

Since $\lambda_1 \leq \tilde{\ell} \leq \lambda_2$ a.e. $[\mu]$, $\hat{\ell} = \tilde{\ell}$ a.e. $[\mu]$. Also, $\lambda_1 \leq \hat{\ell} \leq \lambda_2$ on Ω and $\hat{\ell} \in \Lambda$ because of Condition (a). Finally, since $\hat{\ell} = \tilde{\ell}$ a.e. $[\mu]$, and $\hat{\ell}$ is A-measurable, $\hat{\ell} \in L_2$. ||

We now replace Condition (a) by

(A) $\lambda_\tau \in \Lambda$ for all $\tau \in T$ (arbitrary index set) implies

$$\sup_{\tau \in T} \lambda_\tau \ , \ \inf_{\tau \in T} \lambda_\tau \in \Lambda \ .$$

This implies that every function ϕ on Ω for which there exists $\lambda \in \Lambda$ with $\phi \leq \lambda$ has a smallest majorant $\bar{\phi} \in \Lambda$; if no such $\lambda \in \Lambda$ exists, we set $\bar{\phi} \equiv \infty$. Similarly, if there exists $\lambda \leq \phi$, $\lambda \in \Lambda$, then ϕ possesses a largest minorant $\underline{\phi} \in \Lambda$; if no such $\lambda \in \Lambda$ exists, we set $\underline{\phi} \equiv -\infty$. From Lemma 4.2, we have:

__Lemma 4.3__:

There exists a version of $\hat{\ell}$ for which $\underline{\ell} \leq \hat{\ell} \leq \bar{\ell}$ on Ω .

Note that we can construct such a version on the basis of $\underline{\ell}$ and $\bar{\ell}$ (cf. proof of Lemma 4.2). From now on, we shall only consider those versions of $\hat{\ell}$ that are indicated in Lemma 3.3. Suppose Condition (A) holds and (b) is satisfied for a sequence of measures μ_n . Then we have:

__Theorem 4.4__:

If $\ell_n \in L_2(\mu_n)$ and $\underline{\ell}_n$ and $\bar{\ell}_n$ are uniformly (strongly) consistent estimates of the same $\lambda \in \Lambda$, then $\hat{\ell}_n = P(\ell_n \mid \Lambda \cap L_2)$ is also a uniformly (strongly) consistent estimator of λ .

__Proof__:

By Lemma 4.1 and 4.2, $|\underline{\ell}_n - \lambda| \leq \varepsilon_n$ and $|\bar{\ell}_n - \lambda| \leq \varepsilon_n'$ on Ω implies $|\hat{\ell}_n - \lambda| \leq \max\,(\varepsilon_n, \varepsilon_n')$. ||

Theorem 4.4 can be used to prove the (strong) consistency of $\hat{r}_n(x)$, $r_n^*(x)$

and $\tilde{r}_n(x)$ when the basic estimator

$$r_n(\xi_i) = \frac{F_n(w_{i+1,n}) - F_n(w_{i,n})}{(w_{i+1,n} - w_{i,n})gG^{-1}F_n(\xi_i)}$$

is (strongly) consistent. For example, if f is a uniformly continuous density function and

$$w_{i+1,n} - w_{i,n} = cn^{-\alpha} \qquad 0 < \alpha < 1$$

then $r_n(x)$ is strongly uniformly consistent. [Cf. Nadarya (1965), Theorem 1.]

To utilize Theorem 4.4, we need to verify the uniform strong consistency of $\underline{r}_n$ and $\bar{r}_n$. Since it is not obvious that this condition will be realized in general, we must utilize a device employed by Marshall and Proschan (1965). Let $\mathcal{F} = \{F \mid F \underset{c}{\leq} G\}$ where G is specified and $\mathcal{F}^M = \{F \mid F \underset{c}{\leq} G$ and $r(x) \leq M\}$ where $r(x) = f(x)/gG^{-1}F(x)$. Let $\hat{r}_n^M$ be the isotonic estimator of r subject to $\hat{F}_n^M$ in $\mathcal{F}^M$. Assuming $r_n(x)$ is uniformly strongly consistent, we can show that $\underline{r}_n^M$ and $\bar{r}_n^M(x)$ are uniformly strongly consistent in the same way that the Glivenko-Cantelli lemma proves that $F_n(x)$ is uniformly strongly consistent. Furthermore, $\hat{r}_n^M$ converges in a natural way to $\hat{r}_n$ as $M \to \infty$.

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