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D. Middleton

Bayes Ambiguity Functions: Some Simple Applications to Resolution and Radar Countermeasures

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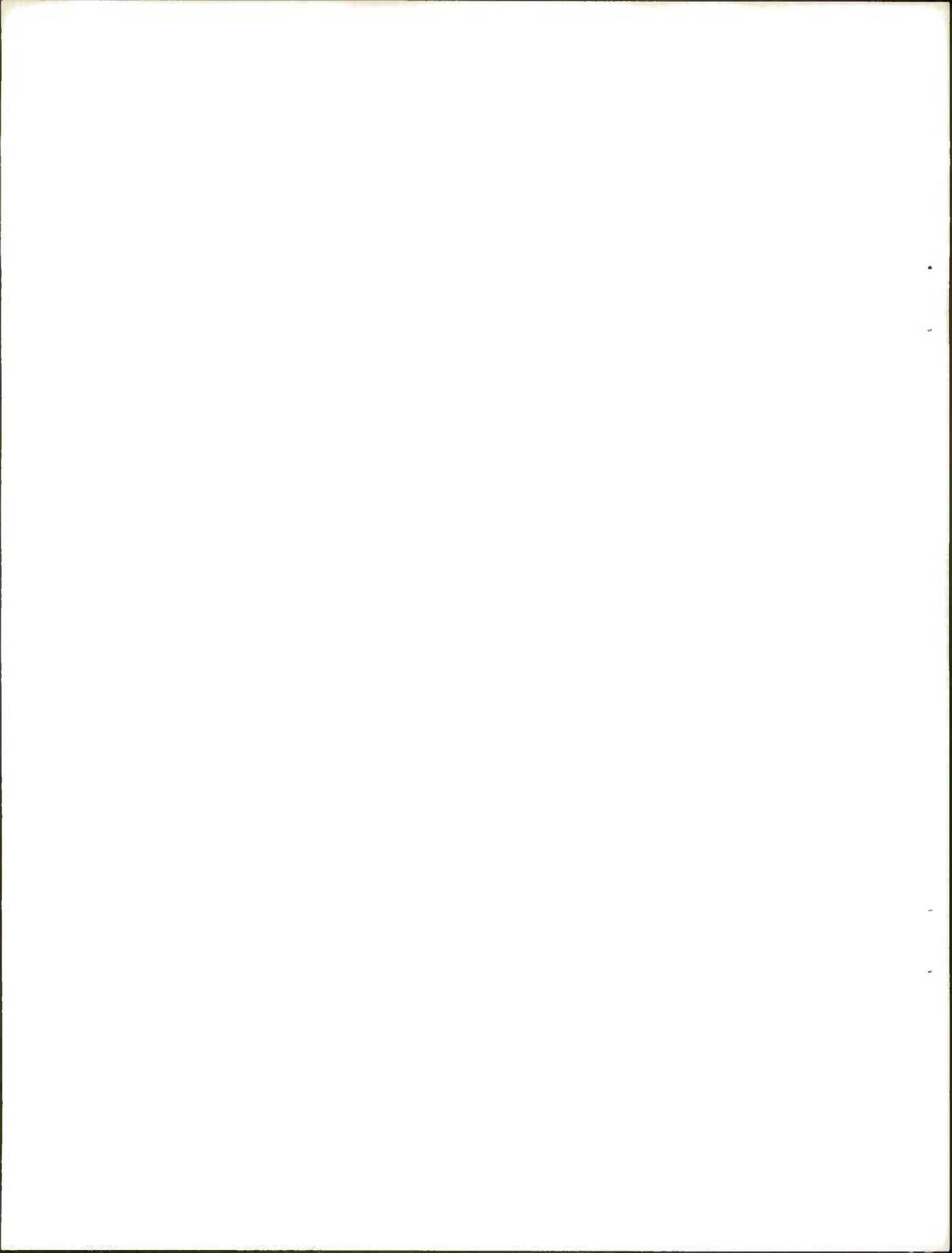
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ABSTRACT

Bayes ambiguity functions are defined as an important parameter governing the performance of optimum (i.e., Bayes or minimum average risk) systems. Bayes ambiguity functions are generalizations of the classical ambiguity functions of Woodward and are specifically derived from an appropriate decision process. It is shown here that it is the real part of the ambiguity function that is significant, rather than its modulus. Optimum target resolution is formulated as a detection problem involving the two hypothesis states H_0: "unresolved" signals versus H_1: "resolved" signals, and general conditions for the qualitative utility of the ambiguity functions are discussed. These latter are: additive gaussian noise and threshold operation; otherwise the ambiguity function is an inadequate description of system performance. The analysis is extended to a number of situations involving interfering signals, such as electronic countermeasures (ECM) and is illustrated with simple examples showing quantitatively, as well as qualitatively, the typical rôles played by the Bayes ambiguity function in a variety of ECM applications. It is emphasized that one must also consider the probability of correct and incorrect decisions, in conjunction with the properties of the ambiguity functions, to achieve a reliable measure of expected performance.

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1.   INTRODUCTION

The classical ambiguity function of Woodward[1] is an established tool for the design of signals in radar and communication problems. As is well known, its principal application in radar is to the resolution of targets in a multi-target environment. The ambiguity function provides a quantitative method for selecting signal waveforms (at the transmitter) which, at the receiver, enables one to separate or in some appropriate sense, distinguish the returns from a desired target from the signals reflected from other, undesired targets. This is accomplished primarily on the basis of two principal parameters of the targets in question: range (or time delay) and velocity (or doppler frequency). Loosely stated, two targets are "resolved" if the differences in range and velocity are such as to make the ambiguity function for the pair of signal returns some sufficiently small fraction of this ambiguity function's maximum value.

The advantage of using an ambiguity function in detection and resolution situations is its comparative analytic simplicity: it is readily calculated in most cases and it provides useful "pictures" whereby performance can be qualitatively estimated, for various choices of signal waveforms and modulations[2,3]. In these respects it is analogous to the signal-to-noise ratio as a partial measure of expected system behavior. The signal-to-noise ratio is a second-order statistic[4] (of the system's

output). It can likewise be calculated where more general statistical measures (such as error probabilities, etc.) can only at best be approximated or simulated, and it does relate various of the chief system parameters. On the other hand, the principal limitation of both the ambiguity function and the (output) signal-to-noise ratio is that they are incomplete measures of system performance. They omit the decision process, which in signal detection and estimation is the system's desired output, and which itself is crucial in the specification of optimum and near-optimum systems for these purposes[5].

Earlier work, with the partial exception of Siebert's important paper[2], has for the most part focused on the structure and properties of the "classical" ambiguity function and its quasi-qualitative uses for signal design purposes. Our effort here appears to be new[2a], chiefly with the introduction of the generalized or Bayes ambiguity functions, their relation to the defining decision process, and application to interference studies. Accordingly, the general purpose of this study is to show how the useful design concept of the ambiguity function can be incorporated into the more complete description of the system and evaluation of its performance based on decision-theoretic methods[5]. In more detail, our aim is to consider a number of radar detection situations in a variety of environments, including electronic counter-measures (ECM), and to attempt to relate, quantitatively, ambiguity functions to optimal system structure and performance. [Related topics, for subsequent analysis, include waveform design to enhance ECM, and signal waveform selection

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to minimize its effects, under various operating constraints (e.g., fixed total energy, bandwidth, etc.).] The criterion of optimality is the usual one of minimum average risk or cost of decision[5], which leads to the so-called Bayes systems[4, 5]. Ambiguity functions associated with such systems are accordingly called "Bayes ambiguity functions." Apart from the solution of a few typical and illustrative problems from this viewpoint, our ultimate aim is to provide greater insight into the rôle that the ambiguity function plays in the decision process, its advantages and limitations as a working tool for the practical system design and waveform selection, and to strengthen its effectiveness in these respects.

This paper is organized as follows: Section 2 presents some preliminary remarks and results concerning the classical ambiguity function. No attempt at completeness is intended; just enough background is given to provide a framework for the subsequent analysis and discussion. Sections 3-5 are the meat of the work and consist of a formulation of the resolution problem (Section 3); illustrative examples of radar resolution (Section 4) and detection (Section 5) subject to ECM. These illustrations are deliberately chosen to have somewhat limited application in order to ensure analytic simplicity. The chief results, however, include the Bayesian ambiguity function in this instance, its detailed structure, its rôle in the decision process, and a variety of important features which enables us to draw some important general conclusions about this extension of the ambiguity concept. These, in turn,

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are summarized and discussed in the closing Section 6, along with a short list of

further questions to be examined.

2. SOME PRELIMINARY REMARKS ON AMBIGUITY FUNCTIONS[*]

Let us begin with a narrow-band, complex signal $\hat{S}(t; \underline{\theta})$, which we write

$$\hat{S}(t; \underline{\theta}) = \hat{S}_o(t, \underline{\theta}) e^{i\omega_o t} \; ; \; \omega_o = 2\pi f_o \, , \tag{2.1}$$

where f_o is the carrier frequency, $\hat{S}_o$ is a complex envelope, slowly-varying vis-à-vis $\exp(i\omega_o t)$, and $\underline{\theta}$ is a set of structure parameters that characterize the waveform. In terms of a real envelope, E, and phase, ϕ, the complex envelope $\hat{S}_o$ is

$$\hat{S}_o(t, \underline{\theta}) = E(t, \underline{\theta}) e^{i\phi(t, \underline{\theta})} \, . \tag{2.2}$$

The (normalized) classical ambiguity function of Woodward is now defined[1] as

$$\chi(\tau, \nu) \equiv \int_{-\infty}^{\infty} \hat{S}_o(t, \underline{\theta}) \, \hat{S}_o(t + \tau, \underline{\theta})^* \, e^{-2\pi i \nu t} \, dt \, \Big/ \int_{-\infty}^{\infty} |\hat{S}_o(t, \underline{\theta})|^2 \, dt \tag{2.3}$$

$$\equiv \int_{-\infty}^{\infty} \hat{s}_o(t, \underline{\theta}) \, \hat{s}_o(t + \tau, \underline{\theta})^* \, e^{-2\pi i \nu t} \, dt \, , \tag{2.4}$$

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[*] A glossary of symbols is included at the end of the paper.

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where

$$\hat{s}_o(t, \underline{\theta}) = \hat{S}_o(t, \underline{\theta}) \left/ \left(\int_{-\infty}^{\infty} |\hat{S}_o(t, \underline{\theta})|^2 \, dt \right)^{\frac{1}{2}} \right. \tag{2.4a}$$

is the <u>normalized</u> complex envelope. It is understood that the integrals in (2.3) - (2.4a) are finite. This means that if the data interval (T) is infinite, periodic signals are excluded on the infinite interval and $\hat{S}_o$ must be such that $\hat{S}_o(\pm \infty)$ vanishes properly. In practice, however, the data on observation periods $(t_o, t + T)$ are always finite, so that

$$\hat{S}_o(t, \underline{\theta})_T = \hat{S}_o(t, \underline{\theta}), \; (t_o < t < t_o + T); = 0 \text{ elsewhere}, \tag{2.5}$$

and Equations (2.3) - (2.4a) are bounded, since $\hat{S}_o \in T$ has*, at worst, integrable singularities.

Physically, we can interpret $\hat{S}_o(t + \tau) e^{2\pi i \nu t}$ as the received signal's (complex) envelope after reflection from a target which introduces a delay or shift in range by τ and a phase change, or doppler displacement, of $2\pi \nu t$ radians, with respect to another, otherwise identical signal. If $\hat{S}_o(t)$ is the (complex) envelope of a target return at time t, then $\hat{S}_o(t + \tau) e^{2\pi i \nu t}$ is the return from an identical target displaced from the former in range (τ) and velocity (ν). Thus, $\chi(\tau, \nu)$ is

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* The symbol, ϵ, means "in the interval."

a complex autocorrelation function with two parameters, range and velocity. In the instance where $\hat{S}_o(t)$ is the envelope of a locally generated signal, $\chi(\tau, \nu)$ may be interpreted as a complex cross-correlation function, between the locally generated signal (envelope) and that of a target return whose range is $(\sim \tau/2)$ and which is moving relative to the receiver with a velocity $(\sim \nu)$. Care must accordingly be taken, when we use χ, to ascertain the intended interpretation of the signals in question.

Some of the more important and easily established properties of $\chi(\tau, \nu)$ are

(i)   <u>Maximum value</u>: $|\chi(0, 0)| = 1$ (2.6)

       [This is an immediate consequence of (2.3).]

(ii)   <u>Fourier transform</u>: $\displaystyle\int_{-\infty}^{\infty} \chi(\tau, \nu)\, e^{2\pi i \nu t}\, d\nu = \hat{S}_o(t)\, \hat{S}_o(t + \tau)^*$;   (2.7)

(iii)   <u>Conservation of volume</u>: $\displaystyle\iint_{-\infty}^{\infty} |\chi(\tau, \nu)|^2\, d\tau\, d\nu = 1$.   (2.8)

This last says that the total volume under the surface $|\chi(\tau, \nu)|$ is bounded, and since the signal is such that the various integrals (2.3), (2.4), e.g., χ, are bounded and continuous, it follows from (2.6) and (2.8) that the $|\chi(\tau, \nu)|-$ surface must eventually fall off to zero in the τ, ν plane. [Equation (2.8) is independent of the particular form of χ.] Usually there is one large "spike," of maximum height unity,

at $(0, 0)$, with other "spikes" for some $(\tau, \nu) \neq (0, 0)$. The precise form of the $|\chi(-, \nu)|$ — surface depends, of course, on that of the signal envelope $\hat{S}_o$.

As a very simple example let us consider a narrow-band rectangular pulse of duration τ_o:

$$\hat{S}(t) = A_o e^{i\omega_o t}, \quad -\tau_o/2 < t < \tau_o/2; = 0, \text{ elsewhere}. \tag{2.9}$$

Accordingly, we have

$$\int_{-\infty}^{\infty} |\hat{S}_o(t)|^2 dt = \int_{-\tau_o/2}^{\tau_o/2} A_o^2 dt = A_o^2 \tau_o, \tag{2.10}$$

and so (2.4) becomes

$$\chi(\tau, \nu) = (1/\tau_o) \int_{-\tau_o/2}^{\tau_o/2 - |\tau|} e^{-2\pi i \nu t} dt, \quad \tau_o > |\tau| > 0 \tag{2.11a}$$

$$= 0, \quad |\tau| > \tau_o$$

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$$\therefore \chi(\tau, \nu) = e^{\pi i \nu \tau_0} \left(\frac{1 - e^{-2\pi i \nu (\tau_0 - |\tau|)}}{2\pi i \nu \tau_0} \right),$$

(2.11b)

$$= e^{\pi i \nu |\tau|} \left[1 - \frac{|\tau|}{\tau_0} \right] \frac{\sin \pi \nu (\tau_0 - |\tau|)}{\pi \nu \tau_0 (1 - |\tau|/\tau_0)},$$

$$= e^{\pi i \nu |\tau|} \frac{\sin \pi \nu (\tau_0 - |\tau|)}{\pi \nu \tau_0};$$

all for $0 < |\tau| < \tau_0$; (2.11c)

$$= 0, \quad |\tau| > \tau_0.$$

And so we have directly

$$|\chi(\tau, \nu)|^2 = \frac{\sin^2 \pi \nu (\tau_0 - |\tau|)}{(\pi \nu \tau_0)^2}, \quad 0 < |\tau| < \tau_0; = 0, \tau \text{ elsewhere}, \quad (2.12)$$

with the properties (2.6) - (2.8) easily verified.

Here we have explicitly

$$|\chi(\tau, 0)| = 1 - |\tau|/\tau_0, \quad 0 < |\tau| < \tau_0; = 0, \quad |\tau| > \tau_0; \quad (2.12a)$$

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$$|\chi(0, \nu)| = \frac{\sin \pi \nu \tau_o}{\pi \nu \tau_o} \quad . \tag{2.12b}$$

The ambiguity function described above is an (auto-) ambiguity function, since the signals are the same. We should therefore write for (2.3), $\chi_{ss}(\tau, \nu)$. This, and the interpretation of the ambiguity function as a measure of separation in frequency and delay of two signals, permit us to generalize (2.4) to the concept of a (cross-) ambiguity function

$$\chi_{12}(\tau, \nu) \equiv \int_{-\infty}^{\infty} \hat{S}_{01}(t) \, \hat{S}_{02}(t + \tau)^* \, e^{-2\pi i \nu t} \, dt \left/ \sqrt{E_{01} \, E_{02}} \right. , \tag{2.13}$$

where the normalizations are

$$E_{01, 02} = \int_{-\infty}^{\infty} |\hat{S}_{01, 02}(t)|^2 \, dt \quad . \tag{2.13a}$$

Thus, rewriting with the help of the normalized signal

$$\hat{s}_{01}(t) = \hat{S}_{01}(t) \cdot E_{01}^{-\frac{1}{2}}, \text{ etc.}, \tag{2.14}$$

we have in a more compact form

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$$\chi_{12}(\tau, \nu) = \int_{-\infty}^{\infty} \hat{s}_{01}(t) \, \hat{s}_{02}(t + \tau)^* \, e^{-2\pi i \nu t} \, dt \, . \tag{2.15}$$

This cross-ambiguity function has just the same sort of interpretation as that given above for $\chi(\tau, \nu)$, cf. Equation (2.5) et seq. We see that χ_{12} is a kind of complex cross-correlation function, between $\hat{s}_{01}(t)$ and $\hat{s}_{02}(t)^*$ delayed by τ and undergoing an (angular) frequency shift $2\pi i \nu t$, e.g.,

$$\hat{s}_1(t) \equiv \hat{s}_{01}(t); \, \hat{s}_2(t; \tau, \nu) = \hat{s}_{02}(t + \tau) \, e^{2\pi i \nu t} \, , \tag{2.16a}$$

$$\therefore \chi_{12}(\tau, \nu) = \int_{-\infty}^{\infty} \hat{s}_1(t) \, \hat{s}_2(t; \tau, \nu)^* \, dt \, . \tag{2.16b}$$

By Schwartz's inequality[6] we see that

$$|\chi_{12}(0, 0)| \leq 1 \, , \tag{2.17}$$

with the equality only if $\hat{s}_{01} = \hat{s}_{02}$. We also easily show that

$$\int_{-\infty}^{\infty} \chi_{12}(\tau, \nu) \, e^{2\pi i \nu t} \, d\nu = \hat{s}_{01}(t) \, \hat{s}_{02}(t + \tau)^* \, , \tag{2.18}$$

like (2.7). It can be demonstrated that

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$$\int\limits_{-\infty}^{\infty}\int |\chi_{12}(\tau, \nu)|^2 \, d\tau \, d\nu < 1 \qquad (2.19)$$

also, but the volume under the surface $|\chi_{12}|^2$, is no longer necessarily conserved cf. (2.8).

From Equation (2.12) we may say that if a signal is displaced in delay τ and frequency ν from itself (at $\tau = 0$, $\nu = 0$), such that $|\chi(\tau, \nu)|$ is (for all such τ, ν) less than some prechosen number, then these two signals are "resolved"--i.e., are "distinguishable" in some observational sense. Thus, for all (τ, ν) such that

$$\chi(\tau, \nu) \le \chi_0 , \qquad (2.20)$$

a region is defined where the two signals are said to be "distinguished," "resolved," or "separated." In the example calculated above, (2.12), this "resolution" region is the entire (τ, ν) plane outside the closed curve bounding the region for which $\chi(\tau, \nu) > \chi_0$, for at least some (τ, ν). Similar remarks apply directly to the (cross-) ambiguity function χ_{12}.

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3. A REFORMULATION OF THE "RESOLUTION" PROBLEM[*]: BAYES AMBIGUITY FUNCTIONS

The central point about the conventional use of the ambiguity function to resolve two signals, or "targets," is the fact that the decision process: "two signals are (or are not) resolved, or distinguished," based on χ ($\leq$ some arbitrary threshold χ_0), is in no direct way related to the actual processing of the received data. One simply agrees that two signals are resolved, or resolvable, if $\chi < \chi_0$ over appropriate regions of (τ, ν). As remarked in the INTRODUCTION, this is similar to the use of the signal-to-noise ratio: $(S/N)_{out}^2$ versus $(S/N)_{in}^2$, in evaluating detector performance in the older theories[4]. In both instances the actual decision process based on the received data does not enter into or influence the calculation of these partial criteria of performance: data processing and decision-making are "uncoupled."

Furthermore, the classical ambiguity function (above) is defined in the absence of the inevitable background noise that accompanies practical operation, and which makes decision-making here, in reality, an act carried out in the face of uncertainty. All this strongly suggests that a more realistic formulation of the problem, which now includes these critical factors of decision-making and accompanying noise, will be illuminating and provide greater insight into the concepts of, and bounds on, the use of ambiguity functions as useful, partial criteria of system performance and signal design.

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[*] See Section 5.0 of Reference 2 for an earlier (nondecision treatment).

Let us therefore reformulate the resolution problem in terms which couple together the actual data processing and decision-making in the face of uncertainty. We choose the now standard concept of average risk or cost to describe the expected performance of our systems[5]. Optimum (or Bayes) systems are those that minimize this average risk, and the decision-theoretical formulation of our problem allows us to determine and compare optimal systems with suboptimum ones for the common purpose in the usual fashion[7].

We now specify the resolution problem involving two (real, narrow-band) signals as a detection problem, involving the test of two hypotheses against each other

H_1: $S_1(0, 0) + S_1(\tau_1, \nu_1) + N$:   a "reference" signal $S_1(0, 0)$ and an otherwise similar signal, displaced from $S_1(0, 0)$ by τ_1, ν_1 in delay and frequency;

$$(3.1)$$

H_2: $S_1(0, 0) + S_1(\tau_2, \nu_2) + N$:   same as H_1, but the "displaced" signal now has a displacement (τ_2, ν_2).

If we set $(\tau_1, \nu_1) = (0, 0)$, $(\tau_2 \rightarrow \tau, \nu_2 \rightarrow \nu)$, we have the decision situation:

$$H_1: 2S_1(0, 0) + N \text{ versus } H_2: S_1(0, 0) + S_1(\tau, \nu) + N, \qquad (3.2)$$

The resolution question is thus the decision problem of deciding which state occurs, H_1, where the two signals completely overlap, or H_2, where there is some

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separation in τ and ν. This is done in the familiar way by assigning costs to the various possible outcomes (H_1 true, H_1 false, etc.), and comparing a suitable test statistic (receiver output) against a threshold, $\mathcal{K}$ (which is a ratio of the above costs). If the threshold is exceeded we decide H_2, otherwise H_1. With a suitable choice of threshold we can ensure high probabilities of correct decisions. Thus, if the threshold is exceeded, we say that for the particular piece of data in hand, the two signals are resolved, with the reverse decision if the threshold is not exceeded. Moreover, for (3.2) if the probability $P(H_2)$ that H_2 occurs is sufficiently large [and $\therefore P(H_1)$ is sufficiently small] we say that the two signals are "resolved," for the values of (τ, ν) given. <u>Expected</u> performance is thus measured in terms of the probabilities of correct decisions as to H_1 and H_2: "no resolution" for the former and "resolution" for the latter. Otherwise, the signals are not resolved. Clearly $P(H_1)$, $P(H_2)$ are functions of (τ, ν). The subjective element here is the choice of the values of $P(H_2)$ and ultimately of (τ, ν), at which we say that resolution occurs. This however, is more fundamental than the choice of threshold, χ_0, in the conventional use of the ambiguity function χ, since it represents the overall decision process and includes the effects of uncertainties occasioned by the presence of background noise.

Useful generalizations of this approach are immediately made: (i) we can use (3.1) to distinguish between pairs of partially resolved signals (in noise); (ii) we can

consider the important case of different signal waveforms, e.g.,

$$H_1: \ S_1(0, \ 0) + S_2(\nu_1, \ \tau_1) + N \text{ versus } H_2: \ S_1(0, \ 0) + S_2(\nu_2, \ \tau_2) + N. \quad (3.3)$$

Resolution is achieved, as before, if $P(H_2)$, or $P(H_1)$, is sufficiently large.

So far we have said nothing about the appearance of ambiguity functions in these decision problems (3.1) - (3.3). Remembering from (2.16b) that the ambiguity function is a species of (complex, first order, i.e., $< S_1 \times S_2 >$) correlation function, we are naturally led to look for such forms in both the structure and the evaluation of performance of these detectors, operating under the hypothesis states (3.1) - (3.3). Accordingly, when such first-order correlation functions appear in the structure and evaluation of optimum (Bayes) systems, we call them Bayes ambiguity functions.

Let us consider optimum (i.e., Bayes) systems and two principal modes of operation:

A.   Threshold detection: Here the input signal-to-noise ratio is small, but the processing gain[8] is large enough to ensure high probabilities of correct decisions. This is the usual limiting situation in detection, for which one seeks optimal receivers[9]. Here the background noise statistics dominate in determining both optimum structure and expected performance.

B.   Strong Input Signals: In this case the input signal-to-noise level is high, and processing gains may be of the order of unity for effective performance. Now, however, it is the statistics of the signal that control

performance, when optimum structures and performance are desired, and these may be very complicated. [Usually, strict system optimality is not critical here--one generally employs the optimum threshold system obtain in A above.]

If now we look at the general structures for threshold performance[10], we see that, generally, <u>all</u> orders of signal correlations, $\langle S_1 \times S_2^2 \rangle$, $\langle S_1^2 \times S_2 \rangle$, etc., appear so that the ambiguity forms $[\langle S_1 \times S_2 \rangle]$ are but one of many parameters of the system. Similarly, the decision probabilities, based as they are on appropriate statistics of the threshold structure, contain all orders of signal correlation as well. For stronger input signals these effects are enhanced. The resulting conclusion is that, not unexpectedly,

1.   The ambiguity function (Bayes or "classical"), above, in the case of <u>general</u> <u>signal</u> <u>and</u> <u>general</u> <u>background</u> <u>noise</u>, is not a system parameter that gives an adequate description of structure and performance.

However, it is very important to note that:

2.   In the common situation of <u>additive</u> <u>signals</u> <u>and</u> <u>gaussian</u> (or gaussian-derived[*]) <u>noise</u>, for optimum and near optimum threshold operation the type of signal correlation characterized by Bayes ambiguity functions is a major parameter of system structure and performance. It may

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[*] Originally gaussian noise, subject to modification by the input signal, or passage through zero-memory non-linear devices, etc.

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therefore be regarded as significant in the description of the system and

in the associated problems of signal design (choice of optimum waveform).

The reason for this, (2), is basically because both structure and performance depend,

significantly, at worst on second-order statistics of the noise and the signal in the

threshold situation. (All our examples in succeeding sections exhibit this explicitly.)

Because of (1), then, we must be cautious in our use of ambiguity functions as a per-

formance description, but because of (2) we can expect the ambiguity function to be a

useful, partial criterion of system behavior in the many important cases obeying the

additive, gaussian, and threshold conditions of operation.

If we regard $S_1(0, 0)$ in Equations (3.1) - (3.3) as a desired target return,

and $S_1(\tau, \nu)$, $S_2(\tau, \nu)$, etc., as interfering or jamming signals, it is evident that

the resolution problem defined above has the equivalent alternative interpretation in

an ECM context as determination of the presence of a desired signal in noise alone

vis-à-vis the presence of the same signal subject to a jamming signal $S_1(\tau, \nu)$ or

$S_2(\tau, \nu)$, etc. A variety of ECM models is clearly possible, and will be discussed

in Section 5. Again, subject to the conditions of Conclusion (1) above we may expect

that the concept of ambiguity function will be of little use here, whereas if the condi-

tions of (2) above apply, appropriate forms of the ambiguity functions may be of

significant aid in describing system behavior.

4.   A RESOLUTION EXAMPLE

Let us illustrate the above remarks with an explicit calculation. We choose the resolution situation embodied in Equation (3.2) and in order to simply the analysis and obtain explicit results, we further assume

(i)   additive gaussian noise, independent of the signals;

(ii)   the (real) signals are completely known, and narrow-band, so that

        detection is coherent.

(iii)   all signals have the same strength.

Next let

$$
\begin{cases}
C_1^{(1)} = \text{cost of deciding, correctly, that } H_1 \text{ occurs when in fact it does;} \\[2em]
C_2^{(1)} = \text{cost of deciding, incorrectly, that } H_2 \text{ occurs when } H_1 \text{ is the} \\ \qquad \text{true state;} \\[2em]
C_2^{(2)} = \text{cost of deciding, correctly, that } H_2 \text{ occurs when } H_2 \text{ actually} \\ \qquad \text{does;} \\[2em]
C_1^{(2)} = \text{cost of deciding, incorrectly, that } H_1 \text{ occurs when } H_2 \text{ actually} \\ \qquad \text{does.}
\end{cases}
\tag{4.1}
$$

(Here the superscript represents the true hypothesis state, while the subscript indicates the type of decision that is made.) The optimum detector of H_2 versus H_1 is

easily shown to be[*] the ratio of the probability densities associated with the two

states H_1, H_2, viz:

$$\Lambda_{21} = \frac{p_2 \left< F_n (\underset{\sim}{V}|S_0, S_1) \right>_{S_0, S_1}}{p_1 \left< F_n (\underset{\sim}{V}|S_0) \right>_{S_0}}$$

(4.2a)

$$= \frac{p_2 \cdot e^{-\frac{1}{2}(\widetilde{\underset{\sim}{v}} - a_{01} \widetilde{\underset{\sim}{s}}_1 - a_{01} \widetilde{\underset{\sim}{s}}_2) \, \underset{\sim}{k}_N^{-1} (\underset{\sim}{v} - a_{01} \underset{\sim}{s}_1 - a_{01} \underset{\sim}{s}_2)}}{p_1 \cdot e^{-\frac{1}{2}(\widetilde{\underset{\sim}{v}} - 2 a_{01} \widetilde{\underset{\sim}{s}}_1) \, \underset{\sim}{k}_N^{-1} (\underset{\sim}{v} - 2 a_{01} \underset{\sim}{s}_1)}}$$

or

$$\therefore \Lambda_{21} = \mu_{21} \exp \left\{ - \widetilde{\underset{\sim}{v}} \, \underset{\sim}{k}_N^{-1} (a_{01} \underset{\sim}{s}_1 - a_{01} \underset{\sim}{s}_2) + 2 a_{01}^2 \widetilde{\underset{\sim}{s}}_1 \, k_N^{-1} \underset{\sim}{s}_1 \right.$$

(4.2b)

$$\left. - \frac{a_{01}^2}{2} \, (\widetilde{\underset{\sim}{s}}_1 + \widetilde{\underset{\sim}{s}}_2) \, \underset{\sim}{k}_N^{-1} (\underset{\sim}{s}_1 + \underset{\sim}{s}_2) \right\} ,$$

where $\mu_{21} \equiv p_2/p_1$, is the ratio of a priori probabilities associated with H_2 and H_1, respectively, and

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[*] Reference 5, Sections (2.2), (2.3), extended to two signal classes.

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$$\underset{\sim}{s}_2 = \underset{\sim}{s}_1(\tau, \nu), \text{ with } \underset{\sim}{s}_1 = \underset{\sim}{s}_1(0, 0). \tag{4.2c}$$

The p.d.'s $F_n(\underset{\sim}{V}|S_0)$, etc., in (4.2a) are the a priori probability densities of the data given signal S_0, etc. As before, $\underset{\sim}{k}_N$ is the normalized covariance matrix of the noise, e.g., $\underset{\sim}{k}(t_j, t_k) = [\overline{N(t_j)N(t_k)}/\overline{N^2}]$, and $\underset{\sim}{v}$, $\underset{\sim}{s}$ are normalized data and signals, and a_{01} is a normalized signal amplitude, e.g.,

$$\underset{\sim}{v} = \underset{\sim}{V}/\sqrt{\psi} \; ; \; \underset{\sim}{s} = \underset{\sim}{S}\sqrt{2}/A_{01} \; ; \; a_{01} = A_{01}/\sqrt{2\psi} \; ; \; \psi = \overline{N^2} \, (\overline{N=0}), \text{ etc.,} \tag{4.2d}$$

cf., page 32, reference 5. Writing

$$\Phi_{11}^{(s)} \equiv \underset{\sim}{\tilde{s}}_1 \underset{\sim}{k}_N^{-1} \underset{\sim}{s}_1 \; ; \; \Phi_{22}^{(s)} = \underset{\sim}{\tilde{s}}_2 \underset{\sim}{k}_N^{-1} \underset{\sim}{s}_2 \; ; \; \Phi_{12}^{(s)} = \Phi_{21}^{(s)} = \underset{\sim}{\tilde{s}}_1 \underset{\sim}{k}_N^{-1} \underset{\sim}{s}_2 \tag{4.3}$$

$$\Phi_v \equiv \underset{\sim}{\tilde{v}} \underset{\sim}{k}_N^{-1} (\underset{\sim}{s}_1 - \underset{\sim}{s}_2), \tag{4.4}$$

we can express the optimum detector (4.2a, b) more compactly as

$$\Lambda_{21} = \mu_{21} \exp\left\{-\Phi_v a_{01} + 2a_{01}^2 \Phi_{11}^{(s)} - \frac{a_{01}^2}{2}\left[\Phi_{11}^{(s)} + \Phi_{22}^{(s)} + 2\Phi_{12}^{(s)}\right]\right\}, \tag{4.5}$$

or, as is often the more convenient form, by

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$$x_{21}^* \equiv \log \Lambda_{21} = \log \mu_{21} - a_{01} \Phi_v + 2 a_{01}^2 \Phi_{11}^{(s)} - \frac{a_{01}^2}{2} \left[\Phi_{11}^{(s)} + 2 \Phi_{12}^{(s)} + \Phi_{22}^{(s)} \right] .$$

$$(4.5a)$$

From this it is at once evident that the data processing consists of a cross-correlation of the received data, $\underline{v}$, with the difference of the "reference" signal s_1 and the "shifted" signal s_2. The other terms represent (precomputed) biases, which must be used with respect to the decision threshold. The decision process is

decide H_2: e.g., a "resolved" signal, in noise, vis-à-vis the reference, if $\quad x_{21}^* \geq \log \mathcal{K}_{21}$;

$$, \quad (4.6)$$

decide H_1: e.g., no resolution $\qquad x_{21}^* < \log \mathcal{K}_{21}$

where $\mathcal{K}_{21}$ is the threshold, formed by the various

$$\mathcal{K}_{21} = \frac{C_2^{(1)} - C_1^{(1)}}{C_1^{(2)} - C_2^{(2)}} \quad (>0) . \tag{4.7}$$

[We must have $C_2^{(1)} > C_1^{(1)}$, $C_1^{(2)} > C_2^{(2)}$, i.e., "failure" costs more than "success."]

4.1 Expected Performance

In order to evaluate how well this optimum system resolves $S_1 = S_1(\tau, \nu)$ vis-à-vis $S_0 = S_1(0, 0)$, we must next determine the error probabilities, or equally

effectively, the probabilities of correct decisions. These are the (conditional) error probabilities

$$\beta_1^{(1)} = 1 - \beta_2^{(1)} = 1 - \int_\Gamma \langle F_n (\underset{\sim}{V}|\underset{\sim}{S}_1) \rangle_{S_1} \delta(\gamma_2|\underset{\sim}{V}) \, d\underset{\sim}{V} \qquad (4.8a)$$

$$\beta_2^{(2)} = 1 - \beta_1^{(2)} = 1 - \int_\Gamma \langle F_n (\underset{\sim}{V}|\underset{\sim}{S}_1, \underset{\sim}{S}_2) \rangle_{S_1, S_2} \delta(\gamma_1|\underset{\sim}{V}) \, d\underset{\sim}{V} , \qquad (4.8b)$$

where $\delta(\gamma_2|\underset{\sim}{V}) + \delta(\gamma_1|\underset{\sim}{V}) = 1$ are the decision rules (probabilities) governing the decisions $\gamma_2: H_2$, $\gamma_1: H_1$ based on the received data $\underset{\sim}{V}$, in the usual way[*]. Here $\beta_2^{(1)}$ is the (conditional) error probability of deciding H_2, i.e., "resolution," when H_1 is truly the case, etc. The probability densities F_n are

$$H_1: \; F_n (\underset{\sim}{V}|\underset{\sim}{S}_0) = \frac{\left\{ \exp\left[-\tfrac{1}{2}(\underset{\sim}{\tilde{v}} - 2a_{01}\underset{\sim}{\tilde{s}}_1) \, k_N^{-1} (\underset{\sim}{v} - 2a_{01} s_1) \right] \right\}}{(2\pi)^{n/2} \sqrt{\det k_N}} \qquad (4.9a)$$

$$H_2: \; F_n (\underset{\sim}{V}|S_0, S_1) = \frac{\left\{ \exp\left[-\tfrac{1}{2}(\underset{\sim}{\tilde{v}} - 2a_{01}\underset{\sim}{\tilde{s}}_2) \, k_N^{-1} (\underset{\sim}{\tilde{v}} - a_{01}\underset{\sim}{\tilde{s}}_1 - a_{01}\underset{\sim}{\tilde{s}}_2) \right] \right\}}{(2\pi)^{n/2} \sqrt{\det k_N}} .$$
$$\qquad (4.9b)$$

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[*] Reference 5, Equation (2.10a, b) extended to the two-signal case.

The averages $<\ >_{S_1}$, $<\ >_{S_1, S_2}$ here drop out, as all parameters of the signals are assumed known. We need first the characteristic functions of x^*_{12}, viz:

$$F_1(i\xi | H_1) = <e^{i\xi x^*_{12}}>_{H_1} \; ; \; F_1(i\xi | H_2) = <e^{i\xi x^*_{12}}>_{H_2}, \qquad (4.10)$$

where $<\ >_{H_1}$ indicates the average with respect to the H_1-density, Equation (4.9a), etc. These averages are easily found with the help of Equation (7.26), Reference 4, to be

$$F_1(i\xi | H_1) = \exp\left[i\xi\left(\log \mu_{21} - \frac{a^2_{01}}{2} \Psi_{12} \right) - \frac{a^2_{01} \xi^2}{2} \Psi_{12} \right], \qquad (4.11)$$

with

$$\Psi_{12} \equiv \Phi^{(s)}_{11} - 2 \Phi^{(s)}_{12} + \Phi^{(s)}_{22}, \qquad (4.11a)$$

and

$$F_1(i\xi | H_2) = \exp\left[i\xi\left(\log \mu_{21} + \frac{a^2_{01}}{2} \Psi_{12} \right) - \frac{a^2_{01}}{2} \xi^2 \Psi_{12} \right]. \qquad (4.12)$$

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The d.d.'s of x_{12}^* with respect to H_1 and H_2 are the Fourier transforms of (4.10), respectively, and become [cf. Equation (2.45), (2.46), Reference 5].

$$W_1(x_{12}^*|H_1) = [\,2\pi a_{01}^2\,\Psi_{12}\,]^{-\frac{1}{2}}\;e^{-(x_{12}^* - \log \mu_{21} + a_{01}^2\,\Psi_{12}/2)^2 \big/ 2a_{01}^2\,\Psi_{12}},$$

(4.13a)

$$W_1(x_{12}^*|H_2) = [\,2\pi a_{01}^2\,\Psi_{12}\,]^{-\frac{1}{2}}\;e^{-(x_{12}^* - \log \mu_{21} - a_{01}^2\,\Psi_{12}/2)^2 \big/ 2a_{01}^2\,\Psi_{12}}.$$

(4.13b)

From this and the fact that the conditional error probabilities for this optimum system are given by

$$\beta_2^{(1)*} = \int_{\log \mathcal{K}_{21}}^{\infty} W_1(x_{12}^*|H_1)\,dx_{12}^*; \quad \beta_1^{(2)*} = \int_{-\infty}^{\log \mathcal{K}_{21}} W_1(x_{12}^*|H_2)\,dx_{12}^*,$$

(4.14)

enables us to write finally [cf., Equation (2.47), Reference 5]

$$\left\{ \begin{array}{c} \beta_2^{(1)*} \\ \beta_1^{(2)*} \end{array} \right\} = \tfrac{1}{2}\left\{ 1 - \text{\textcircled{H}}\left[\frac{a_{01}\sqrt{\Psi_{12}}}{2\sqrt{2}} \pm \frac{\log(\mathcal{K}_{21}/\mu_{21})}{\sqrt{2}\,a_{01}\sqrt{\Psi_{12}}} \right] \right\},$$

(4.15)

where $\text{\textcircled{H}}$ is the familiar error function

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$$\bigcirc\!\!\!\!H\ (x) = \frac{2}{\sqrt{\pi}}\ \int_0^x e^{-t^2}\, dt\,, \quad \bigcirc\!\!\!\!H\ (-x) = -\ \bigcirc\!\!\!\!H\ (x). \tag{4.16}$$

The desired (total) probabilities of <u>correct decisions</u> are now

$$H_1\colon\ p_1(1-\beta_2^{(1)*}) = p_1\beta_1^{(1)*} = \frac{p_1}{2}\left\{1 + \bigcirc\!\!\!\!H\left[\frac{a_{01}\sqrt{\Psi_{12}}}{2\sqrt{2}} + \frac{\log(\mathcal{K}_{21}/\mu_{21})}{\sqrt{2}\,a_{01}\sqrt{\Psi_{12}}}\right]\right\},$$

$$\tag{4.17a}$$

$$H_2\colon\ p_2(1-\beta_1^{(2)*}) = p_2\beta_2^{(2)*} = \frac{p_2}{2}\left\{1 + \bigcirc\!\!\!\!H\left[\frac{a_{01}\sqrt{\Psi_{12}}}{2\sqrt{2}} - \frac{\log(\mathcal{K}_{21}/\mu_{21})}{\sqrt{2}\,a_{01}\sqrt{\Psi_{12}}}\right]\right\}.$$

$$\tag{4.17b}$$

Thus, $p_1\beta_1^{(1)*}$ is the probability (for the optimum system) that, for the given system parameters: (τ, ν) for signal S_2, a_{01}, sample size (n), and threshold $\mathcal{K}_{21}$, as well as a priori probabilities $p_1 + p_2 = 1$, one has decided correctly on a state of complete overlap, i.e., <u>no resolution</u>. Similarly, $p_2\beta_2^{(2)*}$ is the probability of correctly deciding that the two signals S_1, S_2 are "resolved."

An alternative form, which is often more illuminating, relates the (conditional) probability $\beta_2^{(2)*}$ of correctly resolving the two signals, to the (conditional) probability $\beta_2^{(1)*}$ of incorrectly stating that the two signals are resolved. This is obtained by eliminating the threshold expression $\log(\mathcal{K}_{21}/\mu_{21})$, with the help of (4.15), viz:

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$$\log(\mathcal{K}_{21}/\mu_{21}) = \sqrt{2}\, a_{01} \sqrt{\Psi_{12}} \left(H\right)^{-1} (1 - 2\beta_2^{(1)*}) - \frac{\Psi_{12}\, a_{01}^2}{2} \ . \tag{4.18}$$

With this in (4.17b) we get directly the desired relation for the (conditional) proba-
bility of correct resolution:

$$\beta_2^{(2)*} = \tfrac{1}{2} \left\{ 1 + \left(H\right) \left[a_{01} \sqrt{\Psi_{12}/2} - \left(H\right)^{-1} (1 - 2\beta_2^{(1)*}) \right] \right\} \ . \tag{4.19}$$

Let us suppose that $\beta_2^{(1)*}$ is required to be small $[\beta_2^{(1)*} > 0$, of course, for
finite sample sizes and thresholds]. Then $\left(H\right)^{-1}$ is a positive number, $0(< \infty)$, so
that to make $\beta_2^{(2)*}$ large we must have $a_{01} \sqrt{\Psi_{12}/2}$ large enough to offset $[-\left(H\right)^{-1}]$
and be $0(2 - 3$ or more$)$, so that $\left(H\right) \doteq 1$. Thus, we see that for fixed $\beta_2^{(1)*}$, the
value of the desired "resolution" probability is controlled by

$$\sigma_{12}^2 \equiv \frac{a_{01}^2 \Psi_{12}}{2} = \frac{a_{01}^2}{2} \left[\Phi_{11}^{(s)} - 2\Phi_{12}^{(s)} + \Phi_{22}^{(s)} \right] \ . \tag{4.20}$$

We can achieve large σ_{12}^2 if either, or both, the input signal-to-noise ratio a_{01}^2
and the structure factor Ψ_{12} are large. Note that

$$\Psi_{12} = (\tilde{\underset{\sim}{s}}_1 - \tilde{\underset{\sim}{s}}_2)\, \underset{\sim}{k}_N^{-1} (\underset{\sim}{s}_1 - \underset{\sim}{s}_2) > 0 \tag{4.21}$$

is a positive definite quadratic form[*], so that for $a_{01}^2 > 0$, we have $\sigma_{12}^2 > 0$.

4.2   Continuous Sampling: Bayes Ambiguity Functions

To illustrate the rôle of the ambiguity functions, let us now replace the discrete, sampled forms above by the appropriate continuous functionals. From the Appendix (Reference 5) we easily see that now

$$\Phi_{11}^{(s)} \rightarrow \Phi_{11}^{(T)} = \psi \int_T s_1(t - \epsilon_o, \underset{\sim}{\theta}_o) \, X_T \, (t; \epsilon_o, \underset{\sim}{\theta}_o)_1 \, dt \, ; \qquad (4.22a)$$

$$\Phi_{22}^{(s)} \rightarrow \Phi_{22}^{(T)} = \psi \int_T s_2(t - \epsilon_o, \underset{\sim}{\theta}_o) \, X_T \, (t; \epsilon_o, \underset{\sim}{\theta}_o)_2 \, dt \, ; \qquad (4.22b)$$

$$\Phi_{12}^{(s)} \rightarrow \Phi_{12}^{(T)} = \psi \int_T s_2(t - \epsilon_o, \underset{\sim}{\theta}_o) \, X_T \, (t; \epsilon_o, \underset{\sim}{\theta}_o)_1 \, dt$$

$$(4.22c)$$

$$\left(= \psi \int_T s_1(t - \epsilon_o, \underset{\sim}{\theta}_o) \, X_T \, (t; \epsilon_o, \underset{\sim}{\theta}_o)_2 \, dt \right),$$

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[*] This is true, unless $\underset{\sim}{s}_1 = \underset{\sim}{s}_2$, where of course, there is complete overlap of the two signals in H_2, and $\beta_2^{(2)*} = \beta_2^{(1)*}$; also, $\beta_1^{(1)*} = \beta_2^{(2)*} = \beta_1^{(2)*} = \beta_2^{(2)*} = \frac{1}{2}$, $\therefore p_2 \beta_2^{(2)*} + p_1 \beta_1^{(1)*} = \frac{1}{2}$ and $p_2 \beta_1^{(2)*} + p_1 \beta_2^{(1)*} = \frac{1}{2}$: the probability of correct or incorrect decisions as to resolution or not $(H_2$ versus $H_1)$ are equal and $\frac{1}{2}$; either state H_2, or H_1, is equally likely on the average, and since $\underset{\sim}{s}_1 = \underset{\sim}{s}_2$ there is no "resolution."

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where X_{T1}, X_{T2} are the (real) solutions of the basic integral equations

$$\int_T K_N(t, u) \, X_T \, (u, \epsilon_o, \theta_o)_{1, 2} \, du = s_{1, 2}(t - \epsilon_o, \theta_o), \; t \in T. \qquad (4.23)$$

Now, since $s_{1, 2}$ are real, narrow-band signals, we may expect that X_T has a narrow-band structure also. In any case, we can define a complex reference "signal"

$$\hat{Z}_{T1, 2} \equiv |\hat{Z}_{T1, 2}| \, e^{i\omega_o t - j\psi_Z}; \; \hat{Z}_{T01, 2} = \left(|\hat{Z}_{T1, 2}| e^{-i\psi_Z} \right)_{1, 2}, \qquad (4.24)$$

which is the solution of

$$\int_T K_N(t, u) \, \hat{Z}_T \, (u; \epsilon_o, \theta)_{1, 2} \, du = \hat{s}_{1, 2}(t - \epsilon_o, \theta_o) = \hat{s}_{01, 2} \, e^{i\omega_o t}, \; t \in T,$$
$$(4.25)$$

so that

$$\text{Re} \, \hat{Z}_T = X_T = |\hat{Z}_T| \cos(\omega_o t - \psi_Z), \text{ etc.}; = 0, \; t \notin T. \qquad (4.26)$$

We can now express $\Phi_{11}^{(T)}$ etc., in complex form by the following argument:

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$$\Phi_{11}^{(T)} = \psi \int_T \operatorname{Re} \hat{s}_1 \cdot \operatorname{Re} \hat{Z}_{T1} \, dt = \psi \int_T |\hat{s}_{01}| \cos(\omega_o t - \psi'_{s1})$$

$$\text{(4.27)}$$

$$\cdot |\hat{Z}_T| \cos(\omega_o t - \psi_{Z1}) \, dt,$$

with $\psi'_1 = \psi_1 + \omega_o \epsilon_o$. Expanding the cosine product and observing because of the narrow-band character of $|\hat{s}_1|$ and $|\hat{Z}_{T1}|$, and slowly-varying nature of $|\hat{s}_{01}|$, $|\hat{Z}_{T1}|$, ψ'_s, ψ_Z, that the rapidly oscillating term in $\cos(2\omega_o t - \psi'_s - \psi_Z)$ yields an ignorable contribution to the integral, we obtain

$$\Phi_{11}^{(T)} \doteq \frac{\psi}{2} \int_T |\hat{s}_{01}\| \hat{Z}_{T1}| \cos(\psi'_{s1} - \psi_{Z1}) \, dt$$

$$\doteq \frac{\psi}{2} \operatorname{Re} \int_T |\hat{s}_{01}\| \hat{Z}_{T1}| \, e^{i\omega_o t - i\psi'_{s1} - i\omega_o t + i\psi_{Z1}} \, dt$$

$$= \frac{\psi}{2} \operatorname{Re} \int_T \hat{s}_{01}(t, \boldsymbol{\theta}_o) \, \hat{Z}_{T01}(t; \epsilon_o, \boldsymbol{\theta}_o)^* \, dt \left. \begin{array}{c} \\ \\ \\ \\ \\ \end{array} \right\}$$

$$= \frac{\psi}{2} \operatorname{Re} \int_T \hat{Z}_{T01}(t) \, \hat{s}_{01}(t)^* \, dt, \qquad \text{(4.28)}$$

so that $\Phi_{11}^{(T)}$ is now expressed as the (real part of the) product of the complex signal envelope and the complex envelope of the "reference" signal, $\hat{Z}_T$, cf. (4.24).

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Similarly, we get for (4.22b, c)

$$\Phi_{22}^{(T)} \doteq \frac{\psi}{2} \, \mathrm{Re} \int_T \hat{s}_{02}(t) \, \hat{Z}_{T02}(t; ..)^* \, dt \ , \tag{4.29a}$$

$$\Phi_{12}^{(T)} \doteq \frac{\psi}{2} \, \mathrm{Re} \int_T \hat{s}_{01}(t) \, \hat{Z}_{T02}(t; ..)^* \, dt = \frac{\psi}{2} \, \mathrm{Re} \int_T \hat{s}_{02}(t) \, \hat{Z}_{T01}(t; ..)^* \, dt \ , \tag{4.29b}$$

where $\hat{s}_{01}$, $\hat{s}_{02}$, $\hat{Z}_{T02}$, $\hat{Z}_{T01}$ are the complex envelopes of $\hat{s}_1$, $\hat{s}_2$, $\hat{Z}_{T1}$, $\hat{Z}_{T2}$.

At this point let us normalize $\hat{s}_{01}$, $\hat{s}_{02}$ and $\hat{Z}_{T01, 2}$, according to (2.4a).

Letting

$$\hat{z}_{T01, 2} \equiv \hat{Z}_{T01, 2} \Bigg/ \left(\int |\hat{Z}_{T01, 2}(t)|^2 \, dt \right)^{\frac{1}{2}} ; \ (\hat{Z}_T = 0, \ t \notin T \pm) \ , \tag{4.30a}$$

$$\hat{s}_{01, \, 2\text{-norm}} \equiv \hat{S}_{01, \, 2}(t) \Bigg/ \left(\int_{-\infty}^{\infty} |\hat{S}_{01, \, 2}(t)|^2 \, dt \right)^{\frac{1}{2}} \ , \tag{4.30b}$$

be the normalized signals, we can now express $\Phi_{11}^{(T)}$, (4.28), etc., with the help of

the normalizations of (2.4), (2.15) for the respective ambiguity functions, alterna-

tively as

$$\Phi_{11}^{(T)} = \frac{\psi}{2} \left\{ \int_T |\hat{Z}_{T01}(t)|^2 \, dt \cdot \int_{-\infty}^{\infty} |\hat{S}_{01}(t)|^2 \, dt \right\}^{\frac{1}{2}} \cdot \mathrm{Re} \, \chi_{\hat{s}_1, \, \hat{z}_1}(0, 0) \ , \tag{4.31a}$$

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$$\Phi_{22}^{(T)} = \frac{\psi}{2} \left\{ \int_T |\hat{Z}_{T02}(t)|^2 \, dt \cdot \int_{-\infty}^{\infty} |\hat{S}_{02}(t)|^2 \, dt \right\}^{\frac{1}{2}} \cdot \operatorname{Re} \chi_{\hat{s}_2, \hat{z}_2}(0, 0),$$

$$\text{(4.31b)}$$

$$\Phi_{12}^{(T)} = \frac{\psi}{2} \left\{ \int_T |\hat{Z}_{T02}(t)|^2 \, dt \cdot \int_{-\infty}^{\infty} |\hat{S}_{01}(t)|^2 \, dt \right\}^{\frac{1}{2}} \cdot \operatorname{Re} \chi_{\hat{s}_1, \hat{z}_2}(0, 0; \tau, \nu),$$

$$\text{(4.31c)}$$

$$= \frac{\psi}{2} \left\{ \int_T |\hat{Z}_{T01}(t)|^2 \, dt \cdot \int_{-\infty}^{\infty} |\hat{S}_{02}(t)|^2 \, dt \right\}^{\frac{1}{2}} \operatorname{Re} \chi_{\hat{z}_1, \hat{s}_2}(0, 0; \tau, \nu).$$

Thus, we see that it is the real part of the ambiguity function that enters into these expressions[*]. The real part of the ambiguity function is the critical factor deter-mining the magnitude of the quadratic forms $\Phi_{11}^{(T)}$, etc., which determine the key parameter $\Psi_{12} \to \Psi_{12}^{(T)}$, which in turn governs the probabilities of correct and in-correct decisions, $\beta_2^{(2)*}$, etc., cf. (4.19).

We remark that $\chi_{\hat{s}_1, \hat{z}_1}(0, 0)$ does not depend on (τ, ν) here, while $\Phi_{22}^{(T)}$, $\Phi_{12}^{(T)}$ depend, in general, implicitly on the (τ, ν) of $\hat{S}_2 = \hat{S}_1(\tau, \nu)$. In $\Phi_{22}^{(T)}$ this dependence is weak, entering only as small contributions from the "end-effects" (at $t = 0-$, $T+$) in the solution $\hat{Z}_{T2}$ of (4.25). However, in $\Phi_{12}^{(T)}$ the dependence on (τ, ν) is critical, as we shall note below in the important special case of white noise backgrounds, for example.

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[*] This is because χ is a quadratic (or signal energy) form and we deal always with real signals.

The (cross-) ambiguity functions [above (4.31)] are normalized and have the specific forms

$$\chi_{\hat{s}_1, \hat{z}_1}(0, 0) = \int_T \hat{s}_{01}(t)_{norm} \hat{z}_{01}(t)^* dt; \tag{4.32a}$$

$$\chi_{\hat{s}_2, \hat{z}_2}(0, 0) = \int_T \hat{s}_{02}(t)_{norm} \hat{z}_{T02}(t; \tau, \nu)^* dt \tag{4.32b}$$

$$= \int_T \hat{s}_{01}(t + \tau)_{norm} e^{2\pi i \nu t} \hat{z}_{T02}(t; \tau, \nu)^* dt;$$

$$\chi_{\hat{s}_1, \hat{z}_2}(0, 0; \tau, \nu) = \int_T \hat{s}_{01}(t)_{norm} \hat{z}_{T02}(t; \tau, \nu)^* dt \tag{4.32c}$$

$$= \int_T \hat{z}_{T01}(t) \hat{s}_{01}(t + \tau)^*_{norm} e^{-2\pi i \nu t} dt = \chi_{\hat{z}_1, \hat{s}_2}(0, 0; \tau, \nu),$$

where, of course, $\hat{z}_{T01}$, $\hat{z}_{T02}$ are obtained from the solution of (4.25) in conjunction with (4.30a) above. These ambiguity functions are called <u>Bayes</u> <u>ambiguity</u> <u>functions</u> here, because they arise in the determination of the Bayes, or optimum (detector) system--for resolution, in this particular instance [cf. Section 3 earlier]. They represent a more general type of ambiguity function because they are generated as one of the significant parameters of optimum system structure and performance

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through Bayes criteria and decision rules [cf. Section 3]. In this respect they are analogous to Bayes matched filters[11]. They subsume the earlier, "classical" forms and in special instances are identical with them [cf. Section (4.3) following].

4.3   White Noise Backgrounds

With white noise backgrounds, characteristic of most system noise in the receiver and certain types of external interference, we obtain an interesting simplification of the preceding results. We have

$$K_N(t, u) = \frac{W_o}{2} \delta(t - u), \tag{4.33}$$

where W_o is the intensity spectral density of the noise. Applied to (4.25) we get at once

$$\hat{z}_{T1, 2} = \frac{2}{W_o} \hat{s}_{1, 2}(t - \epsilon_o, \underset{\sim}{\theta}), \quad t \in T; = 0, t \notin T \tag{4.34a}$$

$$= \frac{2}{W_o} \hat{s}_{01, 2}(t, \underset{\sim}{\theta}_o) e^{i\omega_o t} = \frac{2}{W_o} |\hat{s}_{01, 2}(t, \underset{\sim}{\theta}_o)| e^{i\omega_o(t - \epsilon_o) - i\psi_s(t, \underset{\sim}{\theta})} \tag{4.34b}$$

$$\therefore \hat{z}_{T01, 2} = \frac{2}{W_o} |\hat{s}_{01, 2}(t, \underset{\sim}{\theta}_o)| e^{-i\psi'_s(t, \underset{\sim}{\theta}_o)} e^{i\omega_o t}; \quad \psi'_s = \psi_s + \omega_o \epsilon_o, \tag{4.35a}$$

and

$$\therefore X_{T1,\,2} = \frac{2}{W_o}\ |\hat{s}_{01,\,2}|\ \cos[\,\omega_o t - \psi'_s(t,\,\theta_0)\,]\ ,\ t\,\epsilon\,T\,. \tag{4.35b}$$

Substituting these relations into (4.31) gives

$$a_{01}^2\ \Phi_{11}^{(T)} = \frac{A_{01}^2}{2W_o}\ \cdot \int_{-\infty}^{\infty}|\hat{s}_{01}(t)|_T^2\ dt\ \cdot\ \mathrm{Re}\ \chi_{\hat{s}_1,\,\hat{z}_1}(0,\,0), \tag{4.36a}$$

$$[\ |\hat{s}_{01}(t)|_T = 0,\ (t\,\notin\,T),\ \mathrm{etc.}\,],$$

$$a_{01}^2\ \Phi_{22}^{(T)} = \frac{A_{01}^2}{2W_o}\ \int_{-\infty}^{\infty}|\hat{s}_{02}(t)|_T^2\ dt\ \cdot\ \mathrm{Re}\ \chi_{\hat{s}_2,\,\hat{z}_2}(0,\,0), \tag{4.36b}$$

$$a_{01}^2\ \Phi_{12}^{(T)} = \frac{A_{01}^2}{2W_o}\left(\int_{-\infty}^{\infty}|\hat{s}_{01}(t)|_T^2\ dt \int_{-\infty}^{\infty}|\hat{s}_{02}(t)|_T^2\ dt\right)^{\frac{1}{2}}\ \mathrm{Re}\ \chi_{\hat{s}_1,\,\hat{z}_2}(0,\,0;\,\tau,\,\nu). \tag{4.36c}$$

Now, because of our original choice of signals, viz.,

$$\hat{s}_{02}(t) = \hat{s}_{01}(t+\tau)\,e^{2\pi i\nu t}, \tag{4.37}$$

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we see that

$$\int_{-\infty}^{\infty} |\hat{s}_{02}(t)|_T^2 \, dt = \int_{-\infty}^{\infty} |\hat{s}_{01}(t + \tau) e^{2\pi i \nu t}|_T^2 \, dt = \int_{-\infty}^{\infty} |\hat{s}_{01}(t')|_T^2 \, dt' , \quad (4.38)$$

and so Equation (4.32a) becomes the (auto-) ambiguity function

$$\chi_{11}(0, 0) = \frac{2}{W_o} \int_T \hat{s}_{01}(t) \hat{s}_{01}(t)^* \, dt \Bigg/ \Bigg[\left(\frac{2}{W_o} \right)^2 \int_T |\hat{s}_{01}(t)|^2 \, dt$$

$$(4.39)$$

$$\cdot \int_T |\hat{s}_{01}(t)|^2 \, dt \Bigg]^{\frac{1}{2}} = 1 ,$$

as expected. Similarly, we get from (4.32b), the (auto-) ambiguity functions

$$\chi_{22}(0, 0) = \int_T \hat{s}_{01}(t + \tau) e^{2\pi i \nu t} \cdot \frac{2}{W_o} \hat{s}_{01}(t + \tau)^* e^{-2\pi i \nu t} \, dt$$

$$\div \frac{2}{W_o} \int_{-\infty}^{\infty} |\hat{s}_{01}(t + \tau)|_T^2 \, dt \qquad (4.40)$$

$$= 1 ,$$

again as expected, and $\chi_{22}(0, 0)$ is independent of (τ, ν). However, for $\chi_{\hat{s}_1, \hat{z}_2}(0, 0; \tau, \nu)$, (4.32c), we get here

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$$\chi_{\hat{s}_1, \hat{z}_2}(0, 0; \tau, \nu) = \int_T \hat{s}_{01}(t)\, \hat{s}_{01}(t + \tau)^* \, e^{-2\pi i \nu t} \, dt \Bigg/ \int_{-\infty}^{\infty} |\hat{s}_{01}(t)|_T^2 \, dt$$

$$\text{(4.41a)}$$

$$= \chi_{11}(\tau, \nu). \tag{4.41b}$$

Accordingly, the (cross-) ambiguity function $\chi_{\hat{s}_1, \hat{z}_2}$ in this white noise case

becomes the (auto-) ambiguity function $\chi_{11}(\tau, \nu)$ of the original signal, $\hat{s}_1$.

Furthermore, we see that $\chi_{11}(\tau, \nu)$ here is not only Bayes, but is identical with

the classical ambiguity function of Woodward, where also $\hat{z}_{02} = \hat{s}_{01}(t + \tau)^* \, e^{-2\pi i \nu t}$

may be interpreted in familiar fashion[2] as a matched filter response to the returned

signal from a target delayed by τ and shifted in doppler by ν. Putting Equations

(4.39) - (4.41) into Equations (4.36a, b, c) gives us finally, for these signals and

white noise,

$$a_{01}^2 \, \Phi_{11}^{(T)} = a_{02}^2 \, \Phi_{22}^{(T)} = \frac{A_{01}^2}{2W_o} \int_T |\hat{s}_{01}(t)|^2 \, dt : \tag{4.42}$$

and

$$a_{01}^2 \, \Phi_{12}^{(T)} = \frac{A_o^2}{2W_o} \int_T |\hat{s}_{01}(t)|^2 \, dt \cdot \operatorname{Re} \chi_{11}(\tau, \nu), \tag{4.43}$$

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so that our basic parameters $\Psi_{12}^{(T)}$, or σ_{12}^2, becomes

$$a_0^2 \, \Psi_{12}^{(T)} = \frac{A_0^2}{W_o} \int_T |\hat{s}_{01}(t)|^2 \, dt \cdot [\, 1 - \text{Re} \, \chi_{11}(\tau, \, \nu) \,] \qquad (4.44a)$$

and

$$\therefore \sigma_{12}^2 = \frac{A_{01}^2}{2W_o} \int_T |\hat{s}_{01}(t)|^2 \, dt \cdot [\, 1 - \text{Re} \, \chi_{11}(\tau, \, \nu) \,]$$

$$(4.44b)$$

$$= \left(\int_T \frac{|\hat{S}_{01}(t)|^2 \, dt}{W_o} \right) \cdot \{ \, 1 - \text{Re} \, \chi_{11}(\tau, \, \nu) \} \, ,$$

where $\hat{S}_{01}$ is the complex envelope of the (unnormalized) reference signal . (Note that for $(\tau, \, \nu) = (0, \, 0)$, $\chi_{11}(0, \, 0)$ is unity; and so σ_{12}^2 vanishes, as expected, with a resulting minimum probability of correct decisions as to resolution, as indicated by (4.20), viz: $\beta_2^{(2)*} = \beta_2^{(1)*}$ [which is required to be small $(\ll \beta_2^{(2)*})$, ordinarily, for effective operation].

4.4 Optimum Resolution

Next, let us suppose that the signal energy available in $(0, \, T)$ is fixed and, again, that the background noise is white. Thus, the only way to increase σ_{12}^2, and hence $\beta_2^{(2)*}$, the probability of correctly deciding that the two signals are resolved

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(for given probability $\beta_2^{(1)*}$ of incorrectly deciding this), cf. (4.19), is to make

Re $\chi_{11}(\tau, \nu)$ as small as possible, by Equation (4.44a, b). This broadly agrees

with our intuitive or "classical" notions of "resolution" being achieved when $|\chi_{11}|$

is small enough [cf. remarks in Section 2, 3 above]. However, note again, that

when "resolution" is formulated on an (optimum) decision basis-à-la-Bayes--it is

Re $\chi_{11}(\tau, \nu)$ that is controlling, not $|\chi_{11}|$ above. Accordingly, we must consider

not only $|\chi_{11}|$ but the angle of χ_{11} as well, e.g.,

$$\text{Re } \chi_{11}(\tau, \nu) = \text{Re} \left\{ |\chi_{11}(\tau, \nu)| \, e^{i\phi_\chi(\tau, \nu)} \right\} = |\chi_{11}(\tau, \nu)| \cos \phi_\chi(\tau, \nu).$$
(4.45)

It is this $\cos \phi_\chi$ term which must be included in our inspection and interpretation of

χ_{11} and $|\chi_{11}|$.

Similar remarks apply for the more general case of colored noise backgrounds.

It is $\Phi_{12}^{(T)}$ that must be made as small as possible, to ensure the largest $\beta_2^{(2)*}$ con-

sistent with given $\beta_2^{(1)*}$, as we can see at once from (4.31) in (4.20). Again, it is

the real part of the ambiguity functions that is pertinent here.

As a numerical example, let us consider again the white-noise case of Section

(4.3) above, and let us require the conditional probability of incorrectly deciding the

"resolved state" H_2 to be $\beta_2^{(1)*} = 5 \times 10^{-2}$. Then $(H)^{-1}(1 - 2\beta_2^{(1)*}) = (H)^{-1}(0.90)$

$= 1.163$. Furthermore, let us also require that $\beta_2^{(2)*} = 0.990$. From (4.19) and

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(4.44a) we easily find that $\sigma_{12}^2 = (2.808)^2 = 7.88$. Now, if sample size T, waveform $\hat{S}_{01}$, and noise density W_o are such that $\int_T |\hat{S}_{01}(t)|^2 dt/W_o = 8.00$, say, we see that

$$\operatorname{Re} \chi_{11}(\tau, \nu) = 1.5 \cdot 10^{-2}, \tag{4.45}$$

for these conditional probabilities. This fixes the domain of (τ, ν): such that $\operatorname{Re} \chi_{11}(\tau, \nu) \leq 1.5 \cdot 10^{-2}$, for a (conditional) probability of correctly deciding that the signals are resolved $\beta_2^{(2)*} \geq 0.990$. In the case of the single rectangular pulse of Equation (2.9), we find that (4.44b) is specifically

$$\sigma_{12}^2 = \frac{A_0^2 \tau_o}{W_o} \left\{ 1 - \frac{\sin \pi \nu(\tau_o - |\tau|)}{\pi \nu \tau_o} \cos \pi \nu \tau \right\} ; 0 < |\tau| < \tau_o$$

$$, \tag{4.46}$$

$$= \frac{A_0^2 \tau_o}{W_o} \qquad\qquad , |\tau| > \tau_o,$$

with $A_0^2 \tau_o / W_o = 8.00$ in the present case, so that we require

$$\operatorname{Re} \chi_{11}(\tau, \nu) = \frac{\sin \pi \nu(\tau_o - |\tau|)}{\pi \nu \tau_o} \cos \pi \nu \tau \overset{<}{=} 1.5 \times 10^{-2}, \ 0 < |\tau| < \tau_o;$$

$$\tag{4.47}$$

$$= 0, \ |\tau| > \tau_o$$

for the present numerical choices. Note that although the conditional probability $\beta_2^{(2)*}$ is nearly unity, the total probability $P(H_2)$ of correct decision on H_2 ("resolution") may be quite small, since the total probability is $p_2 \beta_2^{(2)*}$. Thus, if $p_2 = 0.10$, $P(H_2) = p_2 \beta_2^{(2)*} = 0.099$, while on the other hand, if $p_2 = 0.8$, say, we see that $P(H_2) = 0.792$, which may be regarded as quite large in some applications. Observe that $\beta_2^{(2)*} = \beta_2^{(1)*} = 5 \cdot 10^{-2}$ here if $(\tau, \nu) = (0, 0)$: the (conditional) probability of correct decision as to resolution is equal to the corresponding (preset) error probability, which is taken to be very small. Thus, as expected, we have little accuracy of decision when there is complete overlap under these conditions. Other useful values may be obtained in similar fashion.

5.   SOME SIMPLE ECM EXAMPLES

In the preceding we have illustrated the use and significance of Bayes ambiguity

functions in a typical resolution situation. Now let us examine their uses in a

optimum detection example involving interfering signals. We distinguish four typical

situations, two of which we shall examine in some detail below. These are

$$\underline{\text{Case I:}} \qquad H_1: \ S + S_J + N \text{ vs. } H_o: \ S + N \qquad\qquad (5.1)$$

We wish to decide whether an interfering signal (S_J) is present
or not (and if it is, how effective it is in concealing the desired
signal S: this is basically a signal extraction problem, which
we shall not consider further here).

$$\underline{\text{Case II:}} \qquad H_1: \ S + S_J + N \text{ vs. } H_o: \ N \qquad\qquad (5.2)$$

This is the same as (5.1), but now the desired signal is not
necessarily known to be present.

$$\underline{\text{Case III:}} \qquad H_1: \ S + S_J + N \text{ vs. } H_o: \ S_J + N \qquad\qquad (5.3)$$

Here we wish to determine (or conceal)the presence of a desired
signal in (or by) an interfering signal ("obliteration").

$$\underline{\text{Case IV:}} \qquad H_1: \ S + N \text{ vs. } H_o: \ S_J + N \qquad\qquad (5.4)$$

In this case we wish to distinguish between the presence of the
desired signal against that of the interference ("decoy").

An accompanying background noise is denoted by N, which is usually gaussian, white, and statistically independent of S and S_J. The interference S_J may be deterministic, entirely random (e.g., another gaussian) noise process, or a combination of such elements. The general task here is to obtain optimum detection from a hierarchy of viewpoints:

(i)   given a class of S_J, determine the best detector for desired signals S:

(ii)   given a class of desired signals S, determine the best detector for S

in the presence of S_J, <u>and</u> then find the subclass of S_J, subject to one or

more reasonable constraints, which minimizes this "maximum" perfor-

mance for S:

(iii)   given (ii), find the subclass of desired signals S (again with suitable

constraints) which now must effectively operate against the subclass of

S_J.

If $R(\sigma, \delta | S, S_J)$ is the average risk for detection with S, S_J in various combina-

tions, we may express (i) - (iii) symbolically by [12]

(i) $\min\limits_{\delta \to \delta^*} R = R^*(\sigma, \delta^* | S, S_J); \delta^* =$ Bayes decision rule; $\qquad$ (5.5)

(ii) $\max\limits_{S_J} (R^* + \lambda_1 f_1 (S_J)) = R^{*|\dagger};$ $\qquad$ (5.6)

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(iii) $\min\limits_{S}\ \max\limits_{S_J}\ \ (R^{*\,|\dagger} + \lambda_2 f_2(S)) = R^{*\,|\dagger\,|-}$;   (5.7)

where λ_1, λ_2 are undetermined multipliers for the constraints f_1, f_2. With

random signals, max or min over S, S_J is replaced by max or min over the respec-

tive distributions $\sigma(S)$, $\sigma(S_J)$ governing the signal waveforms.

Let us now illustrate the approach with the promised examples, which although

not very realistic in detail, show qualitatively (and quantitatively) how Bayes

ambiguity functions enter into the evaluation of a detectors's effectiveness against,

and/or vulnerability to, interfering signals. We select Cases III and IV, Equations

(5.3) and (5.4) and for analytic simplicity postulate that the S_J, as well as the desired

signals, S, are deterministic and except for possible presence or absence are other-

wise completely known at the receiver. We proceed as in Section 4 to determine the

optimum detector structures and probabilities of error and detection, under the

additional assumption of independent gaussian noise backgrounds.

5.1   Examples 1 and 1a (Case III)

Let us begin with the appropriate a priori distribution for the various H_o, H_1

states in Cases III and IV. We have

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Case III:

$$H_o: \quad F_n(\underset{\sim}{V}|\underset{\sim}{S}_J) = \frac{e^{-\frac{1}{2}(\underset{\sim}{\widetilde{v}} - a_J \underset{\sim}{\widetilde{s}}_J) \underset{\sim}{k}_N^{-1} (\underset{\sim}{v} - a_J \underset{\sim}{s}_J)}}{(2\pi)^{n/2} \sqrt{\det k_N}} \tag{5.8a}$$

$$H_1: \quad F_n(\underset{\sim}{V}|\underset{\sim}{S}_J + S) = \frac{e^{-\frac{1}{2}(\underset{\sim}{\widetilde{v}} - a_J \underset{\sim}{\widetilde{s}}_J - a_o \underset{\sim}{\widetilde{s}}) \underset{\sim}{k}_N^{-1} (\underset{\sim}{v} - a_J \underset{\sim}{s}_J - a_o \underset{\sim}{s})}}{(2\pi)^{n/2} \sqrt{\det k_N}} \; . \tag{5.8b}$$

[For Case IV: Equation (5.8a) applies directly for H_o, while H_1 here is obtained from (5.8b) on setting $S_J = 0$]. Here a_o^2, a_J^2 are input signal-to-noise ratios, cf. (4.2d), and $\underset{\sim}{s}$ and $\underset{\sim}{v}$ are normalized as before, cf. (4.2d). The optimum detector structures (for detecting the desired signal, S) are at once (in logarithmic form):

$$(x_3^* \equiv) \log \Lambda_{III} = \log \mu_{10} + a_o \underset{\sim}{\widetilde{v}} \underset{\sim}{k}_N^{-1} \underset{\sim}{s} - a_o a_J \underset{\sim}{\widetilde{s}}_J \underset{\sim}{k}_N^{-1} \underset{\sim}{s} - \tfrac{1}{2} a_o^2 \underset{\sim}{\widetilde{s}} \underset{\sim}{k}_N^{-1} \underset{\sim}{s} \, , \tag{5.9}$$

$$(x_4^* \equiv) \log \Lambda_{IV} = \log \mu_{10} - \underset{\sim}{\widetilde{v}} \underset{\sim}{k}_N^{-1} (a_J \underset{\sim}{s}_J - a_o \underset{\sim}{s}) + \tfrac{1}{2} a_J^2 \underset{\sim}{\widetilde{s}}_J \underset{\sim}{k}_N^{-1} \underset{\sim}{s}_J - \tfrac{1}{2} a_o^2 \underset{\sim}{\widetilde{s}} \underset{\sim}{k}_N^{-1} \underset{\sim}{s} \, , \tag{5.10}$$

cf. (4.5a), with $\mu_{10} = p_1/p_o$, the ratio of a priori probabilities associated with the states H_o, H_1, as before.

Writing

$$\Phi_{SS} \equiv \underset{\sim}{s} \underset{\sim}{k}_N^{-1} \underset{\sim}{s} ; \quad \Phi_{SJ} \equiv \underset{\sim}{\widetilde{s}} \underset{\sim}{k}_N^{-1} \underset{\sim}{s}_J = \Phi_{JS} ; \quad \Phi_{JJ} \equiv \underset{\sim}{\widetilde{s}}_J \underset{\sim}{k}_N^{-1} \underset{\sim}{s}_J \, , \tag{5.11}$$

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and evaluating the various characteristic functions of x_3^*, x_4^* as in (4.10), (4.11) we get finally for Case III,

$$F(i \, \xi \, | H_o)_3 = \exp \left[i \, \xi \left(\log \mu_{21} - \frac{a_o^2}{2} \, \Phi_{SS} \right) - \tfrac{1}{2} \, a_o^2 \, \Phi_{SS} \, \xi^2 \right], \qquad (5.12a)$$

$$F(i \, \xi \, | H_1)_3 = \exp \left[i \, \xi \left(\log \mu_{21} + \frac{a_o^2}{2} \, \Phi_{SS} \right) - \tfrac{1}{2} \, a_o^2 \, \Phi_{SS} \, \xi^2 \right]. \qquad (5.12b)$$

It is really not surprising that the effects of the interference S_J do not appear in the statistics (Equations 5.12a, b) of the structure x_3^* and hence in the evaluation of performance. This is because we assume that S_J is known completely at the receiver, so that optimal processing simply requires that we process new data $\underset{\sim}{v}' = \underset{\sim}{v} - a_J \, \underset{\sim}{s}_J$, obtained by subtracting the perfectly known $\underset{\sim}{s}_J$ from the raw data $\underset{\sim}{v}$, according to

$$x_3^{*'} = \log \mu_{21} + a_o \, \widetilde{\underset{\sim}{v}}' \, \underset{\sim}{k}_N^{-1} \, \underset{\sim}{s} - \tfrac{1}{2} \, a_o^2 \, \widetilde{\underset{\sim}{s}} \, k_N^{-1} \, \underset{\sim}{s} = x_3^*, \qquad (5.13)$$

[which is just (5.9), of course]. Our optimal detector is thus invariant of S_J under these circumstances, and performance is described in detail by the material of Section (2.5), Reference 5. [However, as soon as S_J is in any way unknown at the receiver, the problem is no longer trivial, and the results will depend on the interference.]

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{Example 1a: A nontrivial modification of Case III here is to consider the back-ground noise to be the interference. Then, the probability of correctly detecting the desired signal in the noise is easily shown to be [cf. Equation (2.49c), Reference 5]

$$P_D(H_1) = P_1 \, \beta_1^{(1)*} = \frac{P_1}{2} \left\{ 1 + \textcircled{H} \left[\sigma_S - \textcircled{H}^{-1} \left[1 - 2\beta_1^{(0)*} \right] \right] \right\} , \qquad (5.14)$$

where

$$\sigma_S^2 = a_o^2 \, \Phi_{SS}/2 , \quad \text{and} \quad \beta_1^{(0)*} = \text{(conditional) false-alarm probability.} \qquad (5.14a)$$

In general, $\Phi_{SS} \rightarrow \Phi_{SS}^{(T)}$ for continuous sampling, where $\Phi_{SS}^{(T)} = \Phi_{11}^{(T)}$, Equation (4.31a) here, if narrow-band signals are used. Thus, $\Phi_{SS}^{(T)} \sim [\text{Re } \chi_{SS}(0, 0) = 1]$. If the signal is delayed by τ and has a frequency shift ν, i.e., $s = s_2$, cf. Section 4, $\Phi_{SS} = \Phi_{22}^{(T)}$, Equation (4.31b), and Φ_{SS} is at most weakly dependent on (τ, ν). In any case, whatever the signal (and the receiver is optimal and therefore "matched" to it), increasing the level of the interference lowers P_D, as expected. Also, we see again that it is the real part of the (Bayes) ambiguity functions [here at $(0, 0)$] that appears in the critical system parameter, σ_S, determining performance.}

5.2 Example 2 (Case IV)

Turning next to Case IV, which will not be trivial in S_J, we find the charac-teristic functions for x_4^*, Equations (5.10), now to be

$$F(i\,\xi\,|H_o)_4 = \exp\left\{i\,\xi\,(\log\mu_{10} - \Psi_{SJ}\,a_o^2/2) - \tfrac{1}{2}\,a_o^2\,\Psi_{SJ}\,\xi^2\right\}, \tag{5.15}$$

with

$$\Psi_{SJ} \equiv \Phi_{SS} - 2\left(\frac{a_J}{a_o}\right)\Phi_{SJ} + \left(\frac{a_J}{a_o}\right)^2\Phi_{JJ}, \tag{5.16}$$

and

$$F(i\,\xi\,|H_1)_4 = \exp\left\{i\,\xi\,(\log\mu_{10} + \Psi_{SJ}\,a_o^2/2) - \tfrac{1}{2}\,a_o^2\,\Psi_{SJ}\,\xi^2\right\}. \tag{5.17}$$

Consequently, the probability of correctly detecting the desired signal is [cf. (2.49c). Reference 5] directly

$$P_D(H_1) = \frac{p_1}{2}\left\{1 + \boxed{H}\left(\sqrt{a_o^2\,\Psi_{SJ}/2} - \boxed{H}^{-1}\,[\,1 - 2\beta_1^{(0)*}\,]\right)\right\}. \tag{5.18}$$

Here Ψ_{SJ} is entirely analogous to Ψ_{12}, cf. Equations (4.11a), (4.20), [and equal to it, if $a_J = a_o$]. With continuous sampling (and narrow-band signals) we see that $\Phi_{11}^{(T)}$, Equation (4.31a), becomes $\Phi_{SS}^{(T)}$; $\Phi_{22}^{(T)}$, Equation (4.31b) becomes $\Phi_{JJ}^{(T)}$; and Equation (4.31c) is equivalent to $\Phi_{SJ}^{(T)}$, with appropriate modifications of notation and interpretation.

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Again, it is the cross-term, $\Phi_{SJ}^{(T)}$, that primarily controls the "match" or "mismatch" of the interference S_J with the desired signal S. Another important factor is the input jam-to-signal ratio a_J^2/a_o^2, cf. (5.16). For a given value of a_J/a_o there will be certain values of $\Phi_{SJ}^{(T)}$, representing the degree of overlap or cross-correlation between S_J and S, that minimize Ψ_{SJ} and consequently minimize the probability of correct signal detection P_D, according to (5.18). Conversely, there will be other values of Φ_{SJ} that maximize Ψ_{SJ} and P_D, so that from either the viewpoint of the receiver or the interferer we may expect max and min-max solutions, according to Equations (5.5) - (5.7).

A simple case will serve as illustration. Let us assume white noise backgrounds and let us choose for our narrow-band signals the following complex envelopes [cf. (2.11, 2.2)]:

$$\hat{s}_{oJ}(t) = \hat{s}_{o}(t + \tau) e^{2\pi i \nu t}. \tag{5.19}$$

Thus, we choose an interference here that is designed to resemble the desired signal waveform, but to give false information $(\tau, \nu) \neq (0, 0)$, i.e., different range and velocity. Thus, we can use the results of Section (4.3) above directly to write, remembering here that $\mathrm{Re}\, \chi_{JJ}(0, 0) = \mathrm{Re}\, \chi_{SS}(0, 0) = 1$,

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$$\sigma_{SJ}^2 = a_o^2 \, \Phi_{SJ}^{(T)}/2 = \frac{A_o^2}{4W_o} \int_T |\hat{s}_o(t)|^2 \, dt \cdot \left\{ 1 - 2\left(\frac{A_J}{A_o}\right) \text{Re } \chi_{SS}(\tau, \, \nu) + \left(\frac{A_J}{A_o}\right)^2 \right\},$$

$$(5.20)$$

since now $\chi_{SJ}(0, \, 0) = \chi_{SS}(\tau, \, \nu)$, on application of (5.19). With identical levels of "decoy" and desired signals, e.g., $A_o = A_J$, this reduces to our earlier result (4.44b):

$$\sigma_{SJ}^2 = \left(\frac{A_o^2}{2W_o} \int_T |\hat{s}_o(t)|^2 \, dt\right) \cdot \left\{ 1 - \text{Re } \chi_{SS}(\tau, \, \nu) \right\}. \tag{5.20a}$$

From (5.20) it is clear that various trades-off between interference level (A_J/A_o) and signal overlap, Re χ_{SJ}, are possible for either interference against a given signal S, or the choice of desired signal S against a given interference S_J, or a combination of these two situations. For the first case in our present example, where a "decoy" signal is involved, it is not $P_D(H_1)$, (5.18), that we wish to minimize, but $P_D(H_o)$ that we wish to maximize: in other words, for a given desired signal S, we wish to establish a false "target," with $(\tau, \, \nu) \neq (0, \, 0)$, and maximize its probability of detection subject to some (small) fixed probability $P_D(H_1)$ or $\beta_1^{(1)*}$, of correctly detecting the desired signal. Paralleling (4.12) - (4.17) we easily see that [cf. (5.18)]

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$$P_D(H_o) = p_o \beta_o^{(0)*} = \frac{p_o}{2} \left\{ 1 + \text{H} \left[\sigma_{SJ} - \text{H}^{-1} (1 - 2\beta_1^{(1)*}) \right] \right\} . \quad (5.21)$$

This is maximized (or made suitably great) by making σ_{SJ} large. Since (τ, ν) must be appropriately different from zero--in order to establish the false target and velocity--this means (for given signal duration) an appropriate increase in the interference level, A_J, cf. (5.20). And, since we may also not wish the receiver to be able to distinguish the desired from the false target on the basis of signal level, we must then set $A_o = A_J$, and Equation (5.20a) governs the key parameter σ_{SJ}. Now only if the desired input signal level (A_o) is sufficiently large (and/or the signal of sufficiently long duration), so that the coefficient of $\{1 - \text{Re } \chi_{SS}(0, 0)\}$ is likewise large, will $P_D(H_o)$ be near unity and interference effective (to this degree of probability). Conversely, in the second case where S operates against a given interference S_J of the decoy type (5.11), we wish to maximize $P_D(H_1)$, (5.18), [instead of $P_D(H_o)$], subject to an acceptability low value $\beta_1^{(0)*}$. Then it is at once evident from (5.20) and (5.20a) that increasing the desired signal level A_o will accomplish this. Still other variations on these themes are clearly possible, including appropriate signal design for S_J versus S, or S versus S_J, subject to reasonable constraints, etc. We shall however, reserve discussion of this topic to a later study, since our purpose here has been to limit the treatment to general remarks with some simple illustrations.

6.   CONCLUDING REMARKS

We summarize the proceeding analysis and discussion with the following general observations.

(i)   The ambiguity function, in particular the Bayes ambiguity function, is a useful parameter of performance and system structure in the important but restricted class of optimum reception problems involving additive gaussian noise and, generally, threshold reception. Otherwise, it is at best only very partially descriptive of the reception process, and in any case does not give a full picture of system behavior. [See Section 3.]

(ii)   In many radar and other situations when the conditions of (i) apply, it is the real part of the appropriate (Bayes) ambiguity function that is significant*. Thus, the conventional use of the modulus, $|\chi|$, of the ambiguity function in signal design and system evaluation may be noticably incomplete, since one needs Re $\chi = |\chi| \cos \varphi_\chi$, rather than just $|\chi|$ above. Consideration of $|\chi|$ only may be, in some instances (i.e., resolution), too strict. In others it may be quantitatively misleading. The qualitative relationship between $|\chi|$ and Re χ in signal design and selection needs to be studied further, and in each case referred to the decision processes

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* It is the real part because we are dealing always with physically real signals and with signal energies.

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involved [cf. Section 3.] through the various probabilities of correct and incorrect decisions.

(iii) Unlike the classical ambiguity function [cf. (2.4)], the Bayes ambiguity function, $\bar{\chi}$, may be a statistical quantity--i.e., it may be some statistical average of classical forms over one or more random parameters that describe the signal classes in question, or even over the waveforms themselves, in the case of entirely stochastic signals. Again, we expect $\mathrm{Re}\,\bar{\chi}$ to be a significant parameter in determining performance when the conditions of (i) are obeyed.

(iv) Central to an adequate employment of the concept of ambiguity functions in signal design and system performance is its quantitative relationship to the decision process. This is given here for Bayes (i.e., optimum) receivers, and illustrated with a number of elementary examples in detection, resolution, and ECM, to show how these concepts may be applied to more realistic problems.

The principal tasks of the next state of this study are:

(1) To determine the effects of $\mathrm{Re}\,\chi$ on the qualitative use of χ and $|\chi|$ in applications;

(2) To obtain and investigate the rôle of $\mathrm{Re}\,\hat{\chi}$ in more involved and realistic (Bayes--optimum and suboptimum) detection situations, including incoherent reception;

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(3)   To use these ideas and results in specific ECM and counter ECM (ECCM) studies. We emphasize again that one must use the (Bayes) ambiguity function in conjunction with the associated error probabilities and probabilities of correct decisions, as well as such other critical parameters as signal-to-noise ratio and sample size, if a really adequate measure of system performance is to be obtained.

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GLOSSARY OF SYMBOLS

A. A_o = (peak) signal amplitude; pulse amplitude

A_J = (peak) interference signal amplitude

a_o, a_{01}, a_{02} = normalized signal amplitudes

a_J = normalized jamming signal amplitude

a_0^2, a_{01}^2 ... etc. = input signal-to-noise ratios

B. $\beta_2^{(1)}$, $\beta_1^{(2)}$ = conditional error probabilities

$\beta_1^{(1)}$, $\beta_2^{(2)}$ = conditional probabilities of correct decision

$\beta_2^{(1)*}$, $\beta_1^{(2)*}$ = Bayes conditional error probabilities

C. $C_1^{(1)}$, $C_2^{(1)}$, $C_2^{(2)}$, $C_1^{(2)}$ = preassigned costs

χ_o = a "threshold" value of the ambiguity function

$\chi(\tau, \nu)$ = (normalized" ambiguity of functions of Woodward

χ_{11} = (auto-) ambiguity function

χ_{12} = (cross-) ambiguity function

$\chi_{\hat{z}_1, \hat{s}_2}$, $\chi_{\hat{s}_1, \hat{z}_1}$, etc. = normalized (Bayes) ambiguity functions

D. $\delta(\gamma_1 | \underline{V})$, $\delta(\gamma_2 | \underline{V})$ = decision rules (probabilities)

        δ^* = Bayes decision rule

E. $E(t, \underline{\theta})$ = real envelope of a narrow-band wave

        E_{01}, E_{02} = energy of signals 1, 2; normalization factors in the ambiguity function

        ϵ_0 = a fixed epoch (time delay)

        ϵ, $\not\epsilon$ = does, or does not, "lie in the interval"

F. $F_n(\underline{V} | \underline{S})$ = a priori probability density of the data V, given the signal S

        f   = frequency

        f_0 = a carrier frequency

        $F_1(i \xi | H_{0,1})$ = characteristic function associated with the probability density of the detector under hypothesis H_0 or H_1

G. γ_1, γ_2 = decisions

H. $H_{0,1,2}$ = alternative hypothesis states

I.

J.

K. $\underline{K}_N$ = noise covariance matrix

      K_N = noise covariance function

      $\underline{k}_N,\ k_N$ = normalized noise covariance matrix and function

      $\mathcal{K},\ \mathcal{K}_{21}$ = detection thresholds (cost ratios)

L. $\Lambda,\ \Lambda_{12}$ = generalized likelihood ratios

      $\lambda_1,\ \lambda_2$ = undetermined multipliers

M. $\mu,\ \mu_{10},\ \mu_{12},\ \mu_{21}$ = ratios of a priori probabilities as to presence and absence of a signal

N. N = background noise

      $\overline{N}$ = mean noise level

      ν = a frequency shift

O. ω_o = angular carrier frequency

P. $P(H_{0,\ 1,\ 2})$ = probability of the occurrence of the hypothesis states H_0, H_1, H_2

      $P_D(H_0),\ P_D(H_1)$ = probability of detection

      $\phi(t,\ \underline{\theta})$ = a phase

      $P_1,\ P_2$ = a priori probabilities

Q. ψ = mean intensity of background noise = $\overline{N^2}$

$\Phi_{11}^{(s)},\ \Phi_{12}^{(s)},\ \Phi_{22}^{(s)}$

$\left.\vphantom{\begin{array}{c}1\\1\\1\end{array}}\right\}$ = quadratic forms involving the signal

$\Phi_{11}^{(T)},\ \Phi_{22}^{(T)},\ \Phi_{12}^{(T)}$

$\Phi_{SJ},\ \Phi_{SS}$ = quadratic forms

Φ_V = quadratic form involving the data

$\Psi_{12},\ \Psi_{SJ}$ = a combination of signal quadratic forms

ψ_Z = a phase of complex reference signal

$\psi_S,\ \psi_S'$ = phase of an input signal

R.

S. $\underline{S},\ \underline{S}_1,\ \underline{S}_2$ = signal vectors

         $\underline{s}$ = normalized input vector (real)

         $S(t),\ S(t;\underline{\theta})$ = signal waveforms

         S_J = jamming signal

         $\hat{S}(t;\underline{\theta})$ = complex signal waveform

         $\hat{S}_o,\ \hat{S}_o(t;\underline{\theta}),\ \hat{S}_{01},\ \hat{S}_{02}$ = complex signal envelopes

         $s(t),\ \hat{s}(t),\ \hat{s}_{01}(t),$ etc.   = normalized signal waveforms; $\hat{S}$ is complex

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	$(S/N)^2$	= signal-to-noise power ratio
T.	T	= observation interval
	t	= time
	τ	= a delay (in range)
	τ_o	= pulse duration
	$\underline{\theta}$	= a set of (possibly random) signal parameters
	t_o	= an initial time
	Ⓗ	= error function
U.		
V.	V(t)	= received data waveform
	v(t)	= normalized data waveform
	$\underline{V}, \underline{v}$	= data and normalized data vectors
W.	W_o	= intensity density of white noise spectrum
X.	X_T, X_{T1}, X_{T2}	= solutions of an integral equation
Y.		

z. $\hat{z}_{T1}, \hat{z}_{T2}$ = complex reference signals

$\hat{z}_{T01}, \hat{z}_{T02}$ = complex envelopes of reference signals

$\hat{z}_{T01}, \hat{z}_{T02}$ = normalized complex envelopes of reference signals

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13. ABSTRACT

Bayes ambiguity functions are defined as an important parameter governing the performance of optimum (i.e., Bayes or minimum average risk) systems. Bayes ambiguity functions are generalizations of the classical ambiguity functions of Woodward and are specifically derived from an appropriate decision process. It is shown here that it is the real part of the ambiguity function that is significant, rather than its modulus. Optimum target resolution is formulated as a detection problem involving the two hypothesis states H_0: "unresolved" signals versus H_1: "resolved" signals, and general conditions for the qualitative utility of the ambiguity functions are discussed. These latter are: additive gaussian noise and threshold operation; otherwise the ambiguity function is an inadequate description of system performance. The analysis is extended to a number of situations involving interfering signals, such as electronic countermeasures (ECM) and is illustrated with simple examples showing quantitatively, as well as qualitatively, the typical rôles played by the Bayes ambiguity function in a variety of ECM applications. It is emphasized that one must also consider the probability of correct and incorrect decisions, in conjunction with the properties of the ambiguity functions, to achieve a reliable measure of expected performance.

14. KEY WORDS

Bayes ambiguity functions	target resolution
radar countermeasures	electronic countermeasures
ambiguity functions	radar detection
radar signals	