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# USAAVLABS TECHNICAL REPORT 68-18A

# PREDICTION OF ROTOR INSTABILITY AT HIGH FORWARD SPEEDS

## VOLUME I

## STEADY FLIGHT DIFFERENTIAL EQUATIONS OF MOTION FOR A FLEXIBLE HELICOPTER BLADE WITH CHORDWISE MASS UNBALANCE

By

Peter J. Arcidiacono

February 1969

## U. S. ARMY AVIATION MATERIEL LABORATORIES
## FORT EUSTIS, VIRGINIA

CONTRACT DA 44-177-AMC-332(T)
UNITED AIRCRAFT CORPORATION
SIKORSKY AIRCRAFT DIVISION
STRATFORD, CONNECTICUT





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DEPARTMENT OF THE ARMY
U S ARMY AVIATION MATERIEL LABORATORIES
FORT EUSTIS VIRGINIA 23604

This contract was initiated to determine the aeroelastic stability limits of articulated and unarticulated helicopter rotor systems at high forward speeds. The four primary modes of aeroelastic instability (classical flutter, stall flutter, torsional divergence, and flapping or flatwise bending instability) were investigated. The possibility of a flap-lag instability suggested by Dr. Maurice I. Young of the Vertol Division, The Boeing Company, was investigated as a special case of flapping instability.

The results are published as a five-volume set; the subject of each volume is as follows:

| | |
|---|---|
| Volume I | Equations of Motion |
| Volume II | Classical Flutter |
| Volume III | Stall Flutter |
| Volume IV | Torsional Divergence |
| Volume V | Flapping Instability |

These reports have been reviewed by the U. S. Army Aviation Materiel Laboratories. These reports, which are published for the exchange of information and the stimulation of ideas, are considered to be technically sound with regard to technical approach, results, conclusions, and recommended parameter ranges for accurate usage.

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# PREDICTION OF ROTOR INSTABILITY AT HIGH FORWARD SPEEDS

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Volume I

## Steady Flight Differential Equations of Motion for a Flexible Helicopter Blade With Chordwise Mass Unbalance

By

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Prepared by

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for

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FORT EUSTIS, VIRGINIA

## SUMMARY

The purposes of this research program were to extend or develop analytical methods for determining rotor blade aeroelastic stability limits and to perform stability calculations over a range of design and operating variables for articulated and nonarticulated configurations. The usefulness of simpler analytical methods is investigated by comparing results with operating boundaries from the more elaborate analysis.

In this volume the differential equations of motion for a linearly twisted rotor blade having chordwise mass unbalance and operating under steady flight conditions are derived. The motions include flapping and lagging for the articulated blade, as well as flatwise, edgewise, and torsional deformations for the articulated and nonarticulated blades. The fully coupled aerodynamic forcing functions are based on quasi-steady theory. The differential equations of motion are expanded in terms of the uncoupled vibratory modes of the blade in order to facilitate their numerical solution on a digital computer.

# FOREWORD

The work presented in this volume is part of an effort which is contained in five volumes. The work was performed under Contract DA 44-177-AMC-332(T) with the U. S. Army Aviation Materiel Laboratories, Fort Eustis, Virginia. The work was monitored for USAAVLABS by Mr. Joseph McGarvey.

This volume contains a presentation of work performed by Mr. Peter Arcidiacono, of the United Aircraft Research Laboratories. The resulting equations of motion were prepared and programmed for computer solution by Mr. Russell Berquist of Sikorsky Aircraft. The method of computer solution was generated independently for earlier versions of the equations of motion, and was extended under this contract to include non-coincident blade elastic axis and section center of gravity locations.

Volume II presents a linearized discrect azimuth classical flutter analysis for rotor blades, with an appropriate parameter variation study, a comparison with test data, and a comparison with results calculated by using the method of Volume I.

Volume III describes a stall flutter analysis based on the calculation of aerodynamic work during a cycle of blade torsional vibration, using two-dimensional unsteady airfoil test data. The analysis was used to generate stall flutter boundaries.

Volume IV contains the results of a study of static torsional divergence. A set of design charts were generated and the effects of a range of parameter variations are presented. The results of the static divergence calculation are compared with results calculated by using the method of Volume I.

Volume V is concerned with flapping, flatwise bending, and coupled flap-lag instability. A single degree of freedom flapping or flatwise bending analysis was used to investigate a wide range of parameters. Comparisons were made with the results of the more elaborate method of Volume I. The method of Volume I was also used to determine the coupled flap-lag response of a rotor to a number of sudden control changes.

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## LIST OF SYMBOLS

$a$ — acceleration of any point on the rotor blade, $ft/sec^2$

$a_0$ — distance from mid-chord to pitch axis divided by $c/2$, positive when pitch axis is downstream (see Eq. (131))

$c$ — blade chord, ft

$c_d$ — section drag coefficient, $d/(1/2\,\rho U^2 c)$

$c_{m_{c/4}}$ — section pitching moment coefficient about the quarter-chord, $m_{c/4}/1/2\,\rho\, U^2 c^2$

$c_\ell$ — section lift coefficient, $\ell/(1/2\,\rho\, U^2 c)$

$C$ — rotary viscous damping coefficient, ft-lb-sec/rad

$C_{0\text{-}114pj}$ — blade modal constants (see Appendix IV)

$C^*$ — linear viscous damping coefficient, lb-sec/ft

$d$ — section drag force per unit span, lb/ft

$e$ — $x_2$ coordinate of coincident flap-lag hinge, ft

$e_A$ — $y_{10}$ coordinate of centroid of spar area, ft

$E$ — Young's modulus of elasticity, $lb/ft^2$

$g$ — acceleration of gravity, $ft/sec^2$

$G$ — shear modulus of elasticity, $lb/ft^2$

$i, j, k$ — unit vectors

$I_y, I_z$ — spar area moments of inertia about an axis parallel to $z_{10}$ and $y_{10}$ but passing through the spar and counterweight centroid, $ft^4$

$I_a$ — mass moment of inertia of blade about flap and lag hinges (see Eq. (93)), $lb\text{-}ft\text{-}sec^2$

$J$ — torsional stiffness constant of blade section, $ft^4$

$k$ — radius of gyration of blade section mass, ft

$k_A$ — radius of gyration of spar, ft

$\ell$ — lift force per unit span, lb/ft

| | |
|---|---|
| $m$ | blade mass per unit span, lb-sec$^2$/ft$^2$ |
| $m_A$ | section aerodynamic pitching moment per unit span about elastic axis, ft-lb/ft |
| $m_{c/4}$ | steady-state section pitching moment per unit span, ft-lb/ft |
| $m_{cw}$ | counterweight mass per unit span, lb-sec$^2$/ft$^2$ |
| $m_d$ | quasi-steady section damping moment per unit span (see Eq. (130), ft-lb/ft |
| $m_0$ | blade mass per unit span at some reference station, lb-sec$^2$/ft$^2$ |
| $M$ | moment, ft-lb |
| $M_A$ | moment due to aerodynamic forces, ft-lb |
| $M_B$ | blade mass, lb-sec$^2$/ft |
| $M_D$ | moment due to dynamic forces (including moments due to flap damper, lag damper, and pushrod), ft-lb |
| $M_e$ | moment due to elastic deformation of blade, ft-lb |
| $p$ | vector from origin of reference frame to point on rotor blade, ft |
| $q_{v_{(\ )}}$ | amplitude of ( )$^{th}$ edgewise deflection mode (equals blade tip deflection in $z_6$ direction divided by $R$ when $\gamma_{\omega_{(\ )}}$ at tip is defined as 1.0) |
| $q_{w_{(\ )}}$ | amplitude of ( )$^{th}$ flatwise deflection mode (equals blade tip deflection in $z_6$ direction divided by $R$ when $\gamma_{\omega_{(\ )}}$ at tip is defined as 1.0) |
| $q_{\theta_{(\ )}}$ | amplitude of ( )$^{th}$ elastic torsional mode (equals elastic twist angle about $x_9$ axis in radians at tip when $\gamma_{\theta_{(\ )}}$ at tip is defined as 1.0) |
| $r$ | blade spanwise coordinate, measured from flap hinge in $x_5$ direction, ft |
| $r_T$ | value of $r$ at blade tip, ft |
| $r_{cg}$ | value of $r$ at blade center of gravity, ft |
| $r_{o_{cw}}$ | value of $r$ at inboard end of counterweight, ft |
| $R$ | rotor radius ( $e + r_T$ ), ft |

$S$ shear force per unit span, lb/ft

$S_A$ aerodynamic shear force per unit span, lb/ft

$S_D$ dynamic shear force per unit span, lb/ft

$t$ time, sec

$U$ resultant velocity of blade section ( $\sqrt{U_T^2 + U_P^2}$ ), ft/sec

$U_P$ velocity component of blade section, normal to $x_{10}$ axis and $U_T$ , ft/sec

$U_T$ velocity component of blade section parallel to the plane of rotation ( $x_3 - y_3$ plane) and normal to $x_{10}$ , ft/sec

$v$ velocity, ft/sec

$v_e$ local edgewise elastic displacement of elastic axis in $y_6$ direction, ft

$v_0$ translational velocity of axis system, ft/sec

$V$ forward velocity of aircraft, ft/sec

$w_e$ local flatwise elastic displacement of elastic axis in $z_6$ direction, ft

$x, y, z$ axis system coordinates

$\alpha_s$ rotor shaft angle of attack, angle between $z_1$ axis and remote airstream velocity vector, positive when $z_1$ axis tilted aft, radians

$\alpha_r$ local section angle of attack, radians

$\beta$ blade flap angle (see Figure 1), radians

$\gamma_{v_{( )}}$ ( )th edgewise uncoupled mode shape, deflection shape assumed by a rotating blade when vibrating at its ( )th uncoupled edgewise frequency

$\gamma_{w_{( )}}$ ( )th flatwise uncoupled mode shape

$\gamma_{\theta_{( )}}$ ( )th torsional uncoupled mode shape

$\delta$ blade lead angle (see Figure 1), radians

$\delta_3$ pitch flap coupling angle, deg

$\Delta$ nondimensional spanwise distance over which a concentrated force is applied to the blade

$\Delta e_{A_{CW}}$ distance between chordwise location of counterweight and chordwise location of spar centroid, positive when former is between centroid and leading edge, ft

$\epsilon$ local blade section pitch angle (see Figure 5), radians

$\theta$ local blade pitch angle due to control system input and built-in twist (see Eq. (147)), radians

$\theta_B$ local blade pitch angle due to built-in linear twist (see Eq. (148)), radians

$\theta_e$ local elastic twist angle (see Figure 1), radians

$\theta_0$ blade pitch angle due to control system input (see Eq. (149)), radians

$\theta_1$ rate of change of $\theta_B$ with respect to $\bar{r}$, radians

$\Theta$ total local blade pitch angle (see Eq. (147)), radians

$\lambda_s$ ratio of relative air velocity component in $z_1$ direction to $\Omega R$ (see Eq. (138))

$\lambda_1, \lambda_2$ bending rotation angles (see Figure 1), radians

$\mu$ ratio of relative air velocity component in $x_1$ direction to $\Omega R$ (see Eq. (137))

$\nu$ rotor induced velocity, ft/sec

$\xi$ dummy variable of integration along $x_5$ axis, ft

$\rho$ air density, lb-sec$^2$/ft$^4$

$\phi$ local inflow angle (see Figure 5), radians

$\psi$ blade azimuth angle (see Figure 1), radians

$\omega$ angular velocity of axis system, rad/sec

$\omega_{v_{()}}$ uncoupled natural frequency of ( )$^{th}$ edgewise vibratory mode, rad/sec

$\omega_{w_{()}}$ uncoupled natural frequency of ( )$^{th}$ flatwise vibratory mode, rad/sec

$\omega_{\theta_{()}}$ uncoupled natural frequency of ( )$^{th}$ torsional vibratory mode, rad/sec

$\Omega$ rotor angular velocity about $z_1$ axis, rad/sec

## SUBSCRIPTS

| | |
|---|---|
| 1-10 | indicates axis system under consideration |
| C/4 | indicates quantity evaluated at 25% chord |
| 3C/4 | indicates quantity evaluated at 75% chord |
| c.g. | indicates quantity evaluated at local chordwise center of gravity |
| ea | indicates quantity evaluated at elastic axis |
| FD | indicates flap damper or quantity evaluated at flap damper attachment point, as appropriate |
| $i, i', m$ | flatwise mode subscripts (maximum value = 5) |
| $j, j', k$ | torsional mode subscripts (maximum value = 3) |
| LD | indicates lag damper or quantity evaluated at lag damper attachment point, as appropriate |
| PR | indicates pushrod or quantity evaluated at pushrod attachment point, appropriate |
| $p, p', n$ | edgewise mode subscripts (maximum value = 2) |
| $\bar{r} = 0$ | indicates quantity evaluated at $\bar{r} = 0$ |
| $x, y, z$ | indicates quantity evaluated in $x$, $y$, or $z$ direction (Eqs. (29) and (30) |
| $x_{(\ )}, y_{(\ )}, z_{(\ )}$ | indicates quantity evaluated in $x$, $y$, or $z$ direction of ( ) axis system |
| $y_{10_{cg}} = 0$ | indicates quantity evaluated at $y_{10_{cg}} = 0$ |

## DERIVATIVE NOTATION

| | |
|---|---|
| $d(\ )/dr$ | indicates derivative of ( ) with respect to $r$ |
| $(\ )'$ | indicates derivative of ( ) with respect to $\bar{r}$ |
| $(\overset{\circ}{\ })$ | indicates derivative of ( ) with respect to $t$ |
| $(\overset{\times}{\ })$ | indicates derivative of ( ) with respect to $\bar{t}$ (i.e., $\psi$ ) |
| $\frac{d}{dt}$ | differential operator defined on page 9 |

MISCELLANEOUS

$\vec{(\ )}$ indicates vector quantity

$\overline{(\ )}$ indicates quantity nondimensionalized through the use of factors, R , $m_0$ , and $\Omega$

$\triangleq$ indicates an approximate equality

# INTRODUCTION

The development of analyses for the prediction of rotor performance, blade motions, stresses and loads has been progressing for many years. The equations of motion derived in this part represent an important additional step in the refinement and generalization of these methods.

The analysis used to develop the performance charts of Reference 1 determines rigid blade flapping motion by a step-by-step timewise integration of the flapping equation of motion. The solution finally converges to a cyclic pattern if a steady-state condition is being analyzed. The analysis is fully capable of handling a transient condition, however, since the equations of motion are solved with arbitrary starting values.

Following the development of a rigid blade analysis, continued improvements in helicopter performance objectives and the need for prediction of flexible blade loads and stresses made the development of a solution for flexible blade motions necessary. The result of this development utilized rotating blade natural vibration modes as elastic degrees of freedom. The use of these orthogonal or "normal" modes gave rise to the designation "Normal Mode Transient Analysis". The analysis is similar to the rigid blade analysis mentioned above, in that it is essentially a so-called starting value problem, in which the differential equations of motion are integrated on the basis of some set of starting boundary values. When a steady-state condition is being analyzed the integration proceeds in small but finite timewise steps; after a number of rotor revolutions, the predicted motions will become cyclic within a desired tolerance. This is the usual solution desired, and performance, load, and stress calculations are usually based on these cyclic motions. On the other hand, the prediction of rotor behavior following an arbitrary initial disturbance is a basic capability.

For the purposes of this investigation, the above analysis was extended to provide for noncoincident blade elastic and center-of-gravity axes. The basic differential equations are supplied in detail, in order to define and document all assumptions completely. The provision for noncoincident axes opens entirely new areas of investigation to the Normal Mode Transient Analysis.

## ASSUMPTIONS

Nondimensional quantities are used extensively throughout this report. Nondimensionalizing factors are rotor radius, rotor angular velocity, and mass per unit span at some blade reference station. The principal assumptions which were made in the analysis presented herein are listed below.

1. The aircraft is in steady flight with constant rotor angular velocity.

2. The blade has an elastic axis so that blade deflections can be considered as the superposition of two orthogonal translations of and a rotation about the elastic axis.

3. Quasi-steady aerodynamic theory is applicable with the exception that apparent-mass aerodynamic effects are assumed to be negligible.

4. Radial flow effects on aerodynamic forces are negligible.

5. Principal blade flexibility effects can be accounted for by considering only five flatwise, two edgewise, and three torsional vibratory modes.

6. Blade flap and lag hinges are coincident for articulated rotors.

7. The local center of gravity is assumed to lie on the major principal axis of the section.

8. The blade is linearly twisted along its span.

9. The following quantities can be assumed to be small in comparison to unity:

    a. Flap and lead angles (in radians) and their derivatives.

    b. Ratio of elastic deflections to rotor radius and their derivatives.

    c. Ratios of chordwise distances (i.e., chord, center-of-gravity offset, etc.) to rotor radius.

    d. Built-in twist (in radians).

    e. Ratio of flap-lag hinge radial distance from a center of rotation to rotor radius.

    f. Reciprocal of Froude number based on rotor radius ( $g/\Omega^2R$ ).

    g. Ratios of blade thickness dimensions to chord.

10. On the basis of assumptions 8 and 9, the following types of terms in the equations noted can be neglected as higher order:

a. Flatwise and edgewise bending equations:

(1) Second-order products of elastic coordinates.

(2) Third-order products of elastic coordinates, chordwise distances, flap angle, lead angle, built-in twist, and flap-lag hinge offset.

b. Torsional equation:

(1) Third-order products of elastic coordinates.

(2) Fourth-order products involving elastic coordinates, chordwise distances, flap angle, lead angle, built-in twist, and flap-lag hinge offset.

c. Flap angle and lead angle equations:

Second-order terms involving products of elastic coordinates, chordwise distances, and built-in twist.

d. Section velocity equations:

(1) Second-order products of elastic coordinates.

(2) Third-order products involving the elastic coordinates, chordwise distances, flap angle, lead angle, built-in twist, and flap-lag hinge offset as factors.

e. All equations:

The spanwise component of acceleration due to gravity.

## AXIS SYSTEMS

The axis systems employed in the analysis are shown schematically in Figure 1 and are discussed below.

NONROTATING ROTOR HUB AXIS SYSTEM: $x_1, y_1, z_1$

This axis system has its origin at the rotor hub center and is nonrotating with respect to the aircraft. $z_1$ lies along the rotor shaft, positive up; $x_1$ is normal to the rotor shaft, lies in the plane formed by $z_1$ and the remote air velocity vector, and is positive aft; $y_1$ is orthogonal to $x_1$ and $z_1$ and is positive toward the advancing side of the rotor. This axis system is Newtonian in nature, having at most a steady translational velocity.

ROTATING ROTOR HUB AXIS SYSTEM: $x_2, y_2, z_2$

This axis system has its origin at the rotor hub center and rotates with the rotor. Its coordinates are related to those of the "1" axis system above by the following equations:

$$x_1 = x_2 \cos\psi - y_2 \sin\psi \tag{1}$$

$$y_1 = y_2 \cos\psi + x_2 \sin\psi \tag{2}$$

$$z_1 = z_2 \tag{3}$$

OFFSET AXIS SYSTEM: $x_3, y_3, z_3$

This axis system has its origin at the coincident flap and lead-lag hinge radial position. Its coordinates are related to those of the "2" axis system by the following equations:

$$x_2 = x_3 + e \tag{4}$$

$$y_2 = y_3 \tag{5}$$

$$z_2 = z_3 \tag{6}$$

BLADE LEAD ANGLE AXIS SYSTEM: $x_4, y_4, z_4$

This axis system rotates about the $z_3$ axis with an angular velocity $d\delta/dt$. Its coordinates are related to those of the "3" axis system by the following equations:

$$x_3 = x_4 \cos\delta - y_4 \sin\delta \tag{7}$$

$$y_3 = y_4 \cos\delta + x_4 \sin\delta \tag{8}$$

$$z_3 = z_4 \tag{9}$$

Note that $\delta$ is equal to the built-in lead angle for nonarticulated rotors.

RIGID BLADE AXIS SYSTEM: $x_5, y_5, z_5$

This axis system rotates about the $y_4$ axis with an angular velocity $d\beta/dt$. Its coordinates are related to those of the "4" axis system by the following equations:

$$x_4 = x_5 \cos\beta - z_5 \sin\beta \tag{10}$$

$$y_4 = y_5 \tag{11}$$

$$z_4 = z_5 \cos\beta + x_5 \sin\beta \tag{12}$$

Note that the flap angle $\beta$ is equal to the built-in coning angle for nonarticulated rotors.

RIGID BLADE SECTION AXIS SYSTEM: $x_6, y_6, z_6$

This section axis system is obtained by a translation along $x_5$ through a distance $r$ and a rotation about $x_5$ through the local rigid blade pitch angle $\theta$, where $\theta$ includes the pitch angle resulting from control system inputs as well as that resulting from built-in twist. The "6" axis coordinates are related to those of the "5" axis system by the following equations:

$$x_5 = x_6 + r \qquad (13)$$

$$y_5 = y_6 \cos\theta - z_6 \sin\theta \qquad (14)$$

$$z_5 = z_6 \cos\theta + y_6 \sin\theta \qquad (15)$$

BENDING TRANSLATION AXIS SYSTEM: $x_7, y_7, z_7$

This axis system is obtained by bending translations of the elastic axis in the $y_6$ and $z_6$ directions (section principal axis directions before elastic deformation). The "7" axis coordinates are related to the "6" axis coordinates by the following equations:

$$x_6 = x_7 \qquad (16)$$

$$y_6 = y_7 + v_e \qquad (17)$$

$$z_6 = z_7 + w_e \qquad (18)$$

Eq. (16) implies that the change in length of the blade due to elastic displacements is negligible.

BENDING ROTATION AXIS SYSTEM: $x_9, y_9, z_9$

This axis system is obtained from two small rotations which occur when the blade undergoes the elastic bending translations, $w_e$ and $v_e$. The rotations are assumed to be small, so that the order in which they occur is of no consequence. The "9" axis system coordinates are related to those of the "7" axis system by the following equations:

$$x_7 = x_9 - z_9\lambda_1 - y_9\lambda_2 \qquad (19)$$

$$z_7 = z_9 + x_9\lambda_1 \qquad (20)$$

$$y_7 = y_9 + x_9 \lambda_2 \tag{21}$$

where

$$\lambda_1 = \frac{dw_e}{dr} + v_e \frac{d\theta}{dr} \tag{22}$$

$$\lambda_2 = \frac{dv_e}{dr} - w_e \frac{d\theta}{dr} \tag{23}$$

Since $\lambda_1$ and $\lambda_2$ are very small angles, terms involving their products are neglected.

BLADE SECTION AXIS SYSTEM: $x_{10}, y_{10}, z_{10}$

The final blade section axis system is obtained by a rotation about the $x_9$ axis through the elastic twist angle $\theta_e$. The $y_{10}$ and $z_{10}$ axes lie in the plane of the airfoil section of the deformed blade, $y_{10}$ coinciding with the major principal axis (which is also assumed to be coincident with the reference chord line of the airfoil section). The "10" axis coordinates are related to those of the "9" axis system by the following equations:

$$x_9 = x_{10} \tag{24}$$

$$y_9 = y_{10} - z_{10}\theta_e \tag{25}$$

$$z_9 = z_{10} + y_{10}\theta_e \tag{26}$$

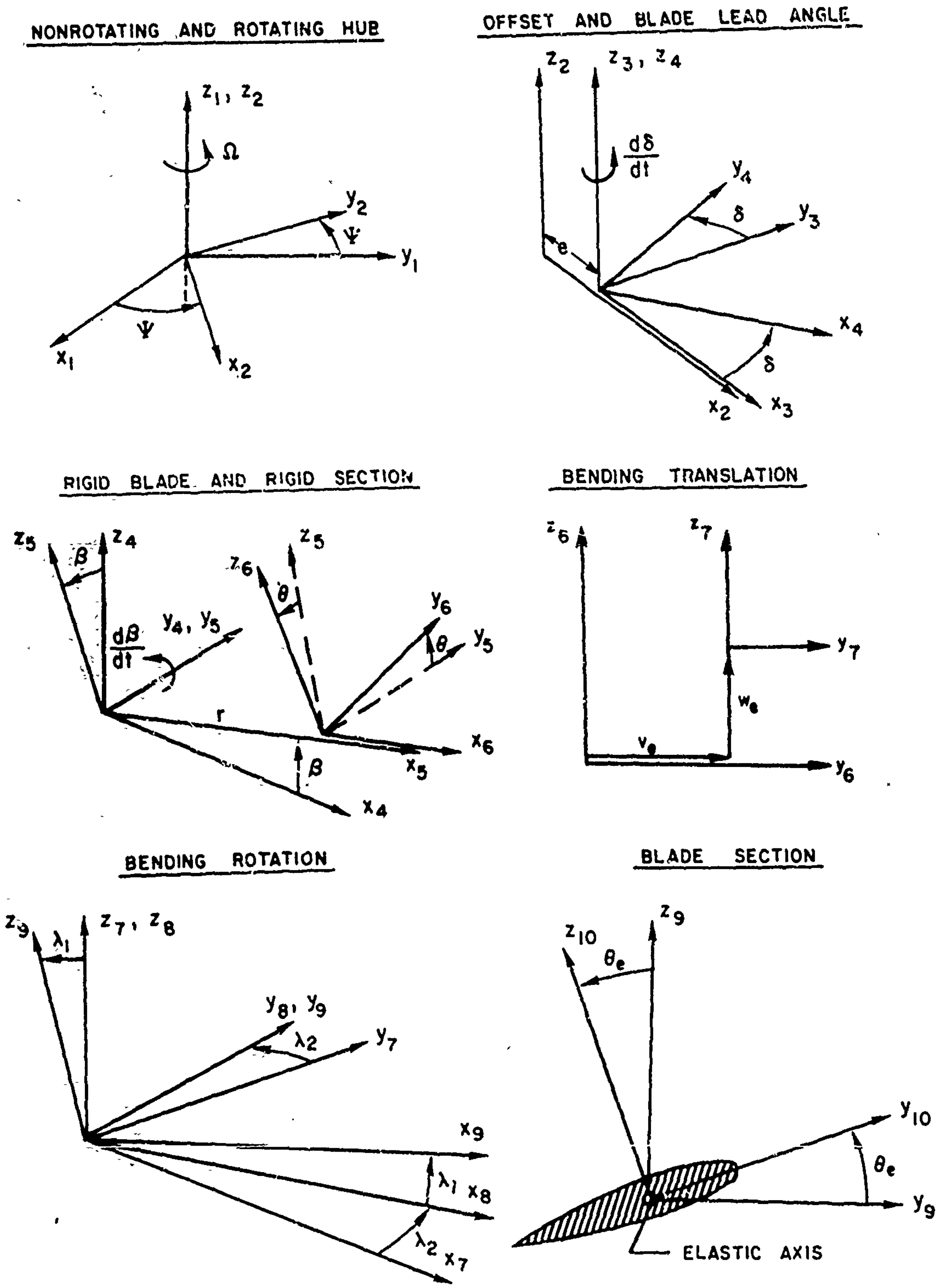


Figure 1. Axis Systems.

## VELOCITY AND ACCELERATION VECTORS IN THE RIGID BLADE AXIS SYSTEM

To develop the blade equations of motion, the velocity and acceleration vectors for a blade mass element in an appropriate reference axis system are required. A convenient reference system and one used herein is the Rigid Blade Axis System ("5" system). Inasmuch as this is a reference frame moving relative to Newtonian space, care must be exercised to account for all velocity and acceleration components. The approach employed herein assumes that the linear and angular velocity vectors of the Nonrotating Rotor Hub Axis System ("1" system) are known along with the gravity vector (from specification of the trim flight condition of the aircraft). The velocity and acceleration vectors in any other axis system can then be derived by classical vector techniques as discussed, for example, in Reference 2 (Article 12.3). The velocity and acceleration of any point defined by coordinates measured in a frame of reference which is both translating and rotating can be shown to be

$$\vec{v} = \vec{v}_0 + \frac{\mathcal{d}\vec{P}}{\mathcal{d}t} + \vec{\omega} \times \vec{P} \tag{27}$$

$$\vec{a} = \frac{\mathcal{d}\vec{v}}{\mathcal{d}t} + \vec{\omega} \times \vec{v} + \vec{g} \tag{28}$$

where

$\vec{P}$ is the vector from the origin of the reference frame to the point in question, $x\vec{i} + y\vec{j} + z\vec{k}$

$\vec{v}_0$ is the reference frame translational velocity vector, $v_{0_x}\vec{i} + v_{0_y}\vec{j} + v_{0_z}\vec{k}$

$\vec{\omega}$ is the reference frame angular velocity vector, $\omega_x\vec{i} + \omega_y\vec{j} + \omega_z\vec{k}$

$\dfrac{\mathcal{d}}{\mathcal{d}t}$ is defined as $\dfrac{d(\ )_x}{dt}\vec{i} + \dfrac{d(\ )_y}{dt}\vec{j} + \dfrac{d(\ )_z}{dt}\vec{k}$

$\vec{g}$ is the gravity acceleration vector

$\vec{i},\vec{j},\vec{k}$ are the usual unit vectors

Eqs. (27) and (28) can be expanded, if the above definitions are used, to yield the following general expressions for $\vec{v}$ and $\vec{a}$ :

$$\vec{v} = \vec{i}(v_{O_x} + \mathring{x} + \omega_y z - \omega_z y) + \vec{j}(v_{O_y} + \mathring{y} + \omega_z x - \omega_x z) + \vec{k}(v_{O_z} + \mathring{z} + \omega_x y - \omega_y x) \tag{29}$$

$$\begin{aligned}\vec{a} = {} & \vec{i}\Big[\mathring{v}_{O_x} - \omega_z v_{O_y} + \omega_y v_{O_z} + \overset{\circ\circ}{x} - 2\omega_z \mathring{y} + (\mathring{\omega}_y + \omega_x \omega_z)z \\ & + 2\omega_y \mathring{z} + (-\mathring{\omega}_z + \omega_x \omega_y)y - (\omega_y^2 + \omega_z^2)x + g_x\Big] \\ & + \vec{j}\Big[\mathring{v}_{O_y} - \omega_x v_{O_z} + \omega_z v_{O_x} + \overset{\circ\circ}{y} - 2\omega_x \mathring{z} + (\mathring{\omega}_z + \omega_y \omega_x)x \\ & + 2\omega_z \mathring{x} + (-\mathring{\omega}_x + \omega_y \omega_z)z - (\omega_z^2 + \omega_x^2)y + g_y\Big] \\ & + \vec{k}\Big[\mathring{v}_{O_z} - \omega_y v_{O_x} + \omega_x v_{O_y} + \overset{\circ\circ}{z} - 2\omega_y \mathring{x} + (\mathring{\omega}_x + \omega_z \omega_y)y \\ & + 2\omega_x \mathring{y} + (-\mathring{\omega}_y + \omega_z \omega_x)x - (\omega_x^2 + \omega_y^2)z + g_z\Big]\end{aligned} \tag{30}$$

where $(\,^\circ) \equiv \dfrac{d(\;)}{dt}$

Eqs. (29) and (30) apply to any axis system. For application to the Rigid Blade Axis System all quantities should be subscripted 5, and expressions for $\vec{v}_{O_5}$ , $\vec{\omega}_5$ , $x_5$ , $y_5$ , and $z_5$ are required. These are derived below.

The velocity vectors $\vec{v}_{O_5}$ and $\vec{\omega}_5$ can be obtained from the corresponding vectors for the "1" axis system ( $\vec{v}_{O_1}$ and $\vec{\omega}_1$ ) which, for steady flight, are

$$\vec{v}_{O_1} = v_{O_{x_1}} \vec{i}_1 + v_{O_{z_1}} \vec{k}_1 \tag{31}$$

and

$$\vec{\omega}_1 = 0 \tag{32}$$

When Eqs. (31) and (32), and the coordinate relations of Eqs. (1) through (12) are employed, and when Eq. (29) is successively applied, the following expressions for $\vec{v}_{O_5}$ and $\vec{\omega}_5$ result:

$$\vec{v}_{O_5} = \vec{i}_5\left\{v_{O_{x_1}}\cos\psi\left(1-\frac{\delta^2}{2}-\frac{\beta^2}{2}\right)-\delta v_{O_{x_1}}\sin\psi+e\Omega\delta+\beta v_{O_{z_1}}\right\}$$

$$+\ \vec{j}_5\left\{-v_{O_{x_1}}\left(1-\frac{\delta^2}{2}\right)\sin\psi+e\Omega-v_{O_{x_1}}\delta\cos\psi\right\}$$

$$+\ \vec{k}_5\left\{v_{O_{z_1}}\left(1-\frac{\beta^2}{2}\right)-v_{O_{x_1}}\beta\cos\psi+v_{O_{x_1}}\beta\,\delta\sin\psi\right\} \qquad (33)$$

$$\vec{\omega}_5 = \vec{i}_5\left[\beta(\Omega+\mathring{\delta})\right]-\mathring{\beta}\,\vec{j}_5+\left[\Omega\left(1-\frac{\beta^2}{2}\right)+\mathring{\delta}\right]\vec{k}_5 \qquad (34)$$

If Eqs. (13) through (26) are used, the coordinates $x_5$, $y_5$ and $z_5$ can be expressed in terms of the blade section coordinates ( $x_{10}$, $y_{10}$, $z_{10}$ ) as follows (assuming blade sections having infinitesimal spanwise thickness so that $x_{10}=0$ ).

$$x_5 = r - z_{10}\lambda_1 - y_{10}\lambda_2 \qquad (35)$$

$$y_5 = (v_e + y_{10} - z_{10}\theta_e)\cos\theta - (w_e + z_{10} + y_{10}\theta_e)\sin\theta \qquad (36)$$

$$z_5 = (w_e + z_{10} + y_{10}\theta_e)\cos\theta + (v_e + y_{10} - z_{10}\theta_e)\sin\theta \qquad (37)$$

Substitution of Eqs. (33) through (37) into Eq. (30) yields the following expression for the acceleration vector in the "5" axis system:

$$
\begin{aligned}
\vec{a}_5 = \; & \vec{i}_5 \Big\{ -e\Omega^2 - z_{10}\frac{d\overset{\circ\circ}{w}_e}{dr} - y_{10}\frac{d\overset{\circ\circ}{v}_e}{dr} - 2\Omega \Big[ (\mathring{v}_e - z_{10}\mathring{\theta}_e)\cos\theta - (v_e + y_{10} \\
& - z_{10}\theta_e)\mathring{\theta}\sin\theta - (\mathring{w}_e + y_{10}\mathring{\theta}_e)\sin\theta - (w_e + z_{10} + y_{10}\theta_e)\mathring{\theta}\cos\theta \Big] \\
& - 2\mathring{\delta}\Big[\mathring{v}_e\cos\theta - (v_e + y_{10})\mathring{\theta}\sin\theta - \mathring{w}_e\sin\theta - (w_e + z_{10})\mathring{\theta}\cos\theta\Big] \\
& + (-\overset{\circ\circ}{\beta} + \beta\Omega^2)\Big[(w_e + z_{10})\cos\theta + (v_e + y_{10})\sin\theta\Big] - 2\mathring{\beta}\Big[\mathring{w}_e\cos\theta \\
& - (w_e + z_{10})\mathring{\theta}\sin\theta + \mathring{v}_e\sin\theta + (v_e + y_{10})\mathring{\theta}\cos\theta\Big] - \overset{\circ\circ}{\delta}\Big[(v_e + y_{10})\cos\theta \\
& - (w_e + z_{10})\sin\theta\Big] - r\Big[\mathring{\beta}^2 + \Omega^2(1 - \beta^2) + 2\Omega\mathring{\delta} + \mathring{\delta}^2\Big] \\
& - \Omega^2\Big[-z_{10}\frac{dw_e}{dr} - y_{10}\frac{dv_e}{dr}\Big] + g_{x_5}\Big\} + \vec{j}_5\Big\{e\Omega^2\delta + (\overset{\circ\circ}{v}_e - z_{10}\overset{\circ\circ}{\theta}_e)\cos\theta \\
& - 2(\mathring{v}_e - z_{10}\mathring{\theta}_e)\mathring{\theta}\sin\theta - (v_e + y_{10} - z_{10}\theta_e)(\overset{\circ\circ}{\theta}\sin\theta + \mathring{\theta}^2\cos\theta) - (\overset{\circ\circ}{w}_e \\
& + y_{10}\overset{\circ\circ}{\theta}_e)\sin\theta - 2(\mathring{w}_e + y_{10}\mathring{\theta}_e)\mathring{\theta}\cos\theta - (w_e + z_{10} + y_{10}\theta_e)(\overset{\circ\circ}{\theta}\cos\theta - \mathring{\theta}^2\sin\theta) \\
& - 2\beta\Omega\Big[\mathring{w}_e\cos\theta - (w_e + z_{10})\mathring{\theta}\sin\theta + \mathring{v}_e\sin\theta + (v_e + y_{10})\mathring{\theta}\cos\theta\Big] \\
& + r(\overset{\circ\circ}{\delta} - 2\beta\mathring{\beta}\Omega) + 2\Omega\Big[-z_{10}\frac{d\mathring{w}_e}{dr} - y_{10}\frac{d\mathring{v}_e}{dr}\Big] - 2\mathring{\beta}\Omega\Big[(w_e + z_{10})\cos\theta \\
& + (v_e + y_{10})\sin\theta\Big] - \Omega^2\Big[(v_e + y_{10} - z_{10}\theta_e)\cos\theta - (w_e + z_{10} + y_{10}\theta_e)\sin\theta\Big] \\
& - 2\Omega\mathring{\delta}\Big[(v_e + y_{10})\cos\theta - (w_e + z_{10})\sin\theta\Big] + g_{y_5}\Big\} + \vec{k}_5\Big\{\beta\Omega^2 e \\
& + (\overset{\circ\circ}{w}_e + y_{10}\overset{\circ\circ}{\theta}_e)\cos\theta - 2(\mathring{w}_e + y_{10}\mathring{\theta}_e)\mathring{\theta}\sin\theta - (w_e + z_{10} + y_{10}\theta_e)(\overset{\circ\circ}{\theta}\sin\theta + \mathring{\theta}^2\cos\theta) \\
& + (\overset{\circ\circ}{v}_e - z_{10}\overset{\circ\circ}{\theta}_e)\sin\theta + 2(\mathring{v}_e - z_{10}\mathring{\theta}_e)\mathring{\theta}\cos\theta + (v_e + y_{10} - z_{10}\theta_e)(\overset{\circ\circ}{\theta}\cos\theta - \mathring{\theta}^2\sin\theta) \\
& + 2\beta\Omega\Big[\mathring{v}_e\cos\theta - (v_e + y_{10})\mathring{\theta}\sin\theta - \mathring{w}_e\sin\theta - (w_e + z_{10})\mathring{\theta}\cos\theta\Big] \\
& + r\Big[\overset{\circ\circ}{\beta} + \beta(\Omega^2 + 2\mathring{\delta}\Omega)\Big] + g_{z_5}\Big\}
\end{aligned} \tag{38}
$$

## EQUATIONS OF MOTION FOR BLADE ELASTIC DEGREES OF FREEDOM

Details of the derivation of the equations of motion for the blade elastic degrees of freedom are given below. Briefly, the equations were derived from consideration of the equilibrium of aerodynamic, dynamic, and elastic moments at a given section of the blade. The resulting moment equilibrium equations were differentiated as required to express them in terms of the local blade loadings; the elastic deflections of the blade were replaced by summations of appropriate uncoupled blade vibratory modes, of which a finite number were retained in the final analysis. Stiffness terms, both structural and centrifugal, appearing in the equations were replaced by equivalent natural frequency terms, and the orthogonality properties of the assumed modes were employed where possible to eliminate modal dynamic coupling terms.

### MOMENT EQUILIBRIUM EQUATIONS

The basic moment equilibrium equations for any blade section are of the following general form:

$$(M_e)_{x_{10}} = (M_A)_{x_{10}} + (M_D)_{x_{10}} \tag{39}$$

$$(M_e)_{y_{10}} = (M_A)_{y_{10}} + (M_D)_{y_{10}} \tag{40}$$

$$(M_e)_{z_{10}} = (M_A)_{z_{10}} + (M_D)_{z_{10}} \tag{41}$$

where $M_e$ , $M_A$ and $M_D$ represent moments due to elastic deformation, applied aerodynamic forces, and dynamic body forces respectively. It is convenient to express the latter two moments with reference to the Rigid Blade, or "5" Axis System. When the axis transformation relationships given in Eqs. (1) through (26) are employed, Eqs. (39) through (41) are as follows:

$$(M_e)_{x_{10}} = M_{x_5} + \lambda_2 \left[M_{y_5} \cos\theta + M_{z_5} \sin\theta\right] + \lambda_1 (M_{z_5} \cos\theta - M_{y_5} \sin\theta) \tag{42}$$

$$(M_e)_{y_{10}} = M_{y_5}(\cos\theta - \theta_e \sin\theta) + M_{z_5}(\sin\theta + \theta_e \cos\theta) - \lambda_2 M_{x_5} \tag{43}$$

$$(M_e)_{z_{10}} = M_{z_5}(\cos\theta - \theta_e \sin\theta) - M_{y_5}(\sin\theta + \theta_e \cos\theta) - \lambda_1 M_{x_5} \tag{44}$$

where $M_{x_5}$, $M_{y_5}$, and $M_{z_5}$ represent the sum of the aerodynamic and dynamic moment components in the $x_5$, $y_5$, and $z_5$ direction, respectively (e.g., $M_{x_5} = (M_A)_{x_5} + (M_D)_{x_5}$). Solving for $M_{x_5}$ from Eq. (42), substituting the result into Eqs. (43) and (44), neglecting higher order terms, and defining a total pitch angle

$$\Theta = \theta + \theta_e \tag{45}$$

yields the following forms of Eqs. (43) and (44):

$$(M_e)_{y_{10}} = M_{y_5}\cos\Theta + M_{z_5}\sin\Theta - \lambda_2 (M_e)_{x_{10}} \tag{46}$$

$$(M_e)_{z_{10}} = M_{z_5}\cos\Theta - M_{y_5}\sin\Theta - \lambda_1 (M_e)_{x_{10}} \tag{47}$$

It is now convenient to nondimensionalize Eqs. (42), (46), and (47) by using as nondimensionalizing factors the rotor radius, $R$, the rotor angular velocity, $\Omega$, and a reference mass per unit length, $m_0$. All such nondimensionalized quantities will hereafter be indicated by means of a bar (e.g., $\bar{r} = r/R$, $\bar{M}_{y_5} = M_{y_5}/m_0\Omega^2R^3$, etc.); further, differentiation with respect to nondimensional time will be indicated by ( $^{*}$ ), while differentiation with respect to nondimensional radial distance will be indicated by ( )'. If these conventions are employed and Eqs. (22) and (23) are used, the moment equilibrium equations are then

$$(\bar{M}_e)_{x_{10}} = \bar{M}_{x_5} + (\bar{v}_e{}' - \theta'\bar{w}_e)(\bar{M}_{y_5}\cos\theta + \bar{M}_{z_5}\sin\theta) + (\bar{w}_e{}' + \theta'\bar{v}_e)(\bar{M}_{z_5}\cos\theta - \bar{M}_{y_5}\sin\theta) \tag{48}$$

$$(\bar{M}_e)_{y_{10}} = \bar{M}_{y_5} \cos\Theta + \bar{M}_{z_5} \sin\Theta - (\bar{M}_e)_{x_{10}} \left( \bar{v}_e{}' - \bar{w}_e \theta' \right) \qquad (49)$$

$$(\bar{M}_e)_{z_{10}} = \bar{M}_{z_5} \cos\Theta - \bar{M}_{y_5} \sin\Theta - (\bar{M}_e)_{x_{10}} \left( \bar{w}_e{}' + \bar{v}_e \theta' \right) \qquad (50)$$

The moments associated with the elastic deformation of a section of a beam having a finite rate of twist along its span and a center of tension displaced in the chordwise direction from the elastic axis have been derived in Reference 3. Expressing the results of Reference 3 in terms of the $w_e$ and $v_e$ elastic coordinates employed in this analysis and neglecting higher order terms in accordance with assumptions 8 and 10 lead to the following expressions for the elastic deformation moments:

$$(\bar{M}_e)_{y_{10}} = -\overline{EI}_y \left( \bar{w}_e{}'' + 2\bar{v}_e{}' \theta' \right) \qquad (51)$$

$$(\bar{M}_e)_{z_{10}} = \overline{EI}_z \left( \bar{v}_e{}'' - 2\bar{w}_e{}' \theta' \right) - \bar{e}_A \int_{\bar{r}}^{\bar{r}_T} \bar{S}_{x_5} \, d\bar{\xi} + \Delta \bar{e}_{A_{cw}} \int_{\bar{r}_{o_{cw}}}^{\bar{r}} \bar{S}_{x_5} \frac{\bar{m}_{cw}}{\bar{m}} \, d\bar{\xi} \qquad (52)$$

$$(\bar{M}_e)_{x_{10}} = \overline{GJ} \, \theta_e{}' + (\theta' + \theta_e{}') \, \bar{k}_A{}^2 \int_{\bar{r}}^{\bar{r}_T} \bar{S}_{x_5} \, d\bar{\xi} \qquad (53)$$

The last term in Eq. (52) is not contained in the results of Reference 3 but rather arises when, for blade balance purposes, a counterweight is employed which is so mounted as to be restrained radially by the blade's tip structure and laterally by the blade's local structure. This type of counterweight is then constrained to bend with the blade and is in compression under the action of centrifugal force.

Combining Eqs. (51) through (53) with Eqs. (48) through (50) and neglecting higher order terms in accordance with assumption 10, results in the final form of the moment equilibrium equations below:

$$\overline{GJ}\,\theta_e' + (\theta' + \theta_e')\bar{k}_A^2 \int_{\bar{r}}^{\bar{r}_T} \bar{S}_{x_5}\, d\bar{\xi} = \bar{M}_{x_5} + (\bar{v}_e' - \theta'\bar{w}_e)(\bar{M}_{y_5}\cos\theta + \bar{M}_{z_5}\sin\theta) + (\bar{w}_e' + \theta'\bar{v}_e)(\bar{M}_{z_5}\cos\theta - \bar{M}_{y_5}\sin\theta) \tag{54}$$

$$-\overline{EI}_y\,(\bar{w}_e'' + 2\bar{v}_e'\theta') = \bar{M}_{y_5}\cos\Theta + \bar{M}_{z_5}\sin\Theta - (\bar{M}_e)_{x_{10}}(\bar{v}_e' - \bar{w}_e\theta') \tag{55}$$

$$\overline{EI}_z\,(\bar{v}_e'' - 2\bar{w}_e'\theta') - \bar{e}_A \int_{\bar{r}}^{\bar{r}_T} \bar{S}_{x_5}\, d\bar{\xi} + \Delta\bar{e}_{A_{cw}} \int_{\bar{r}_{o_{cw}}}^{\bar{r}} \bar{S}_{x_5} \frac{\bar{m}_{cw}}{\bar{m}}\, d\bar{\xi} = \bar{M}_{z_5}\cos\Theta - \bar{M}_{y_5}\sin\Theta - (\bar{M}_e)_{x_{10}}(\bar{w}_e' + \bar{v}_e\theta') \tag{56}$$

The last two terms in Eqs. (55) and (56) are always comparatively small, and are neglected hereafter.

## LOADING EQUILIBRIUM EQUATIONS

### Flatwise Loading Equilibrium Equation

The desired flatwise loading equilibrium equation can be obtained by differentiating Eq. (55) twice with respect to the spanwise coordinate $\bar{r}$ (if techniques described in Reference 4, pp. 352-353 are used). The resulting equation is

$$-\left[\overline{EI}_y\left(\overline{w}_e'' + 2\overline{v}_e'\theta'\right)\right]'' = \Theta''\left[-\overline{M}_{y_5}\sin\Theta + \overline{M}_{z_5}\cos\Theta\right]$$

$$+ (\Theta')^2\left[-\overline{M}_{y_5}\cos\Theta - \overline{M}_{z_5}\sin\Theta\right] + 2\Theta'\left[-\overline{M}_{y_5}'\sin\Theta + \overline{M}_{z_5}'\cos\Theta\right]$$

$$+ \overline{M}_{y_5}''\cos\Theta + \overline{M}_{z_5}''\sin\Theta \qquad (57)$$

To proceed further, expressions for the "5" axis moment components are required. These are derived in Appendix I along with their derivatives with respect to $\bar{r}$. The derivatives of $M_{y_5}$ and $M_{z_5}$ in Eq. (57) are replaced by the expressions given in Eqs. (110) through (114) of Appendix I. Then by employing the relations

$$\bar{z}_{5_{ea}}'\sin\Theta + \bar{y}_{5_{ea}}'\cos\Theta \triangleq -\overline{w}_e\theta' + \overline{v}_e' \qquad (58)$$

$$\bar{z}_{5_{ea}}'\cos\Theta - \bar{y}_{5_{ea}}'\sin\Theta \triangleq \overline{w}_e' + \overline{v}_e\theta' \qquad (59)$$

$$\bar{y}_{5_{ea}}''\sin\Theta - \bar{z}_{5_{ea}}''\cos\Theta \triangleq -\overline{w}_e'' + (\theta')^2\overline{w}_e - 2\overline{v}_e'\theta' - \overline{v}_e\theta'' \qquad (60)$$

$$\bar{y}_{5_{ea}}''\cos\Theta + \bar{z}_{5_{ea}}''\sin\Theta \triangleq \overline{v}_e'' - \overline{v}_e(\theta')^2 - 2\overline{w}_e'\theta' - \overline{w}_e\theta'' \qquad (61)$$

and by neglecting higher order terms in accordance with assumption 10, Eq. (57) becomes

$$-\left[\overline{EI}_y\left(\overline{w}_e'' + 2\overline{v}_e'\theta'\right)\right]'' = -2\Theta'\left\{\sin\Theta\int_{\bar r}^{\bar r_T}\bar S_{z_5}\,d\bar\xi + \cos\Theta\int_{\bar r}^{\bar r_T}\bar S_{y_5}\,d\bar\xi\right\} - \overline{w}_e''\int_{\bar r}^{\bar r_T}\bar S_{x_5}\,d\bar\xi$$

$$-\bar S_{z_5}\cos\Theta + \bar S_{y_5}\sin\Theta + \bar S_{x_5}\left(\overline{w}_e' + \bar v_e\theta'\right) + 2\overset{x}{\theta}\cos\theta\left[\bar m\bar k_{y_{10}}^2\right]'$$

$$+\Theta'\bar y_{10_{cg}}(\bar S_D)_{x_5,y_{10_{cg}}=0} + \Theta''\left[-\bar M_{y_5}\sin\Theta + \bar M_{z_5}\cos\Theta\right]$$

$$+(\Theta')^2\left[-\bar M_{y_5}\cos\Theta - \bar M_{z_5}\sin\Theta\right] + 2\Theta'\left[\sin\Theta(\bar m_A)_{y_5} - (\bar m_A)_{z_5}\cos\Theta\right]$$

$$-(m_A)'_{y_5}\cos\Theta - (m_A)'_{z_5}\sin\Theta$$

(62)

The flatwise loading equation can now be expressed in the desired form if the $\Theta''$ and $(\Theta')^2$ terms are expanded through use of Eqs. (45), (55) and (56), and if higher order terms are neglected in accordance with assumption 10.

$$-\left[\overline{EI}_y\left(\overline{w}_e'' + 2\overline{v}_e'\theta'\right)\right]'' = -2\Theta'\left\{\sin\Theta\int_{\bar r}^{\bar r_T}\bar S_{z_5}\,d\bar\xi + \cos\Theta\int_{\bar r}^{\bar r_T}\bar S_{y_5}\,d\bar\xi\right\} - \overline{w}_e''\int_{\bar r}^{\bar r_T}\bar S_{x_5}\,d\bar\xi$$

$$-\bar S_{z_5}\cos\Theta + \bar S_{y_5}\sin\Theta + \bar S_{x_5}\left(\overline{w}_e' + \bar v_e\theta'\right) + 2\overset{x}{\theta}\cos\theta\left[\bar m\bar k_{y_{10}}^2\right]'$$

$$+\Theta'\bar y_{10_{cg}}(\bar S_D)_{x_5,y_{10_{cg}}=0} - \theta_e''\bar e_A\int_{\bar r}^{\bar r_T}\bar S_{x_5}\,d\bar\xi + \Delta\bar e_{A_{cw}}\theta_e''\int_{\bar r_{0_{cw}}}^{\bar r}\bar S_{x_5}\frac{\bar m_{cw}}{\bar m}\,d\bar\xi$$

$$+2\Theta'\left[(\bar m_A)_{y_5}\sin\Theta - (\bar m_A)_{z_5}\cos\Theta\right] - (\bar m_A)'_{y_5}\cos\Theta - (\bar m_A)'_{z_5}\sin\Theta$$

(63)

Edgewise Loading Equilibrium Equation

The desired edgewise loading equilibrium equation is obtained by differentiating Eq. (56) twice with respect to $\bar{r}$ . The resulting equation is

$$\left[\overline{EI}_z(\bar{v}_e'' - 2\bar{w}_e'\theta') - \bar{e}_A\int_{\bar{r}}^{\bar{r}_T}\bar{S}_{x_5}d\xi + \Delta\bar{e}_{A_{cw}}\int_{\bar{r}_{o_{cw}}}^{\bar{r}}\bar{S}_{x_5}\frac{\overline{m}_{cw}}{\overline{m}}d\xi\right]''$$

$$= -\Theta''\left[\overline{M}_{z_5}\sin\Theta + \overline{M}_{y_5}\cos\Theta\right] + (\Theta')^2\left[-\overline{M}_{z_5}\cos\Theta + \overline{M}_{y_5}\sin\Theta\right]$$

$$- 2\Theta'\left[\overline{M}_{z_5}'\sin\Theta + \overline{M}_{y_5}'\cos\Theta\right] + \overline{M}_{z_5}''\cos\Theta - \overline{M}_{y_5}''\sin\Theta$$

(64)

Substituting (from Appendix I) for the derivatives of $M_{y_5}$ and $M_{z_5}$ , using Eqs. (58) to (61) as required, and neglecting higher order terms where necessary, allows Eq. (64) to be expressed as

$$\left[EI_z(\bar{v}_e'' - 2\bar{w}_e'\theta') - \bar{e}_A\int_{\bar{r}}^{\bar{r}_T}\bar{S}_{x_5}d\xi + \Delta\bar{e}_{A_{cw}}\int_{\bar{r}_{o_{cw}}}^{\bar{r}}\bar{S}_{x_5}\frac{\overline{m}_{cw}}{\overline{m}}d\xi\right]''$$

$$= -\Theta''\left[\overline{M}_{z_5}\sin\Theta + \overline{M}_{y_5}\cos\Theta\right] + (\Theta')^2\left[-\overline{M}_{z_5}\cos\Theta + \overline{M}_{y_5}\sin\Theta\right]$$

$$-2\Theta'\left\{-\sin\Theta\int_{\bar{r}}^{\bar{r}_T}\bar{S}_{y_5}d\xi + \cos\Theta\int_{\bar{r}}^{\bar{r}_T}\bar{S}_{z_5}d\xi\right\} + \bar{v}_e''\int_{\bar{r}}^{\bar{r}_T}\bar{S}_{x_5}d\xi + \bar{S}_{y_5}\cos\Theta$$

$$+ \bar{S}_{z_5}\sin\Theta - (-\bar{w}_e\theta' + \bar{v}_e')\bar{S}_{x_5} + \left[\bar{y}_{10_{cg}}(\bar{S}_D)_{x_5, y_{10_{cg}}=0}\right]' - 2\overset{x}{\theta}\sin\theta(\bar{k}_{z_{10}}^2\overline{m})'$$

$$+ 2\Theta'\left[(\overline{m}_A)_{z_5}\sin\Theta + (\overline{m}_A)_{y_5}\cos\Theta\right] - (\overline{m}_A)_{z_5}'\cos\Theta + (\overline{m}_A)_{y_5}'\sin\Theta$$

(65)

Or, expanding the $\Theta''$ and $(\Theta')^2$ terms through the use of Eqs. (45), (55), and (56) and neglecting higher order terms permits Eq. (65) to be placed in the desired form, as follows:

$$\left[EI_z(\bar{v}_e'' - 2\bar{w}_e'\theta') - \bar{e}_A \int_{\bar{r}}^{\bar{r}_T} \bar{S}_{x_5} d\bar{\xi} + \Delta\bar{e}_{A_{cw}} \int_{\bar{r}_{0_{cw}}}^{\bar{r}} \bar{S}_{x_5} \frac{\bar{m}_{cw}}{\bar{m}} d\bar{\xi}\right]''$$

$$= -2\Theta'\left\{-\sin\Theta \int_{\bar{r}}^{\bar{r}_T} \bar{S}_{y_5} d\bar{\xi} + \cos\Theta \int_{\bar{r}}^{\bar{r}_T} \bar{S}_{z_5} d\bar{\xi}\right\} + \bar{v}_e'' \int_{\bar{r}}^{\bar{r}_T} \bar{S}_{x_5} d\bar{\xi} + \bar{S}_{y_5}\cos\Theta$$

$$+ \bar{S}_{z_5}\sin\Theta - (-\bar{w}_e\theta' + \bar{v}_e')\bar{S}_{x_5} + \left[\bar{y}_{10_{cg}} (\bar{S}_D)_{x_5, y_{10_{cg}}=0}\right]' - 2\overset{x}{\theta}\sin\theta\,(\bar{k}_{z_{10}}^2 \bar{m})'$$

$$+ 2\Theta'\left[(\bar{m}_A)_{z_5}\sin\Theta + (\bar{m}_A)_{y_5}\cos\Theta\right] - (\bar{m}_A)_{z_5}'\cos\Theta + (\bar{m}_A)_{y_5}'\sin\Theta \qquad (66)$$

Torsional Loading Equilibrium Equation

The desired torsional loading equation is obtained by differentiating Eq. (54) once with respect to $\bar{r}$ . Prior to differentiation, however, it is convenient (1) to substitute for $\bar{M}_{x_5}$ with the use of Eq. (106) of Appendix I, and (2) to simplify the $\bar{M}_{y_5}$ and $\bar{M}_{z_5}$ terms in Eq. (54) with the use of the definitions below:

$$\bar{y}_{5_{eo}}' = \bar{v}_e'\cos\theta - \theta'\bar{w}_e\cos\theta - \bar{w}_e'\sin\theta - \theta'\bar{v}_e\sin\theta \qquad (67)$$

$$\bar{z}_{5_{eo}}' = \bar{v}_e'\sin\theta - \theta'\bar{w}_e\sin\theta + \bar{w}_e'\cos\theta + \theta'\bar{v}_e\cos\theta \qquad (68)$$

The resulting expression for the torsional moment equation is then

$$\overline{GJ}\,\theta_e' + (\theta' + \theta_e')\bar{k}_A^2 \int_{\bar{r}}^{\bar{r}_T} \bar{S}_{x_5} d\bar{\xi} = \int_{\bar{r}}^{\bar{r}_T} \Big\{ (\bar{m}_A)_{x_5} + \bar{S}_{z_5}\big(\bar{y}_{5_{eo}} - \bar{y}_{5_{eo}}(\bar{r})\big)$$

$$- \bar{S}_{y_5}\big(\bar{z}_{5_{eo}} - \bar{z}_{5_{eo}}(\bar{r})\big)\Big\} d\bar{\xi} + \int_{\bar{r}}^{\bar{r}_T} \ddot{\bar{y}}_{10_{cg}} \left[(\bar{S}_D)_{z_5}\cos\Theta - (\bar{S}_D)_{y_5}\sin\Theta\right]_{y_{10_{cg}}=0} d\bar{\xi}$$

$$- \int_{\bar{r}}^{\bar{r}_T} \bar{m}(\bar{k}_{z_{10}}^2 + \bar{k}_{y_{10}}^2)(\overset{xx}{\Theta} + \overset{x}{\beta}) d\bar{\xi} + \int_{\bar{r}}^{\bar{r}_T} m(\bar{k}_{y_{10}}^2 - \bar{k}_{z_{10}}^2)\Big[\frac{1}{2}(1 + 2\overset{x}{\delta})\sin 2\theta$$

$$+ (\theta_e - \overset{x}{\beta})\cos 2\theta\Big] d\bar{\xi} - 2\int_{\bar{r}}^{\bar{r}_T} \bar{m}\,\bar{k}_{z_{10}}^2 \overset{x}{\bar{v}}_e'\sin\theta\, d\bar{\xi} - 2\int_{\bar{r}}^{\bar{r}_T} \bar{m}\,\bar{k}_{y_{10}}^2 \overset{x}{\bar{w}}_e'\cos\theta\, d\bar{\xi}$$

$$+ \bar{M}_{y_5}\bar{y}_{5_{eo}}' + \bar{M}_{z_5}\bar{z}_{5_{eo}}' \qquad (69)$$

Note that all quantities appearing in the various integrands above are functions of the dummy variable of integration $\xi$, unless explicitly indicated to be a function of $\bar{r}$. Differentiating Eq. (69) with respect to $\bar{r}$ and using the definitions of $y''_{5_{eq}}$ and $z''_{5_{eq}}$ along with Eqs. (55) and (56) to eliminate $\bar{M}_{z_5}$ and $\bar{M}_{y_5}$ results in the final torsional loading equation below:

$$
\begin{aligned}
\left[\overline{GJ}\,\theta_e' + (\theta' + \theta_e')\bar{k}_A^2 \int_{\bar{r}}^{\bar{r}_T} \bar{S}_{x_5}\, d\bar{\xi}\right]' = &-(\bar{m}_A)_{x_5} - \bar{y}_{10_{cg}}\left[(\bar{S}_D)_{z_5}\cos\Theta - (\bar{S}_D)_{y_5}\sin\Theta\right]_{y_{10_{cg}}=0} \\
&+ \bar{m}(\bar{k}_{z_{10}}^2 + \bar{k}_{y_{10}}^2)(\overset{xx}{\Theta} + \overset{x}{\beta}) - \bar{m}(\bar{k}_{y_{10}}^2 - \bar{k}_{z_{10}}^2)\left[\frac{1}{2}(1 + 2\overset{x}{\delta})\sin 2\theta + (\theta_e - \overset{x}{\beta})\cos 2\theta\right] \\
&+ 2\bar{m}\bar{k}_{z_{10}}^2 \sin\theta(\overset{x}{\bar{v}}_e{}' - \overset{x}{\theta}\bar{w}_e') + 2\bar{m}\bar{k}_{y_{10}}^2 \cos\theta(\overset{x}{\bar{w}}_e{}' + \overset{x}{\theta}\bar{v}_e') + \bar{y}_{10_{cg}}(\bar{S}_D)_{x_5, y_{10_{cg}}=0}(\bar{w}_e' + \bar{v}_e\theta') \\
&+ (\bar{w}_e'' + 2\bar{v}_e'\theta')\left[-\bar{e}_A \int_{\bar{r}}^{\bar{r}_T} \bar{S}_{x_5}\, d\bar{\xi} + \Delta\bar{e}_{A_{cw}} \int_{\bar{r}_{0_{cw}}}^{\bar{r}} \frac{\bar{m}_{cw}}{\bar{m}} \bar{S}_{x_5}\, d\bar{\xi}\right] \\
&+ (\overline{EI}_z - \overline{EI}_y)(\bar{w}_e''\bar{v}_e'' - 2\bar{w}_e''\bar{w}_e'\theta' + 2\bar{v}_e''\bar{v}_e'\theta')
\end{aligned}
\tag{70}
$$

MODAL EQUATIONS OF MOTION

The preceding sections of this report have been directed toward the derivation of the fundamental differential equations of motion (Eqs. (63), (66), and (70)) governing the forced response of the elastic motion of a twisted rotating beam. These equations have generally been derived in terms of the local dynamic and aerodynamic shear forces and moments. Expressions for the dynamic shear forces are given in Eqs. (102), (103) and (108) of Appendix I, while the aerodynamic shear forces and mechanical damping forces are derived in Appendixes II and III respectively. Examination of the dynamic and aerodynamic shear force expressions indicate that closed-form solutions to the blade equations of motion are not possible because of the nonlinearities present. Therefore solutions must be obtained by approximate means. The particular approach employed herein is a modal approach wherein the elastic coordinates $w_e$, $v_e$, and $\theta_e$, are expressed as finite series summations of assumed radial shape functions (mode shapes) with each suitably scaled by time-dependent generalized coordinates. The series expansions employed are

$$\bar{w}_e(\bar{r},\psi) = \sum_{n=1}^{5} \gamma_{w_n}(\bar{r})\, q_{w_n}(\psi) \tag{71}$$

$$\bar{v}_e(\bar{r},\psi) = \sum_{m=1}^{2} \gamma_{v_m}(\bar{r})\, q_{v_m}(\psi) \tag{72}$$

$$\theta_e(\bar{r},\psi) = \sum_{k=1}^{3} \gamma_{\theta_k}(\bar{r}) q_{\theta_k}(\psi) \tag{73}$$

Note that the number of terms retained in the expansions of $w_e$, $v_e$, and $\theta_e$ are 5, 2, and 3, respectively. The radial shape functions (hereinafter referred to as mode shapes) employed in this analysis correspond to the natural vibratory mode shapes of an untwisted blade mounted on an unaccelerated rotor hub and operating at zero pitch, flap, and lead angles and at a rotor angular velocity, $\Omega$. As indicated in the analysis which follows, the substitutions of Eqs. (71) through (73) into Eqs. (63), (66), and (70) permit these equations of motion to be expanded into a system of simultaneous differential equations with the generalized coordinates as the unknowns. Further, because of the orthogonality properties of the particular mode shapes employed, the resulting system of equations can, to a large extent, be dynamically uncoupled as far as the blade elastic degrees of freedom are concerned. As a result, solution of the equations through numerical integration techniques is greatly facilitated. Details of the procedures followed are given below.

Flatwise Modal Equation

Substituting Eqs. (71) through (73) into Eq. (63), setting $(\bar{m}_A)_{y_5}$, $(\bar{m}_A)_{z_5}$ and $(\bar{S}_A)_{x_5}$ equal to zero in accordance with the results of Appendix II, and employing the uncoupled free vibration form of Eq. (63), i.e.,

$$\left[\overline{EI}_y \gamma''_{w_n}\right]'' + \bar{m}\gamma'_{w_n}(\bar{e}+\bar{r}) - \gamma''_{w_n}\int_{\bar{r}}^{\bar{r}_T} \bar{m}(\bar{e}+\bar{\xi})\,d\bar{\xi} = \bar{m}\gamma_{w_n}\bar{\omega}_{w_n}^2 \tag{74}$$

permits Eq. (63) to be expressed in the following form:

$$-\overline{m}\sum_n \gamma_{w_n} q_{w_n}\overline{\omega}_{w_n}^{\,2} - 2\left[\overline{EI}_y \theta' \sum_m \gamma_{v_m}{}' q_{v_m}\right]'' = -2\Theta'\left\{\sin\Theta \int_{\overline{r}}^{\overline{r}_T} \overline{S}_{z_5}\, d\overline{\xi}\right.$$

$$\left. + \cos\Theta \int_{\overline{r}}^{\overline{r}_T} \overline{S}_{y_5}\, d\overline{\xi}\right\} - \sum_n \gamma_{w_n}'' q_{w_n}\left[\int_{\overline{r}}^{\overline{r}_T} (\overline{S}_D)_{x_5} d\overline{\xi} - \int_{\overline{r}}^{\overline{r}_T} \overline{m}(\overline{e}+\overline{\xi})\, d\overline{\xi}\right]$$

$$- \overline{S}_{z_5}\cos\Theta + \overline{S}_{y_5}\sin\Theta + \sum_n \gamma_{w_n}{}' q_{w_n}\left[(\overline{S}_D)_{x_5} - \overline{m}(\overline{e}+\overline{r})\right] + (\overline{S}_D)_{x_5}\theta' \sum_m \gamma_{v_m} q_{v_m}$$

$$+ 2\overset{x}{\theta}\cos\theta\,(\overline{m}\,\overline{k}_{y_{10}}^{\,2})' + \Theta'\overline{y}_{10_{cg}}(\overline{S}_D)_{x_5, y_{10_{cg}}} = 0 - \theta_e''\,\overline{e}_A \int_{\overline{r}}^{\overline{r}_T} (\overline{S}_D)_{x_5}\, d\overline{\xi}$$

$$+ \Delta\overline{e}_{A_{cw}}\,\theta_e'' \int_{\overline{r}_{o_{cw}}}^{\overline{r}} \frac{\overline{m}_{cw}}{\overline{m}} (\overline{S}_D)_{x_5}\, d\overline{\xi} \qquad (75)$$

Note that in Eq. (75) and in succeeding equations, a shorthand summation notation is used whereby the summation limits are not explicitly indicated but rather are understood to be those of Eqs. (71) through (73).

The integrals containing the $z_5$ and $y_5$ shears can be simplified considerably by noting that when the response of a beam is represented by a summation of modes, the shears at any point on the beam may be expressed in terms of the modal inertial loadings and, hence, in terms of the modal amplitudes and associated modal natural frequencies (Reference 5, p. 641). Thus, the following relations are valid for a limited mode analysis:

$$\int_{\overline{r}}^{\overline{r}_T} \overline{S}_{z_5}\, d\overline{\xi} \cong \sum_n \overline{\omega}_{w_n}^{\,2} q_{w_n} \int_{\overline{r}}^{\overline{r}_T} \overline{m}\gamma_{w_n} \cos\theta\, d\overline{\xi} + \sum_m (1+\overline{\omega}_{v_m}^{\,2}) q_{v_m} \int_{\overline{r}}^{\overline{r}_T} \overline{m}\gamma_{v_m} \sin\theta\, d\overline{\xi} \qquad (76)$$

$$\int_{\overline{r}}^{\overline{r}_T} \overline{S}_{y_5}\, d\overline{\xi} \cong -\sum_n \overline{\omega}_{w_n}^{\,2} q_{w_n} \int_{\overline{r}}^{\overline{r}_T} \overline{m}\gamma_{w_n} \sin\theta\, d\overline{\xi} + \sum_m (1+\overline{\omega}_{v_m}^{\,2}) q_{v_m} \int_{\overline{r}}^{\overline{r}_T} \overline{m}\,\gamma_{v_m} \cos\theta\, d\overline{\xi} \qquad (77)$$

The 1's in the second summations above result from the centrifugal shears associated with the uncoupled edgewise modal displacement. Employing Eqs. (76) and (77) and the definitions of $\bar{S}_{z_5}$ , $\bar{S}_{y_5}$ , $(\bar{S}_D)_{x_5}$ , $(\bar{S}_D)_{x_5}\, y_{10cg} = 0$ and $\theta$ from Appendixes I and II permits Eq. (75) to be expressed in the following form (after higher order terms have been eliminated):

$$
\begin{aligned}
&\bar{m}\sum_n \gamma_{w_n} q_{w_n} \bar{\omega}_{w_n}^2 + 2\left[\overline{EI}_y \sum_m \gamma_{v_m}' q_{v_m} \theta_1\right]'' = 2\theta_1 \Big\{ \sum_n \bar{\omega}_{w_n}^2 q_{w_n} \int_{\bar{r}}^{\bar{r}_T} \bar{m}\gamma_{w_n} \big(-\theta_B \\
&\quad + \sum_k \gamma_{\theta_k}(\bar{r}) q_{\theta_k} + \theta_B(\bar{r})\big) d\bar{\xi} + \sum_m (1+\bar{\omega}_{v_m}^2) q_{v_m} \int_{\bar{r}}^{\bar{r}_T} \bar{m}\gamma_{v_m} d\bar{\xi} \Big\} \\
&\quad + 2\sum_k \gamma_{\theta_k}' q_{\theta_k} \Big\{ \sum_n \bar{\omega}_{w_n}^2 q_{w_n} \int_{\bar{r}}^{\bar{r}_T} \bar{m}\gamma_{w_n} (-\theta_B + \theta_B(\bar{r}))\, d\bar{\xi} + \sum_m (1+\bar{\omega}_{v_m}^2) q_{v_m} \int_{\bar{r}}^{\bar{r}_T} \bar{m}\gamma_{v_m} d\bar{\xi} \Big\} \\
&\quad + \sum_n \gamma_{w_n}'' q_{w_n} \int_{\bar{r}}^{\bar{r}_T} \bar{m}\left[2\bar{\xi}\overset{x}{\delta} - 2\bar{y}_{10_{cg}} \overset{x}{\theta}_0 \sin\theta_0\right] d\bar{\xi} - \bar{m}\sum_n \gamma_{w_n}' q_{w_n}\left[2\bar{r}\overset{x}{\delta} - 2\bar{y}_{10_{cg}} \overset{x}{\theta}_0 \sin\theta_0\right] \\
&\quad - \bar{m}\bar{r}\theta_1 \sum_m \gamma_{v_m} q_{v_m} - 2\overset{x}{\theta}_0 \cos\theta_0 (\overline{mk}_{y_{10}}^2)' - (\theta_1 + \sum_k \gamma_{\theta_k}' q_{\theta_k})\bar{m}\bar{y}_{10_{cg}}\bar{r} \\
&\quad + (\bar{S}_A - \bar{S}_{FD} - \bar{S}_{PR})_{z_5} \cos\Theta - (\bar{S}_A - \bar{S}_{LD})_{y_5} \sin\Theta - \bar{m}\Big\{ \bar{e}\beta\cos\theta_0 - \bar{e}\delta\sin\theta_0 \\
&\quad + \sum_n \gamma_{w_n}\Big[\overset{xx}{q}_{w_n} + q_{w_n}\big(-\overset{x}{\theta}{}_0^2 - 2\beta\overset{x}{\theta}_0 + \overset{x}{\beta}\sin 2\theta_0 - \tfrac{1}{2}\langle 1 - \cos 2\theta_0 + 2\theta_B \sin 2\theta_0\rangle \\
&\quad - \overset{x}{\delta}\langle 1 - \cos 2\theta_0\rangle\big)\Big] + (\bar{y}_{10_{cg}} + \sum_m \gamma_{v_m} q_{v_m})\Big[\overset{x}{\beta}(1 - \cos 2\theta_0) + \tfrac{1}{2}(\sin 2\theta_0 + 2\theta_B \cos 2\theta_0) \\
&\quad + \overset{x}{\delta}\sin 2\theta_0 + \overset{xx}{\theta}_0\Big] + \sum_m \gamma_{v_m}\overset{x}{q}_{v_m}\left[2\overset{x}{\theta}_0 + 2\beta\right] + \sum_m \gamma_{v_m}' \overset{x}{q}_{v_m} 2\bar{y}_{10_{cg}} \sin\theta_0 \\
&\quad + \sum_k \gamma_{\theta_k} q_{\theta_k}\left[\bar{y}_{10_{cg}} \cos 2\theta_0 - \bar{r}(\overset{xx}{\beta} + \beta)\sin\theta_0 - \bar{r}\overset{xx}{\delta}\cos\theta_0\right] + \bar{y}_{10_{cg}} \sum_k \gamma_{\theta_k} \overset{xx}{q}_{\theta_k} \\
&\quad - \sum_k \gamma_{\theta_k}'' q_{\theta_k}\left[-\bar{e}_A \int_{\bar{r}}^{\bar{r}_T} \bar{m}\bar{\xi}\, d\bar{\xi} + \Delta\bar{e}_{A_{cw}} \int_{\bar{r}_{\rho_{cw}}}^{\bar{r}} \bar{m}_{cw}\bar{\xi}\, d\bar{\xi}\right] + \bar{r}(\overset{xx}{\beta} + \beta)(\cos\theta_0 - \theta_B \sin\theta_0) \\
&\quad + 2\beta\overset{x}{\delta}\bar{r}\cos\theta_0 + \bar{g}_{z_5}\cos\theta_0 - \bar{g}_{y_5}\sin\theta_0 - \bar{r}\overset{xx}{\delta}(\sin\theta_0 + \theta_B\cos\theta_0) + 2\bar{r}\beta\overset{x}{\beta}\sin\theta_0 \Big\}
\end{aligned}
\tag{78}
$$

The equations of motion for each flatwise mode can be obtained by considering the work done during a virtual displacement of each mode. Formally, this invloves multiplication of Eq. (78) by $\gamma_{w_i}$, integration over the blade span, and application of the orthogonality property of the assumed uncoupled natural modes, namely,

$$\sum_n \int_0^{\bar{r}_T} \bar{m}\,\gamma_{w_n}\,\gamma_{w_i}\,d\bar{r} = \int_0^{\bar{r}_T} \bar{m}\,\gamma_{w_i}^{2}\,d\bar{r} \tag{79}$$

This procedure yields the following equation of motion for the i^th flatwise mode:

$$\begin{aligned}
0 = {} & \int_0^{\bar{r}_T} \gamma_{w_i}\left[(\bar{S}_A)_{z_5}\cos\Theta - (\bar{S}_A)_{y_5}\sin\Theta\right]d\bar{r} - \bar{M}_{PR}\cos\theta_{PR}(\gamma_{w_i}')_{\bar{r}=0} + \sin\theta_{LD}\bar{M}_{LD}(\gamma_{w_i}')_{\bar{r}=0} \\
& - \cos\theta_{FD}\bar{M}_{FD}(\gamma_{w_i}')_{\bar{r}=0} - C_{1_i}\Big[\overset{xx}{q}_{w_i} + q_{w_i}\Big\langle \bar{\omega}_{w_i}^{2} - \overset{x}{\theta}_0^{\,2} - 2\beta\overset{x}{\theta}_0 + \overset{x}{\beta}\sin 2\theta_0 - \frac{1}{2}(1 \\
& + 2\overset{x}{\delta})(1-\cos 2\theta_0)\Big\rangle\Big] + \sum_{i'} q_{w_{i'}}\Big[-2\theta_1\bar{\omega}_{w_{i'}}^{2}C_{9_{i,i'}} + \theta_1\sin 2\theta_0 C_{68_{i,i'}} + 2\overset{x}{\delta}(C_7 - C_4)_{i,i'} \\
& - 2\overset{x}{\theta}_0\sin\theta_0(C_{65} - C_{69})_{i,i'}\Big] + \sum_p q_{v_p}\Big\{C_{3_p}\Big[-\overset{xx}{\theta}_0 - \overset{x}{\beta}(1-\cos 2\theta_0) - \frac{1}{2}(1+2\overset{x}{\delta})\sin 2\theta_0\Big] \\
& - \theta_1\cos 2\theta_0(C_5 - 0.75C_3)_{i,p} - C_{27_{i,p}} + 2\theta_1(1+\bar{\omega}_{v_p}^{2})C_{8_{i,p}} - \theta_1 C_{5_{i,p}}\Big\} \\
& + \sum_p \overset{x}{q}_{v_p}\Big[-2\sin\theta_0 C_{64_{i,p}} - 2(\overset{x}{\theta}_0 + \beta)C_{3_{i,p}}\Big] + 2\sum_p\sum_j q_{\theta_j}\,q_{v_p}(1+\bar{\omega}_{v_p}^{2})C_{67_{i,p,j}} \\
& - \sum_j \overset{xx}{q}_{\theta_j}C_{62_{i,j}} + \sum_j q_{\theta_j}\Big[C_{57_{i,j}} - C_{62_{i,j}}\cos 2\theta_0 + C_{56_{i,j}}\Big\langle(\overset{xx}{\beta}+\beta)\sin\theta_0 + \overset{xx}{\delta}\cos\theta_0\Big\rangle \\
& - C_{61_{i,j}}\Big] - C_{10_i}\Big[\bar{e}(\beta\cos\theta_0 - \delta\sin\theta_0) + \bar{g}_{z_5}\cos\theta_0 - \bar{g}_{y_5}\sin\theta_0\Big] - C_{12_i}\Big[(\overset{xx}{\beta} \\
& + \beta + 2\beta\overset{x}{\delta})\cos\theta_0 + (2\beta\overset{x}{\beta} - \overset{xx}{\delta})\sin\theta_0\Big] + C_{13_i}\Big[(\overset{xx}{\beta}+\beta)\sin\theta_0 + \overset{xx}{\delta}\cos\theta_0\Big] \\
& - C_{63_i}\Big[\overset{xx}{\theta}_0 + \overset{x}{\beta}(1-\cos 2\theta_0) + \frac{1}{2}(1+2\overset{x}{\delta})\sin 2\theta_0\Big] - \theta_1\cos 2\theta_0(C_{60_i} - 0.75C_{63_i}) - \theta_1 C_{60_i}
\end{aligned} \tag{80}$$

where the various modal integration constants (e.g., $C_{1_i}$ ) are defined in Appendix IV. Eq. (80) above is the final flatwise modal equation of motion. Note that in this equation, as in the final edgewise and torsional modal equations to follow in the next sections, the modal subscript notation has, for convenience, been standardized so that the $i$ subscript refers to flatwise modal quantities, the $p$ subscript to edgewise modal quantities, and the $j$ subscript to torsional modal quantities. Where products of flatwise modes appear, $i'$ subscripts are introduced to differentiate between terms associated with the various summations. The $p'$ and $j'$ subscripts serve similar purposes for the edgewise and torsional modes. Furthermore, Eqs. (155), (156), and (161) have been substituted for the flap damper, lag damper, and pushrod forces, respectively; and the integrations over $\bar{r}$ have been performed to obtain the corresponding moment contributions defined below:

$$\bar{M}_{LD} = \frac{C_{LD}}{m_0 \Omega R^3} \left[ \overset{x}{\delta} + \sum_p (\gamma_{v_p}')_{\bar{r}=0} (\overset{x}{q}_{v_p} \cos\theta_{LD} - q_{v_p} \overset{x}{\theta}_0 \sin\theta_{LD}) - \sum_i (\gamma_{w_i}')_{\bar{r}=0} (\overset{x}{q}_{w_i} \sin\theta_{LD} + q_{w_i} \overset{x}{\theta}_0 \cos\theta_{LD}) \right] \quad (81)$$

$$\bar{M}_{FD} = \frac{C_{FD}}{m_0 \Omega R^3} \left[ \overset{x}{\beta} + \sum_i (\gamma_{w_i}')_{\bar{r}=0} (\overset{x}{q}_{w_i} \cos\theta_{FD} - q_{w_i} \overset{x}{\theta}_0 \sin\theta_{FD}) + \sum_p (\gamma_{v_p}')_{\bar{r}=0} (\overset{x}{q}_{v_p} \sin\theta_{FD} + \overset{x}{\theta}_0 \cos\theta_{FD}) \right] \quad (82)$$

$$\bar{M}_{PR} = \tan\delta_3 \, (\overline{GJ})_{\bar{r}=0} \sum_j (\gamma_{\theta_j}')_{\bar{r}=0} \, q_{\theta_j} \quad (83)$$

Edgewise Modal Equation

The desired edgewise modal equation is derived in a manner entirely analogous to that followed in deriving the flatwise modal equation. The free vibration identity and the orthogonality condition for the uncoupled edgewise modes used in the derivation are

$$\left[ \overline{EI}_z \gamma_{v_m}'' \right]'' + \gamma_{v_m}' \bar{m} (\bar{e} + \bar{r}) - \gamma_{v_m}'' \int_{\bar{r}}^{\bar{r}_T} \bar{m} (\bar{e} + \bar{r}) \, d\bar{\xi} - \bar{m} \gamma_{v_m} = \bar{m} \gamma_{v_m} \bar{\omega}_{v_m}^2 \quad (84)$$

$$\sum_m \int_0^{\bar{r}_T} \bar{m} \gamma_{v_m} \gamma_{v_p} \, d\bar{r} = \int_0^{\bar{r}_T} \bar{m} \gamma_{v_p}^2 \, d\bar{r} \quad (85)$$

The resulting edgewise modal equation is (with the standardized i , j , p modal subscripts)

$$0 = \int_0^{\bar{r}_T} \gamma_{v_p} \left[ (\bar{S}_A)_{y_5} \cos\Theta + (\bar{S}_A)_{z_5} \sin\Theta \right] d\bar{r} - (\gamma_{v_p}')_{\bar{r}=0} \bar{M}_{PR} \sin\theta_{PR}$$

$$-(\gamma_{v_p}')_{\bar{r}=0} \sin\theta_{FD} \bar{M}_{FD} - (\gamma_{v_p}')_{\bar{r}=0} \cos\theta_{FD} \bar{M}_{LD} - C_{14_p} \left[ \overset{xx}{q}_{v_p} + q_{v_p} \left\langle \bar{\omega}_{v_p}^2 \right. \right.$$

$$
\begin{aligned}
&- \overset{x}{\theta}_0{}^2 - 2\beta\overset{x}{\theta}_0 - \overset{x}{\beta}\sin 2\theta_0 - \frac{1}{2}(1+2\overset{x}{\delta})(1+\cos 2\theta_0) + 1\Big\rangle \Big] \\
&+ \sum_{p'} q_{v_{p'}}\Big[ -C_{34_{p,p'}} \sin 2\theta_0 + 2\overset{x}{\delta}\,(C_{20}-C_{18})_{p,p'} - 2\overset{x}{\theta}_0 \sin\theta_0 (C_{77}+C_{51}+C_{98})_{p,p'} \\
&+ 2\theta_1\, C_{21_{p,p'}} (1+\bar{\omega}_{v_p}{}^2)\Big] + \sum_{p'} \overset{x}{q}_{v_{p'}}\, 2\cos\theta_0\, (C_{51}+C_{96}+2C_{79})_{p,p'} \\
&-2 \sum_i \sum_j q_{\theta_j} q_{w_i} \bar{\omega}_{w_i}^2\, C_{75_{i,p,j}} + \sum_i q_{w_i}\Big[ C_{26_{i,p}} - 2\theta_1 \bar{\omega}_{w_i}^2 C_{22_{i,p}} \\
&- C_{3_{i,p}} \Big\langle -\overset{xx}{\theta}_0 - \overset{x}{\beta}\,(1+\cos 2\theta_0) + \frac{1}{2}(1+2\overset{x}{\delta})\sin 2\theta_0 \Big\rangle + \theta_1 C_{5_{i,p}} \\
&- \theta_1 \cos 2\theta_0 (C_5 - 0.75C_3)_{i,p} - 2\overset{x}{\theta}_0 \cos\theta_0 (C_{53}+C_{99}+C_{81})_{i,p} \Big] \\
&+ 2\sum_i \overset{x}{q}_{w_i}\Big[ C_{3_{i,p}} (\overset{x}{\theta}_0 + \beta) - \sin\theta_0 (C_{53}+C_{99}+C_{81})_{i,p}\Big] + 2\overset{x}{\theta}_0 \sum_j \overset{x}{q}_{\theta_j} C_{74_{p,j}} \\
&+ \sum_j q_{\theta_j}\Big[ -C_{74_{p,j}} \sin 2\theta_0 - C_{100_{p,j}} \Big\langle (\overset{xx}{\beta}+\beta)\cos\theta_0 - \overset{xx}{\delta}\sin\theta_0 \Big\rangle\Big] - C_{15_p}\Big[\bar{e}(\beta\sin\theta_0 \\
&+ \delta\cos\theta_0) + \bar{g}_{z_5}\sin\theta_0 + \bar{g}_{y_5}\cos\theta_0\Big] - C_{47_p}\Big[(\overset{xx}{\beta}+\beta+2\beta\overset{x}{\delta})\sin\theta_0 \\
&+ (\overset{xx}{\delta} - 2\beta\overset{x}{\beta})\cos\theta_0\Big] - C_{33_p}\Big[(\overset{xx}{\beta}+\beta)\cos\theta_0 - \overset{xx}{\delta}\sin\theta_0\Big] + C_{70_p}\Big[\overset{x}{\theta}_0{}^2 + 2\beta\overset{x}{\theta}_0 \\
&+ \overset{x}{\beta}\sin 2\theta_0 + \frac{1}{2}(1+2\overset{x}{\delta})(1+\cos 2\theta_0)\Big] - \sin 2\theta_0 C_{76_p} + \bar{e}(C_{50}+C_{72} \\
&+ C_{23})_p + (1+2\overset{x}{\delta})(C_{52}+C_{73}+C_{70}+C_{24})_p - 2\overset{x}{\theta}_0 \sin\theta_0 C_{91_p} \qquad (86)
\end{aligned}
$$

The modal integration constants are defined in Appendix IV.

## Torsional Modal Equation

The torsional modal equation is derived as follows: First, Eqs. (71) through (73), (147), and (148) are substituted into Eq. (70) and $\bar{k}_{y_{10}}{}^2$ is neglected compared to $\bar{k}_{z_{10}}{}^2$ (a valid assumption for blades having thin airfoil-type cross sections). Next, the torsional free vibration identity for uncoupled vibration,

$$
\Big[ \overline{GJ}\,\gamma_{\theta_k}{}' + \gamma_{\theta_k}{}' \bar{k}_A{}^2 \int_{\bar{r}}^{\bar{r}_T} \bar{m}(\bar{e}+\bar{\xi})\,d\bar{\xi} \Big]' - \bar{m}\bar{k}_{z_{10}}{}^2 \gamma_{\theta_k} = -\bar{m}\bar{k}_{z_{10}}{}^2 \gamma_{\theta_k}\, \bar{\omega}_{\theta_k}{}^2 \qquad (87)
$$

and the shear force definitions of Appendix I are substituted into the resulting equation. The final torsional modal equation is then obtained by multiplying the torsional loading equation (as modified by the above operations) by $\Omega_{\theta_j}$ , by integrating over the blade in the $\bar{r}$ direction, and by applying the orthogonality condition for the assumed uncoupled torsional vibratory modes, namely,

$$\sum_k \int_0^{\bar{r}_T} \bar{m}\bar{k}_{z_{10}}^2 \gamma_{\theta_j} \gamma_{\theta_k} \, d\bar{r} = \int_0^{\bar{r}_T} \bar{m}\bar{k}_{z_{10}}^2 \gamma_{\theta_j}^2 \, d\bar{r} \tag{88}$$

The resulting torsional modal equation is (with standardized modal subscript notation being used)

$$\begin{aligned}
0 = {} & \int_0^{\bar{r}_T} \gamma_{\theta_j} (\bar{m}_A)_{x_5} d\bar{r} + C_{37_j} + \cos 2\theta_0 C_{80_j} - \sum_i \overset{xx}{q}_{w_i} C_{62_{i,j}} - C_{36_j} \overset{xx}{q}_{\theta_j} \\
& + C_{36_j} q_{\theta_j} (1 - \bar{\omega}_{\theta_j}^2 - \cos 2\theta_0) - C_{44_j} \left[ \overset{xx}{\theta}_0 + \frac{1}{2} (1 + 2\overset{x}{\delta}) \sin 2\theta_0 + \overset{x}{\beta} (1 - \cos 2\theta_0) \right] \\
& - \sum_i \sum_p q_{w_i} q_{v_p} C_{46_{i,p,j}} + \theta_1 \sum_i \sum_{i'} q_{w_i} q_{w_{i'}} C_{38_{i,i',j}} + \theta_1 \sum_p \sum_{p'} q_{v_p} q_{v_{p'}} C_{39_{p,p',j}} \\
& + \sum_i q_{w_i} \Big[ 2\overset{x}{\theta}_0 \sin\theta_0 C_{110_{i,j}} - C_{86_{i,j}} - 2\overset{x}{\delta} C_{88_{i,j}} - \bar{e} C_{83_{i,j}} - (1 + 2\overset{x}{\delta}) C_{84_{i,j}} - C_{62_{i,j}} \langle -\overset{x}{\theta}_0^{\,2} \\
& - 2\beta\overset{x}{\theta}_0 + \overset{x}{\beta} \sin 2\theta_0 - \frac{1}{2} (1 + 2\overset{x}{\delta})(1 - \cos 2\theta_0) \rangle + \sin 2\theta_0 C_{95_{i,j}} \Big] \\
& - \sum_p \overset{x}{q}_{v_p} \left[ 2\sin\theta_0 C_{82_{p,j}} + 2C_{74_{p,j}} (\overset{x}{\theta}_0 + \beta) \right] + \sum_p q_{v_p} \Big[ -2\theta_1 C_{87_{p,j}} - \theta_1 C_{85_{p,j}} - C_{74_{p,j}} \langle \overset{xx}{\theta}_0 \\
& + \overset{x}{\beta} (1 - \cos 2\theta_0) + \frac{1}{2} (1 + 2\overset{x}{\delta}) \sin 2\theta_0 \rangle - \cos 2\theta_0 C_{114_{p,j}} \Big] \\
& - \sum_i \sum_p 2C_{111_{i,p,j}} q_{w_i} (\overset{x}{q}_{v_p} \cos\theta_0 - q_{v_p} \overset{x}{\theta}_0 \sin\theta_0) + \sum_i \sum_{i'} 2C_{112_{i,i',j}} q_{w_i} (\overset{x}{q}_{w_{i'}} \sin\theta_0 \\
& - q_{w_{i'}} \overset{x}{\theta}_0 \cos\theta_0) - C_{92_j} \left[ \bar{e} (\beta\cos\theta_0 - \delta\sin\theta_0) + \bar{g}_{z_5} \cos\theta_0 - \bar{g}_{y_5} \sin\theta_0 \right] \\
& - C_{93_j} \left[ (\overset{xx}{\beta} + \beta + 2\beta\overset{x}{\delta}) \cos\theta_0 - (\overset{xx}{\delta} - 2\beta\overset{x}{\beta}) \sin\theta_0 \right] + C_{94_j} \Big[ (\overset{xx}{\beta} + \beta) \sin\theta_0 \\
& + \overset{xx}{\delta} \cos\theta_0 \Big] + \sum_{j'} q_{\theta_{j'}} C_{113_{j,j'}} \left[ (\overset{xx}{\beta} + \beta) \sin\theta_0 - \overset{xx}{\delta} \cos\theta_0 \right]
\end{aligned} \tag{89}$$

## EQUATIONS OF MOTION FOR BLADE FLAP ANGLE AND LEAD ANGLE DEGREES OF FREEDOM

### FLAP ANGLE EQUATION

The flap angle equation can be obtained directly from Eq. (98) by setting the $\bar{r}$ integration limit to zero to obtain the moment about the flap hinge. This, in turn, is equated to zero. With the elimination of $\bar{r}$ items in the integrand, $\bar{\xi}$ (the dummy variable of integration in the $\bar{r}$ direction) can be replaced by $\bar{r}$ to yield the following equation:

$$0 = \int_0^{\bar{r}_T} \Big\{ (\bar{m}_A)_{y_5} - \bar{S}_{z_5}\bar{r} + \bar{S}_{x_5}\bar{z}_{5_{ea}} + \bar{y}_{10_{cg}} \sin\Theta\, (\bar{S}_D)_{x_5, y_{10_{cg}}=0}$$
$$- 2\,\overset{x}{\theta}\,\bar{m}\,(\bar{k}_{y_{10}}^{2}\cos^2\theta + \bar{k}_{z_{10}}^{2}\sin^2\theta) \Big\}\, d\bar{r} \qquad (90)$$

Substituting the shear force and moment expressions from Appendixes I through III, and the modal expansion equations (Eqs. (71) through (73)), neglecting higher order terms in accordance with previously defined assumptions, and noting that the uncoupled flatwise and edgewise mode shapes are, for small hinge offset, essentially orthogonal with the function $\bar{r}$, results in the following desired flap angle equation of motion:

$$0 = -\bar{M}_{FD} + \int_0^{\bar{r}_T} (\bar{S}_A)_{z_5}\bar{r}\, d\bar{r} - \bar{M}_B\bar{r}_{cg}(\bar{e}\beta + \bar{g}_{z_5}) - \bar{I}_B\Big[\overset{xx}{\beta} + \beta(1 + 2\overset{x}{\delta})\Big]$$
$$- C_{97}\Big[\overset{xx}{\theta}_0\cos\theta_0 - \overset{x}{\theta}{}_0^{2}\sin\theta_0 - 2\beta\overset{x}{\theta}_0\sin\theta_0 - (1 + 2\overset{x}{\delta})\sin\theta_0\Big]$$
$$- \tan\delta_3(\overline{GJ})_{\bar{r}=0}\sum_j (\gamma'_{\theta_j})_{\bar{r}=0}\, q_{\theta_j} - \bar{e}\Big[\sum_i q_{w_i} C_{10_i}\cos\theta_0$$
$$+ \sum_p q_{v_p} C_{15_p}\sin\theta_0 + \sin\theta_0 C_{96}\Big] \qquad (91)$$

where

$$\bar{M}_B\,\bar{r}_{cg} = \int_0^{\bar{r}_T} \bar{m}\bar{r}\, d\bar{r} \qquad (92)$$

$$\bar{I}_B = \int_0^{\bar{r}_T} \bar{m}\bar{r}^2 d\bar{r} \qquad (93)$$

LEAD ANGLE EQUATION

The lead angle equation is obtained by setting moments about the $z_4$ axis to zero. This condition, expressed in terms of "5" axis moment components, is

$$(M_{z_4})_{\bar r=0} = 0 = (M_{z_5})_{\bar r=0}(1-\beta^2/2) + (M_{x_5})_{\bar r=0}\beta \tag{94}$$

Neglecting the latter term as higher order (because of the small magnitude of $M_{x_5}$ and using Eq. (105) to evaluate the $z_5$ moment at the root of the blade yields

$$0 = (1-\beta^2/2)\left\{\int_0^{\bar r_T}\left[(\bar m_A)_{z_5} + \bar S_{y_5}\bar r - \bar S_{x_5}\bar y_{5_{ea}} - \bar y_{10_{cg}}\cos\Theta\,(\bar S_D)_{x_5,\, y_{10_{cg}}=0} + \dot\theta\,\bar m\sin 2\theta\,(\bar k_{z_{10}}^2 - \bar k_{y_{10}}^2)\right]d\bar r\right\} \tag{95}$$

With a procedure analogous to that followed in developing Eq. (91) the following lead angle equation is obtained:

$$\begin{aligned}0 = {} & -\bar M_{LD} + (1-\beta^2/2)\int_0^{\bar r_T}\bar r(\bar S_A)_{y_5}\,d\bar r - (\bar e\delta + \bar g_{y_5})\bar M_B\,\bar r_{cg} - \bar I_B(\ddot\delta - 2\beta\dot\beta) \\ & + C_{97}(\ddot\theta_0\sin\theta_0 + \dot\theta_0^2\cos\theta_0 + 2\beta\dot\theta_0\cos\theta_0 + 2\dot\beta\sin\theta_0) - \bar e\Big[\cos\theta_0\sum_p q_{v_p}C_{15_p} \\ & - \sin\theta_0\sum_i q_{w_i}C_{10_i} + \cos\theta_0 C_{96}\Big]\end{aligned} \tag{96}$$

## SUMMARY OF PRINCIPAL EQUATIONS

In view of the length of the analysis described in this report, a summary of the principal equations is given here.

The blade equations of motion are as follows:

Flap Angle - Eq. (91)

Lead Angle - Eq. (96)

$i^{th}$ Flatwise Bending Mode - Eq. (80)

$p^{th}$ Edgewise Bending Mode - Eq. (86)

$j^{th}$ Torsional Mode - Eq. (89)

The above equations of motion are given in terms of section aerodynamic forces and moments, flap damper, lag damper and pushrod moments, and modal natural frequencies and integration constants. The section aerodynamic forces and moments are given by Eqs. (141), (142), and (146); the damper and pushrod moments are given by Eqs. (81) through (83); and the modal integration constants are defined in Appendix IV. The required uncoupled blade mode shapes and frequencies can be determined from an appropriate eigenvalue-eigenvector analysis.

## CONCLUDING REMARKS

The differential equations outlined in this volume are an extension of an existing Sikorsky Aircraft advanced method for the prediction of rotor loads, stresses, and performance. In most cases, these predictions will be good approximations of actual rotor behavior. In some instances, however, unsteady aerodynamic effects will be an important factor. Therefore, the task of including unsteady aerodynamic effects in the Normal Mode Transient Analysis should receive further attention.

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## APPENDIX I
## AERODYNAMIC AND DYNAMIC MOMENTS AND DERIVATIVES IN THE RIGID BLADE AXIS SYSTEM

The expression for the acceleration vector developed previously (Eq. (38)) is employed herein, along with the force and moment diagrams presented in Figures 2 through 4, to determine expressions for the total aerodynamic and dynamic moments acting at a given blade station. These moments, for convenience, are expressed with reference to the Rigid Blade Axis System ("5" axis system).

### MOMENT ABOUT $y_5$ AXIS

The moment at a blade section at $r$ about an axis which is parallel to the $y_5$ axis and which passes through the elastic axis of the section can be obtained by integrating the elemental moment contributions due to the aerodynamic and dynamic forces acting on all sections of the blade outboard of the section under consideration. Thus, in Figure 2, the desired moment at station $r$ is

$$M_{y_5} = (M_A)_{y_5} + (M_D)_{y_5} = \int_r^{r_T} \Big\{ (m_A)_{y_5} - \Big[ (S_A)_{z_5} + (S_{LD})_{z_5} + (S_{FD})_{z_5} + (S_{PR})_{z_5} \Big] (x_{5_{ea}} - x_{5_{ea}}(r)) + (\bar{S}_A)_{x_5} (z_{5_{ea}} - z_{5_{ea}}(r)) \Big\} d\xi$$

$$+ \int_r^{r_T}\int_c \Big\{ a_{z_5}\big[x_5 - x_{5_{ea}}(r)\big] - a_{x_5}\big[z_5 - z_{5_{ea}}(r)\big] \Big\} dm\, d\xi \qquad (97)$$

where $(S_A)_{z_5}$ and $(S_A)_{x_5}$ are the aerodynamic forces per unit span in the $z_5$ and $x_5$ directions, (acting at the elastic axis) respectively, and $(m_A)_{y_5}$ is the $y_5$ component of the aerodynamic moment per unit span. The terms $(S_{FD})_{z_5}$ , $(S_{LD})_{z_5}$ , $(S_{PR})_{z_5}$ represent the shear forces per unit span in the $z_5$ direction introduced by a flap damper, a lag damper, and the control system pushrod. The precise nature of these forces is described on p. 49 . All quantities on the right-hand side of Eq. (97) are functions of the dummy spanwise variable, $\xi$ , unless explicitly indicated to be a function of $r$ (i.e., the radial location of the blade section for which the moment is being determined). Also, the differential, $dm$ , denotes the local blade mass per unit area while the integral subscripted $c$ symbolically indicates integration over the chord of the blade section.

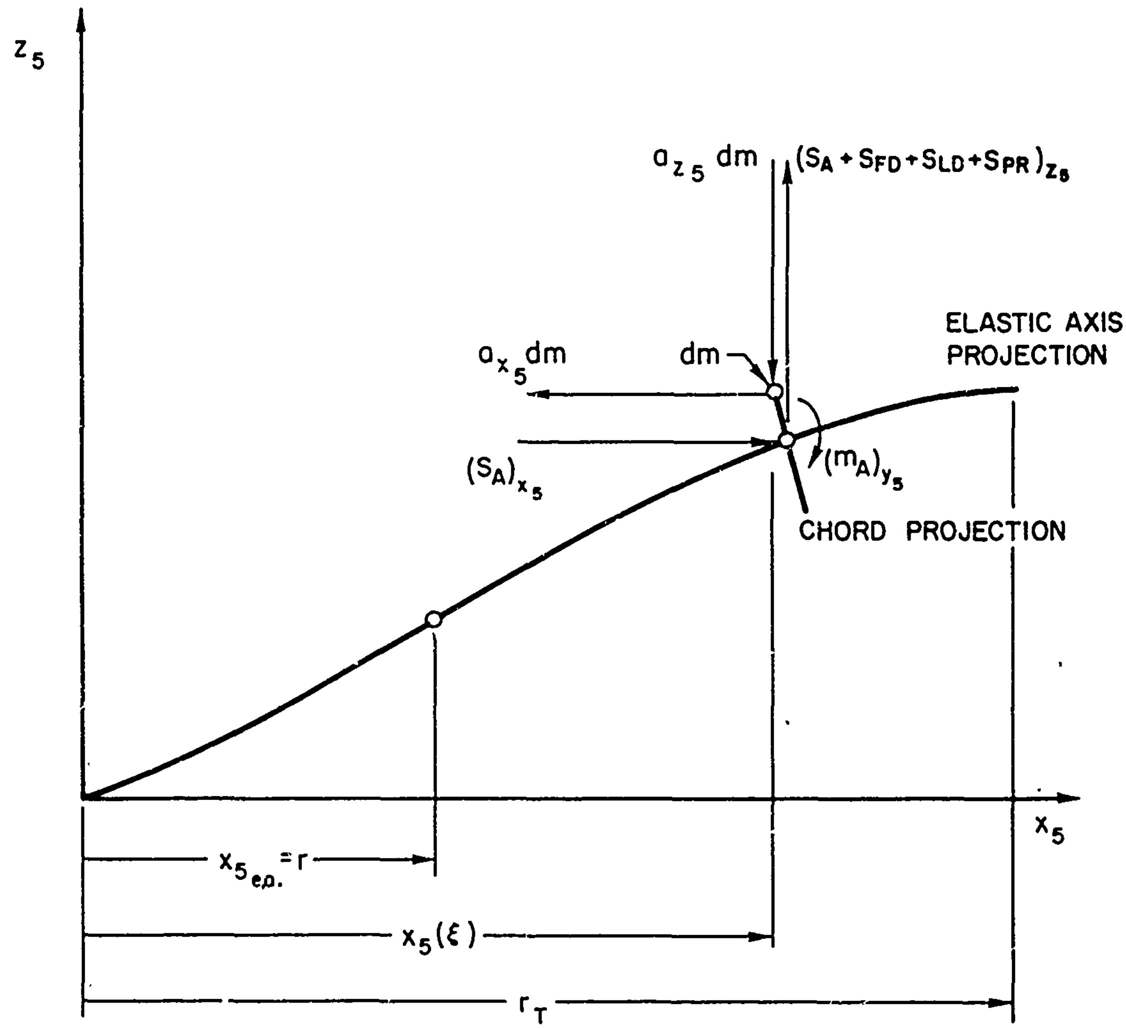


Figure 2. Section Forces and Moments - View Normal to $x_5$ - $z_5$ Plane.

Substituting Eqs. (35) through (38) into Eq. (97) and performing the integrations over the chord, yields the following:

$$
\begin{aligned}
M_{y_5} = & \int_r^{r_T} \Big\{ (m_A)_{y_5} - \left[ (S_A)_{z_5} + (S_{LD})_{z_5} + (S_{PR})_{z_5} + (S_{FD})_{z_5} \right] (x_{5_{ea}} - x_{5_{ea}}(r)) \\
& + (S_A)_{x_5} (z_{5_{ea}} - z_{5_{ea}}(r)) \Big\} d\xi + \int_r^{r_T} m(\xi - r) \Big\{ e\Omega^2 \beta + (\ddot{w}_e + y_{10_{cg}} \ddot{\theta}_e) \cos\theta \\
& - 2(\dot{w}_e + y_{10_{cg}} \dot{\theta}_e) \dot{\theta} \sin\theta - (w_e + y_{10_{cg}} \theta_e)(\ddot{\theta} \sin\theta + \mathring{\theta}^2 \cos\theta) + \ddot{v}_e \sin\theta \\
& + 2\dot{v}_e \dot{\theta} \cos\theta + (v_e + y_{10_{cg}})(\ddot{\theta} \cos\theta - \mathring{\theta}^2 \sin\theta) + 2\beta\Omega \left[ \dot{v}_e \cos\theta \right. \\
& \left. - (v_e + y_{10_{cg}}) \dot{\theta} \sin\theta - \dot{w}_e \sin\theta - w_e \mathring{\theta} \cos\theta \right] + r \left[ \ddot{\beta} + \beta(\Omega^2 + 2\mathring{\delta}\Omega) \right] \\
& + g_{z_5} \Big\} d\xi - \int_r^{r_T} m \left[ w_e \cos\theta + v_e \sin\theta - w_e(r) \cos\theta(r) \right. \\
& \left. - v_e(r) \sin\theta(r) \right] \left\{ -e\Omega^2 + 2\Omega y_{10_{cg}} \mathring{\theta} \sin\theta - r(\Omega^2 + 2\Omega\mathring{\delta}) + g_{x_5} \right\} d\xi \\
& - 2\Omega \int_r^{r_T} m k_{y_{10}}{}^2 \dot{\theta} \cos^2\theta \, d\xi + \int_r^{r_T} m y_{10_{cg}} \theta_e \cos\theta \, r \, \Omega^2 d\xi - \int_r^{r_T} m y_{10_{cg}} \sin\theta \Big\{ -e\Omega^2 \\
& - 2\Omega \left[ \mathring{v}_e \cos\theta - v_e \dot{\theta} \sin\theta - \mathring{w}_e \sin\theta - w_e \mathring{\theta} \cos\theta \right] - r(\Omega^2 + 2\Omega\dot{\delta}) + g_{x_5} \Big\} d\xi \\
& - 2\Omega \int_r^{r_T} m k_{z_{10}}{}^2 \dot{\theta} \sin^2\theta \, d\xi
\end{aligned}
\tag{98}
$$

where the following definitions apply:

| | |
|---|---|
| $\int_c dm \equiv m$ | (mass per unit length in $r$ direction) |
| $\int_c z_{10} \, dm \equiv 0$ | (i.e., blade section c.g. lies on the $y_{10}$ axis -- this is true for symmetric sections, and errors introduced for non-symmetric sections are believed to be negligible) |
| $\int_c y_{10} \, dm \equiv m (y_{10})_{cg}$ | (first moment of section mass about elastic axis) |
| $\int_c y_{10} z_{10} \, dm \equiv 0$ | (i.e., the $y_{10}$ axis is an axis of symmetry) |
| $\int_c y_{10}^2 \, dm \equiv m k_{z_{10}}^2$ | (second moment of section mass about $z_{10}$ axis) |

$$\int_c z_{10}^2 \, dm \equiv m k_{y_{10}}^2 \qquad \text{(second moment of section mass about } y_{10} \text{ axis)}$$

Equation (98), after nondimensionalization, becomes

$$\bar{M}_{y_5} = \int_{\bar{r}}^{\bar{r}_T} \Big\{ (\bar{m}_A)_{y_5} - \bar{S}_{z_5}(\bar{\xi} - \bar{r}) + \bar{S}_{x_5}(\bar{z}_{5_{ea}} - \bar{z}_{5_{ea}}(\bar{r})) + \bar{y}_{10_{cg}}(\sin\theta + \theta_e \cos\theta)(\bar{S}_D)_{x_5, y_{10_{cg}}=0} - 2\dot{\theta}\bar{m}(\bar{k}_{y_{10}}^2 \cos^2\theta + \bar{k}_{z_{10}}^2 \sin^2\theta) \Big\} d\bar{\xi} \tag{99}$$

where the following definitions have been made:

$$\bar{S}_{z_5} \equiv (\bar{S}_A)_{z_5} + (\bar{S}_{FD})_{z_5} + (\bar{S}_D)_{z_5} + (\bar{S}_{LD})_{z_5} + (\bar{S}_{PR})_{z_5} \tag{100}$$

$$\bar{S}_{x_5} = \bar{S}_{A_{x_5}} + \bar{S}_{D_{x_5}} \tag{101}$$

$$(\bar{S}_D)_{x_5} = \bar{m}\Big\{ \bar{e} + 2(\dot{\bar{v}}_e \cos\theta - \bar{v}_e \dot{\theta}\sin\theta - \dot{\bar{w}}_e \sin\theta - \bar{w}_e \dot{\theta}\cos\theta) + \bar{r}(1 + 2\dot{\delta}) - \bar{g}_{x_5} - 2\bar{y}_{10_{cg}}\dot{\theta}\sin\theta \Big\} \tag{102}$$

$$\begin{aligned}(\bar{S}_D)_{z_5} = -\bar{m}\Big\{ & \bar{e}\beta + (\ddot{\bar{w}}_e + \bar{y}_{10_{cg}}\ddot{\theta}_e)\cos\theta - 2(\dot{\bar{w}}_e + \bar{y}_{10_{cg}}\dot{\theta}_e)\dot{\theta}\sin\theta \\ & -(\bar{w}_e + \bar{y}_{10_{cg}}\theta_e)(\ddot{\theta}\sin\theta + \dot{\theta}^2\cos\theta) + \ddot{\bar{v}}_e \sin\theta + 2\dot{\bar{v}}_e\dot{\theta}\cos\theta \\ & +(\bar{v}_e + \bar{y}_{10_{cg}})(\ddot{\theta}\cos\theta - \dot{\theta}^2\sin\theta) + 2\beta\left[\dot{\bar{v}}_e\cos\theta - (\bar{v}_e + \bar{y}_{10_{cg}})\dot{\theta}\sin\theta \right. \\ & \left. - \dot{\bar{w}}_e \sin\theta - \bar{w}_e\dot{\theta}\cos\theta\right] + \bar{r}\left[\ddot{\beta} + \beta(1 + 2\dot{\delta})\right] + \bar{g}_{z_5} \Big\}\end{aligned} \tag{103}$$

$$\bar{z}_{5_{ea}} - \bar{z}_{5_{ea}}(\bar{r}) = \bar{w}_e \cos\theta + \bar{v}_e \sin\theta - \bar{w}_e(\bar{r})\cos\theta(\bar{r}) - \bar{v}_e(\bar{r})\sin\theta(\bar{r}) \tag{104}$$

When $(\bar{S}_D)_{x_5}$ is defined as in Eq. (102), it is recognized that upon substitution of Eq. (102) into Eq. (99) certain products will be neglected as higher order in accordance with assumption 10.

MOMENTS ABOUT $z_5$ AND $x_5$ AXES

If a procedure is followed that is exactly analogous to that followed in developing the $y_5$ moment and if Figures 3 and 4 are utilized, the following expressions for the moments about section axes parallel to the $z_5$ and $x_5$ axes can be obtained:

$$\bar{M}_{z_5} = (\bar{M}_A)_{z_5} + (\bar{M}_D)_{z_5} = \int_{\bar{r}}^{\bar{r}_T} \Big\{ (\bar{m}_A)_{z_5} + \bar{S}_{y_5}(\bar{\xi} - \bar{r}) - \bar{S}_{x_5}(\bar{y}_{5_{eo}} - \bar{y}_{5_{eo}}(\bar{r}))$$

$$- \bar{y}_{10_{cg}} (\cos\theta - \theta_e \sin\theta)(\bar{S}_D)_{x_5, \bar{y}_{10_{cg}}=0} + \overset{x}{\theta}\,\bar{m} \sin 2\theta (\bar{k}_{z_{10}}^{\,2} - \bar{k}_{y_{10}}^{\,2}) \Big\} d\xi \tag{105}$$

$$\bar{M}_{x_5} = (\bar{M}_A)_{x_5} + (\bar{M}_D)_{x_5} = \int_{\bar{r}}^{\bar{r}_T} \Big\{ (\bar{m}_A)_{x_5} + \bar{S}_{z_5}(\bar{y}_{5_{eo}} - \bar{y}_{5_{eo}}(\bar{r})) - \bar{S}_{y_5}(\bar{z}_{5_{eo}} - \bar{z}_{5_{eo}}(\bar{r}))$$

$$+ \bar{y}_{10_{cg}} \Big[ (\bar{S}_D)_{z_5, y_{10_{cg}}=0} (\cos\theta - \theta_e \sin\theta) - (\bar{S}_D)_{y_5, y_{10_{cg}}=0} (\sin\theta + \theta_e \cos\theta) \Big]$$

$$+ \bar{m}(\bar{k}_{z_{10}}^2 + \bar{k}_{y_{10}}^2)(-\overset{xx}{\theta} - \overset{xx}{\theta}_e - \overset{x}{\beta}) + \bar{m}(\bar{k}_{y_{10}}^2 - \bar{k}_{z_{10}}^2)\Big[\tfrac{1}{2}(1 + 2\overset{x}{\delta})\sin 2\theta$$

$$+ (\theta_e - \overset{x}{\beta})\cos 2\theta\Big] - 2\bar{m}\Big[\bar{k}_{z_{10}}^2 \overset{x'}{\bar{v}}_e \sin\theta + \bar{k}_{y_{10}}^2 \overset{x'}{\bar{w}}_e \cos\theta\Big] \Big\} d\bar{\xi} \tag{106}$$

where

$$\bar{S}_{y_5} \equiv (\bar{S}_A)_{y_5} + (\bar{S}_D)_{y_5} + (\bar{S}_{LD})_{y_5} + (\bar{S}_{FD})_{y_5} + (\bar{S}_{PR})_{y_5} \tag{107}$$

$$(\bar{S}_D)_{y_5} = -\bar{m}\Big\{ \bar{e}\delta + \overset{xx}{\bar{v}}_e \cos\theta - 2\overset{x}{\bar{v}}_e \overset{x}{\theta} \sin\theta - (\bar{v}_e + \bar{y}_{10_{cg}})(\overset{xx}{\theta}\sin\theta + \overset{x^2}{\theta}\cos\theta)$$

$$- (\overset{xx}{\bar{w}}_e + \bar{y}_{10_{cg}} \overset{xx}{\theta}_e)\sin\theta - 2(\overset{x}{\bar{w}}_e + \bar{y}_{10_{cg}} \overset{x}{\theta}_e)\overset{x}{\theta}\cos\theta - (\bar{w}_e + \bar{y}_{10_{cg}}\theta_e)(\overset{xx}{\theta}\cos\theta - \overset{x^2}{\theta}\sin\theta)$$

$$-2\beta\Big[\overset{x}{\bar{w}}_e \cos\theta - \bar{w}_e \overset{x}{\theta}\sin\theta + \overset{x}{\bar{v}}_e \sin\theta + (\bar{v}_e + \bar{y}_{10_{cg}})\overset{x}{\theta}\cos\theta\Big] + \bar{r}(\overset{xx}{\delta} - 2\beta\overset{x}{\beta})$$

$$-2\bar{y}_{10_{cg}} \overset{x'}{\bar{v}}_e - 2\overset{x}{\beta}\Big[\bar{w}_e \cos\theta + (\bar{v}_e + \bar{y}_{10_{cg}})\sin\theta\Big] - (1 + 2\overset{x}{\delta})\Big[(\bar{v}_e + \bar{y}_{10_{cg}})\cos\theta$$

$$- \bar{w}_e \sin\theta\Big] + \bar{y}_{10_{cg}} \theta_e \sin\theta \div \bar{g}_{y_5}\Big\} \tag{108}$$

$$\bar{y}_{5_{eo}} - \bar{y}_{5_{eo}}(\bar{r}) = \bar{v}_e \cos\theta - \bar{w}_e \sin\theta - \bar{v}_e(\bar{r})\cos\theta(\bar{r}) + \bar{w}_e(\bar{r})\sin\theta(\bar{r}) \tag{109}$$

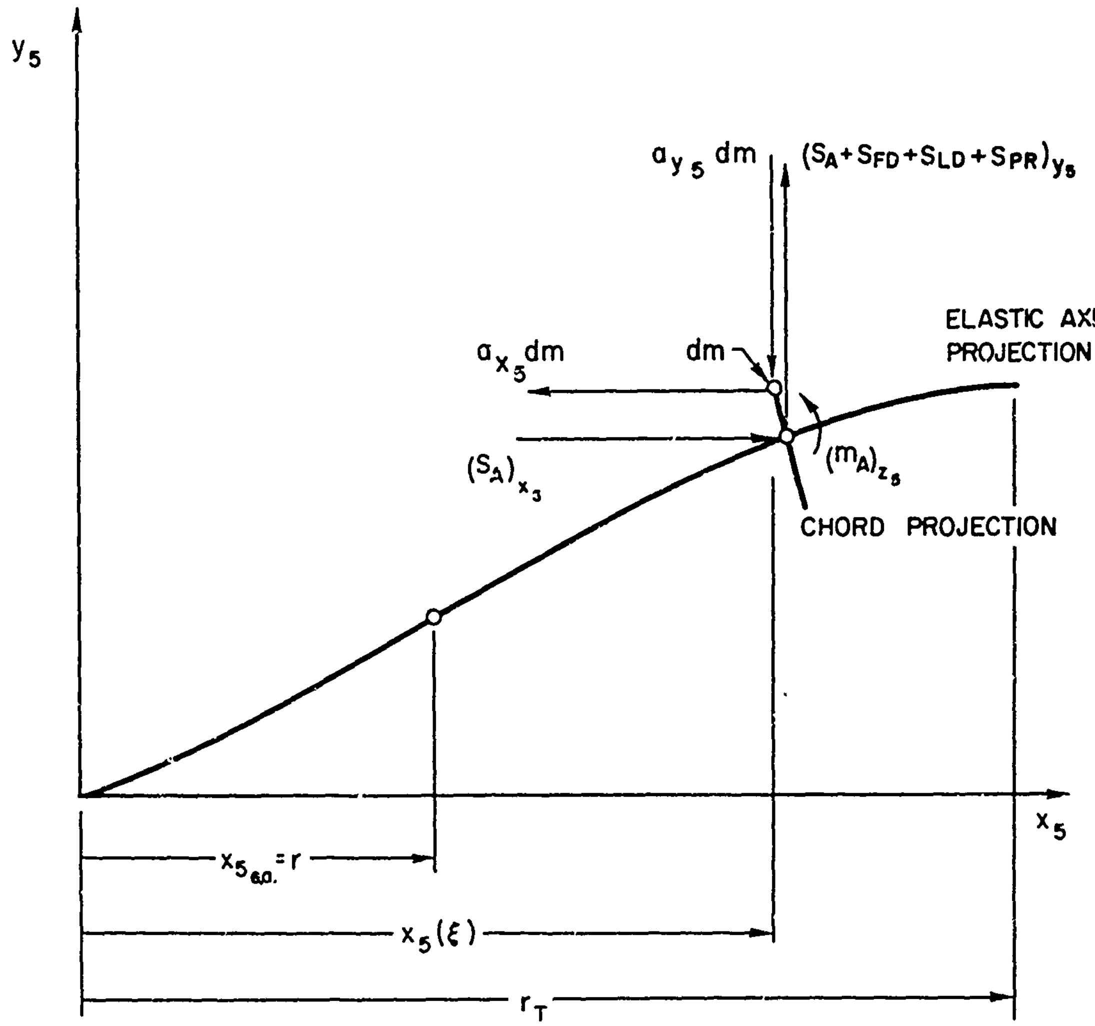


Figure 3. Section Forces and Moments – View
Normal to $x_5$ - $y_5$ Plane.

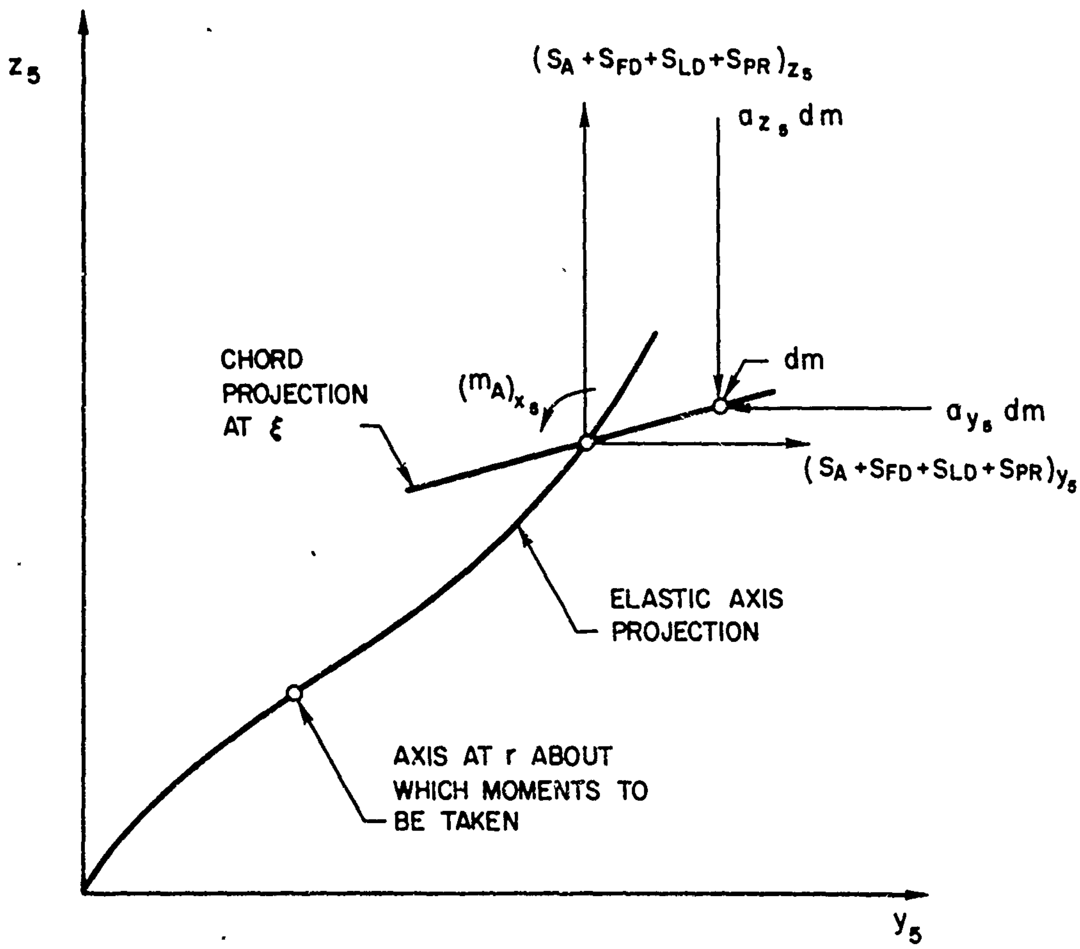


Figure 4. Section Forces and Moments - View Normal to $y_5$ - $z_5$ Plane.

In Eq. (107), $(\bar{S}_{LD})_{y_5}$ , $(\bar{S}_{FD})_{y_5}$ , and $(\bar{S}_{PR})_{y_5}$ represent $y_5$ components of the lag damper, flap damper, and pushrod forces, respectively. These are given in Appendix III.

SPANWISE DERIVATIVES OF MOMENTS

The following expressions, which represent the first and second derivatives with respect to $\bar{r}$ of Eqs. (99) and (105) and the corresponding first derivative of Eq. (106), are also required in the derivation of the final modal equations of motion:

$$\bar{M}_{y_5}' = \int_{\bar{r}}^{\bar{r}_T} \left\{ \bar{S}_{z_5} - \bar{z}_{5_{ea}}'(\bar{r})\,\bar{S}_{x_5} \right\} d\bar{\xi} - \bar{y}_{10_{cg}} \sin\Theta\,(\bar{S}_D)_{x_5, y_{10_{cg}}=0} - (\bar{m}_A)_{y_5} + 2\overset{x}{\theta}\,\bar{m}\left(\bar{k}_{y_{10}}^2 \cos^2\theta + \bar{k}_{z_{10}}^2 \sin^2\theta\right) \tag{110}$$

$$\bar{M}_{y_5}'' = -\bar{S}_{z_5} - \bar{z}_{5_{ea}}'' \int_{\bar{r}}^{\bar{r}_T} (\bar{S}_D)_{x_5}\, d\bar{\xi} + \bar{z}_{5_{ea}}' (\bar{S}_D)_{x_5} - \Big[ \bar{y}_{10_{cg}} \sin\Theta\,(\bar{S}_D)_{x_5, y_{10_{cg}}=0} + (\bar{m}_A)_{y_5} - 2\overset{x}{\theta}\,\bar{m}\left(\bar{k}_{y_{10}}^2 \cos^2\theta + \bar{k}_{z_{10}}^2 \sin^2\theta\right)\Big]' \tag{111}$$

$$\bar{M}_{z_5}' = \int_{\bar{r}}^{\bar{r}_T} \left\{ -\bar{S}_{y_5} + \bar{y}_{5_{ea}}'(\bar{r})\,\bar{S}_{x_5} \right\} d\bar{\xi} + \bar{y}_{10_{cg}} \cos\Theta\,(\bar{S}_D)_{x_5, y_{10_{cg}}=0} - (\bar{m}_A)_{z_5} - \bar{m}\overset{x}{\theta}\sin 2\theta\left(\bar{k}_{z_{10}}^2 - \bar{k}_{y_{10}}^2\right) \tag{112}$$

$$\bar{M}_{z_5}'' = \bar{S}_{y_5} - \bar{y}_{5_{ea}}' \bar{S}_{x_5} + \bar{y}_{5_{ea}}'' \int_{\bar{r}}^{\bar{r}_T} \bar{S}_{x_5} d\bar{\xi} + \Big[ \bar{y}_{10_{cg}} \cos\Theta\,(\bar{S}_D)_{x_5, y_{10_{cg}}=0} - (\bar{m}_A)_{z_5} - \bar{m}\overset{x}{\theta}\sin 2\theta\left(\bar{k}_{z_{10}}^2 - \bar{k}_{y_{10}}^2\right)\Big]' \tag{113}$$

$$\begin{aligned}\bar{M}_{x_5}' = & -\bar{y}_{5_{ea}}' \int_{\bar{r}}^{\bar{r}_T} \bar{S}_{z_5} d\bar{\xi} + \bar{z}_{5_{ea}}' \int_{\bar{r}}^{\bar{r}_T} \bar{S}_{y_5} d\bar{\xi} - (\bar{m}_A)_{x_5} - \bar{y}_{10_{cg}} \Big[ (\bar{S}_D)_{z_5} \cos\Theta \\ & - (\bar{S}_D)_{y_5} \sin\Theta \Big]_{y_{10_{cg}}=0} + \bar{m}\left(\bar{k}_{z_{10}}^2 + \bar{k}_{y_{10}}^2\right)\left(\overset{xx}{\theta} + \overset{xx}{\theta_e} + \overset{x}{\beta}\right) - \bar{m}\Big(\bar{k}_{y_{10}}^2 \\ & - \bar{k}_{z_{10}}^2\Big)\left[\frac{1}{2}\left(1 + 2\overset{x}{\delta}\right)\sin 2\theta + \left(\theta_e - \overset{x}{\beta}\right)\cos 2\theta\right] + 2\bar{m}\,\bar{k}_{z_{10}}^2 \overset{x}{v_e}{}' \sin\theta + 2\bar{m}\,\bar{k}_{y_{10}}^2 \overset{x}{w_e}{}' \cos\theta\end{aligned} \tag{114}$$

where the angle $\Theta$ is given by Eq. (45).

## APPENDIX II
## AERODYNAMIC SHEAR FORCES AND MOMENTS

To determine the aerodynamic shear forces and moments acting on the rotor blade, a blade element approach was followed that utilized the quasi-steady aerodynamic theory as developed in Reference 5. Quasi-steady theory neglects the effect of the wake generated by the blade, and its use is dictated by the lack of a generalized, variable-inflow theory applicable to rotating wings in forward flight. In the analysis herein, the apparent-mass terms appearing in quasi-steady theory have also been neglected as second order. The primary differences between the resulting quasi-steady aerodynamic expressions and those predicted by classical steady aerodynamic theory are that (1) the aerodynamic forces are determined by the velocity components at the 75% chord point and (2) an aerodynamic damping moment in pitch, $m_d$ , is predicted. The use of the blade element approach implies that radial flow effects are also neglected.

When the aerodynamic shear forces and moments are derived, reference will be made to Figure 5, which shows a cross section of the blade as well as the relative air velocity components $U_T$ and $U_P$ . The tangential velocity vector $U_T$ is parallel to the plane of rotation of the rotor and normal to the local $x_{10}$ axis while $U_P$ is normal to both the local $x_{10}$ axis and $U_T$ . $U_T$ and $U_P$ thus lie in the plane formed by the blade section axes, $y_{10}$ and $z_{10}$ . Within the assumptions of small elastic rotation of the blade section, the incidence angle $\epsilon$ is equal to the local total pitch angle $\Theta$ . The aerodynamic force and moment components in the "10" axis system are then

$$(S_A)_{z_{10}} = (\ell\cos\phi + d\sin\phi)\cos\Theta - (\ell\sin\phi - d\cos\phi)\sin\Theta \tag{115}$$

$$(S_A)_{y_{10}} = (\ell\sin\phi - d\cos\phi)\cos\Theta + (\ell\cos\phi + d\sin\phi)\sin\Theta \tag{116}$$

$$(S_A)_{x_{10}} = 0 \tag{117}$$

$$(m_A)_{z_{10}} = 0 \tag{118}$$

$$(m_A)_{y_{10}} = 0 \tag{119}$$

$$(m_A)_{x_{10}} \triangleq m_{c/4} + (\ell\cos\phi + d\sin\phi)y_{10_{c/4}}\cos\Theta - (\ell\sin\phi - d\cos\phi)y_{10_{c/4}}\sin\Theta + m_d \tag{120}$$

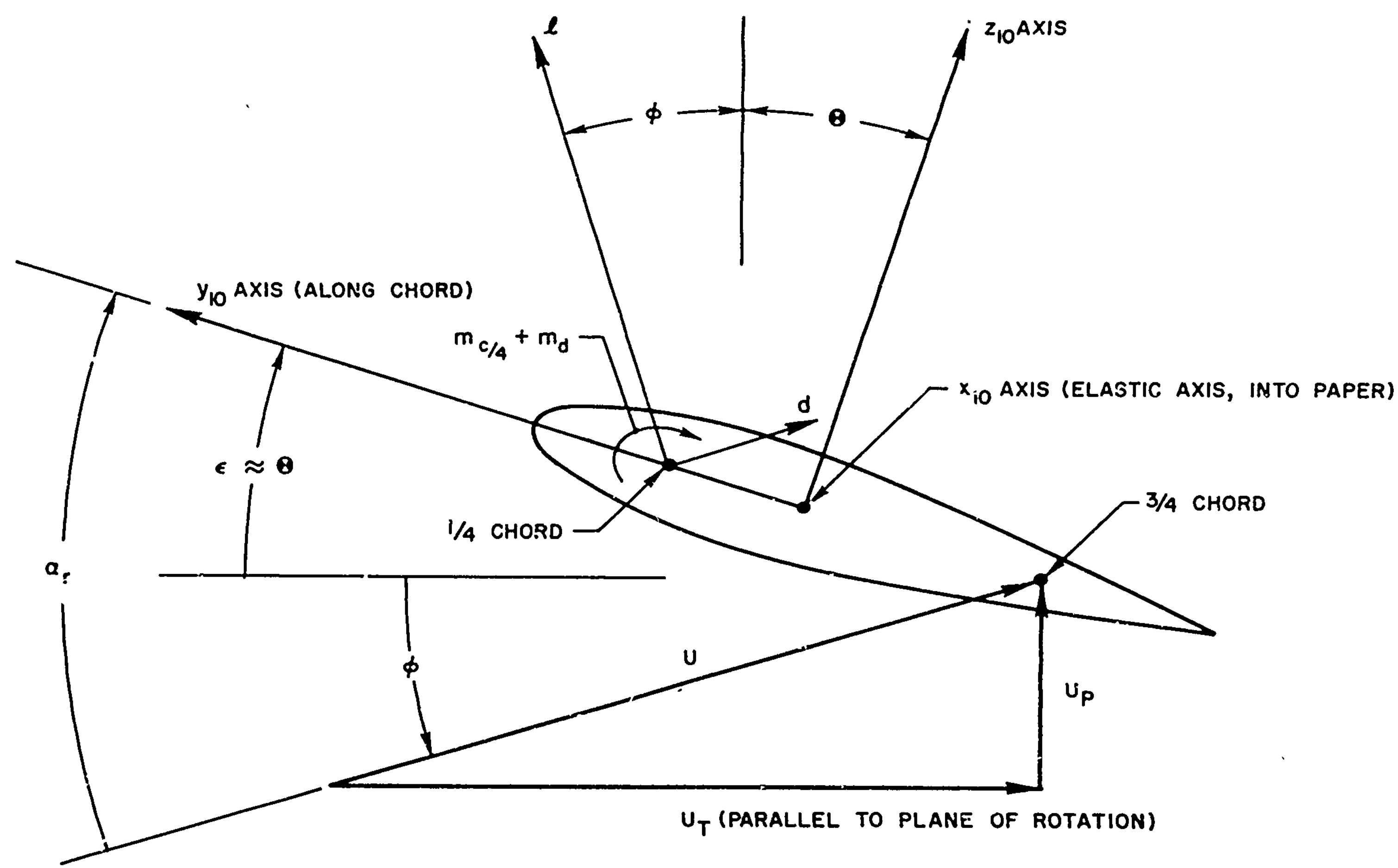


Figure 5. Blade Section Aerodynamics.

The above forces and moments can be resolved into the desired "5" axis components (using previously defined axis relationships) to obtain, after nondimensionalization, the following:

$$(\bar{S}_A)_{z_5} = \bar{l}\cos\phi + \bar{d}\sin\phi \tag{121}$$

$$(\bar{S}_A)_{y_5} = \bar{l}\sin\phi - \bar{d}\cos\phi \tag{122}$$

$$(\bar{S}_A)_{x_5} = -(\bar{l}\cos\phi + \bar{d}\sin\phi)(\lambda_1\cos\Theta + \lambda_2\sin\Theta) - (\bar{l}\sin\phi - \bar{d}\cos\phi)(-\lambda_1\sin\Theta + \lambda_2\cos\Theta) \tag{123}$$

$$(\bar{m}_A)_{z_5} = \left[\bar{m}_{c/4} + \bar{m}_d + (\bar{l}\cos\phi + \bar{d}\sin\phi)\bar{y}_{10_{c/4}}\cos\Theta - (\bar{l}\sin\phi - \bar{d}\cos\phi)\bar{y}_{10_{c/4}}\sin\Theta\right]\left[\lambda_1\cos\Theta + \lambda_2\sin\Theta\right] \tag{124}$$

$$(\bar{m}_A)_{y_5} = \left[\bar{m}_{c/4} + \bar{m}_d + (\bar{l}\cos\phi + \bar{d}\sin\phi)\bar{y}_{10_{c/4}}\cos\Theta - (\bar{l}\sin\phi - \bar{d}\cos\phi)\bar{y}_{10_{c/4}}\sin\Theta\right]\left[\lambda_2\cos\Theta - \lambda_1\sin\Theta\right] \tag{125}$$

$$(\bar{m}_A)_{x_5} = \bar{m}_{c/4} + \bar{m}_d + (\bar{l}\cos\phi + \bar{d}\sin\phi)\bar{y}_{10_{C/4}}\cos\Theta - (\bar{l}\sin\phi - \bar{d}\cos\phi)\bar{y}_{10_{C/4}}\sin\Theta \tag{126}$$

In the above, $\bar{\ell}$ is the local lift, $\bar{d}$ the local drag, $\bar{m}_{c/4}$ the local pitching moment about the 25% chord predicted by steady-state aerodynamic theory, and $\bar{m}_d$ the damping moment predicted by quasi-steady theory. With use of conventional aerodynamic notation, the following expressions are readily derived:

$$\bar{\ell} = \frac{\ell}{m_0 \Omega^2 R} = \frac{1}{2}\left(\frac{\rho R^2}{m_0}\right)\left(\frac{U}{\Omega R}\right)^2\left(\frac{c}{R}\right)c_\ell \qquad (127)$$

$$\bar{d} = \frac{d}{m_0 \Omega^2 R} = \frac{1}{2}\left(\frac{\rho R^2}{m_0}\right)\left(\frac{U}{\Omega R}\right)^2\left(\frac{c}{R}\right)c_d \qquad (128)$$

$$\bar{m}_{c/4} = \frac{m_{c/4}}{m_0 \Omega^2 R^2} = \frac{1}{2}\left(\frac{\rho R^2}{m_0}\right)\left(\frac{U}{\Omega R}\right)^2\left(\frac{c}{R}\right)^2 c_{m_{c/4}} \qquad (129)$$

$$\bar{m}_d = \frac{m_d}{m_0 \Omega^2 R^2} = -\frac{\pi}{8}\left(\frac{\rho R^2}{m_0}\right)\left(\frac{c}{R}\right)^3\left(\frac{U}{\Omega R}\right)\left(\frac{1}{2} - a_0\right)\mathring{\Theta} \qquad (130)$$

In Eq. (130) the quantity $a_0$ represents the nondimensional distance (in terms of 50% chord lengths) from the mid-chord to the pitching axis of the blade section. The quantity $a_0$ is positive when the pitch axis is downstream of the mid-chord. For a rotor blade, the downstream direction depends on whether the blade is in conventional or reverse flow. Hence, $a_0$ is

$$a_0 = \begin{cases} \frac{2}{c} x_{0_{c/4}} - \frac{1}{2} & \text{(conventional flow)} \\ \frac{1}{2} - \frac{2}{c} x_{0_{c/4}} & \text{(reverse flow)} \end{cases} \qquad (131)$$

The aerodynamic coefficients in Eqs. (127) to (130) are functions of the section angle of attack and Mach number, as determined by the velocity components at the 75% chord point. From Figure 5, the angle of attack is

$$\alpha_r = \Theta + \tan^{-1}\frac{U_P}{U_T} \tag{132}$$

The relative air velocity components are given by

$$\frac{U_T}{\Omega R} = \left(\overline{V}y_{10}\right)_{\frac{3}{4}c}\cos\Theta - \left(\overline{V}z_{10}\right)_{\frac{3}{4}c}\sin\Theta \tag{133}$$

$$\frac{U_P}{\Omega R} = -\left(\overline{V}z_{10}\right)_{\frac{3}{4}c}\cos\Theta - \left(\overline{V}y_{10}\right)_{\frac{3}{4}c}\sin\Theta \tag{134}$$

Or, expressing the "10" axis velocity components in terms of "5" axis components, Eqs. (133) and (134) are

$$\frac{U_T}{\Omega R} = \left(\overline{V}y_5\right)_{\frac{3}{4}c} + \left(\overline{V}x_5\right)_{\frac{3}{4}c}\left(\lambda_1\sin\Theta - \lambda_2\cos\Theta\right) \tag{135}$$

$$\frac{U_P}{\Omega R} = -\left(\overline{V}z_5\right)_{\frac{3}{4}c} + \left(\overline{V}x_5\right)_{\frac{3}{4}c}\left(\lambda_1\cos\Theta + \lambda_2\sin\Theta\right) \tag{136}$$

By using Eq. (29) to determine $(\overline{V}z_5)_{3/4\,c}$, etc., in terms of "1" axis velocity components, by neglecting higher order terms, and by noting that $\overline{V}_{0x_1}$ and $\overline{V}_{0z_1}$ can be written more familiarly as

$$\overline{V}_{0x_1} = -\frac{V\cos\alpha_s}{\Omega R} = -\mu \tag{137}$$

$$\overline{V}_{0z_1} = -\frac{(V\sin\alpha_s - \nu)}{\Omega R} = -\lambda_s \tag{138}$$

Eqs. (135) and (136) can be rewritten as

$$
\begin{aligned}
\frac{U_T}{\Omega R} = \; & \mu\left[\left(1-\frac{\delta^2}{2}\right)\sin\psi + \delta\cos\psi\right] + \bar{e} + \bar{r}\left(1-\frac{\beta^2}{2}+\overset{x}{\delta}\right) + \overset{x}{\bar{v}}_e\cos\theta \\
& -\left(\bar{v}_e + \bar{y}_{10_{\frac{3}{4}c}}\right)\overset{x}{\theta}\sin\theta - \left(\overset{x}{\bar{w}}_e + \bar{y}_{10_{\frac{3}{4}c}}\overset{x}{\theta}_e\right)\sin\theta - \left(\bar{w}_e + \bar{y}_{10_{\frac{3}{4}c}}\theta_e\right)\overset{x}{\theta}\cos\theta \\
& - \bar{y}_{10_{\frac{3}{4}c}}\bar{v}_e' - \beta\left[\bar{w}_e\cos\theta + \left(\bar{v}_e + \bar{y}_{10_{\frac{3}{4}c}}\right)\sin\theta\right] \\
& +\left(\bar{w}_e'\sin\theta - \bar{v}_e'\cos\theta\right)\left[-\mu(\cos\psi - \delta\sin\psi) - \lambda_s\beta - \bar{y}_{10_{\frac{3}{4}c}}\cos\theta\right] \\
& - \theta'\left[\bar{v}_e\sin\theta + \bar{w}_e\cos\theta\right]\mu\cos\psi
\end{aligned}
\tag{139}
$$

$$
\begin{aligned}
\frac{U_P}{\Omega R} = \; & \lambda_s\left(1-\frac{\beta^2}{2}\right) - \mu\beta(\cos\psi - \delta\sin\psi) - \left(\overset{x}{\bar{w}}_e + \bar{y}_{10_{\frac{3}{4}c}}\overset{x}{\theta}_e\right)\cos\theta \\
& + \left(\bar{w}_e + \bar{y}_{10_{\frac{3}{4}c}}\theta_e\right)\overset{x}{\theta}\sin\theta - \overset{x}{\bar{v}}_e\sin\theta - \left(\bar{v}_e + \bar{y}_{10_{\frac{3}{4}c}}\right)\overset{x}{\theta}\cos\theta \\
& - \beta\left[\left(\bar{v}_e + \bar{y}_{10_{\frac{3}{4}c}}\right)\cos\theta - \bar{w}_e\sin\theta\right] - \bar{r}\overset{x}{\beta} \\
& + \left(\bar{w}_e'\cos\theta + \bar{v}_e'\sin\theta\right)\left[-\mu(\cos\psi - \delta\sin\psi) - \lambda_s\beta - \bar{y}_{10_{\frac{3}{4}c}}\cos\theta\right] \\
& - \theta'\mu\cos\psi\left(\bar{v}_e\cos\theta - \bar{w}_e\sin\theta\right)
\end{aligned}
\tag{140}
$$

Now, inasmuch as the aerodynamic terms $(\bar{S}_A)_{x_5}$ , $(\bar{m}_A)_{y_5}$ , $(\bar{m}_A)_{z_5}$ involve elastic coordinate factors, they can be neglected as higher order. Thus, the final aerodynamic force and moment expressions are (after substitution of Eqs. (127) to (130) into (121) to (125))

$$
(\bar{S}_A)_{z_5} = \frac{1}{2}\left(\frac{\rho R^2}{m_0}\right)\bar{c}\left(\frac{U}{\Omega R}\right)^2\left(c_\ell\cos\phi + c_d\sin\phi\right) \tag{141}
$$

$$
(\bar{S}_A)_{y_5} = \frac{1}{2}\left(\frac{\rho R^2}{m_0}\right)\bar{c}\left(\frac{U}{\Omega R}\right)^2\left(c_\ell\sin\phi - c_d\cos\phi\right) \tag{142}
$$

$$(\bar{S}_A)_{x_5} \triangleq 0 \tag{143}$$

$$(\bar{m}_A)_{y_5} \triangleq 0 \tag{144}$$

$$(\bar{m}_A)_{z_5} \triangleq 0 \tag{145}$$

$$\begin{aligned}(\bar{m}_A)_{x_5} = \frac{1}{2}\left(\frac{\rho R^2}{m_0}\right)\Big\{ & \bar{c}^2\left(\frac{U}{\Omega R}\right)^2 c_{m_{\frac{c}{4}}} - \frac{\pi}{4}\left(\frac{U}{\Omega R}\right)\bar{c}^3\,\overset{*}{\Theta}\left(\frac{1}{2} - a_0\right) \\ & + \bar{y}_{10_{\frac{c}{4}}}\bar{c}\left(\frac{U}{\Omega R}\right)^2\Big[\cos\Theta\,(c_\ell\cos\phi + c_d\sin\phi) \\ & - \sin\Theta\,(c_\ell\sin\phi - c_d\cos\phi)\Big]\Big\}\end{aligned} \tag{146}$$

The following pitch angle relationships are also required in the aerodynamic expressions:

Total section pitch angle

$$\Theta = \theta + \theta_e = \theta_0 + \theta_B + \theta_e \tag{147}$$

Built-in pitch angle

$$\theta_B = \theta_1(\bar{e} + \bar{r} - 0.75) \tag{148}$$

Control system pitch angle

$$\theta_0 = \theta_{\frac{3}{4}R} - A_{1_S}\cos\psi - B_{1_S}\sin\psi - \tan\delta_3\left[\beta + (\bar{w}_e')_{\bar{r}=0}\right] \tag{149}$$

## APPENDIX III
## FLAP DAMPER, LAG DAMPER, AND PUSHROD FORCES

### FLAP DAMPER FORCE

In this analysis, it is assumed that the flap damper force is in the $z_4$ direction and is located at a distance $r_{FD}$ from the flapping hinge. The damper thus provides a pure moment about the flapping axis $y_4$, which can be expressed as

$$M_{FD} = r_{FD} \cos\beta \, (S_{FD})_{z_4,T} \tag{150}$$

where $(S_{FD})_T$ is the total damper force in pounds. For a pure viscous damper

$$(S_{FD})_{z_4,T} = -C_{FD}^{*}\left(\frac{dz_{4_{eo}}}{dt}\right)_{FD} \tag{151}$$

where $C_{FD}^{*}$ is the linear damping coefficient of the damper in units of lb-sec/ft and the bracketed expression represents the approximate velocity of the blade in the $z_4$ direction at the point of attachment of the lag damper. By nondimensionalizing, and if Eqs. (12), (36) and (37) are used, and if higher order terms are neglected, Eq. (151) becomes

$$(\bar{S}_{FD})_{z_4,T} = -\frac{C_{FD}^{*}}{m_0\Omega R}\Big[\bar{r}\overset{x}{\beta} + \overset{x}{\bar{w}}_e \cos\theta - \bar{w}_e \overset{x}{\theta}\sin\theta + \overset{x}{\bar{v}}_e \sin\theta + \bar{v}_e \overset{x}{\theta}\cos\theta\Big]_{FD} \tag{152}$$

Resolving the above force into "5" axis components and neglecting higher order terms yields the following:

$$(\bar{S}_{FD})_{z_5,T} \triangleq (\bar{S}_{FD})_{z_4,T} \tag{153}$$

$$(\bar{S}_{FD})_{y_5,T} = 0 \tag{154}$$

To obtain the flap damper force per unit span $(\bar{S}_{FD})_{z_5}$, it is assumed that the total damper force is applied uniformly over the infinitesimal interval $\Delta$. Thus,

$$(\bar{S}_{FD})_{z_5} = \begin{cases} \dfrac{(\bar{S}_{FD})_{z_5,T}}{\Delta} & \text{for } \left(\bar{r}_{FD} - \dfrac{\Delta}{2}\right) \le \bar{r} \le \left(\bar{r}_{FD} + \dfrac{\Delta}{2}\right) \\ 0 & \text{for } \left(\bar{r}_{FD} + \dfrac{\Delta}{2}\right) < \bar{r} < \left(\bar{r}_{FD} - \dfrac{\Delta}{2}\right) \end{cases} \tag{155}$$

## LAG DAMPER FORCE

The required lag damper force per unit span, $(\bar{S}_{LD})_{y_5}$, can be obtained from an analysis similar to that for the flap damper assuming, however, that the damper force acts in the $y_5$ direction. The following result is obtained:

$$(\bar{S}_{LD})_{y_5} = \begin{cases} \dfrac{(\bar{S}_{LD})_{y_5,T}}{\Delta} & \text{for } \left(\bar{r}_{LD} - \dfrac{\Delta}{2}\right) \le \bar{r} \le \left(\bar{r}_{LD} + \dfrac{\Delta}{2}\right) \\ 0 & \text{for } \left(\bar{r}_{LD} + \dfrac{\Delta}{2}\right) < \bar{r} < \left(\bar{r}_{LD} - \dfrac{\Delta}{2}\right) \end{cases} \tag{156}$$

$$(\bar{S}_{LD})_{z_5} = 0 \tag{157}$$

where

$$(\bar{S}_{LD})_{y_5,T} = \frac{(S_{LD})_{y_5,T}}{m_0 \Omega^2 R^2} \triangleq \frac{C_{LD}^{*}}{m_0 \Omega R} \Big[ \bar{r}\,\overset{\times}{\delta} + \overset{\times}{\bar{v}}_e \cos\theta - \bar{v}_e\,\overset{\times}{\theta} \sin\theta - \overset{\times}{\bar{w}}_e \sin\theta - \bar{w}_e\,\overset{\times}{\theta} \cos\theta \Big]_{LD} \tag{158}$$

PUSHROD FORCE

Rotor blades are generally restrained in pitch (i.e., the motion about the $x_5$ axis) by the moment exerted about the $x_5$ axis due to the pushrod of the control system. For conventional rotors, the pushrod force passes through the $y_5$ axis and is parallel to the $z_4$ axis. A pure pitching moment is thus provided. For rotors having pitch-flap coupling, the pushrod is displaced from the $y_5$ axis in the direction of $x_5$ by a distance $r_{PR}$. In this position, the force of the pushrod exerts a moment about the $y_5$ axis and thus influences the flapping and bending motions of the blade as well as the torsional motion.

Pushrod effects in the torsional equation are introduced by representing the torsional deformation by a series of uncoupled modes determined on the basis of appropriate root boundary conditions (i.e., fixed or partially fixed). To determine the effect of the pushrod force in the blade flapping and bending equations, an expression for the pushrod force is required. With the representation of the torsional response of the blade in terms of uncoupled modes, the required force is approximately given by

$$(\bar{S}_{PR})_{z_{5,T}} = -\frac{\left[\overline{GJ}\sum_j \gamma_{\theta_j}{}' q_{\theta_j}\right]_{\bar{r}=0}}{\bar{y}_{5_{PR}}} \tag{159}$$

$$(\bar{S}_{PR})_{y_{5,T}} = 0 \tag{160}$$

where $y_{5_{PR}}$ is the distance from the $x_5$ axis to the pushrod. The corresponding pushrod force per unit span is

$$(\bar{S}_{PR})_{z_5} = \begin{cases} \dfrac{(\bar{S}_{PR})_{z_{5,T}}}{\Delta} & \text{for} \left(\bar{r}_{PR} - \dfrac{\Delta}{2}\right) \leq \bar{r} \leq \left(\bar{r}_{PR} + \dfrac{\Delta}{2}\right) \\ 0 & \text{for} \left(\bar{r}_{PR} + \dfrac{\Delta}{2}\right) < \bar{r} < \left(\bar{r}_{PR} - \dfrac{\Delta}{2}\right) \end{cases} \tag{161}$$

## APPENDIX IV
## MODAL CONSTANTS

The various constants appearing in Eqs. (80), (86), (89), (91) and (96) are given below:

$$C_0 = \frac{C}{m_0 \Omega_0 R^3} \tag{162}$$

$$C_{1_i} = \int_0^{\bar{r}_T} \bar{m}\, \gamma_{w_i}^2 d\bar{r} \tag{163}$$

$$C_{3_{i,p}} = \int_0^{\bar{r}_T} \bar{m}\, \gamma_{w_i} \gamma_{v_p}\, d\bar{r} \tag{164}$$

$$C_{4_{i,i'}} = \int_0^{\bar{r}_T} \bar{m}\, \gamma_{w_i} \gamma_{w_{i'}}' \bar{r}\, d\bar{r} \tag{165}$$

$$C_{5_{i,p}} = \int_0^{\bar{r}_T} \bar{m}\, \gamma_{w_i} \gamma_{v_p} \bar{r}\, d\bar{r} \tag{166}$$

$$C_{7_{i,i'}} = \int_0^{\bar{r}_T} \gamma_{w_i} \gamma_{w_{i'}}'' \int_{\bar{r}}^{\bar{r}_T} \bar{m} \bar{\xi}\, d\bar{\xi}\, d\bar{r} \tag{167}$$

$$C_{8_{i,p}} = \int_0^{\bar{r}_T} \gamma_{w_i} \int_{\bar{r}}^{\bar{r}_T} \bar{m}\, \gamma_{v_p} d\bar{\xi}\, d\bar{r} \tag{168}$$

$$C_{9_{i,i'}} = \theta_1 \int_0^{\bar{r}_T} \gamma_{w_i} \left[ \int_{\bar{r}}^{\bar{r}_T} \bar{m}\, \gamma_{w_{i'}} \bar{\xi}\, d\bar{\xi} - \bar{r} \int_{\bar{r}}^{\bar{r}_T} \bar{m}\, \gamma_{w_{i'}} d\bar{\xi} \right] d\bar{r} \tag{169}$$

$$C_{10_i} = \int_0^{\bar{r}_T} \bar{m}\, \gamma_{w_i} d\bar{r} \tag{170}$$

$$C_{12_i} = \int_0^{\bar{r}_T} \bar{m}\, \gamma_{w_i} \bar{r}\, d\bar{r} \tag{171}$$

$$C_{13_i} = \theta_1 \int_0^{\bar{r}_T} \bar{m}\, \bar{r}\, \gamma_{w_i} (\bar{r} - 0.75)\, d\bar{r} \tag{172}$$

$$C_{14_p} = \int_0^{\bar{r}_T} \bar{m}\, \gamma_{v_p}^2 d\bar{r} \tag{173}$$

$$C_{15_p} = \int_0^{\bar{r}_T} \bar{m}\, \gamma_{v_p} d\bar{r} \tag{174}$$

$$C_{18_{p,p'}} = \int_0^{\bar{r}_T} \bar{m}\, \gamma_{v_p} \gamma_{v_{p'}}' \bar{r}\, d\bar{r} \tag{175}$$

$$C_{20_{p,p'}} = \int_0^{\bar r_T}\gamma_{v_p}\gamma''_{v_{p'}}\int_{\bar r}^{\bar r_T}\bar m\bar\xi\, d\bar\xi\, d\bar r \tag{176}$$

$$C_{21_{p,p'}} = -\theta_1\int_0^{\bar r_T}\gamma_{v_p}\int_{\bar r}^{\bar r_T}\bar m\gamma_{v_{p'}}(\bar\xi-\bar r)\, d\bar\xi\, d\bar r \tag{177}$$

$$C_{22_{i,p}} = \int_0^{\bar r_T}\gamma_{v_p}\int_{\bar r}^{\bar r_T}\bar m\gamma_{w_i}\, d\bar\xi\, d\bar r \tag{178}$$

$$\begin{aligned} C_{23_p} = {} & \int_0^{\bar r_T}\gamma_{v_p}\left[\bar e_A''\int_{\bar r}^{\bar r_T}\bar m\, d\bar\xi - 2\bar e_A'\bar m - \bar e_A\bar m'\right]d\bar r + \int_0^{\bar r_T}\gamma_{v_p}\left[\Delta\bar e_{A_{cw}}''\int_{\bar r}^{\bar r_{o_{cw}}}\bar m_{cw}\, d\bar\xi\right. \\ & \left. - 2\,\Delta\bar e_{A_{cw}}'\bar m_{cw} - \Delta\bar e_{A_{cw}}\bar m_{cw}'\right]d\bar r \end{aligned} \tag{179}$$

$$\begin{aligned} C_{24_p} = {} & \int_0^{\bar r_T}\gamma_{v_p}\left[\bar e_A''\int_{\bar r}^{\bar r_T}\bar m\bar\xi\, d\bar\xi - 2\bar e_A'\bar m\bar r - \bar e_A(\bar m'\bar r + \bar m)\right]d\bar r + \int_0^{\bar r_T}\gamma_{v_p}\left[\Delta\bar e_{A_{cw}}''\int_{\bar r}^{\bar r_{o_{cw}}}\bar m_{cw}\bar\xi\, d\bar\xi\right. \\ & \left. -2\Delta\bar e_{A_{cw}}'\bar m_{cw}\bar r - \Delta\bar e_{A_{cw}}(\bar m_{cw}'\bar r + \bar m_{cw})\right]d\bar r \end{aligned} \tag{180}$$

$$C_{26_{i,p}} = 2\theta_1\int_0^{\bar r_T}\gamma_{v_p}(\overline{EI}_z\gamma_{w_i})''\, d\bar r \triangleq -2\theta_1\int_0^{\bar r_T}\gamma'_{v_p}\left[(\overline{EI}_z)'\gamma'_{w_i} + \overline{EI}_z\gamma''_{w_i}\right]d\bar r \tag{181}$$

$$C_{27_{i,p}} = 2\theta_1\int_0^{\bar r_T}\gamma_{w_i}(\overline{EI}_y\gamma'_{v_p})''\, d\bar r \triangleq -2\theta_1\int_0^{\bar r_T}\gamma'_{w_i}\left[(\overline{EI}_y)'\gamma'_{v_p} + \overline{EI}_y\gamma''_{v_p}\right]d\bar r \tag{182}$$

$$C_{33_p} = \theta_1\left[\int_0^{\bar r_T}\bar m\gamma_{v_p}\bar r^2\, d\bar r - 0.75\,C_{47_p}\right] \tag{183}$$

$$C_{34_{p,p'}} = \theta_1\int_0^{\bar r_T}\bar m\gamma_{v_p}\gamma_{v_{p'}}(\bar r - 0.75)\, d\bar r \tag{184}$$

$$C_{36_j} = \int_0^{\bar r_T}\bar m\bar k_{z_{10}}^2\gamma_{\theta_j}^2\, d\bar r \tag{185}$$

$$C_{37_j} = \theta_1\int_0^{\bar r_T}\gamma_{\theta_j}\left[\bar k_A^2\int_{\bar r}^{\bar r_T}\bar m\bar\xi\, d\bar\xi\right]'d\bar r \tag{186}$$

$$C_{38_{i,i',j}} = 2\int_0^{\bar r_T}\gamma_{\theta_j}\gamma'_{w_i}\gamma''_{w_{i'}}(\overline{EI}_z - \overline{EI}_y)\, d\bar r \tag{187}$$

$$C_{39_{j,p,p'}} = -2\int_0^{\bar r_T}\gamma_{\theta_j}\gamma'_{v_p}\gamma''_{v_{p'}}(\overline{EI}_z - \overline{EI}_y)\, d\bar r \tag{188}$$

$$C_{44_j} = \int_0^{\bar r_T}\bar m\bar k_{z_{10}}^2\gamma_{\theta_j}\, d\bar r \tag{189}$$

$$C_{46_{i,p,j}} = \int_0^{\bar{r}_T} \gamma_{\theta_j} \gamma''_{w_i} \gamma''_{v_p} (\overline{EI}_z - \overline{EI}_y)\, d\bar{r} \qquad (190)$$

$$C_{47_p} = \int_0^{\bar{r}_T} \bar{m} \gamma_{v_p} \bar{r}\, d\bar{r} \qquad (191)$$

$$C_{50_p} = \int_0^{\bar{r}_T} \bar{m}' \bar{y}_{10_{cg}} \gamma_{v_p}\, d\bar{r} \qquad (192)$$

$$C_{51_{p,p'}} = \int_0^{\bar{r}_T} \bar{m}' \bar{y}_{10_{cg}} \gamma_{v_p} \gamma_{v_{p'}}\, d\bar{r} \qquad (193)$$

$$C_{52_p} = \int_0^{\bar{r}_T} \bar{m}' \gamma_{v_p} \bar{r} \bar{y}_{10_{cg}}\, d\bar{r} \qquad (194)$$

$$C_{53_{i,p}} = \int_0^{\bar{r}_T} \bar{m}' \gamma_{v_p} \gamma_{w_i} \bar{y}_{10_{cg}}\, d\bar{r} \qquad (195)$$

$$C_{56_{i,j}} = \int_0^{\bar{r}_T} \gamma_{w_i} \bar{m} \gamma'_{\theta_j} \bar{r}\, d\bar{r} \qquad (196)$$

$$C_{57_{i,j}} = \int_0^{\bar{r}_T} \gamma_{w_i} \gamma''_{\theta_j} \left[ \bar{e}_A \int_{\bar{r}}^{\bar{r}_T} \bar{m} \bar{\xi}\, d\bar{\xi} - \Delta\bar{e}_{A_{cw}} \int_{\bar{r}_{o_{cw}}}^{\bar{r}} \bar{m}_{cw} \bar{\xi}\, d\bar{\xi} \right] d\bar{r} \qquad (197)$$

$$C_{59} = \theta_1 \int_0^{\bar{r}_T} \bar{m} \bar{y}_{10_{cg}} \bar{r} (\bar{r} - 0.75)\, d\bar{r} \qquad (198)$$

$$C_{60_i} = \int_0^{\bar{r}_T} \bar{m} \gamma_{w_i} \bar{y}_{10_{cg}} \bar{r}\, d\bar{r} \qquad (199)$$

$$C_{61_{i,j}} = \int_0^{\bar{r}_T} \bar{m} \gamma_{w_i} \gamma'_{\theta_j} \bar{y}_{10_{cg}} \bar{r}\, d\bar{r} \qquad (200)$$

$$C_{62_{i,j}} = \int_0^{\bar{r}_T} \bar{m} \gamma_{w_i} \gamma_{\theta_j} \bar{y}_{10_{cg}}\, d\bar{r} \qquad (201)$$

$$C_{63_i} = \int_0^{\bar{r}_T} \bar{m} \gamma_{w_i} \bar{y}_{10_{cg}}\, d\bar{r} \qquad (202)$$

$$C_{64_{i,p}} = \int_0^{\bar{r}_T} \bar{m} \gamma_{w_i} \gamma'_{v_p} \bar{y}_{10_{cg}}\, d\bar{r} \qquad (203)$$

$$C_{65_{i,i'}} = \int_0^{\bar{r}_T} \gamma_{w_i} \gamma''_{w_{i'}} \int_{\bar{r}}^{\bar{r}_T} \bar{m} \bar{y}_{10_{cg}}\, d\bar{\xi}\, d\bar{r} \qquad (204)$$

$$C_{67_{i,p,j}} = \int_0^{\bar{r}_T} \gamma_{w_i} \gamma_{\theta_j}' \int_{\bar{r}}^{\bar{r}_T} \bar{m} \gamma_{v_p} \, d\bar{\xi} \, d\bar{r} \tag{205}$$

$$C_{68_{i,i'}} = \int_0^{\bar{r}_T} \bar{m} \gamma_{w_i} \gamma_{w_{i'}} (\bar{r} - 0.75) \, d\bar{r} \tag{206}$$

$$C_{69_{i,i'}} = \int_0^{\bar{r}_T} \bar{m} \gamma_{w_i} \gamma_{w_{i'}}' \bar{y}_{10_{cg}} \, d\bar{r} \tag{207}$$

$$C_{70_p} = \int_0^{\bar{r}_T} \bar{m} \gamma_{v_p} \bar{y}_{10_{cg}} \, d\bar{r} \tag{208}$$

$$C_{72_p} = \int_0^{\bar{r}_T} \gamma_{v_p} \bar{m} \bar{y}_{10_{cg}}' \, d\bar{r} \tag{209}$$

$$C_{73_p} = \int_0^{\bar{r}_T} \bar{m} \gamma_{v_p} \bar{y}_{10_{cg}}' \bar{r} \, d\bar{r} \tag{210}$$

$$C_{74_{p,j}} = \int_0^{\bar{r}_T} \bar{m} \gamma_{v_p} \bar{y}_{10_{cg}} \gamma_{\theta_j} \, d\bar{r} \tag{211}$$

$$C_{75_{i,p,j}} = \int_0^{\bar{r}_T} \gamma_{v_p} \gamma_{\theta_j}' \int_{\bar{r}}^{\bar{r}_T} \bar{m} \gamma_{w_i} \, d\bar{\xi} \, d\bar{r} \tag{212}$$

$$C_{76_p} = \theta_1 \int_0^{\bar{r}_T} \bar{m} \gamma_{v_p} \bar{y}_{10_{cg}} (\bar{r} - 0.75) \, d\bar{r} \tag{213}$$

$$C_{77_{p,p'}} = \int_0^{\bar{r}_T} \gamma_{v_p} \gamma_{v_{p'}}'' \int_{\bar{r}}^{\bar{r}_T} \bar{m} \bar{y}_{10_{cg}} \, d\bar{\xi} \, d\bar{r} \tag{214}$$

$$C_{79_{p,p'}} = \int_0^{\bar{r}_T} \bar{m} \gamma_{v_p} \gamma_{v_{p'}}' \bar{y}_{10_{cg}} \, d\bar{r} \tag{215}$$

$$C_{80_j} = -\theta_1 \int_0^{\bar{r}_T} \bar{m} \gamma_{\theta_j} \bar{k}_{z_{10}}^2 (\bar{r} - 0.75) \, d\bar{r} \tag{216}$$

$$C_{81_{i,p}} = \int_0^{\bar{r}_T} \bar{m} \gamma_{v_p} \bar{y}_{10_{cg}} \gamma_{w_i}' \, d\bar{r} \tag{217}$$

$$C_{82_{p,j}} = \int_0^{\bar{r}_T} \bar{m} \cdot \bar{k}_{z_{10}}^2 \gamma_{\theta_j} \gamma_{v_p}' \, d\bar{r} \tag{218}$$

$$C_{83_{i,j}} = \int_0^{\bar{r}_T} \gamma_{\theta_j} \bar{m}\, \bar{y}_{10_{cg}} \gamma_{w_i}' \, d\bar{r} \tag{219}$$

$$C_{84_{i,j}} = \int_0^{\bar{r}_T} \bar{m}\, \bar{y}_{10_{cg}} \gamma_{\theta_j} \gamma_{w_i}' \, \bar{r}\, d\bar{r} \tag{220}$$

$$C_{85_{p,j}} = \int_0^{\bar{r}_T} \bar{m}\, \bar{y}_{10_{cg}} \gamma_{\theta_j} \bar{r}\, \gamma_{v_p} d\bar{r} \tag{221}$$

$$C_{86_{i,j}} = \int_0^{\bar{r}_T} \gamma_{\theta_j} \gamma_{w_i}'' \left[ -\bar{e}_A \int_{\bar{r}}^{\bar{r}_T} \bar{m}(\bar{\xi}+\bar{e})\, d\bar{\xi} + \Delta\bar{e}_{A_{cw}} \int_{\bar{r}_{0_{cw}}}^{\bar{r}} \bar{m}_{cw}(\bar{\xi}+\bar{e})\, d\bar{\xi} \right] d\bar{r} \tag{222}$$

$$C_{87_{p,j}} = \int_0^{\bar{r}_T} \gamma_{\theta_j} \gamma_{v_p}' \left[ -\bar{e}_A \int_{\bar{r}}^{\bar{r}_T} \bar{m}\, \bar{\xi}\, d\bar{\xi} + \Delta\bar{e}_{A_{cw}} \int_{\bar{r}_{0_{cw}}}^{\bar{r}} \bar{m}_{cw}\, \bar{\xi} d\bar{\xi} \right] d\bar{r} \tag{223}$$

$$C_{88_{i,j}} = \int_0^{\bar{r}_T} \gamma_{\theta_j} \gamma_{w_i}'' \left[ -\bar{e}_A \int_{\bar{r}}^{\bar{r}_T} \bar{m}\, \bar{\xi}\, d\bar{\xi} + \Delta\bar{e}_{A_{cw}} \int_{\bar{r}_{0_{cw}}}^{\bar{r}} \bar{m}_{cw}\, \bar{\xi}\, d\bar{\xi} \right] d\bar{r} \tag{224}$$

$$C_{90_p} = \int_0^{\bar{r}_T} \bar{m}\, \bar{y}_{10_{cg}} \gamma_{v_p}' d\bar{r} \tag{225}$$

$$C_{91_p} = \int_0^{\bar{r}_T} \gamma_{v_p} \left( \bar{m}\, \bar{k}_{z_{10}}^2 \right) d\bar{r} \tag{226}$$

$$C_{92_j} = \int_0^{\bar{r}_T} \bar{m}\, \bar{y}_{10_{cg}} \gamma_{\theta_j} \, d\bar{r} \tag{227}$$

$$C_{93_j} = \int_0^{\bar{r}_T} \bar{m}\, \bar{y}_{10_{cg}} \gamma_{\theta_j} \bar{r}\, d\bar{r} \tag{228}$$

$$C_{94_j} = \theta_1 \int_0^{\bar{r}_T} \bar{m}\, \bar{y}_{10_{cg}} \gamma_{\theta_j} \bar{r}(\bar{r} - 0.75)\, d\bar{r} \tag{229}$$

$$C_{95_{i,j}} = \theta_1 \int_0^{\bar{r}_T} \bar{m}\, \bar{y}_{10_{cg}} \gamma_{\theta_j} \, \gamma_{w_i} (\bar{r} - 0.75) d\bar{r} \tag{230}$$

$$C_{96} = \int_0^{\bar{r}_T} \bar{m}\, \bar{y}_{10_{cg}} d\bar{r} \tag{231}$$

$$C_{97} = \int_0^{\bar{r}_T} \bar{m} \bar{r}\, \bar{y}_{10_{cg}} d\bar{r} \tag{232}$$

$$C_{98_{p,p'}} = \int_0^{\bar{r}_T} \bar{m}\, \gamma_{v_p}\, \gamma_{v_{p'}}\, \bar{y}_{10_{cg}}'\, d\bar{r} \tag{233}$$

$$C_{99_{i,p}} = \int_0^{\bar{r}_T} \bar{m}\, \gamma_{w_i}\, \gamma_{v_p}\, \bar{y}_{10_{cg}}'\, d\bar{r} \tag{234}$$

$$C_{100_{p,j}} = \int_0^{\bar{r}_T} \bar{m}\, \gamma_{v_p}\, \gamma_{\theta_j}\, \bar{r}\, d\bar{r} \tag{235}$$

$$C_{110_{i,j}} = \int_0^{\bar{r}_T} \bar{m}\, \bar{k}_{z_{10}}^2\, \gamma_{\theta_j}\, \gamma_{w_i}'\, d\bar{r} \tag{236}$$

$$C_{111_{i,p,j}} = \int_0^{\bar{r}_T} \bar{m}\, \bar{y}_{10_{cg}}\, \gamma_{\theta_j}\, \gamma_{w_i}'\, \gamma_{v_p}\, d\bar{r} \tag{237}$$

$$C_{112_{i,i',j}} = \int_0^{\bar{r}_T} \bar{m}\, \bar{y}_{10_{cg}}\, \gamma_{\theta_j}\, \gamma_{w_{i'}}\, \gamma_{w_i}'\, d\bar{r} \tag{238}$$

$$C_{113_{j,j'}} = \int_0^{\bar{r}_T} \bar{m}\, \bar{y}_{10_{cg}}\, \gamma_{\theta_j}\, \gamma_{\theta_{j'}}\, \bar{r}\, d\bar{r} \tag{239}$$

$$C_{114_{p,j}} = \theta_1 \int_0^{\bar{r}_T} \bar{m}\, \bar{y}_{10_{cg}}\, \gamma_{\theta_j}\, \gamma_{v_p}\, (\bar{r} - 0.75)\, d\bar{r} \tag{240}$$

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In this volume the differential equations of motion for a linearly twisted rotor blade having chordwise mass unbalance and operating under steady flight conditions are derived. The motions include flapping and lagging for the articulated blade, as well as flatwise, edgewise, and torsional deformations for the articulated and nonarticulated blades. The fully coupled aerodynamic forcing functions are based on quasi-steady theory. The differential equations of motion are expanded in terms of the uncoupled vibratory modes of the blade in order to facilitate their numerical solution on a digital computer.

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