BOEING SCIENTIFIC RESEARCH LABORATORIES DOCUMENT D1-82-0829

BOEING

SCIENTIFIC RESEARCH LABORATORIES

AD686592

Birkhoff's Normalization

Andre' Deprit

Jaqués Henrard

J. F. Price

Arnold Rom

D D C
RECEIVED
MAY 13 1969
A

This document has been approved for public release and sale; its distribution is unlimited.

Reproduced by the
CLEARINGHOUSE
for Federal Scientific & Technical
Information Springfield Va. 22151

MATHEMATICS RESEARCH MARCH 1969

D1-82-0829

BIRKHOFF'S NORMALIZATION

by

André Deprit
Jacques Henrard
J. F. Price
Arnold Rom

Mathematical Note No. 590

Mathematics Research Laboratory

BOEING SCIENTIFIC RESEARCH LABORATORIES

March 1969

SUMMARY

Birkhoff's normalizing canonical transformation at an equilibrium of elliptic type with no internal resonance can be built explicitly and recursively, without partial inversions or substitutions, by means of Lie transforms.

Invariant sections and ordinary families of periodic orbits for truncated normalized systems are analyzed in detail.

Introduction

We consider a conservative dynamical system with m degrees of freedom whose equations of motion we write in vector form

$$\dot{q} = \frac{\partial}{\partial p}\mathcal{H}, \qquad \dot{p} = -\frac{\partial}{\partial q}\mathcal{H}, \tag{1}$$

in which q and p indicate vectors with components q_j and p_j $(1 \leq j \leq m)$ respectively. We assume that the origin $(p = 0, q = 0)$ in the phase space is an equilibrium point, and that the Hamiltonian $\mathcal{H}(q,p)$ is an analytic function of the state variables in a neighborhood of the equilibrium. Hence Taylor's expansion of $\mathcal{H}$ in that neighborhood is of the form

$$\mathcal{H}(p,q) = \sum_{n\geq 0} \frac{1}{n!}\,\mathcal{H}_n(p,q), \tag{2}$$

where, for each $n \geq 0$, $\mathcal{H}_n$ is a homogeneous polynomial of degree $n + 2$ in the state variables. Indeed the constant term in the development of $\mathcal{H}$ around the equilibrium does not contribute to the equations (1), and since $(q = 0, p = 0)$ is a solution of (1), the first degree terms in (2) must vanish. Thus the power series for $\mathcal{H}$ begins with quadratic terms.

We suppose that the series (2) has only real coefficients, in particular that the quadratic form $\mathcal{H}_0$ is real. Let $\mathcal{A}$ be the matrix of the linear Hamiltonian system derived from $\mathcal{H}_0$. If λ is an eigenvalue of $\mathcal{A}$, then its complex conjugate $\bar{\lambda}$ and its inverse $-\lambda$ are also eigenvalues of

$\mathcal{A}$. Thus assuming that all eigenvalues of $\mathcal{A}$ are distinct and purely imaginary, we can represent them by the sequence

$$i\lambda_1, i\lambda_2, \ldots, i\lambda_m, -i\lambda_1, -i\lambda_2, \ldots, -i\lambda_m, \tag{3}$$

where $\lambda_1, \lambda_2, \ldots, \lambda_m$ are non zero real numbers. Under these conditions, there exists a canonical transformation

$$\begin{aligned} q_j &= \sum_{1 \le k \le m} \left[c_{jk}\sqrt{I_k} \cos \phi_k + s_{jk}\sqrt{I_k} \sin \phi_k \right], \\ p_j &= \sum_{1 \le k \le m} \left[C_{jk}\sqrt{I_k} \cos \phi_k + S_{jk}\sqrt{I_k} \sin \phi_k \right] \end{aligned} \tag{4}$$

from the original state variables (q,p) to angle coordinates $\phi = (\phi_k)$ and action momenta $I = (I_k)$ such that the quadratic form $\mathcal{H}_0$ converts into

$$\mathcal{H}_0(I,-) = \sum_{1 \le k \le m} \lambda_k I_k. \tag{5}$$

Under the mapping (4), the homogeneous component $\mathcal{H}_n$ transforms into a homogeneous polynomial of degree $n + 2$ in the state variables $\sqrt{I_k}$, its coefficients being trigonometric sums in the arguments ϕ_j with real coefficients. For each $n \ge 1$, the component $\mathcal{H}_n(I,\phi)$ has the *d'Alembert characteristic*. It means that, for a term like

$$\left(\sqrt{I_1}\right)^{p_1}\left(\sqrt{I_2}\right)^{p_2} \cdots \left(\sqrt{I_m}\right)^{p_m} \cos\left(k_1\phi_1 + k_2\phi_2 + \cdots + k_m\phi_m\right)$$

to enter the homogeneous component of degree $p_1 + p_2 + \cdots + p_m$, it is necessary that, for $1 \le j \le m$,

$$|k_j| \leq p_j \qquad \text{and} \quad k_j \equiv p_j \qquad (\text{mod. } 2); \tag{6}$$

the same condition is to be satisfied by any sine term.

Birkhoff's normalization consists in finding for a given $N \geq 1$ a canonical transformation $\theta:(I,\phi) \rightarrow (I^*,\phi^*)$ such that the converted Hamiltonian decomposes into the sum

$$\mathcal{H}(I^*,\phi^*) = \mathcal{N}(I^*,-) + \mathcal{P}^{(N)}(I^*,\phi^*), \tag{7}$$

where $\mathcal{N}$ is a polynomial with real coefficients in the transformed actions $I^* = (I_k^*)$ of degree at most equal to $(N + 2)/2$, and $\mathcal{P}^{(N)}$ is a series in the action amplitudes $\sqrt{I_j^*}$, beginning with terms of at least degree $(N + 3)$, the coefficients being periodic functions of the angles ϕ_k^*.

Birkhoff (1927) has shown that the normalized part $\mathcal{N}(I^*,-)$ of the Hamiltonian is invariant with respect to the various canonical transformations leading to it. Hence it yields dynamical characteristics of the flow of trajectories close to the equilibrium.

By halting the sequence of normalizing transformations after a finite number N of steps, we obtain an asymptotic representation of the phase portrait at the equilibrium as the actions $I_j^* \rightarrow 0$. Practically the original system as described by $\mathcal{H}$ is replaced by its asymptotic approximation truncated after order $N + 2$ as represented by $\mathcal{N}$. But, as the angles ϕ_j^* are ignorable in $\mathcal{N}$, the dynamical system that it describes is integrable, and its structure in phase space is the product

of an m-dimensional torus by the m-dimensional phase space (Arnold and Avez 1967). Now most of the invariant tori of quasi-periodic motions in the flow of trajectories determined by $\mathcal{N}$ will survive the perturbations induced by the remainder $\mathcal{R}^{(N)}$. This may explain why (although Birkhoff's normalization is only exceptionally convergent as $N \to \infty$), it proves useful in revealing the dynamical structures around the equilibrium for the full system $\mathcal{H}$.

1. Normalization by Lie transforms

It is assumed that the basic frequencies $\lambda_1, \lambda_2, \ldots, \lambda_m$ are rationally independent of each other.

Traditionally the normalizing transformation is defined from a generating function

$$W(I^*, \phi) = \sum_{1 \le k \le m} I_k^* \phi_k + \sum_{n \ge 1} W_n(I^*, \phi). \tag{8}$$

by the implicit equations

$$\begin{aligned} \phi_k^* &= \phi_k + \sum_{n \ge 1} \frac{\partial}{\partial I_k^*} W_n, \qquad (1 \le k \le m) \\ I_k &= I_k^* + \sum_{n \ge 1} \frac{\partial}{\partial \phi_k} W_n. \qquad (1 \le k \le m) \end{aligned} \tag{9}$$

Conversion is then made to the complex variables

$$\xi_k = \sqrt{I_k}\, e^{i\phi_k}, \qquad \eta_k = -i\sqrt{I_k}\, e^{-i\phi_k} \qquad (1 \le k \le m) \tag{10}$$

so that the operations of normalization are performed entirely within

the algebra of polynomials in $2m$ letters over the complex numbers. They have been programmed for automatic processing by computers for systems with two degrees of freedom: programs originally set up to deal with resonance cases (Gustavson 1966) have been modified to operate in the regular case (Chai and Kass 1966).

The use of a generating function in order to build a normalizing canonical transformation is reminiscent of a well known procedure in the theory of General Perturbations (Poincaré 1893) that is usually referred to as Von Zeipel's method. But its shortcomings have recently prompted the formulation of a more direct algorithm based on Lie transforms (Deprit 1969). This new formalism simplifies to a large extent Birkhoff's normalization. The present application will in turn check the general procedure. For we had previously normalized the Hamiltonian of the restricted problem of three bodies at L_4 (Deprit *et al.* 1967), and we shall find that the algorithm about to be described indeed restores the already established normalized Hamiltonian.

The basic step is to construct a sequence of functions

$$W_j(I^*,\psi^*) \qquad (1 \leq j \leq N)$$

depending not on mixed variables but on both new angles and new actions. They will be obtained in the course of transforming the Hamiltonian. For reasons of clarity, let the exposition refer to the diagram in Fig. 1

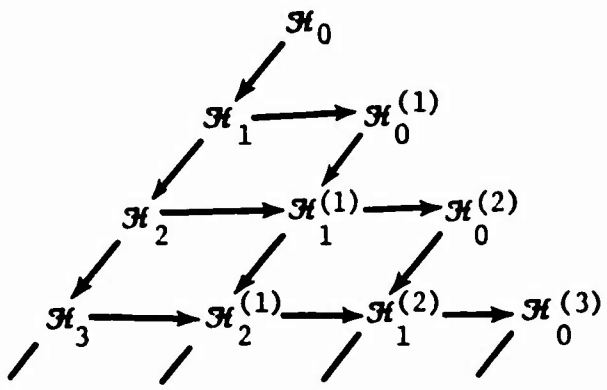

Fig. 1. - Recursive normalization of the Hamiltonian.

The functions entering the diagonal $\mathcal{H}_0, \mathcal{H}_1, \mathcal{H}_2, \ldots$ are the homogeneous components in the initial series (2) as expressed by means of the primitive angles and actions ϕ_j and I_j $(1 \leq j \leq m)$; the functions entering the diagonal $\mathcal{H}_0, \mathcal{H}_0^{(1)}, \mathcal{H}_0^{(2)}, \ldots$ are their normalized forms.

The elements in Table I are to be built recursively according to the law

$$\mathcal{H}_q^{(p)} = \mathcal{H}_{q+1}^{(p-1)} + \sum_{0 \leq j \leq q} \binom{q}{j} \left(\mathcal{H}_{q-j}^{(p-1)} ; W_{j+1} \right) \qquad (11)$$

where $\binom{q}{j}$ is the binomial coefficient

$$\binom{q}{j} = \frac{q(q-1) \cdots (q-j+1)}{1.2. \cdots j}$$

and $\left(\mathcal{H}_{q-j}^{(p-1)} ; W_{j+1} \right)$ is the Poisson bracket

$$\left(\mathcal{H}_{q-j}^{(p-1)} ; W_{j+1} \right) = \sum_{1 \leq k \leq m} \left(\frac{\partial}{\partial \phi_k^*} \mathcal{H}_{q-j}^{(p-1)} \frac{\partial}{\partial I_k^*} W_{j+1} - \frac{\partial}{\partial I_k^*} \mathcal{H}_{q-j}^{(p-1)} \frac{\partial}{\partial \phi_k^*} W_{j+1} \right)$$

Extending a proposition established by Brown and Shook (1933), we can prove that, if f is a homogeneous d'Alembert series of degree p with

respect to the variables $\sqrt{I_1^*}, \sqrt{I_2^*}, \ldots, \sqrt{I_m^*}, \phi_1^*, \phi_2^*, \ldots, \phi_m^*$ and g is a homogeneous d'Alembert series of degree q with respect to the same variables, then the Poisson bracket $(f;g)$ is also a d'Alembert series with respect to these variables, but of degree $p + q - 2$. The fact is that, while the separate partial derivatives with respect to the actions $I_1^*, I_2^*, \ldots, I_m^*$ are not d'Alembert series, in each Jacobian

$$\frac{\partial(f,g)}{\partial(\phi_k^*, I_k^*)} = \frac{1}{2\sqrt{I_k^*}} \frac{\partial(f,g)}{\partial(\phi_k^*, \sqrt{I_k^*})},$$

the divisor $\sqrt{I_k^*}$ disappears, and by exact cancellation of all terms not having the d'Alembert characteristic, the Jacobian of f and g with respect to ϕ_k^* and $\sqrt{I_k^*}$ turns out to be a homogeneous polynomial in the variables $\sqrt{I_1^*}, \sqrt{I_2^*}, \ldots, \sqrt{I_m^*}$ of degree $p + q - 2$ with the d'Alembert characteristic.

This proposition assures the consistency of the scheme we propose here; it is useful as a check on the various parts of a program to be implemented by computer.

As we shall see, the differential operator

$$D(\lambda) = \sum_{1 \leq k \leq m} \lambda_k \frac{\partial}{\partial \phi_k^*}$$

will play an essential role. It is a linear mapping into itself of the algebra $\mathcal{A} = \mathcal{A}(I^*, \phi^*)$ of d'Alembert series with respect to $\sqrt{I_1^*}, \sqrt{I_2^*}, \ldots, \sqrt{I_m^*}, \phi_1^*, \phi_2^*, \ldots, \phi_m^*$. More precisely it is a linear mapping into itself of the vector space $\mathcal{A}^{(p)} = \mathcal{A}^{(p)}(I^*, \phi^*)$ of homogeneous polynomials of

degree p in $\sqrt{I_1^*}, \sqrt{I_2^*}, \ldots, \sqrt{I_m^*}$ having the d'Alembert characteristic and, under the condition that the frequencies $\lambda = (\lambda_j)$ are rationally independent of each other, the kernel of this restriction is the vector space $\mathcal{P}^{(p)} \equiv \mathcal{P}^{(p)}(I^*)$ of homogeneous polynomials in $\sqrt{I_1^*}, \sqrt{I_2^*}, \ldots, \sqrt{I_m^*}$ of degree p with real coefficients. Notice also that

$$\begin{aligned} D(\lambda)\cos\Big(\sum_{1\le k\le m} j_k\phi_k^*\Big) &= -\Big|\sum_{1\le k\le m} j_k\lambda_k\Big|\sin\Big(\sum_{1\le k\le m} j_k\phi_k^*\Big), \\ D(\lambda)\sin\Big(\sum_{1\le k\le m} j_k\phi_k^*\Big) &= \Big|\sum_{1\le k\le m} j_k\lambda_k\Big|\cos\Big(\sum_{1\le k\le m} j_k\phi_k^*\Big), \end{aligned} \tag{12}$$

which makes the inverse operator $(D(\lambda))^{-1}$ a trivial operation over $\mathcal{A}$.

Now let us see how the normalization is carried out recursively. In view of (11),

$$\mathcal{H}_0^{(1)} = \mathcal{H}_1 + \{\mathcal{H}_0; W_1\}, \tag{13}$$

with the understanding that in $\mathcal{H}_0$ and $\mathcal{H}_1$ the letters I and ϕ have been replaced by I^* and ϕ^* respectively. We collect in $\mathcal{H}_0^{(1)}$ the terms of $\mathcal{H}_1$ that do not depend explicitly on the angle coordinates ϕ^*. Of course, since $\mathcal{H}_1$ is an element of $\mathcal{A}^{(3)}$, there are no such terms, and we find that $\mathcal{H}_0^{(1)}$ must be put equal to zero. On the other hand

$$\{\mathcal{H}_0; W_1\} = -D(\lambda)W_1,$$

so that the definition (13) transforms into the linear partial differential equation

$$D(\lambda)W_1 = \mathcal{H}_1. \tag{14}$$

A particular solution

$$W_1 = \frac{1}{D(\lambda)} \mathcal{H}_1$$

of (14) is constructed according to the rules (12), and thus it is an element of $\mathcal{A}^{(3)}$. We neglect to complete it by arbitrary elements of the kernel of $D(\lambda)$; their addition would indeed deny the d'Alembert characteristic to W_1.

Suppose that the first N rows $(N > 1)$ of the triangle in Fig. 1 1 have been computed, that is to say the Hamiltonian has been normalized up to order $N + 1$ and the first $N - 1$ generators $W_1, W_2, \ldots, W_{N-1}$ of the canonical transformation have been computed. At this step, it is convenient to put $W_N = 0$ for the time being, and to compute the elements of the $(N + 1)$-th row of the triangle in Fig. 1 by the formula (11). Denoting them by $\tilde{\mathcal{H}}_q^{(p)}$, we obviously have that, for $p \geq 1$ and $p + q = N$,

$$\mathcal{H}_q^{(p)} = \tilde{\mathcal{H}}_q^{(p)} + (\mathcal{H}_0; W_N) \tag{15}$$

Notice that each element $\tilde{\mathcal{H}}_q^{(p)}$ belongs to the vector space $\mathcal{A}^{(N+2)}$.

Making $p = N$ and $q = 0$ in (15), we find the relation

$$\tilde{\mathcal{H}}_0^{(N)} + (\mathcal{H}_0; W_N) = \mathcal{H}_0^{(N)}. \tag{16}$$

Once more we collect in $\mathcal{H}_0^{(N)}$ the terms of $\tilde{\mathcal{H}}_0^{(N)}$ which do not depend explicitly on the angle coordinates ϕ^*. If N is odd, then $\mathcal{H}_0^{(N)}$ turns out to be zero; but if $N = 2M$ is even, $\mathcal{H}_0^{(N)}$ is a homogeneous polynomial of degree M in the letters $I_1^*, I_2^*, \ldots, I_m^*$ with real coefficients. Let us put

$$\mathcal{P}^{(N)} = \tilde{\mathcal{H}}_0^{(N)} - \mathcal{H}_0^{(N)} \tag{17}$$

so that we can write the identity (16) as the linear partial differential equation

$$D(\lambda)W_N = \mathcal{P}^{(N)}, \tag{18}$$

of which a solution is found in accordance with the rules (12):

$$W_N = \frac{1}{D(\lambda)}\mathcal{P}^{(N)}. \tag{19}$$

Observe that the particular solution (19) belongs to the vector space $\mathcal{G}^{(N+2)}$. If N is odd, completing W_N with elements of the kernel of $D(\lambda)$ would result in depriving the completed solution of its d'Alembert characteristic. But, if $N = 2M$ is even, that characteristic would not disappear by adding to W_N a homogeneous polynomial of degree M in the letters $I_1^*, I_2^*, \ldots, I_m^*$ with real coefficients. Yet, as follows from Birkhoff's proof of invariance with respect to the group of normalizing canonical mappings, this complementary term is not going to contribute subsequently to the normalized Hamiltonian. Therefore we decide to omit it systematically.

Having determined W_N, we complete the elements (15) as follows

$$\mathcal{H}_q^{(p)} = \tilde{\mathcal{H}}_q^{(p)} - \mathcal{P}^{(N)} \qquad (p \geq 1,\ q \geq 0,\ p + q = N) \tag{20}$$

so as to be ready to extend the normalization, if necessary, beyond the order $N + 2$.

So far we have determined the components of the normalized

Hamiltonian, and the generators $(W_1, W_2, \ldots, W_N)$ of the Lie transform. In order to construct formally the normalizing canonical transformation, we have to compute the Lie transforms of the original angle and action variables (ϕ, I). To this effect, we apply the usual triangular algorithm (Deprit 1969) which converts series like

$$\phi_0 \equiv \phi, \quad \phi_n \equiv 0 \qquad \text{for} \quad n \geq 1$$

into functions of the normalizing angle and action variables (ϕ^*, I^*). We proceed in the same manner to express the original state variables (q,p) as given by (4), if we need them expressed in terms of (ϕ^*, I^*).

2. Normalized systems with two degrees of freedom

From now on we restrict ourselves to dynamical systems with two degrees of freedom

As we just described it, Birkhoff's normalization has now become a relatively easy operation to be implemented automatically by computer, and the analysis of an elliptic equilibrium can avail itself freely of this routine technique. Hence, from the methodological standpoint, the emphasis shifts on the problem of how to obtain local information about the phase space at the equilibrium from the normalized Hamiltonian.

Quite naturally we are led to contrast two kinds of dynamical systems. On one hand there is the original system described by the full Hamiltonian $\mathcal{H}(I,\phi)$ which, after transformation, takes the form

(7); on the other hand, at a given order $N + 2$, there is the integrable system described by the principal part $\mathcal{N}(I^*,-)$. Before we indicate how the integrable system is a deformed, yet informative, version of the full system, let us see how we can study in depth the principal part.

The canonical transformation $\theta^*:(q_1,q_2,p_1,p_2) \rightarrow (\phi_1^*,\phi_2^*,I_1^*,I_2^*)$ from the original state variables to the normalizing angle and action variables is singular along the phase manifolds $I_1^* \equiv I_1^*(q_1,q_2,p_1,p_2) = 0$ and $I_2^* \equiv I_2^*(q_1,q_2,p_1,p_2) = 0$.

But, for the normalized system, i.e. the dynamical system represented by the integrable Hamiltonian $\mathcal{N}$, the equilibrium configuration is defined by $I_1^* = I_2^* = 0$. Moreover, the frequencies λ_1 and λ_2 being rationally independent, there emanate from the equilibrium two *natural* families of periodic orbits. The family $\mathcal{G}_1$ associated with the frequency λ_1 lies on the manifold $I_2^* = 0$, and conversely any point of the integral manifold $I_2^* = 0$ belongs to one, and only one, periodic orbit in the family $\mathcal{G}_1$. Likewise the family $\mathcal{G}_2$ that Liapunov's theorem associates with the frequency λ_2 lies on the manifold $I_1^* = 0$, and it is coextensive to that set.

It is easy to show (Deprit *et al*, 1967) that the nontrivial characteristic exponents for the periodic orbits belonging to $\mathcal{G}_1$ and $\mathcal{G}_2$ are respectively the series

$$\pm i\omega_1 \equiv \pm i\left|\frac{\partial}{\partial I_1^*}\mathcal{N}(I_1^*,I_2^*)\right|_{I_2^*=0} = \pm i(\lambda_1 + a_1 I_1^* + a_2 I_1^{*2} + \cdots)$$

$$\pm i\omega_2 \equiv \pm i\left|\frac{\partial}{\partial I_2^*}\mathcal{N}(I_1^*,I_2^*)\right|_{I_1^*=0} = \pm i(\lambda_2 + b_1 I_2^* + b_2 I_2^{*2} + \cdots)$$

in the powers of I_1^* or I_2^*. The independence of λ_1 and λ_2 over the rationals implies of course that neither λ_1 nor λ_2 is equal to zero. Consequently the nontrivial multipliers of the periodic orbits either in $\mathcal{G}_1$ or in $\mathcal{G}_2$ are not equal to one; in other words for these orbits, the algebraic multiplicity of the multiplier +1 is equal to 2, which implies that the gradients of the integrals I_1^* and I_2^* in the original phase space (q_1,q_2,p_1,p_2) at any point of the integral manifold $I_1^* = 0$ and $I_2^* = 0$ are collinear.

Outside the manifolds $I_1^* = 0$ or $I_2^* = 0$, the normalizing canonical transformation θ^* is regular, i.e. its Jacobian matrix is invertible. As a consequence, since the gradients of I_1^* and I_2^* in the phase space (q_1,q_2,p_1,p_2) form two rows of the Jacobian of $(\theta^*)^{-1}$, they must be linearly independent. In other words, along any solution of the normalized system, which is neither the equilibrium nor an orbit of $\mathcal{G}_1$ or $\mathcal{G}_2$, the gradients of the "adelphic" integrals are not collinear.

A torus $(I_1^* \neq 0,\ I_2^* \neq 0)$ is filled by quasi-periodic motions having the frequencies

$$\dot{\phi}_1^* = \frac{\partial \mathcal{N}}{\partial I^*} = \nu_1(I_1^*,I_2^*),$$

$$\dot{\phi}_2^* = \frac{\partial \mathcal{N}}{\partial I_2^*} = \nu_2(I_1^*,I_2^*).$$

Orbits in that torus are transformed into each other by properly adjusting the additive constants in the phase angles ϕ_1^* and ϕ_2^*. In composition with the normalizing canonical transformation θ^*, this translation of origin in the torus constitutes what Whittaker (1916) called the *adelphic transformation*.

Exceptionally the constant values given to the integrals I_1^* and I_2^* can be such that the frequencies are commensurable: there could exist a pair (p_1,p_2) of relatively prime integers such that

$$p_2\nu_1(I_1^*,I_2^*) = p_1\nu_2(I_1^*,I_2^*). \qquad (21)$$

In this case, the quasi-periodic motions degenerate into the *ordinary* families of periodic orbits mentioned by Whittaker (1916). For such orbits, the fact that the adelphic integrals have linearly independent gradients implies that all multipliers are equal to +1; in each torus, all the orbits have the same period.

The totality of motions for the normalized system is aptly summarized in the plane of actions (I_1^*,I_2^*) (see Fig. 2). The origin represents the equilibrium; to each point along the I_1^*-axis corresponds one and only one singular periodic orbit in the family $\mathcal{F}_1$; likewise to each point of the I_2^*-axis corresponds one and only one singular periodic orbit in the family $\mathcal{F}_2$. Elsewhere in the plane of actions, the correspondance between pairs (I_1^*,I_2^*) and orbits of the system is one-to-many: any point (I_1^*,I_2^*) not on the action axes represents

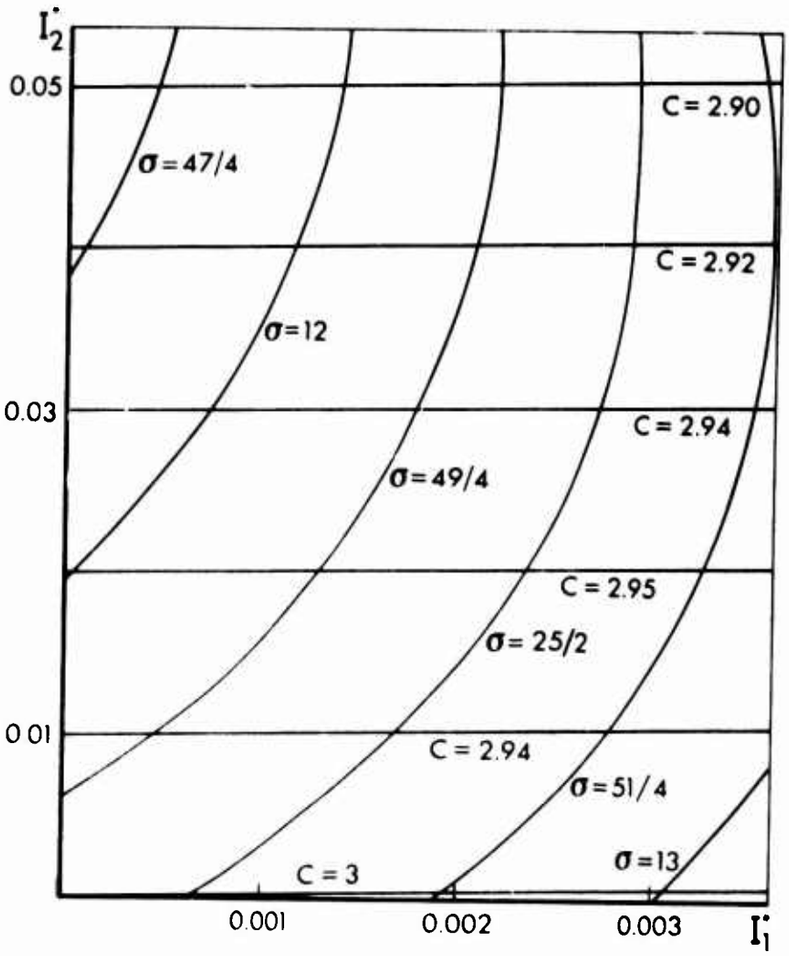

Fig. 2. - The plane of actions for the normalization (truncated at order 14) of the restricted problem around L_4 in the system sun-Jupiter.

the class of quasi-periodic motions transformed into one another by adelphic transformation.

There is a natural network of curves coordinatizing the plane of actions (I_1^*, I_2^*). On one hand the energy manifolds appear in the diagram as the one-parameter family of curves defined by the equation

$$\mathcal{H}(I_1^*, I_2^*) = \text{constant}.$$

On the other hand the ratio of frequencies

$$\rho = \nu(I_1^*, I_2^*)/\nu_2(I_1^*, I_2^*),$$

which we found suggestive to call the rotation number, defines another family of curves. When ρ is rational, each point of such a curve represents the family of ordinary periodic orbits filling the torus (I_1^*, I_2^*). We shall modify Whittaker's definition, and call ordinary family of periodic orbits the two-parameter set $\mathcal{O}(p_2/p_1)$ of ordinary periodic-orbits corresponding to the curve (20) in the plane of actions. Let us recall that, along $\mathcal{O}(p_2/p_1)$, the period is a function of the energy alone (Deprit and Henrard 1968, pp. 62-63).

Much expertise is gained from studying invariant sections through the phase space of a normalized system.

The figures that follow pertain to the normalization (truncated at order 14) for the restricted problem of three bodies around the triangular center of libration L_4 in the system sun-Jupiter. The state variables

are the Cartesian coordinates and velocities in the synodical frame of reference located at L_4. The sections are cut by the hyperplane $x = 0$ (the x-axis is in the direction of the line of syzygies oriented from sun to Jupiter).

A plane section of a three-dimensional energy manifold is a two-dimensional surface, which we project on the coordinate plane $(y,\dot{y})$ by lines parallel to the $\dot{x}$-axis.

One way of making this surface apparent in its projection is by plotting the projection of some of its curves. In the case of an integrable system, the most informative curves are obviously the traces of the invariant tori of quasi periodic orbits in the energy manifold, in particular the tori corresponding to ordinary families of periodic orbits. Fig. 3 displays for a *normalized system* sections of several ordinary families lying on the energy manifold at the equilibrium. The restricted problem at L_4 for the system sun-Jupiter was normalized up to degree 14; the residual in the Hamiltonian was neglected. Then the curves were computed from the series for the state variables $x,y,\dot{x},\dot{y}$ in terms of $\phi_1^*,\phi_2^*,I_1^*,I_2^*$. The integral of energy defines a region of the plane $(y,\dot{y})$ that will not be covered by the projection of the section (the area shaded in Fig. 3). On each curve that projects the intersection of a natural family of periodic orbits with the x-plane passing through the equilibrium, the points at which $\dot{x} \geq 0$ are joined by solid lines--they are located above the projection plane $(y,\dot{y})$; the

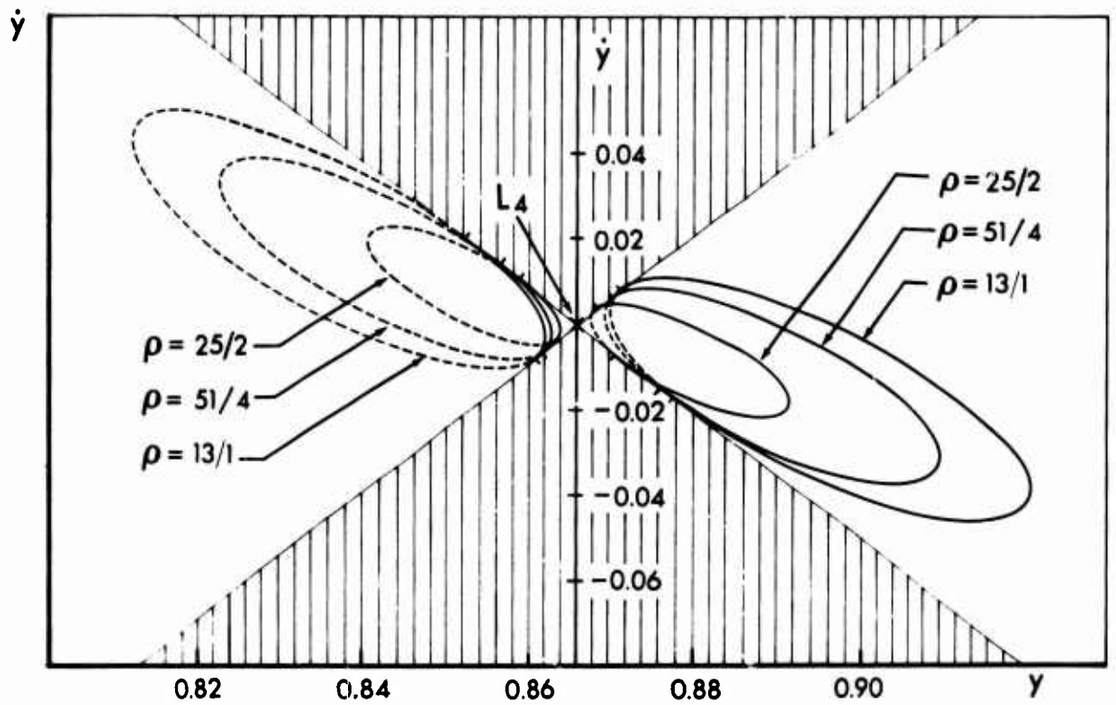

Fig. 3. - Projection on the plane $(y,\dot{y})$ of the section in the energy manifold $C = 3$ by the x-plane going through L_4.

points at which $\dot{x} \leq 0$ are joined by dashed lines. As can be seen from Fig. 3, invariant curves may have intersecting projections. But the points having the same projection correspond to distinct phase states: on one curve, $\dot{x}$ is > 0 whereas, on the other, it is < 0. In numerical explorations of invariant curves, it has been the rule for some time to plot only the intersection points at which $\dot{x}$ is ≥ 0; Fig. 3 makes it obvious that, on adopting this convention, one discards so much information that a correct interpretation of the sections could be in peril. We see for instance that the locus of intersection points with $\dot{x} > 0$ separates into two disconnected arcs located on both sides of L_4; the same is true for the curve of intersection points with $\dot{x} < 0$. However, in each quadrant, the arc corresponding to $\dot{x} > 0$ falls in continuity with the arc corresponding to $\dot{x} < 0$. Once this continuity is perceived, it is immediately rationalized: the plane $(y,\dot{y})$ cuts the torus, leaving part of it visible above it, while the rest remains hidden under the plane on which the section is projected.

In one respect, Fig. 3 may seem odd: the invariant sections do not contain the equilibrium L_4. But this peculiar feature is just another instance of the fact that we are dealing here with the *projection* of a surface onto a plane. Had we made the projection on the plane $(\dot{x},\dot{y})$, we would have found that section curves in the x-hyperplane do indeed go around the equilibrium L_4 (see Fig. 4).

Most analytical treatments of an equilibrium treated so far were concerned with Hamiltonians without gyroscopic terms (linear in the

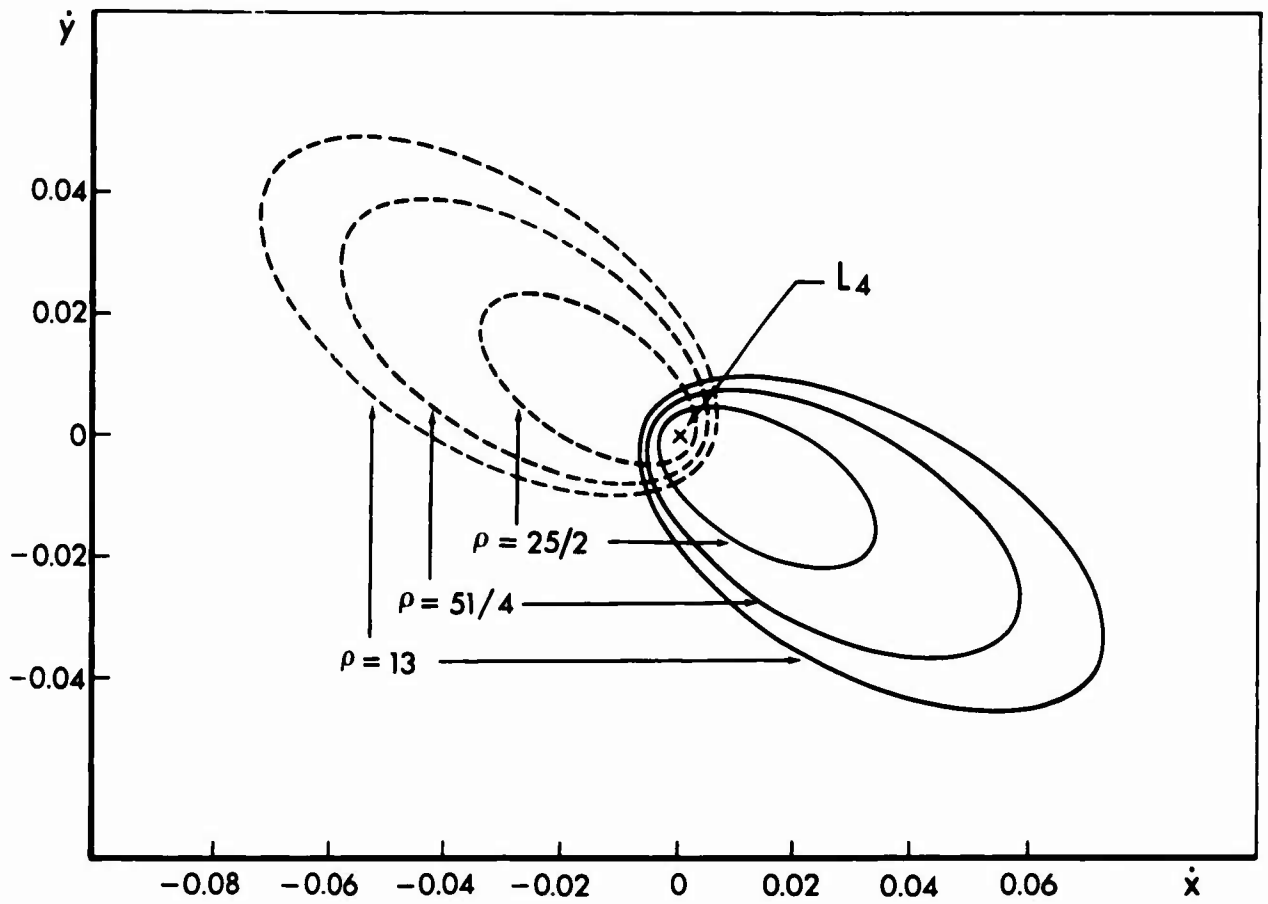

Fig. 4. - Projection on the plane $(\dot{x},\dot{y})$ of the section in the energy manifold $C = 3$ by the x-hyperplane going through L_4.

momenta) wherein the equilibrium constitutes a minimum on the potential surface. But, in the restricted problem, L_4 and L_5 are maxima on the potential surface, whose elliptic character in stability is conferred by the Coriolis forces due to the rotation of the synodical frame of reference. One has to expect that the global structures of the phase space differ substantially on the energy manifolds according as to whether they lie above $(C < 3)$ or below $(C > 3)$ the equilibrium configuration. We illustrate this change of patterns in the normalized system $\mathcal{N}$ in Fig. 5 where we present the projection of the section of the energy manifold $C = 2.9$ by the x-hyperplane passing through L_4.

Since we have in analytical form the general solution for the principal part $\mathcal{N}$, we can without much labor observe the evolution of invariant sections with changes in the Jacobi constant. By way of illustration, we followed the projection on the plane $(y,\dot{y})$ of the section by the x-plane going through L_4 of the energy manifold for the Jacobi constant going from $C = 3.0001015$ to $C = 2.9$. On each manifold we concentrated on the ordinary family of periodic orbits $O(25/2)$ corresponding to the rotation number $\rho = 25/2$. The sequence of curves (see Fig. 6) begin with two points: these are the intersection points of the orbit in the natural family $\mathcal{G}_1 = \mathcal{L}_4^{\ell}$ out of which branch the ordinary family. Then for a while, the invariant section is made up of two disjoint ovals; such is the case at $C = 3$. Later on, the components touch one another, and thereon the section takes

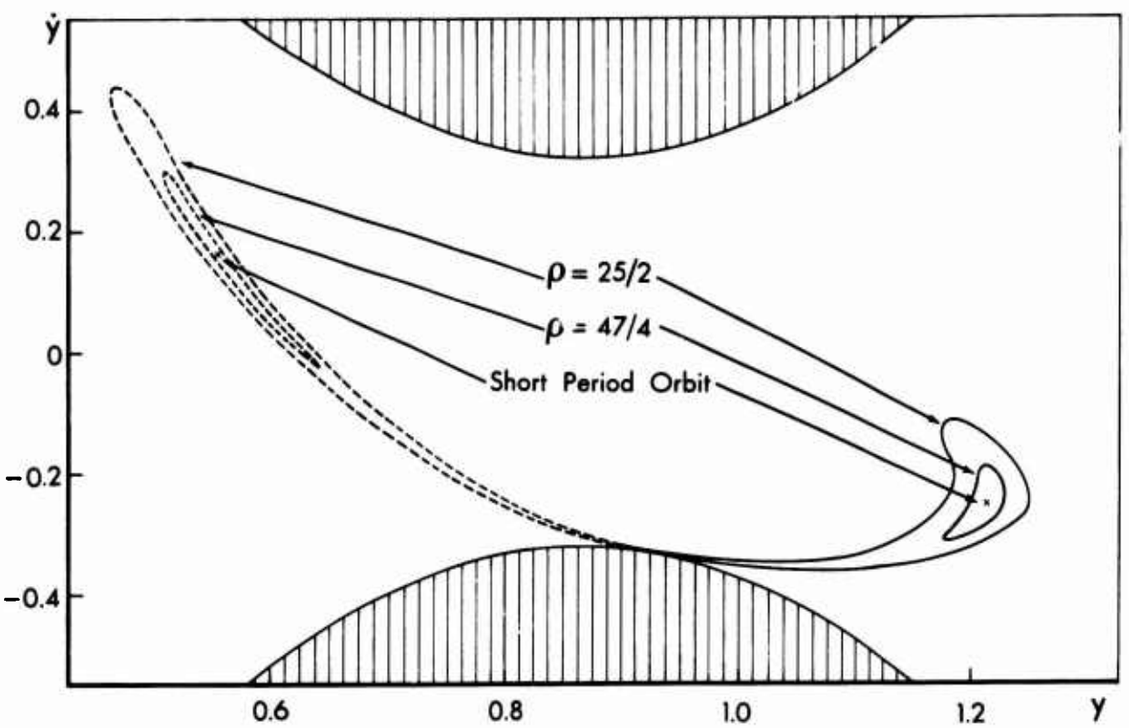

Fig. 5. - Projection on the plane $(y,\dot{y})$ of the section in the energy manifold $C = 2.9$ by the x-hyperplane going through L_4. (Normalized system).

the shape of two hinged crescents. There will come a value of C at which the crescents will separate; each closed component will contract and eventually, at C = 2.559, they will collapse each into a point corresponding to the orbit in the natural family $\mathcal{G}_2 = \mathcal{L}_4^s$ on which the ordinary family terminates.

The same pattern is found for the sections in the ordinary family of periodic orbits $O(12/1)$ (see Fig. 7). We know that the family originates at C = 2.960750380 from an orbit in the natural family $\mathcal{L}_4^s$, and that it terminates at C = 2.635008320 on an orbit in the same family. By the way, in the full restricted problem, the *ordinary* two-parameter family of periodic orbits associated with the commensurability 12/1 dissolves into a *natural* family which is a two lane bridge from $\mathcal{L}_4^s$ to $\mathcal{L}_4^s$. Of the curves drawn in Fig. 7, there remain only finite sets of isolated fixed points, half of them being elliptic (for the stable orbits in the bridge) and the other half being hyperbolic (for the unstable orbits). In this respect, Fig. 7 which refers to the truncated normalized system $\mathcal{N}$ should be compared with the corresponding figure published elsewhere (Deprit and Rabe 1969) for the full restricted problem of three bodies.

So far we have considered invariant sections in the map defined by the original Cartesian coordinates. It is not without interest to analyze them in the normalizing coordinates $(\phi_1^*, \phi_2^*, I_1^*, I_2^*)$ themselves. Thus in Fig. 8, we have plotted for three values of the Jacobi constant C the values of ϕ_1^* and ϕ_2^* of the points at which the periodic orbits in the ordinary family $O(25/2)$ intersect the x-hyperplane

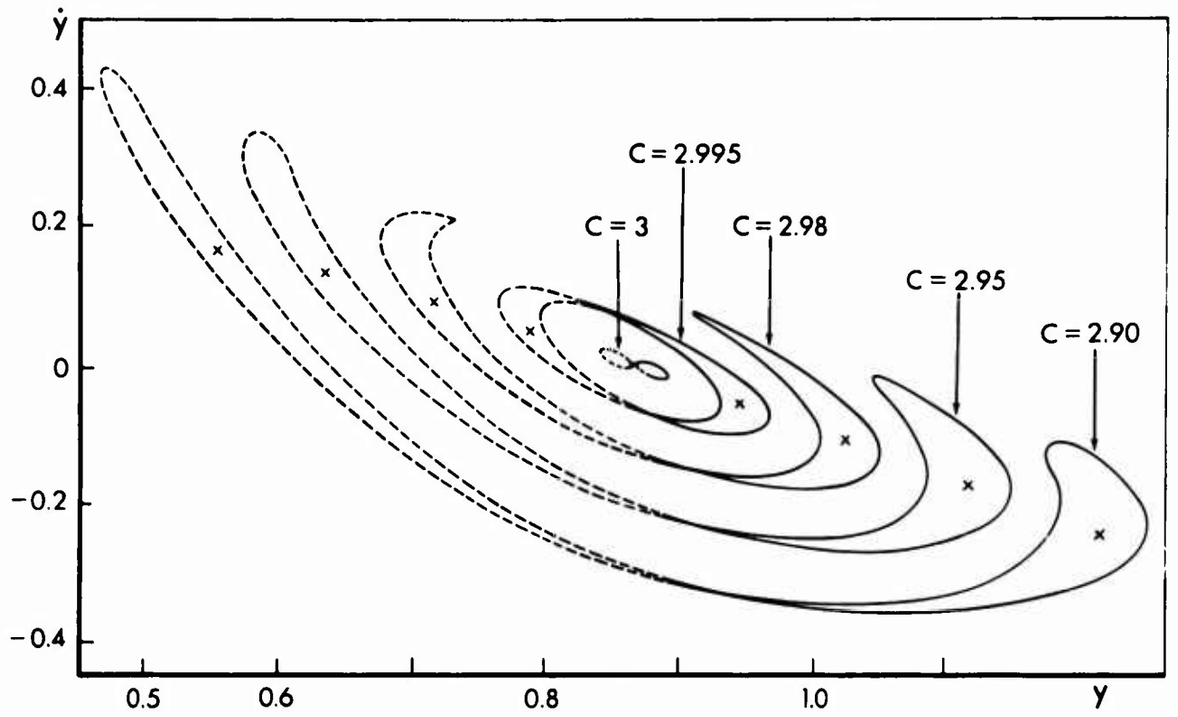

Fig. 6. - Evolution of the projection on the plane $(y,\dot{y})$ of the sections in the ordinary family $O(25/2)$.

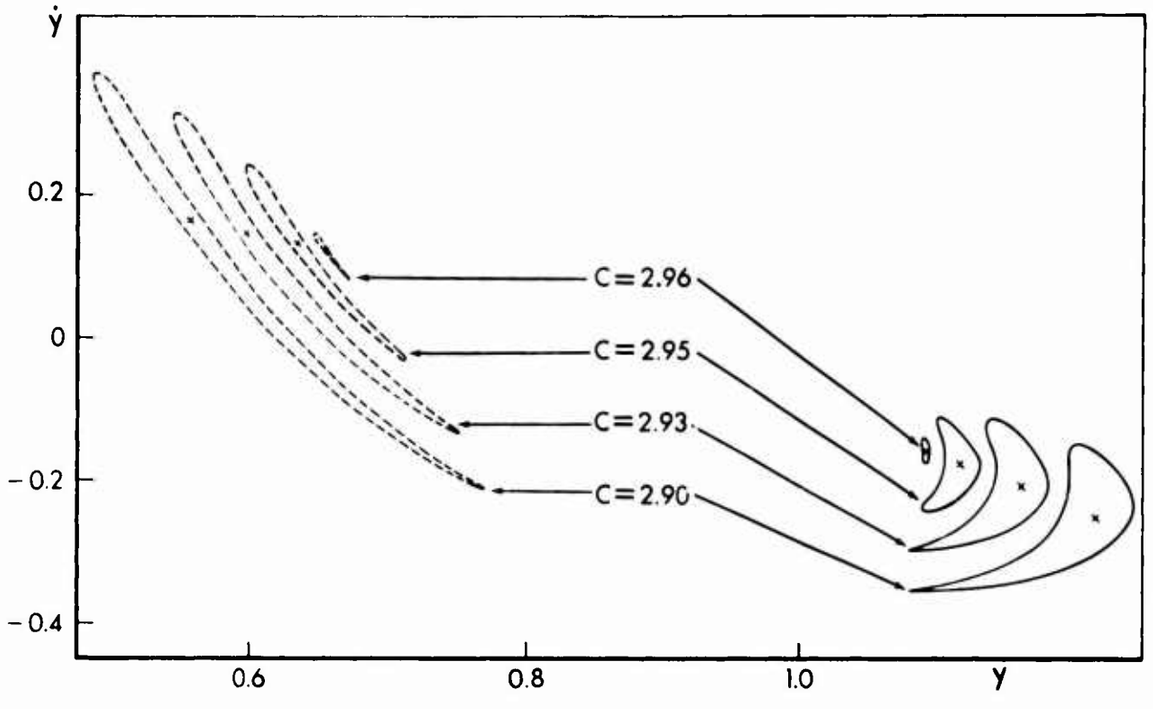

Fig. 7. - Evolution of the projection on the plane $(y,\dot{y})$ of the sections in the ordinary family of periodic orbits $O(12/1)$.

going through L_4. Actually Fig. 8 is but the transform of Fig. 6 by the normalizing mapping $(x,y,p_x,p_y) \rightarrow (\phi_1^*,\phi_2^*,I_1^*,I_2^*)$. At $C = 3.0001015$, i.e. the energy level at which the ordinary family bifurcates from the natural family $\mathcal{L}_4^{\ell}$, the section would consist of two lines $\phi_1^* = \text{constant}$: the family $O(25/2)$ reduces there to an orbit of long period. Somewhere between $C = 2.995$ and $C = 2.9$, the two branches of the intersection curve made a contact, and thereafter the curve reduces to a closed oval. As it expands in the direction of the angles ϕ_1^*, it will eventually touch its image in the congruence modulo 2π which defines the angle ϕ_1^*. From this level of energy onward, the intersection curve will split into two separate arcs. The evolution terminates when these arcs have become two straight lines $\phi_2^* = \text{constant}$. We are then at the energy level $C = 2.559$ at which the ordinary family $O(25/2)$ collapses into a short period orbit circuited 25 times.

The preceeding explorations cause us to question an empirical definition given lately for the rotation number ρ. Given a dynamical system, usually not integrable, the section of an energy manifold by a hyperplane is determined by numerical integration: on an orbit starting from a selection of initial conditions, the program marks the points at which the hyperplane is penetrated. In the absence of any qualitative information about the structure of the phase space around this orbit, Contopoulos (1967) proposes to define the rotation number as the average

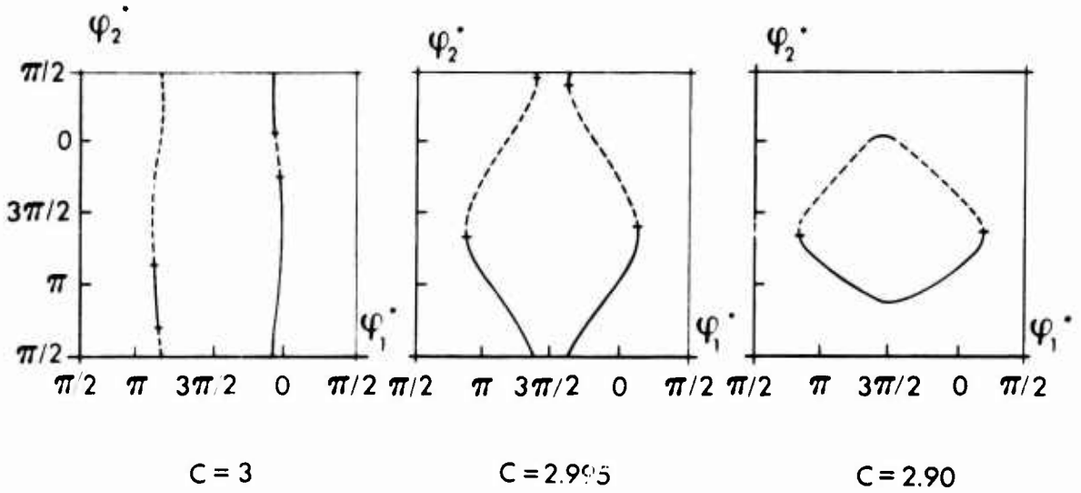

Fig. 8. - Evolution of the projection on the torus $(\varphi_1^*, \varphi_2^*)$ of the sections in the ordinary family $O(25/2)$ by the x-hyperplane going through L_4.

angle between successive pointers to the intersection points from what seems to be the fixed point inside the island circumscribed by the "invariant curve". Let us explain now on the principal part $\mathcal{N}$ for the restricted problem at L_4 how this empirical definition fails to be satisfactory. At the energy level $C = 2.560$ a bit after the family branched out of a short period orbit traveled 25 times, the orbits of $O(25/2)$ have 25 well developed loops, and they intersect the section plane in 25 points. But it takes two full rotations around the fixed point which is the trace of the short period orbit before the crossings come back on the first intersection point. Hence, according to Contopoulos, the rotation number would lie in the vicinity of 25/2. But, as C increases, the loops contract, so that fewer of them are crossing the section hyperplane. For instance at the level $C = 2.9$, we count only 16 intersections on the orbits of $O(25/2)$ after two full rotations: according to Contopoulos' definition, the rotation number would have decreased now to 16/2. Eventually the family approaches a long period orbit traveled twice. A little before $C = 3.0001015$ where the bifurcation takes place, the plane section presents only two crossing points, in which case Contopoulos' rotation number cannot be unambiguously estimated.

3. The resolution power of a normalization

How does the portrait in phase space for the normalized system $\mathcal{N}$ compare with that of the full system? In order to answer that question, it might seem logical to begin by inquiring about the convergence of Birkhoff's normalizing transformations. The few precise results mathematicians have so far established look discouraging. We know that, in an admissible set of power series provided with a natural topology, the subset of Hamiltonians for which Birkhoff's normalizations are divergent is everywhere dense (Siegel 1941), whereas the subset of Hamiltonians for which these transformations are convergent is of first category in the sense of Baire (Siegel 1954). However, Siegel's topology can be weakened sufficiently without yet losing a certain tightness so that the set of Hamiltonian systems for which Birkhoff's normalization converges appears as everywhere dense (Contopoulos 1963).

Such theorems show that so far the mathematical problem of convergence is not yet of physical significance. From a purely experimental point of view, we have learned to use normalization as a tool, much as a physicist uses a microscope. The ultimate structure in the phase space around the equilibrium is yet beyond our reach. Numerical integration of the original equations of motion will provide a detail here and there, but it will leave us totally ignorant of some global structures. For instance we might be interested in locating some invariant tori of quasi-periodic motions surrounding the equilibrium.

Let us assume that we are in the conditions which guarantee their existence for the full Hamiltonian $\mathcal{H}$. In this case, we could act under the assumption that, if the transformation is carried to a sufficiently high degree, the normalized system $\mathcal{N}$ will possess tori in the domain of phase space that we analyze, and that the perturbing residual $\mathcal{P}$ will not wipe them all from the phase space of the full system.

Normalization will reveal some of the features in the phase space around the equilibrium. Of course on the whole it will simplify them. Where it tells the story of two basic frequencies so well locked in a resonance that they generate an ordinary family of periodic orbits, we have to expect that, in the full problem, the ordinary family breaks down into isolated natural families of periodic orbits, some of them being of the elliptic type and thus surrounded by "islands" indicative of tori of quasi-periodic orbits, others being of the hyperbolic type and acting as epicentres in the middle of instability zones. But this distribution of stability islands and instability faults is very likely inscribed in annulus bordered by fringes of quasi-periodic motions with nonzero measure. The normalization is unable to resolve these fine details; yet through the blurring, an experienced investigator might gain a feeling for what complexity there is in the full problem.

A normalization truncated after the first few orders may be outrageously simplifying. The higher the order of truncation is, the more significant the approximation could be. Compare for instance Fig. 12,

13, and 1 of Gustavson (1966) where a progress for order 6 to 8 results in bringing forth more patently new fixed points in the sections, i.e. natural families in the phase space. Of course, like with any asymptotic series, there comes an optimum beyond which the resolution power of the normalization loses its significance. And it is likely that, even before the optimum in truncation is reached, numerical accuracy will have failed. For the normalization will have been obtained by so many operations on the intermediate results that the high order coefficients in the power series will have lost most, if not all, of their relative accuracy; besides coordinates and velocities obtained by adding a large number of small terms soon cease to be meaningful.

Whereas the mathematician spontaneously looks for bounded domains in which trajectories can be guaranteed to flow for ever without leaking or for uniform vicinities in which orbits are predicted to stay for ever close to one another, astronomers and physicists accommodate themselves with more realistic and thus less stringent requirements. In order to build ephemerides for Trojan planets from an analytical theory, it is sufficient to be certain that an intermediate orbit provided by a normalizing transformation about L_4 will stay for a sufficiently long time, although not for ever, reasonably close to an approximately quasi-periodic motion of the restricted problem of three bodies. In this respect we see two reasons why the normalized model may fail:

(i) As we carry the normalization up to an assigned order, *truncation errors* will steadily decrease the accuracy of the normalization as the normalized action integrals move away from their singular values at the equilibrium.

(ii) But also as the order of the normalization goes to infinity, we do not expect that the normalizing transformations converge uniformly in a neighborhood of the equilibrium. Rather we shall now show that there are regions where the structures of the phase space for the full problem differ significantly from the ones predicted by the normalization. In these regions we shall speak of *model errors*.

As we plan to test the normalizing approximation where we expect it to be at its worst, we focused our attention on the ordinary families of periodic orbits revealed by the normalization. Another test would have been to compare for the same initial conditions the solution of the restricted problem and the quasi-periodic motion of its normalization.

Actually the test on ordinary families of periodic orbits has been done under very convincing conditions. Four years before we even contemplated normalizing the restricted problem at L_4, we began computing invariant sections in the neighborhood. The differential and variational equations were integrated simultaneously by recurrent power series (Deprit and Price 1965) so as to assure at least twelve significant figures for the orbit and nine significant figures for the variations for at least several hundred units of time. At the level $C = 3.00007$, we obtained at first several curves strongly suggestive of quasi-periodic

motions; then we began looking for stable periodic orbits belonging to natural families and thus causing swirls to appear in the general stream of quasi-periodic motions. To our dismay, all we could get was several sets of periodic orbits with equal periods and the stability index $Tr(T)$ equal to 2 up to eight decimal figures. Moreover, in each case, the normal variation n_2 solution of Hill's homogeneous equation for the initial conditions $n_2(0) = 0$, $\dot{n}_2(0) = 1$, and the normal variation n_3 solution of Hill's nonhomogeneous equation for the initial conditions $n_3(0) = \dot{n}_3(0) = 0$ checked at the end of the period the relation

$$n_2(T)\dot{n}_3(T) - [\dot{n}_2(T) - 1]n_3(T) = 0,$$

as predicted for an ordinary periodic orbit (Deprit and Henrard 1968).

Initial conditions for these remarkable orbits constitute all the entries of Table I but the last two ones. The integrations have been carried out at two different seasons, hence unfortunately a slight variation of the mass ratio which reflects in the entries of the first row. It has induced small alterations in the corresponding periods.

The abundance of seemingly ordinary families of periodic orbits seemed odd to us at the time we discovered them. For in the opinion of some experts, "ordinary solutions...are never found after extensive search by many investigators" (Bray and Goudas 1967). Normalization of the restricted problem of three bodies was then undertaken to elucidate these results.

Table I. - Periodic orbits about L_4 at C = 3.000/

$x_0 = \frac{1}{2} - \mu$	y_0	$10^2 \times \dot{x}_0$	$10^2 \times \dot{y}_0$	T	T_r	$\dot{y}(T) - \dot{y}_0$	Type
0.499046124643	0.875811000000	0.741285707271	-0.948138431840	157.6060220484	2.	10^{-11}	$\mathcal{B}$(2L,25S)
0.499046124643	0.873103968233	0.514863308876	-0.4	157.6060220484	2.	10^{-11}	$\mathcal{B}$(2L,25S)
0.499046124643	0.88	1.871204074140	-0.418210107500	157.6060220484	2.	10^{-11}	$\mathcal{B}$(2L,25S)
0.499046124643	0.882	2.152621945760	-0.617656719500	157.6060220484	2.	10^{-11}	$\mathcal{B}$(2L,25S)
0.499046125	0.873436700422	0.708500472084	-0.176774247117	157.6060215396	2.	10^{-11}	$\mathcal{B}$(2L,25S)
0.499046125	0.876157017727	1.255265116370	-0.164851001412	157.6060215396	2.	10^{-11}	$\mathcal{B}$(2L,25S)
0.499046125	0.878173947584	1.591629922460	-0.275299811791	157.6060215396	2.	10^{-11}	$\mathcal{B}$(2L,25S)
0.499046125	0.884250078409	2.429579410386	-0.911597719216	157.6060215396	2.	10^{-11}	$\mathcal{B}$(2L,25S)
0.499046124643	0.884083396388	2.568824748034	0	239.5919061022	2.	2×10^{-11}	$\mathcal{B}$(3L,35S)
0.499046124643	0.901495482010	4.741199226697	-2.2	239.5919061022	2.	2×10^{-11}	$\mathcal{B}$(3L,35S)
0.499046125	0.87521279504	1.02748114216	0.37206548500	239.591905382	1.9999999	10^{-10}	$\mathcal{B}$(3L,35S)
0.499046125	0.89363930292	3.94104403951	-0.88803994336	239.591905382	2.	3×10^{-10}	$\mathcal{B}$(3L,35S)
0.499046124643	0.90731560831	5.31078685620	-3.	321.576015385	2.	10^{-10}	$\mathcal{B}$(4L,51S)
0.499046124643	0.88707513748	3.03464717944	0	321.576015383	2.	5×10^{-11}	$\mathcal{B}$(4L,51S)
0.499046125	0.880336072	1.9320286989	0.3951507597	321.5760144	2.	10^{-9}	$\mathcal{B}$(4L,51S)
0.499046125	0.891786454	3.7374755053	-0.3901338478	321.5760144	2.	10^{-9}	$\mathcal{B}$(4L,51S)
0.499046124643	0.888605112188	3.270603375861	0	403.559475504	2.	3×10^{-11}	$\mathcal{B}$(5L,64S)
0.499046124643	0.91000681314	5.76827965747	-3.	403.559475504	2.	10^{-10}	$\mathcal{B}$(5L,64S)
0.499046125	0.8995341525	4.8209615761	-1.038524844	403.5594743	2.	2×10^{-9}	$\mathcal{B}$(5L,64S)
0.499046125	0.8810603377	2.0404415275	0.4479494544	403.5594743	2.	10^{-9}	$\mathcal{B}$(5L,64S)
0.499046125	0.8943891196	4.1339904508	-0.4052928442	485.5426243	2.	10^{-9}	$\mathcal{B}$(6L,77S)
0.499046125	0.8877133841	3.1304588034	0.1318462230	485.5426243	2.	10^{-9}	$\mathcal{B}$(6L,77S)
0.499046124643	0.896953806806	2.6401964382	-3.6960096291	81.9824816427	2.00062536	10^{-11}	$\mathcal{L}_4^{\ell}$
0.499046125	0.917056143709	6.9691551649	-2.9095347433	81.9826562186	1.99937434	10^{-12}	$\mathcal{B}$(13S,14S)

We started with initial conditions which, for the normalized model, correspond to ordinary families of periodic orbits. We integrated numerically the orbits they generate in the restricted problem and stopped the calculation when we reached the estimate of the period predicted by the normalization. Then either, within the assigned tolerance, the final state was equal to the initial one, or there was a discrepancy. In the former case, the orbit was held to be periodic. In the latter, by trial and error, the initial conditions were adjusted so as to produce a periodic orbit.

If its characteristic exponents came all four very close to zero, then we restarted the search for periodic orbits along the invariant curve of the ordinary family of periodic orbits in the normalized problem. This was a cumbersome and time-consuming investigation; the usual differential corrections to converge onto a periodic orbit fail when the characteristic exponents are so definitely close to zero. Thus we had to satisfy ourselves with a few orbits for each invariant case. Anyway, first we recovered in this manner the ordinary families that had been discovered prior to the normalization, and, then, on each of them, we added a few more. It is reasonable to conclude that, in a close neighborhood of L_4, within the accuracy at present available by careful calculations in double precision the structure of the phase space about L_4 cannot be magnified beyond the resolution of a Birkhoff's normalization of the thirteenth order. The ordinary families of the normalized system for the rotation numbers 25/2, 35/3, 51/4, 64/5 and 77/6 could not be broken down into natural families of periodic orbits.

Yet further away from the equilibrium in the phase plane (I_1^*, I_2^*), the torus of the normalized system which is filled with the ordinary periodic orbits corresponding to the rotation number 13/1 is washed away from the restricted problem. In its place, numerical integration detected long and thin islands. But the crossing points on the section curves move so slowly that we did not think it worthwhile to calculate the complete pattern. In one case, we had followed a trajectory for 1600 units of time (i.e. 30000 solar years), and yet we had not even traveled halfway around the island. Instead we applied variational corrections to converge onto the subsiding natural periodic orbits. One of them (see the last two lines of Table I) has unstable characteristic exponents, and we discovered later on by numerical continuation that it is an element of the family $\mathcal{L}_4^{\ell}$ on its way back from its maximum Jacobi constant. The other natural periodic orbit has characteristic exponents of the stable type; we also established later on that it belongs to the bridge $\mathcal{B}(13S,14S)$ of periodic orbits connecting an element of $\mathcal{L}_4^{s}$ traveled 13 times to another element of $\mathcal{L}_4^{s}$ traveled 14 times.

The dissolution of the family $\mathcal{O}(13/1)$ could be rationalized on two grounds: not only does the resonance 13/1 occur in the normalized system further away from the equilibrium than the commensurabilities previously encountered, but also, as indicated by Contopoulos (1967), the width of an island at the resonance m/n is proportional to $\varepsilon^{\frac{1}{2}(m+n-4)}$, where ε is an estimate of the distance to the equilibrium. In the present case, the sum m + n drops to its minimum value when

the rotation number passes by the ratio 13/1, and therefore in an intuitive way, we expect significantly wider islands along the invariant curve of $O(13/1)$.

The breakdown of an ordinary family into a pair of natural families with characteristic exponents of opposite type in stability is but an illustration of familiar statements of Poincaré. What it tells also is that the normalization is only a local tool, and that it fails to reveal the *global* structure of the phase space. Indeed, in the normalized approximation, the ordinary family $O(13/1)$ constitutes a two-parameter bridge connecting a natural orbit of $\mathcal{L}_4^{\ell}$ and a natural orbit of $\mathcal{L}_4^{s}$ travelel thirteen times. But, in the restricted problem, the two natural families which survive the destruction of $O(13/1)$ have certain terminations modified.

Our findings at the energy level C = 3.00007 are presented in Fig. 9.

A similar analysis has been undertaken at the energy level C = 2.90. We find there the same pattern. Close to the fixed point belonging to the short period orbit--an element of the natural family $\mathcal{L}_4^{s}$--the ordinary families of the normalized system subsist in the restricted problem. Such is the case, for instance, of the family $O(35/3)$. But at a greater distance, the invariant curves of the ordinary families break down into interspersed subsets of isolated fixed points corresponding to natural families of periodic orbits. The invariant curve of the ordinary family $O(12/1)$ gives way to the elliptic fixed points

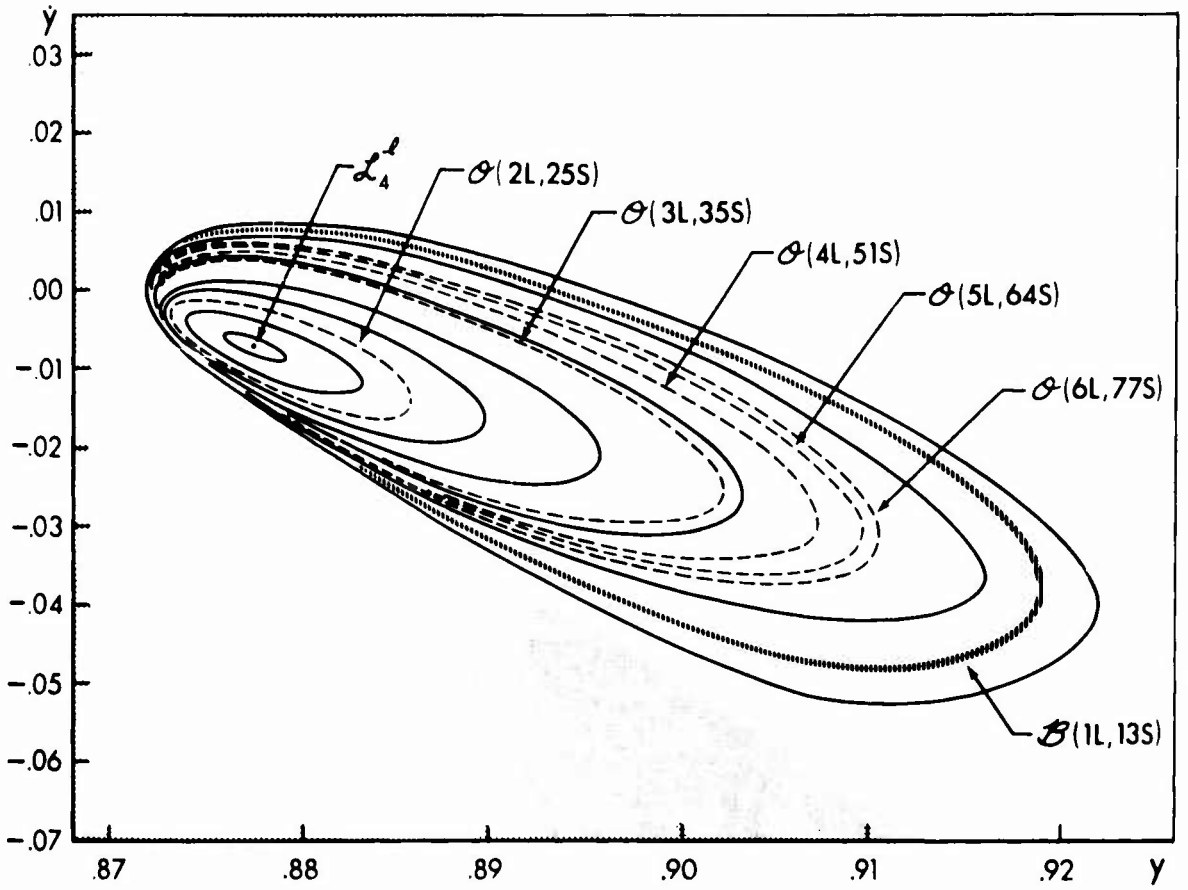

Fig. 9. - Projection on the plane $(y,\dot{y})$ of the section in the energy manifold $C = 3.00007$ by the x-hyperplane going through L_4 (Restricted problem of three bodies for the system sun-Jupiter).

(the small circles on Fig. 10) of a stable periodic orbit and the hyperbolic fixed point (the crosses on Fig. 10) of an unstable periodic orbit; they belong respectively to the stable and the unstable lane of a bridge $\mathcal{B}(12S,12S)$ connecting a short period orbit at $C = 2.960750380$ traveled 12 times to the short period orbit at $C = 2.635008320$ also traveled 12 times (Deprit and Rabe 1969). The ordinary family $\mathcal{O}(25/2)$ dissipates likewise, making place to the stable and unstable lanes of a natural bridge $\mathcal{B}(25S,2L)$ connecting the short period orbit at $C = 2.559466612$ traveled 25 times to the long period orbit at about $C = 3.0003$ traveled twice. Also the curve of $\mathcal{O}(13/1)$ survives only in the elliptic fixed points imprinted by the orbit in the stable leg of the bridge $\mathcal{B}(13S,14S)$, and the hyperbolic fixed points of the unstable orbit in the family $\mathcal{L}_4^{\ell}$ on its way from a maximum C to its termination on the short period orbit at $C = 2.501429345$. Beyond this invariant curve, the normalization truncated beyond order 13 is no longer a valid approximation. In the shaded area of Fig. 10, we have marked by x's the intersection points of the unstable periodic orbit in the return leg of the bridge $\mathcal{B}(18S,19S)$, although we have not tried to elucidate the structure of the phase space in that remote part of the energy manifold.

In this respect we cannot leave our readers unaware of the over-abundant amount of numerical integrations required to produce but the handful of conclusions we came to so far. As an illustration we have blown up in Fig. 11 a minute area of the cross section plotted in Fig. 10. This is a rectangle covering the dissipated invariant curve

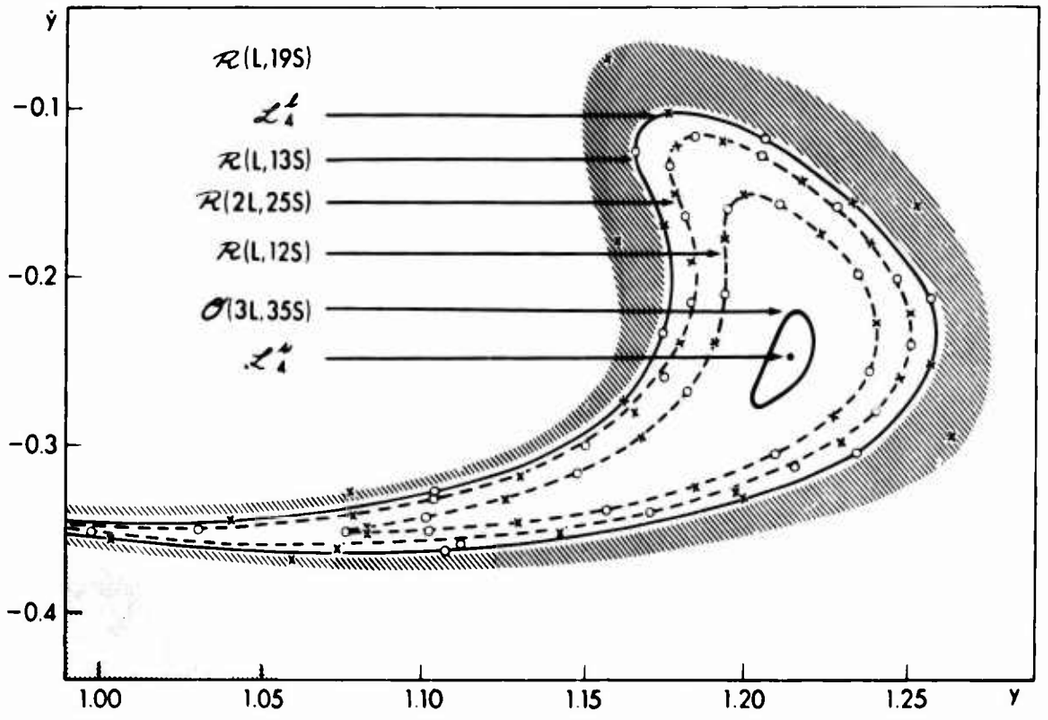

Fig. 10. - Projection on the plane $(y,\dot{y})$ of the section in the energy manifold $C = 2.9$ by the x-hyperplane going through L_4 (Restricted problem of three bodies for the system sun-Jupiter)

relative to $\mathcal{O}(25/2)$ between two successive hyperbolic fixed points belonging to the two-lane bridge $\mathcal{B}(25S,2L)$. The closed curve around the elliptic fixed point demanded that the numerical integration be extended over 5723 units of time (i.e. about 10,000 solar years), yet keeping fixed 10 decimal figures of the Jacobi constant.

At the level $C = 3.00007$, the ordinary family $\mathcal{O}(25/2)$ of the normalized system seemingly subsists in the restricted problem, at least within the constraints of an accuracy limited to the double precision arithmetic of an IBM 7094; but at $C = 2.9$, it has dissolved. The dissolution is progressive as the Jacobi constant recedes from its value $C = 3$ at the equilibrium state. As the natural family $\mathcal{B}(25S,2L)$ moves from its bifurcation out of $\mathcal{L}_4^S$, we noticed that the stability index at first grows larger than 2 along the unstable lane of the bridge, then decreases toward 2 and, starting with $C = 2.99$, it stays equal to 2 up to nine decimal figures (see Fig. 12). At this point the natural family has entered the zone around L_4 where the restricted problem cannot be detached from its normalizing approximation.

The neighborhoods of the equilibrium that we considered often times in this analysis are of course defined in the plane of normalized actions (I_1^*, I_2^*). Mapped back into the original Cartesian phase space, the orbits we had under study usually depart significantly from the textbook concept of an infinitesimal neighborhood of L_4. By way of illustrating this contrast, we have drawn in Fig. 13 the unstable orbit of the bridge $\mathcal{B}(12S,12S)$ that we discovered on the energy manifold $C = 2.9$.

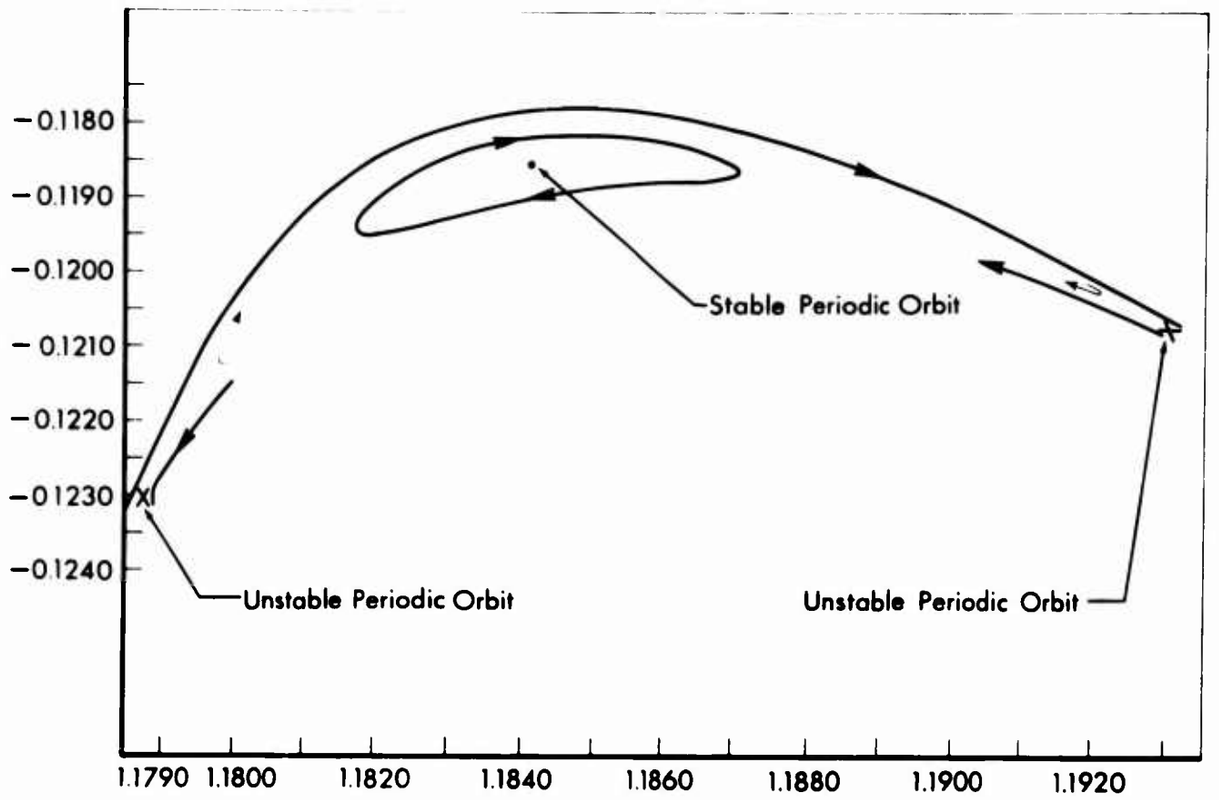

Fig. 11. - Quasi-periodic motions on the energy manifold C = 2.9 in the neighborhood of the two-lane bridge $\mathcal{B}$(25S,2L).

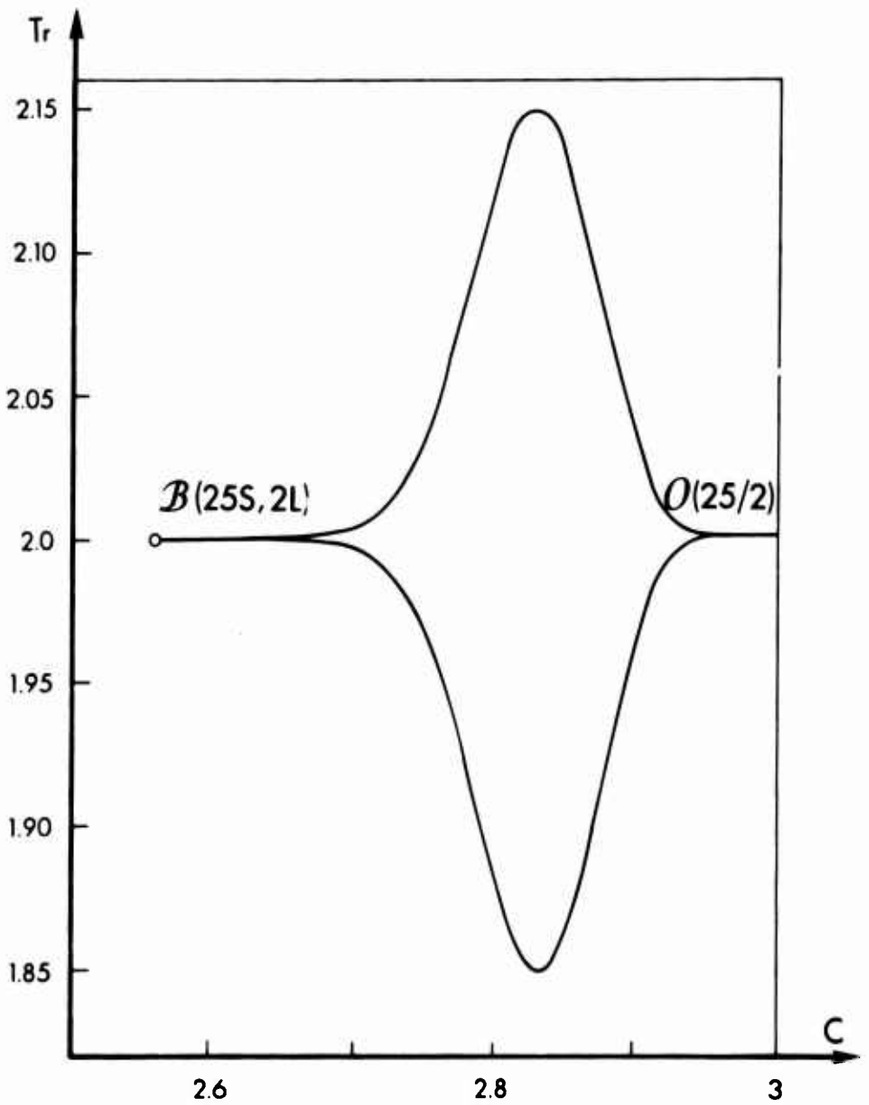

Fig. 12. - Stability curve for the stable and unstable lanes of the bridge $\mathcal{B}(25S,2L)$ showing the progressive blurring from a natural family into the ordinary family $O(25/2)$.

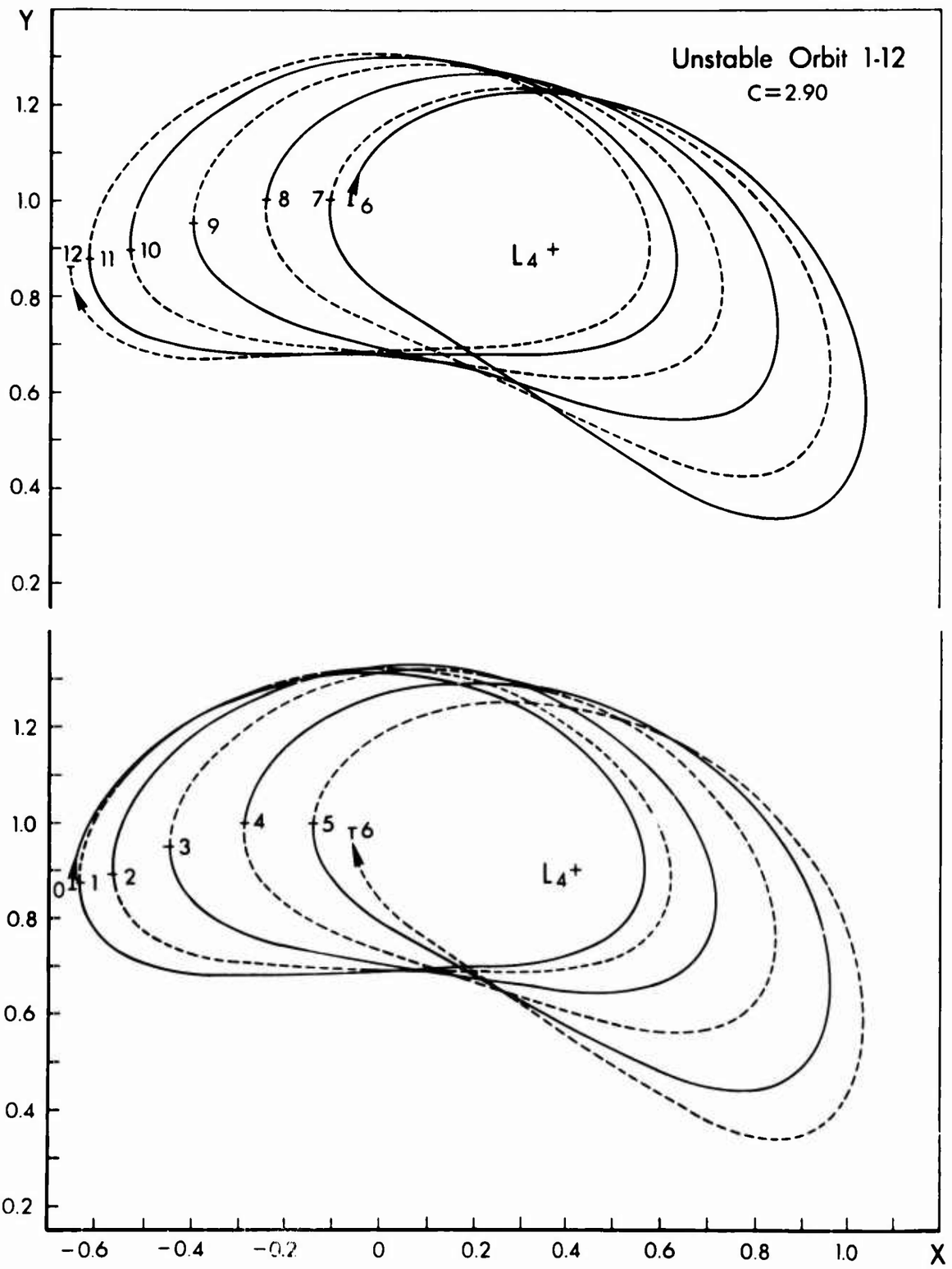

Fig. 13. - An orbit in the unstable lane of the natural bridge $\mathcal{B}(12S,12S)$ for the restricted problem of three bodies.

Conclusions

The relations between a dynamical system having an equilibrium and its truncated Birkhoff's normalization around that critical element may stir as much controversy as the existence of the so-called third integral in galactic dynamics. As a matter of fact the two formalisms bear on the same technique which is to approximate, of course in a conveniently unspecified topology, a dynamical system--integrable or not--by one whose variables are obviously separable. The technique of the third integral consists in building formally an integral that is both independent of the energy and in involution with it; from thereon one relies on Liouville's theorem to provide the capability of constructing a system of separable coordinates. On the other hand, Birkhoff's normalization aims straightforwardly at constructing these separable coordinates, in such a way that, the angle coordinates being ignorable, the action momenta turn out to be the independent integrals in involution sought by the technique of the third integral.

From the mathematical standpoint, the basic concern so far has been with deciding whether or not the third integral exists, or equivalently whether Birkhoff's normalizing transformations form a convergent sequence. Whatever the answer to that question may be, and it may well be in the negative, one cannot but wonder at the many and deep physical insights these mathematically crippled techniques provide in astronomy. In a way numerical experimentations in celestial mechanics compel us to adopt the kind of pragmatism that pervades observational astronomy.

Birkhoff's normalization acts as a lens focused on a dynamical system. It helps in penetrating the complexity beyond the first order appearances. Yet its resolution power is limited. There comes a threshold beyond which the present computer equipment does not allow us to separate the normalized image from the real phase space. Fortunately, at least in the case we have had under special investigation, to learn that there is a hard core in which double precision arithmetic cannot divide the restricted problem from its normalization truncated beyond order 13 is not unwelcome news. At least it offers a way of substantially improving the theory of Trojan planets by building it on Birkhoff's normalization.

On the whole the present study confirms numerically the views held by Contopoulos and others (see for instance Walker and Ford 1969). For librations around the equilibrium with sufficiently small amplitudes, most orbits are not influenced by resonances in a way that could presently be detected by accurate integrations carried over a long time; yet one is already able to detect exceptional motions influenced by isolated resonances. Mathematically speaking these motions are densely populating the area covered by the well behaved majority; however Birkhoff's normalization filters only a few of the exceptional motions, namely those corresponding to the smallest resonances, and thus isolates them. We have shown here that numerical integration, however accurate, could do no better than normalization in this respect.

Appendix 1

The program

The basic information on how we represent Poisson series (3-3) in the computer has been given elsewhere (Deprit and Rom 1968 a, 1968 b). We concentrate here on the specific program which implements Birkhoff's normalization.

The function BCOEF(N,M) computes in double precision arithmetic the binomial coefficient $\binom{N}{M}$. Dr. David Walkup, our colleague at B.S.R.L., is the author of this program.

Before we describe the next subroutines, let us agree on some conventions. We operate exclusively with Poisson series of the type (3-3); elements of a series are of the form

$$x_1^{n_1} x_2^{n_2} x_3^{n_3} \cos(m_1;_1 + m_2;_2 + m_3;_3),$$

$$x_1^{n_1} x_2^{n_2} x_3^{n_2} \sin(m_1;_1 + m_2;_2 + m_3;_3).$$

The first polynomial argument is not used, the second stands for $\sqrt{L}$, and the third for $\sqrt{S}$; the first trigonometric argument refers to ℓ, the second refers to s, and the third is not used.

The subroutine PDS(A,B,J) differentiates the Poisson series A with respect to the polynomial argument x_J and stores the result as the Poisson series B.

The subroutine PDJS(A,B,α,β,γ) applies to the Poisson series A the differential operator

$$\alpha \frac{\partial}{\partial \phi_1} + \beta \frac{\partial}{\partial \phi_2} + \gamma \frac{\partial}{\partial \phi_3}$$

and stores the result as the Poisson series B.

The subroutine PROD3(A,B,C,α) stores as the Poisson series C the result of the operation C + αAB.

These three commands enable us to compute the Jacobian determinants

$$\frac{\partial A}{\partial \phi_1} \frac{\partial B}{\partial x_2} - \frac{\partial A}{\partial x_2} \frac{\partial B}{\partial \phi_1}, \qquad \frac{\partial A}{\partial \phi_2} \frac{\partial B}{\partial x_3} - \frac{\partial A}{\partial x_3} \frac{\partial B}{\partial \phi_2}$$

But, when we compute the Poisson bracket of two functions in L, S, l and s, we should not overlook that x_2 stands for $\sqrt{L}$ and not for L, and that x_3 stands for $\sqrt{S}$ and not for S. Therefore

$$\frac{\partial(A,B)}{\partial(\ell,L)} = \frac{1}{2} \frac{1}{x_2} \frac{\partial(A,B)}{\partial(\phi_1,x_2)},$$

$$\frac{\partial(A,B)}{\partial(s,S)} = \frac{1}{2} \frac{1}{x_3} \frac{\partial(A,B)}{\partial(\phi_2,x_3)}.$$

Therefore we need the subroutine DVLS(A,J) which replaces the Poisson series A by the series A/x_J.

The various partial derivatives of a d'Alembert series and their products do not necessarily possess the d'Alembert characteristic. But the exceptional terms in the products $\frac{\partial A}{\partial \ell} \frac{\partial B}{\partial L}$ and $\frac{\partial A}{\partial L} \frac{\partial B}{\partial \ell}$ cancel one another in the formation of the Jacobian determinant $\frac{\partial(A,B)}{\partial(\ell,L)}$. In the computer, should we proceed in integer arithmetic, all the exceptional terms would disappear in the Jacobian which thus would possess the d'Alembert characteristic exactly. But, as we proceed only in double

precision, the expected cancellations are missed by relatively small amounts. The subroutine DALPIC(A,B) remedies this deficiency: it scans the elements of A checking for each one such that $|m_1| \leq n_2$, $|m_2| \leq n_3$, $n_2 \equiv m_1 \pmod 2$ and $n_3 \equiv m_2 \pmod 2$. The terms which satisfy these properties are stored as the Poisson series B.

The preceding subprograms are called in the subroutine DPBRAC(/A/,/B/,/C/). The Poisson series A and B are assumed to have the d'Alembert characteristic, the resulting Poisson series C is the d'Alembert series

$$C = \frac{\partial(A,B)}{\partial(\ell,L)} + \frac{\partial(A,B)}{\partial(s,S)} .$$

i.e., the Poisson bracket (A;B), of the functions A = A(ℓ,s,L,S) and B = B(ℓ,s,L,S) in the phase space (ℓ,s,L,S).

DPBRAC is the main ingredient of the subroutine LTRAN(F,W,P,Q,N,/ANS/). It is assumed that all Poisson series entering the triangle

$$\begin{array}{ccccccc}
 & & & f_0 & & & \\
 & & f_1 & & f_0^{(1)} & & \\
 & f_2 & & f_1^{(1)} & & f_0^{(2)} & \\
f_3 & & f_2^{(1)} & & f_1^{(2)} & & f_0^{(3)}
\end{array}$$

---

have been defined up to and including those belonging to the (n-1)-th row. Note that these elements are labeled in FORTRAN as follows

F(1,1)

F(1,2) F(2,1)

F(1,3) F(2,2) F(3,1)

F(1,4) F(2,3) F(3,2) F(4,1)

Also it is assumed that the Poisson series $W_1, W_2, \ldots, W_n$ have received a FORTRAN name, W(1),W(2),...,W(N). The subroutine computes the element $f_q^{(p)}$ from the rule

$$f_Q^{(P)} = \sum_{0 \le j \le Q} \binom{Q}{j} \left(f_{Q-j}^{(P-1)}; W_{j+1} \right) + f_{Q+1}^{(P-1)}$$

and stores it as the Poisson series ANS. One should not overlook that LTRAN assumes that input and output series have the d'Alembert characteristic; it cannot be regarded as a general subroutine to compute any element in the triangle of a Lie transform under any canonical transformation.

BIRKOF(NMAX) essentially performs a cycle to generate from the sequence $\mathcal{H}_0, \mathcal{H}_1, \ldots, \mathcal{H}_{NMAX}$ the sequence $\mathcal{H}_0, \mathcal{H}_0^{(1)}, \ldots, \mathcal{H}_0^{(NMAX)}$ that is the normalized Hamiltonian, and the sequence $W_1, W_2, \ldots, W_{NMAX}$ that defines the normalizing transformation. The N-th cycle begins by setting W_N equal to zero; it is accomplished by declaring that W_N--whose FORTRAN designation is W(N+1)--is the Poisson series in formation at the bottom of the stack. As long as no element is entered in W_N, it will represent the null d'Alembert series. Then the provisional elements $\tilde{\mathcal{H}}_{N-1}^{(1)}, \tilde{\mathcal{H}}_{N-2}^{(2)}, \ldots, \tilde{\mathcal{H}}_0^{(N)}$ are formed by calling recursively LTRAN; they are stored in the disk, except $\tilde{\mathcal{H}}_0^{(N)}$ which is kept in core in

the temporary series H. The subroutine COLECT transcribes in $\mathcal{H}_0^{(N)}$ the nontrigonometric terms of $\tilde{\mathcal{H}}_0^{(N)}$, as indicated by the value (0,0,0,0) given initially to its argument TPART (N.B. Nontrigonometric terms in Poisson series of type (3,3) are characterized as having the last four flags of their label word equal to zero). At this point we have obtained a new component of the normalized Hamiltonian.

The computation of W_N begins by storing in P the purely trigonometric part $\mathcal{H}_0^{(N)}$ by means of the subroutine SCREEN. Then IPDJS accomplishes the quadrature

$$W_N = \frac{1}{D(\lambda,\sigma)} P$$

and stores the resulting Poisson series in W_N. It is punched in hexadecimal (HEXOUT) and it is written on the disk (disk address stored in W(N+1)). Unless the program has reached the last component W_N to be obtained, the triangle elements $\tilde{\mathcal{H}}_{N-1}^{(1)}, \tilde{\mathcal{H}}_{N-2}^{(2)}, \ldots, \tilde{\mathcal{H}}_1^{(N-1)}$ are then completed by subtracting from each of them the periodic part P of $\tilde{\mathcal{H}}_0^{(N)}$. The N-step of the cycle ends on a reading of the clock to show the time spent in the operations at that stage.

The subroutine BIRKOF(NMAX) is escorted by another program which produces the Lie transform of the coordinates x,y and the velocities $\dot{x},\dot{y}$. Obviously it is built around a cycle calling recursively the subroutine LTRAN.

Appendix 2

Listings

```
FORTRAN IV    MODEL 44  PS        VERSION 2,  LEVEL 1    DATE  68271

0001                SUBROUTINE BIRKOF(NMAX)
0002                REAL*8 LAMBDA,SIGMA
0003                COMMON/HAM/LAMBDA,SIGMA,F(20,20),W(20)
0004                INTEGER P,Q,TPART(4)/4*0/
0005                DO 100 N=1,NMAX
0006                CALL DEFINE(W(N+1))
0007                CALL GETIME(I1)
0008                Q=N-1
0009                DO  10 J=1,N
0010                CALL LTRAN(F,W,J,Q,20,H)
0011                IF(J.EQ.N) GO TO 12
0012                CALL SEEK(Q)
0013                CALL WRITEP(H,F(J+1,Q+1),-1,0)
0014                CALL ERASE(H)
0015             10 Q=Q-1
0016             12 CALL COLECT(H,F(N+1,1),TPART)
0017                CALL LIST3(F(N+1,1),'H',-N)
0018                CALL HEXOUT(F(N+1,1))
0019                CALL SCREEN(H,P,TPART)
0020                CALL ERASE(H)
0021                CALL IPDJS(P,WN,LAMBDA,SIGMA)
0022                CALL HEXOUT(WN)
0023                CALL SEEK(Q)
0024                CALL ERASE(W(N+1))
0025                CALL WRITEP(WN,W(N+1),-1,200+N)
0026                CALL ERASE(WN)
0027                IF(N.EQ.1) GO TO 101
0028                IF(N.EQ.NMAX) GO TO 101
0029                N1=N-1
0030                Q=N
0031                DO 20 J=1,N1
0032                J1=J+1
0033                CALL DEFINE(TEMP)
0034             ,  CALL ACUM1(P,TEMP,-1.0D0)
0035                CALL  ACUM1(F(Q,J1),TEMP,1.0D0)
0036                CALL SEEK(F(Q,J1))
0037                CALL WRITEP(TEMP,F(Q,J1),-1,0)
0038                CALL ERASE(TEMP)
0039             20 Q=Q-1
0040            101 CALL GETIME(I2)
0041                ETM=I2-I1
0042                ETM=(ETM/192C0.)/60.
0043                PRINT 40,N,ETM
0044             40 FORMAT(5X,'TIME FOR',I3,' = ',F7.2)
0045            100 CALL ERASE(P)
0046                RETURN
0047                END
```

```
FORTRAN IV    MODEL 44  PS        VERSION 2,  LEVEL 1   DATE  68271

 0001               SUBRCUTINE DPBRAC(/F/,/G/,/PB/)
 0002               REAL*8 LS(2,2)/1.D0,0.,0.,1.D0/
 0003               CALL DEFINE(PB)
 0004               DC  10 I= 1,2
 0005               CALL DEFINE(P1)
 0006               CALL PDJS(F,DF,LS(1,I),LS(2,I),0.D0)
 0007               CALL PDS(G,DG,I+1)
 0008               CALL PRCD3(DF,DG,P1,.5D0)
 0009               CALL ERASE(DG)
 0010               CALL ERASE(DF)
 C011               CALL PDS(F,DF,I+1)
 0012               CALL PCJS(G,DG,LS(1,I),LS(2,I),0.D0)
 0013               CALL PROD3(DF,DG,P1,-.5D0)
 C014               CALL ERASE(DF)
 0015               CALL ERASE(DG)
 0016               CALL CRDER(P1)
 0017               CALL DVLS(P1,I)
 C018               CALL ACUM1(P1,PB,1.D0)
 0019          10   CALL ERASE(P1)
 0020               CALL CALPIC(PB,SPB)
 0021               CALL ERASE(PB)
 0022               CALL PXCA(PB,SPB)
 0023               RETURN
 0024               ENC

FORTRAN IV    MODEL 44  PS        VERSION 2,  LEVEL 1   DATE  68271

 0001               SUBRCUTINE LTRAN(F,W,P,Q,N,/ANS/)
 0002               REAL*8 BCOEF
 0003               INTEGER P,Q
 0004               DIMENSICN F(N,N),W(N)
 0005               NJ=Q+1
 0C06               CALL DEFINE(ANS)
 0007               DO  10 J1=1,NJ
 0008               CALL DPBRAC(F(P,Q-J1+2),W(J1+1),B)
 0009               CALL ACUM1(B,ANS,BCOEF(C,J1-1))
 0010          10   CALL ERASE(B)
 0011               CALL ACUM1(F(P,Q+2),ANS,1.D0)
 0012               RETURN
 0013               ENC

FORTRAN IV    MODEL 44  PS        VERSION 2,  LEVEL 1   DATE  68271

 CC01               COUBLE PRECISION FUNCTION BCOEF(N,M)
 C002               IBC=1
 0003               IF(2*M.LE.N)GO TO 20
 CC04               IB=N-M
 C005               GO TC 30
 0006          20   IB=M
 0007          30   IF(IB.LE.0)GO TO 60
 0008               CO 50 I=1,IB
 CC09          50   IBC=((N-I+1)*IBC)/I
 0010          60   BCCEF=IBC
 0011               RETURN
 C012               ENC
```

REFERENCES

Arnold, A. and Avez, G. 1967, Problimes ergodiqnes de la mécanique classique, Gauthier-Villars, Paris.

Birkhoff, G. 1927, Dynamical Systems, American Mathematical Society Colloquium Publications, Volume IX.

Bray, T. and Goudas, C. L. 1967, *Astron. J.*, 72, 202.

Brown, E. and Shook, C. A. 1933, Planetary Theory, Cambridge University Press, Cambridge.

Chai, A. and Kass, L. 1966, AFOSR Scientific Report No. 67-0123.

Contopoulos, G. 1963, *Astron. J.*, 68, 11.

Contopoulos, G. 1967, Colloque sur les nouvelles méthodes de la dynamique stellaire. Editions du CHRS, 223.

Deprit, A. and Price, J. F. 1965, *Astron. J.*, 70, 836.

Deprit, A., Henrard, J. and Rom, A. 1967, *AAS Science and Technology Series*, 11, *Space Flight Mechanics Specialists Symposium*, 85.

Deprit, A. 1969, *Celestial Mechanics*, 1, in print.

Deprit, A. and Henrard, J. 1968, *Adv. in Astronomy and Astrophysics*, 6, 1.

Deprit, A. and Rabe, E. 1969, *Astron. J.*, 74, in print.

Gustavson, F. G. 1966, *Astron. J.*, 71, 687.

Poincare, H. 1893, Les méthodes nouvelles de la mécanique céleste, Gauthier-Villars (Reprinted by Dover Publ. 1957).

Siegel, C. L. 1941, *Ann. of Math.*, 42, 210.

Siegel, C. L. 1954, *Math. Ann.*, 128, 144.

Walker, G. H., and Ford, J. 1969, Manuscript submitted for publication. In private communication from the Georgia Institute of Technology, January 1969.

Whittaker, E. T. 1916, *Proc. Roy. Soc. Edinburgh*, 37, 97.