ERR-AN-059 A
Stress

THIN CYLINDRICAL SHELL UNDER INTERNAL PRESSURE AND CONCENTRATED NORMAL LOAD

J. E. GOLDBERG

31 July 1961

ENGINEERING DEPARTMENT

GENERAL DYNAMICS | ASTRONAUTICS
A DIVISION OF GENERAL DYNAMICS CORPORATION

ERR-AN- 09 A
Stress

THIN CYLINDRICAL SHELL UNDER INTERNAL PRESSURE AND CONCENTRATED NORMAL LOAD

J. E. GOLDBERG

31 July 1961

ENGINEERING DEPARTMENT

This work was supported under General Dynamics
sponsored research program Number 111-9220.

GENERAL DYNAMICS | ASTRONAUTICS

A DIVISION OF GENERAL DYNAMICS CORPORATION

CONTENTS

		PAGE
	ABSTRACT	1
I	INTRODUCTION	1
II	METHOD OF ANALYSIS	1
III	DISCUSSION	5
IV	ILLUSTRATIVE EXAMPLE	6
V	DISTRIBUTION	9

THIN CYLINDRICAL SHELL UNDER

INTERNAL PRESSURE AND

CONCENTRATED NORMAL LOAD

J. E. Goldberg

ABSTRACT

A method is developed in the present report for calculating the normal displacements in a very thin, pressurized cylindrical shell which is subjected to a concentrated normal load. The shell is treated as a membrane and the solution is obtained in the form of a single or one-dimensional Fourier series, the longitudinal coordinate being taken as the independent variable of the trigonometric terms, with the circumferential coordinate being absorbed into the Fourier coefficients.

The solution given in the present report is an alternative to a solution given in a previous report in the form of a double Fourier series.

The report concludes with a numerical example.

INTRODUCTION

The present report is concerned with the problem of determining the displacements in a very thin, pressurized cylindrical shell which is subjected to an "almost concentrated" normal load.

In a previous report[1], a method was given for determining the displacements under the loading stated above by first expanding the "concentrated" load into a double Fourier series. It was assumed that the displacements could also be represented by a double Fourier series of the same trigonometric functions, and it was shown that a definite relation existed between the coefficients of the loading series and the coefficients of the displacement series.

Unfortunately, the series involved in the method previously described is likely to converge quite slowly in certain actual cases and, in these cases, it becomes necessary to compute a large number of terms in order to obtain the required accuracy.

METHOD OF ANALYSIS

We present now a method of analysis which could reduce materially the amount of computation involved in obtaining a solution to the problem stated above.

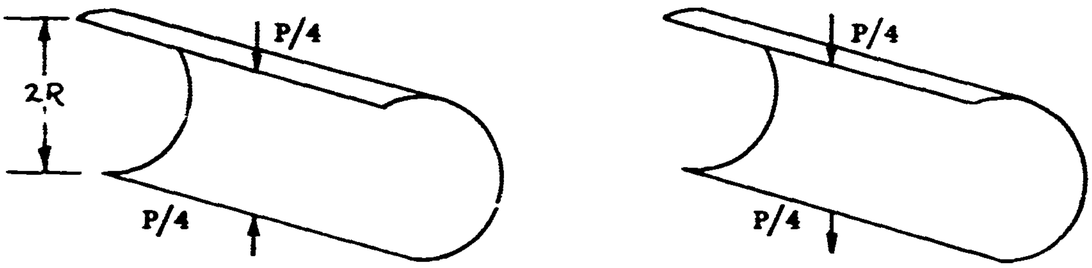
1. Goldberg, J. E., "Theory of Membranes, Deflections and Stresses in Membranes under Concentrated Normal Load", ERR-AN-058, 15 July 1961

We assume that the normal load P is located on the surface of the shell at (in cylindrical coordinates) $\Theta = 0$, $x = c$ where c is measured from one end. Instead of a single load P at that location, we shall assume that we have, at $x = c$, two loading cases as follows:

(a) Symmetrical loads each of magnitude P/2 located at $\Theta = 0, \pi$
(b) Anti-symmetrical loads each of magnitude P/2 located at $\Theta = 0, \pi$

Clearly, the loading obtained by superimposing loadings (a) and (b) is equivalent to the original single load P at $\Theta = 0$.

We will now imagine that the shell is divided into two semi-cylindrical shells by cutting at $\Theta = 0$ and $\Theta = \pi$, and we assume that the several loads P/2 which have been placed on the lines at $\Theta = 0$ and $\Theta = \pi$ have also been split into two equal components.



The problem now becomes that of finding the displacements in a semi-cylindrical "membrane" which, in addition to an internal pressure is loaded along its straight edges as shown.

We assume that the internal pressure is such as to cause a hoop membrane force (i.e., $N_o = pR$) and an axial membrane force $N_o/2$. We shall take w to be the additional deflection (i.e., in addition to that due to internal pressure alone) due to the applied load P.

Within the boundaries of each semi-cylinder the deflection must satisfy the equation

$$\frac{N_o}{R^2}\frac{\partial^2 w}{\partial \theta^2} + \frac{N_o}{2}\frac{\partial^2 w}{\partial x^2} = 0 \qquad (1)$$

We will assume that the deflection may be taken in the form of a series

$$w(x,\theta) = \sum_{n=1}^{\infty} W_n \sin n\pi x/L \qquad (2)$$

where the factors $W_n = W_n(\Theta)$, that is, the coefficients, are functions of the angular position. This series will give zero normal displacement at the circular edges of the semi-cylindrical shell.

We will consider the implications of one term of the series in Equation (2), that is, we will assume that

$$w = w_n = W_n \sin n\pi x/L \tag{3}$$

Substitution of Equation (3) into Equation (1) would lead to

$$\frac{d^2 W_n}{d\theta^2} - k_n^2 W_n = 0 \tag{4}$$

where $$k_n^2 = \frac{R^2}{2}\left(\frac{n\pi}{L}\right)^2 \tag{5}$$

The general solution of Equation (4) is

$$W_n = A_n \sinh k_n\theta + B_n \cosh k_n\theta \tag{6}$$

Along the edge $\theta = 0$,

$$\frac{1}{R}\frac{\partial w_n}{\partial\theta} = \frac{A_n k_n}{R} \sin\frac{n\pi x}{L}$$

And therefore the distributed normal (i.e., radial) force along this edge is

$$f_n = \frac{N_o A_n k_n}{R} \sin\frac{n\pi x}{L} \tag{7}$$

Let us now consider a "concentrated" load $Q = P/4$ which is applied at the point $x = c$ on the edge $\theta = 0$. This "concentrated" load may be assumed to be distributed over a small length $2e$ in the direction of the edge, and we may therefore say that Q is equivalent to a distributed loading along the edge defined as follows:

$$q = \begin{cases} 0 & 0\le x\le (c-e) \\ Q/2e & (c-e)\le x\le (c+e) \\ 0 & (c+e)\le x\le L \end{cases} \tag{8}$$

Such a loading may be represented by a Fourier series:

$$Q \sim \sum_{n=1}^{\infty} q_n \sin n\pi x/L \tag{9}$$

We have now to determine the coefficients q_n by a standard technique. We substitute Equation (8) into Equation (9), multiply both sides of the resulting equation by $\sin i\pi x/L\, dx$ (where i is an arbitrary integer) and integrate. We observe that substitution of Equation (8) into the left side of Equation (9) and performing the operations mentioned above would yield

$$\int_0^L q \sin\frac{i\pi x}{L}\,dx = 0 + \frac{Q}{2e}\int_{c-e}^{c+e} \sin\frac{i\pi x}{L}\,dx + 0$$

If e is taken sufficiently small, the integrand of the term on the right side remains essentially constant[2], and we obtain

$$\int_0^L q \sin \frac{i\pi x}{L}\, dx = \frac{Q}{2e}\int_{c-e}^{c+e} \sin \frac{i\pi c}{L}\, dx$$

$$\int_0^L q \sin \frac{i\pi x}{L}\, dx = Q \sin \frac{i\pi c}{L} \qquad (10)$$

Performing the same operations upon the right side of Equation (9) we have, by use of the orthogonality relations

$$\int_0^L \left(\sum q_n \sin \frac{n\pi x}{L}\right) \sin \frac{i\pi x}{L}\, dx = \int_0^L q_i \sin^2 \frac{i\pi x}{L}\, dx$$

$$= \frac{q_i L}{2} \qquad (11)$$

Equating (10) and (11) and replacing i by n yields

$$q_n = \frac{2Q}{L} \sin \frac{n\pi c}{L} \qquad (12)$$

Hence, the concentrated load Q (=P/4) may be represented by the series

$$Q \sim 2Q/L \sum_{n=1}^{\infty} \left(\sin \frac{n\pi c}{L}\right)\left(\sin \frac{n\pi x}{L}\right) \qquad (13)$$

Thus a concentrated load corresponds to the superposition of a series of sinusiodally varying distributed loads.

We may now compare one term of the series in Equation (13) with the sinusoidal term of Equation (7) and we see that the two terms are identical if

$$A_n = -\frac{2QR}{LN_o k_n} \sin \frac{n\pi c}{L}$$

Recalling that Q = P/4, and using Equation (5) this becomes

$$A_n = \frac{-P}{\sqrt{2\pi N_o}\, n} \sin \frac{n\pi c}{L} \qquad (14)$$

---

2. Alternatively $\frac{Q}{2e}\int_{c-e}^{c+e} \sin \frac{i\pi x}{L}\, dx = \frac{QL}{2ei\pi}\left[\cos \frac{i\pi(c-e)}{L} - \cos \frac{i\pi(c+e)}{L}\right]$

$$= \frac{QL}{ei\pi} \sin \frac{i\pi c}{L} \sin \frac{i\pi e}{L} \approx \frac{QL}{ei\pi}\left(\sin \frac{i\pi c}{L}\right) \frac{i\pi e}{L} \quad \text{if} \quad i\pi e/L \text{ is}$$

fairly small, say one-tenth of a radian or less. Hence, if we can "cut off" our series at a point where this restriction is satisfied, the results should be acceptable.

We return now to consideration of Equation (6), the general solution of the ordinary differential equation for W_n. If the loads P/4 are applied symmetrically, then the deflections along the edges $\theta = 0$ and $\theta = \pi$ are equal. Hence, for this case

$$B_n = A_n \sinh k_n\pi + B_n \cosh k_n\pi$$

or

$$B_n = A_n \sinh k_n\pi \,/(1 - \cosh k_n\pi) \quad \text{(symmetrical)} \qquad (15)$$

If, on the other hand, the loads P/4 are applied antisymmetrically, the deflections along these edges are equal in magnitude but opposite in sign. Hence for the antisymmetrical case

$$B_n = -A_n \sinh k_n\pi - B_n \cosh k_n\pi$$

or

$$B_n = -A_n \sinh k_n\pi \,/(1 + \cosh k_n\pi) \quad \text{(antisymmetrical)} \qquad (16)$$

A concentrated load P/2 at $\theta = 0$, $x = c$ on the semi-shell (corresponding to a single load P on the complete shell) was shown to be equivalent to symmetrical and antisymmetrical loads P/4 on the semi-shell. Therefore we combine both Equations (15) and (16), substitute into Equation (6), thence into Equation (3), and upon using Equation (14) and Equation (2) we obtain the following expression for the additional normal deflections of a very thin pressurized cylindrical shell when subjected to a concentrated normal load P applied at $\theta = 0$, $x = c$:

$$w(x,\theta) = \frac{\sqrt{2}P}{2\pi N_o} \cdot \sum_{n=1}^{\infty} \frac{1}{n} \left(-\sinh k_n\theta + \frac{\cosh k_n\pi \cosh k_n\theta}{\sinh k_n\pi}\right) \sin\frac{n\pi c}{L} \sin\frac{n\pi x}{L} \qquad \text{(A)}$$

or

$$w(x,\theta) = \frac{\sqrt{2}P}{2\pi N_o} \sum_{n=1}^{\infty} \frac{1}{n} \frac{\cosh k_n(\pi - \theta)}{\sinh k_n\pi} \sin\frac{n\pi c}{L} \sin\frac{n\pi x}{L} \qquad (17) \qquad \text{(A)}$$

where

$$k_n = \frac{B_n \pi}{\sqrt{2}\, L}$$

DISCUSSION

The coefficients in Equations (17) are easily evaluated either by use of a table of hyperbolic functions or by use of the fundamental exponential forms of these functions and a table of powers of e. When the argument of these hyperbolic functions becomes large, asymptotic forms may be useful. For example: $\sinh \theta/\cosh \theta \rightarrow 1 - 2e^{-2\theta}$ as θ becomes very large.

Equations (17) are valid when

(a) the shell is very thin
(b) the internal pressure is high
(c) the "concentrated" load is small

When these restrictions are violated, the deflections given by these equations generally are larger than the actual deflections. Rules for establishing the range of validity of the membrane theory, and for estimating the secondary or additional stresses in terms of the displacements given by the above equations were presented in the previous report.

We will not apply Equations (17) to the point directly under the center of the load since a theoretically concentrated load would result in an infinite deflection at the load point. Actually however, we do not deal with theoretically concentrated loads but with loads which are distributed over some finite area. If the area is small, Equations (17) should give reasonably good results beyond the edge of the actual contact area.

It would be well to avoid sharp edges on the attachment which transmits the load to the shell. If possible the slope at the edge of the fitting should not be less than the calculated slope of the shell at the edge of the fitting. Also, the curvature of the fitting should not exceed that which, when applied to the shell, causes extreme fiber stresses of a magnitude such as to raise the stress level above the design allowable stress.

ILLUSTRATIVE EXAMPLE

As an illustrative example, we take a very thin cylindrical shell subjected to a positive internal pressure of sufficient magnitude as to place the shell in the approximate condition of a membrane. The shell is subjected also to a "concentrated" normal load P located at c = L/2. The necessary data include:

$$R = 40 \text{ inches} \qquad k_n = \frac{R\pi}{\sqrt{2}\,L}\, n = .740480\, n$$

$$L = 120 \text{ inches} \qquad k_n \pi = 2.326287\, n$$

We will calculate the normal displacement at the point $x = L/2$, $\theta = 0.1$. Thus, if the fitting which transmits the load to the shell has a dimension of about eight inches in the circumferential direction, is suitably shaped both in profile and plan (probably with an elliptical planform having its major axis in the circumferential direction), and is itself properly designed, the calculated deflection would be expected to pertain to the edge of the fitting.

We observe that since $\sin n\pi c/L = 0$ for all even integral values of n, it is necessary to calculate terms only for the odd integers. Furthermore, since $c = L/2$ and $x = L/2$, then $\sin \frac{n\pi c}{L} \sin \frac{n\pi x}{L} = 1$ for the odd integers. We note also that $\coth k_n\pi$ becomes unity (to the sixth decimal place) when n = 5 and remains at that value for larger n. Consequently, for n equal to or larger than 5,

$$n W_n = \coth k_n\pi \cosh k_n\theta - \sinh k_n\theta = e^{-k_n\theta}$$

Obviously this simplification is possible only when k_n is sufficiently large. Aside from this simplification, the series may be expected to converge nicely if θ (which is a coordinate of the point where the deflection is being computed, not the point where the load is applied) is not too small, i.e., is not too close to the generator upon which the load is applied.

The calculations are carried out in the table below. Because of the simplification mentioned above, $\sinh k_n\theta$ and $\cosh k_n\theta$ are not tabulated for n greater than 5, and the value of nW_n for these values of n is taken directly from a table of exponentials as $e^{-k_n\theta}$. The last column of the table should be multiplied term by term by $\sin \frac{n\pi c}{L} \sin \frac{n\pi x}{L}$ before summing. Since this product was shown to be unity (for odd values of n) for the particular load point and deflection point under consideration, this step has been omitted. In the general case, obviously, a column should be provided for this calculation.

n	$k_n\pi$	$\text{cth } k_n\pi$	$k_n\theta$	$\text{snh } k_n\theta$	$\text{csh } k_n\theta$	nW_n	W_n
1	2.326287	1.019258	.074116	.074116	1.002742	.947890	.947890
3	6.978862	1.000001	.222144	.223976	1.024775	.800800	.266933
5	11.631436	1.000000	.370240	.378757	1.069325	.690568	.138114
7	20.936585		.518336			.619035	.088434
9			.666432			.513538	.057060
11			.814528			.442848	.040247
13			.962624			.381890	.029376
15			1.110720			.329322	.021955
17			1.258816			.283990	.016705
19			1.406912			.244899	.012889
21			1.555008			.211189	.010057
23			1.703104			.182112	.007918
25			1.851200			.157048	.006282
27			1.999296			.135431	.005016
29			2.147392			.116788	.004027

$\Sigma W_n = 1.552903$

The above result shows that, for a sufficiently thin shell having the dimensions stated above and subjected to a sufficiently high internal pressure, the normal deflection at the point $x = L/2$, $\theta = 0.1$ would be

$$w = \frac{\sqrt{2}\ P}{2\pi\ N_o}\ (1.6529)$$ (A)

If the shell and loads do not satisfy the stated conditions, a solution of the above type may be taken as a first approximation.