

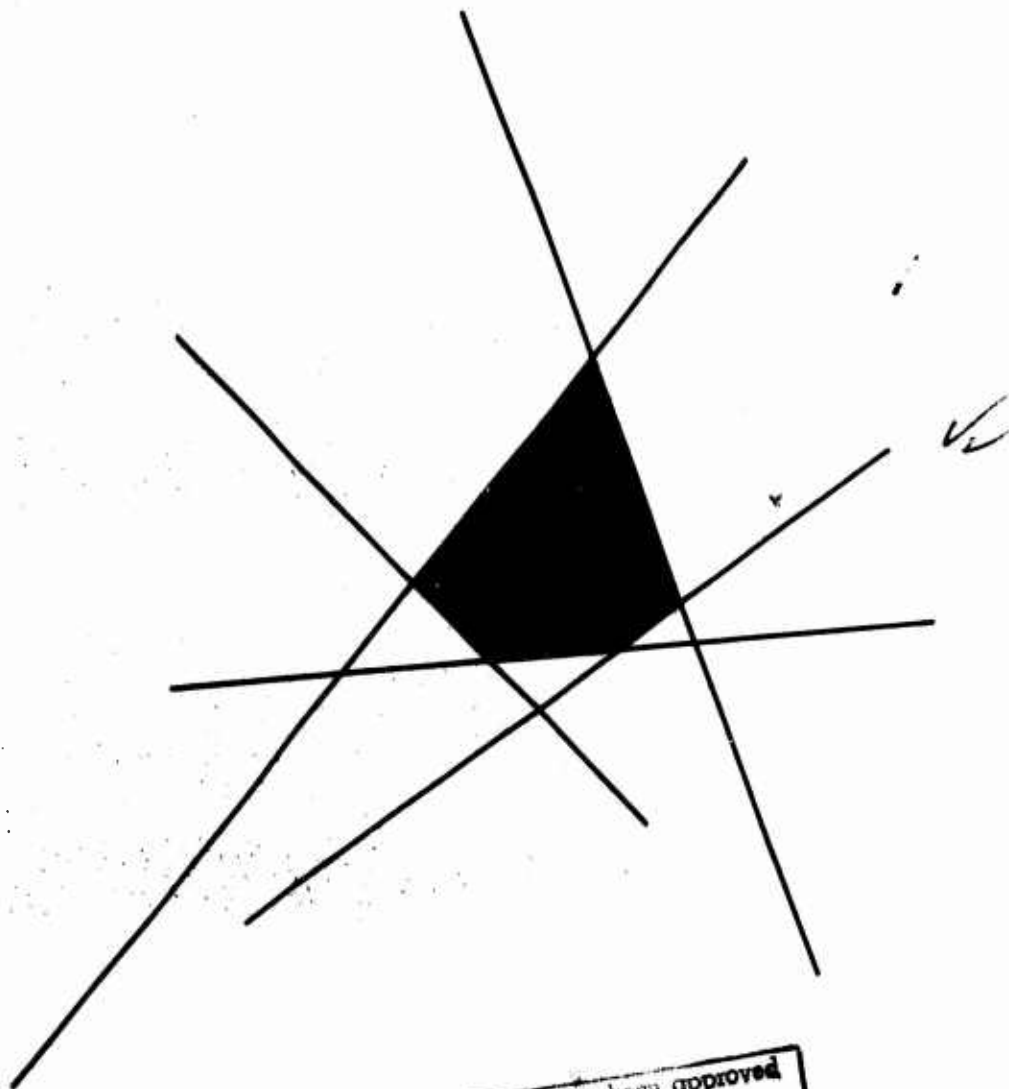
ORC 68-27  
NOVEMBER 1968

# OPTIMAL DISPATCHING OF A POISSON PROCESS

by

SHELDON M. ROSS

AD 685254



**OPERATIONS  
RESEARCH  
CENTER**

This document has been approved  
for public release and where the  
restriction is indicated.

DDC  
APR 15 1969

**UNIVERSITY OF CALIFORNIA • BERKELEY**

Reproduced by the  
CLEARINGHOUSE  
for Federal Scientific & Technical  
Information Springfield Va 22151

OPTIMAL DISPATCHING OF A POISSON PROCESS

by

Sheldon M. Ross  
Department of Industrial Engineering  
and Operations Research  
University of California, Berkeley

November 1968

ORC 68-27

This research has been partially supported by the Office of Naval Research under Contract Nonr-222(83), the National Science Foundation under Grant GP-8695, and the U.S. Army Research Office-Durham under Contract DA-31-124-ARO-D-331 with the University of California. Reproduction in whole or in part is permitted for any purpose of the United States Government.

## ABSTRACT

Items arrive at a processing plant at a Poisson rate  $\lambda$ . At time  $T$ , all items are dispatched from the system. An intermediate dispatch time is to be chosen to minimize the total wait of all items. It is shown that if the dispatch time must be chosen at time 0 then  $T/2$  not only minimizes the expected total wait but it also maximizes the probability that the total wait is less than  $a$  for every  $a > 0$ . If the intermediate dispatch time is allowed to be a (random) stopping time, then it is shown that the policy which dispatches at time  $t$  iff  $N(t) \geq \lambda(T - t)$  is optimal, where  $N(t)$  denotes the number of items present at time  $t$ . The distribution of the optimal dispatch time and the optimal expected total wait are determined. A generalization to the case of a nonhomogeneous Poisson Process, a time lag, and batch arrivals is given. Finally, the case where the process goes on indefinitely and any number of dispatches are allowed (at a cost  $K$  per dispatch) is considered.

# OPTIMAL DISPATCHING OF A POISSON PROCESS

by

Sheldon M. Ross

## 1. INTRODUCTION

Items arrive at a processing plant at a Poisson rate  $\lambda$ . At time  $T$ , all items are dispatched from the system. An intermediate dispatch time, at which all items arriving up to that time are dispatched, is to be chosen so as to minimize the total wait of all items.

## 2. CONSTANT DISPATCH TIME

If we suppose that the intermediate dispatch time must be chosen at time  $0$ , then it is easy to see that the dispatch time minimizing the expected total waiting time is  $T/2$ , and the minimal expected waiting time is  $\lambda T^2/4$ .<sup>†</sup> In fact, we can say more than this, as the following argument shows. Let  $t$  ( $0 < t < T$ ) be any intermediate dispatch time. Then the total waiting time is  $\sum_{i=1}^{N(T)} W_t(i)$ , where  $N(T)$  denotes the number of items arriving in  $(0, T]$ , and

$$(1) \quad W_t(i) = \begin{cases} t - \tau_i & \text{if } \tau_i \leq t \\ T - \tau_i & \text{if } \tau_i > t \end{cases}$$

where  $\tau_i, i = 1, \dots, N(T)$  are the arrival epochs. Now conditional on  $N(T) = n$ , the (unordered) points  $\tau_1, \dots, \tau_n$  are independent and identically distributed (iid) as uniform random variables on  $(0, T)$ . Thus,  $\sum_{i=1}^{N(T)} W_t(i)$  has the same distribution as  $\sum_{i=1}^P Z_t(i)$ , where  $P$  is distributed as a Poisson random variable with parameter  $\lambda T$ , and the  $Z_t(i)$  are iid independent of  $P$  having

<sup>†</sup>The expected number of arrivals in  $(0, T/2)$  is  $\lambda T/2$ , and each arrival is uniformly distributed on  $(0, T/2)$  and thus has expected delay  $T/4$ . Similarly for arrivals in  $(T/2, T)$ .

$$(2) \quad \Pr(Z_t(i) < a) = \begin{cases} 2a/T & \text{for } a \leq t, a + t \leq T \\ \frac{a+T-t}{T} & \text{for } a \leq t, a + t > T \\ \frac{a+t}{T} & \text{for } a > t, a + t \leq T \\ 1 & \text{for } a > t, a + t > T. \end{cases}$$

From (2), it follows that for fixed  $a$ ,  $\Pr(Z_t(i) < a)$  is maximized by  $t = T/2$  and

$$(3) \quad \Pr(Z_{T/2}(i) < a) = \begin{cases} 2a/T & \text{for } a \leq T/2 \\ 1 & \text{for } a > T/2. \end{cases}$$

Since the distribution of  $P$  is independent of  $t$  it follows that not only does  $T/2$  minimize the expected total waiting time, it also maximizes the probability that the total wait is less than  $a$  for every  $a > 0$ .

$\sum_1^{N(T)} W_t(i)$  also has the same distribution as  $\sum_1^{P_1} U_t(i) + \sum_1^{P_2} V_t(i)$ , where  $P_1$  is Poisson  $(\lambda t)$ ,  $P_2$  is Poisson  $(\lambda(T-t))$ ,  $U_t(i)$  are iid with  $\Pr(U_t(i) < a) = \begin{cases} a/t & \text{for } a \leq t \\ 1 & \text{for } a > t \end{cases}$ ,  $V_t(i)$  are iid with  $\Pr(V_t(i) < a) = \begin{cases} a/T-t & \text{for } a \leq T-t \\ 1 & \text{for } a > T-t \end{cases}$ , and all of the  $P_1, P_2, U_t(i), V_t(i)$  are independent.

### 3. DYNAMIC VERSION

The problem becomes more interesting if we allow the intermediate dispatch time  $\Delta$  to be a (random) stopping time.<sup>†</sup> We shall say that a policy (i.e., stopping time) is optimal if it minimizes the expected total wait. Let  $\delta$  be the policy which dispatches at  $\tau = \text{Min} \{t \geq 0 : N(t) \geq \lambda(T-t)\}$ .

<sup>†</sup>This implies that  $\{\Delta \leq t\}$  is independent of  $\{N(s) - N(t), s > t\}$ .

We note that  $N(\tau) = \lambda(T - \tau)$  unless  $N(t)$  has a jump at  $\tau$  (see Figure 1).

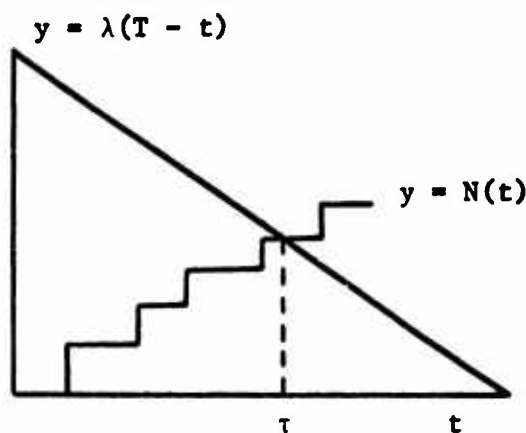


FIGURE 1.

Theorem 1:

$\delta$  is optimal.

Proof:

Let  $\delta_1$  be any policy, and let  $\delta_2$  follow  $\delta_1$  with the exception that if  $N(t) < \lambda(T - t)$  and  $\delta_1$  dispatches at  $t$  then  $\delta_2$  dispatches at

$t + \frac{\lambda(T-t) - N(t)}{\lambda} \equiv t + h$ . The expected total wait from  $t$  under  $\delta_1$  is  $\lambda(T - t)^2/2$ , while the expected wait from  $t$  under  $\delta_2$  is  $N(t) \cdot h + \lambda h^2/2 + \lambda(T - t - h)^2/2 = \lambda(T - t)^2/2$ ; and thus  $\delta_1$  and  $\delta_2$  have the same expected wait.

Now let  $\delta$  denote the policy which dispatches at  $\tau = \text{Min} \{t \geq 0 : N(t) = \lambda(T - t)\}$ . Since  $N(t + h) \geq N(t) = \lambda(T - t - h)$  it follows from the definition of  $\delta_2$  that  $t(\delta_2) \geq \tau$ , where  $t(\delta_2)$  denotes the time at which  $\delta_2$  dispatches. Thus,  $t(\delta_2) = \tau + \epsilon$ , where  $\epsilon$  is nonnegative and random. Therefore, the conditional expected total wait from  $\tau$  under  $\delta_2$  is at least  $N(\tau) \cdot \epsilon + \lambda(T - \tau - \epsilon)^2/2$ , which is strictly greater than  $\lambda(T - \tau)^2/2$  (= the conditional expected total wait under  $\delta$ ) whenever  $\epsilon > 0$ . From this it follows that  $\delta$  is at least as good as  $\delta_2$ .

Q.E.D.

It remains to determine the distribution of the optimal dispatch time and the optimal expected total wait. Now  $\tau = \text{Min} \{t \leq T : N(t) \geq \lambda(T - t)\}$ , and thus

$$(4) \quad F(a) \equiv \Pr\{\tau < a\} = \Pr(N(a) > \lambda(T - a)) \\ = e^{-\lambda a} \sum_{i=[\lambda(T-a)]+1}^{\infty} (\lambda a)^i / i!,$$

where  $[x] = \begin{cases} \text{smallest integer } \leq x, & \text{for } x \geq 0 \\ -1 & \text{for } x < 0 \end{cases}$ . We also note from (4) that

$$(5) \quad a_j \equiv \Pr\left(\tau = \frac{\lambda T - 1}{\lambda}\right) = e^{-(\lambda T - j)} (\lambda T - j)^j / j!, \quad j = 0, 1, \dots, [\lambda T].$$

The optimal expected wait is given by

$$(6) \quad EW = E\left[\sum_{i=1}^{N(\tau)} (\tau - \tau_i)\right] + E\left[\sum_{i=N(\tau)+1}^{N(T)} (T - \tau_i)\right].$$

Now,

$$(7) \quad E\left[\sum_{i=N(\tau)+1}^{N(T)} (T - \tau_i) \mid \tau\right] = \lambda(T - \tau)^2 / 2,$$

and

$$(8) \quad E\left[\sum_{i=1}^{N(\tau)} (\tau - \tau_i) \mid \tau\right] = \begin{cases} j\tau/2 & \text{for } \tau = \frac{\lambda T - 1}{\lambda}, \quad j = 0, 1, \dots, [\lambda T] \\ (j - 1)\tau/2 & \text{for } \tau \in \left(\frac{\lambda T - 1}{\lambda}, \frac{\lambda T - j + 1}{\lambda}\right) \quad j = 1, \dots, [\lambda T], \end{cases}$$

where (8) follows from the observation that (i)  $\tau = \frac{\lambda T - 1}{\lambda}$  is equivalent to  $N\left(\frac{\lambda T - 1}{\lambda}\right) = j$  and (ii)  $\tau \in \left(\frac{\lambda T - 1}{\lambda}, \frac{\lambda T - j + 1}{\lambda}\right)$  implies that  $N(\tau) = j$ , and  $\tau_j = \tau$ .

Thus, by combining (7) and (8) we arrive at

$$(9) \quad EW = \sum_{j=0}^{[\lambda T]} j a_j \frac{\lambda T - j}{2\lambda} + \sum_{j=1}^{[\lambda T]} \int_{x \in \left(\frac{\lambda T - j}{\lambda}, \frac{\lambda T - j + 1}{\lambda}\right)} (j-1)x/2 \, dF(x) + \lambda/2 \int_0^T (T-x)^2 dF(x) ,$$

where  $a_j$  and  $F$  are given by (4) and (5).

Equation (9) is rather unwieldy, but a simple and significant bound can be determined as follows. From (7) and (8) we have that

$$(10) \quad EW \geq E[(N(\tau) - 1)\tau/2] + E(\lambda(T - \tau)^2/2) .$$

However, by definition  $N(\tau) \geq \lambda(T - \tau)$ , and thus

$$(11) \quad \begin{aligned} EW &\geq E[(\lambda(T - \tau) - 1)\tau/2] + E \lambda(T - \tau)^2/2 \\ &= \lambda T^2/2 - \frac{\lambda T + 1}{2} E\tau . \end{aligned}$$

Now it is easily verified that  $N(t) - \lambda t$  is a zero-mean Martingale, and thus by a simple Martingale systems theorem (see [1], Chapter 5, Section 3), we have that  $E[N(\tau) - \lambda\tau] = 0$ . Using this and the fact that  $N(\tau) \leq \lambda(T - \tau) + 1$ , we arrive at

$$(12) \quad E\tau \leq \frac{\lambda T + 1}{2\lambda} .$$

By combining (11) and (12) we get

$$(13) \quad EW \geq \lambda T^2/4 - T/2 - 1/4\lambda .$$

Equation (13) is especially significant as  $\lambda T^2/4$  is just the total expected wait when the (fixed) dispatch time  $T/2$  is used, and thus

$$(14) \quad \lambda T^2/4 \geq EW \geq \lambda T^2/4 - T/2 - 1/4\lambda .$$

It is interesting to note that the gain in choosing a random dispatch time over a fixed dispatch time is of the order of  $T$ , while the wait is of the order of  $T^2$ . Thus if  $T = 100$ ,  $\lambda = 1/2$  then the optimum wait is at least 1199.5, while by using a fixed time procedure the expected wait would be 1250. Since, the optimal procedure must continuously watch the process it hardly seems worth the gain.



#### 4. SOME GENERALIZATIONS

##### a. Nonhomogeneous Poisson Process

One generalization is to allow the arrival stream to be a nonhomogeneous Poisson Process. Letting  $\lambda(t)$  be the arrival rate at  $t$ , we have

##### Theorem 2:

If  $\lambda(t)$  is continuous and nonincreasing on  $[0, T]$ , then the policy which dispatches at the smallest  $t$  such that  $N(t) \geq \lambda(t)(T - t)$  is optimal.

We first give a "pseudo"-proof

##### Proof: (Pseudo)

Suppose  $N(t) < \lambda(t)(T - t)$ . The expected total wait (from  $t$ ) of a policy dispatching at  $t$  is  $\int_t^T (T - x)\lambda(x)dx$ , while the expected total wait (from  $t$ )

of a policy dispatching at  $t + \epsilon$  is  $\int_t^{t+\epsilon} \lambda(x)(t+\epsilon-x)dx + \epsilon N(t) + \int_{t+\epsilon}^T (T - x)\lambda(x)dx$ .

Thus, waiting an additional time  $\epsilon$  is better whenever  $N(t) < \int_t^{t+\epsilon} \lambda(x) \frac{(T-t-\epsilon)}{\epsilon} dx$ .

Since  $\int_t^{t+\epsilon} \lambda(x) \frac{(T-t-\epsilon)}{\epsilon} dx \rightarrow \lambda(t)(T - t)$  as  $\epsilon \rightarrow 0$ , it follows that the inequality

holds for  $\epsilon$  small.

Now, suppose  $N(t) \geq \lambda(t)(T - t)$  and consider a policy which dispatches at  $t + \epsilon$  ( $\epsilon \geq 0$ ,  $\epsilon$  random). The expected total wait from  $t$  of this procedure is

$$\text{at least } \epsilon N(t) + \int_{t+\epsilon}^T \lambda(x)(T - x)dx = \int_t^T \lambda(x)(T - x)dx + \epsilon N(t) - \int_t^{t+\epsilon} \lambda(x)(T - x)dx$$

$$\geq \int_t^T \lambda(x)(T - x)dx$$

by the monotonicity assumption and  $N(t) \geq \lambda(t)(T - t)$ . Thus if we haven't

dispatched before  $t = \min \{u \geq 0 : N(u) \geq \lambda(u)(T - u)\}$ , then we should dispatch at  $t$ . End of pseudo-proof.

It is tempting to replace "end of pseudo-proof" by Q.E.D., as we have shown that (i) if  $N(t) \geq \lambda(t)(T - t)$  then it is optimal to dispatch at  $t$ , and (ii) if  $N(t) < \lambda(t)(T - t)$  then it is not optimal to dispatch at  $t$ . However, the non-optimality of dispatching at  $t$  when  $N(t) < \lambda(t)(T - t)$  only implies that it is optimal not to dispatch when an optimal policy is known to exist. Thus, the pseudo proof could be made rigorous by proving the existence of an optimal policy. However, a direct proof showing that the optimal policy is better than every other policy may be constructed exactly as in Theorem 1.<sup>†</sup>

We note that monotonicity assumption was not used in the first part of the "pseudo" proof, and thus it is never optimal to dispatch at  $t$  when  $N(t) < \lambda(t)(T - t)$ .

#### b. Batch Arrivals and a Time Lag

A second generalization is to assume that there is a time lag  $L (> 0)$  between the time at which the decision to dispatch is made and the time of actual dispatchment. In this case, it can be shown that under the conditions of Theorem 2 the policy which orders a dispatch at the smallest  $t$  for which

$$N(t) + \int_t^{t+L} \lambda(x) dx \geq \lambda(t+L)(T-t-L), \text{ is optimal.}$$

Finally, we may suppose that arrivals consist of batches of items. The batch size is assumed to be a random variable with mean  $\mu$ . In this case, still assuming a time lag  $L$  and also the conditions of Theorem 2 we have that the policy which orders a dispatch at the smallest  $t$  for which

$$N(t) + \mu \int_t^{t+L} \lambda(x) dx \geq \mu \lambda(t+L)(T-t-L), \text{ is optimal.}$$

<sup>†</sup>It is interesting to note that under the conditions of Theorem 2 the optimal fixed dispatch time is that  $t$  such that  $E(N(t)) = \lambda(t)(T - t)$ .

## 5. MULTIPLE DISPATCHES

In this section we suppose that the process continues indefinitely, and we allow any number of dispatches. Each dispatch is assumed to incur a cost  $K$ , and we also assume a cost  $C(i)$  per unit time when there are  $i$  items in the system. (If there is a cost  $C$  per unit of waiting time, then  $C(i) = iC$ ; however, the additional generality does not complicate matters.)

We assume a Poisson arrival rate  $\lambda$  and seek to find a rule minimizing the long-run average cost per unit time. We shall only consider rules whose actions depend solely on the present number in the system. (These are the stationary rules.)

It is well known (see (2)) that, if a cycle is defined as the time between successive dispatches, then the long-run average cost per unit time is just  $E[\text{cost of cycle}]/E[\text{time of cycle}]$ . Thus, the rule which dispatches when there are  $n$  people in the system has a long-run average cost equal to

$$(15) \quad (n/\lambda)^{-1} \left[ K + E \int_0^{\tau_n} C(N(t)) dt \right],$$

which is easily seen to equal

$$(16) \quad \lambda/n [K + C(0)\tau_1 + C(1)(\tau_2 - \tau_1) + \dots + C(n-1)(\tau_n - \tau_{n-1})] = \\ = \frac{\lambda K}{n} + 1/n \sum_{i=0}^{n-1} C(i).$$

The optimal value of  $n$  can then be found from (16).<sup>†</sup> For example, if  $C(i) = iC$

<sup>†</sup>We are supposing that never dispatching is not optimal. Since the average cost in this case would just be  $\lim_{n \rightarrow \infty} \sum_{k=1}^n C(k)/n$ , it is easily seen when this is non-optimal.

then (16) equals  $\lambda K/n + \frac{C(n-1)}{2}$  and treating  $n$  as a continuous variable, we get by differentiable calculus that the optimal  $n$  is one of the two integers adjacent to  $\sqrt{2\lambda K/C}$ .

The above problem was originally formulated in [3] but surprisingly was only solved for certain special cases, as the equivalence between (15) and (16) was not noted.

Finally, let us assume a discount factor  $1 - \alpha$  (i.e., a cost  $b$  incurred at time  $t$  is worth  $be^{-\alpha t}$  at time 0); and look for the rule minimizing the expected total discounted cost. Now, let  $D_i$  be the cost (discounted to the beginning of the cycle) incurred during the  $i^{\text{th}}$  cycle, and let  $T_i$  be the time of the  $i^{\text{th}}$  cycle. Then, the total discounted cost is

$$(17) \quad D_1 + e^{-\alpha T_1} D_2 + e^{-\alpha(T_1+T_2)} D_3 + \dots$$

Now using the fact that the pairs  $(D_i, T_i)$  are independent and identically distributed, we have that the expected total discounted cost equals

$$(18) \quad \frac{ED_1}{1 - Ee^{-\alpha T_1}}$$

For the rule which dispatches whenever there are  $n$  people in the system, (18) equals

$$\begin{aligned}
 & \frac{E \int_0^{\tau_n} C(N(t)) e^{-\alpha t} dt + K E e^{-\alpha \tau_n}}{1 - E(e^{-\alpha \tau_n})} \\
 (19) \quad & = \frac{\sum_{i=1}^n C(i-1) E \left[ \int_{\tau_{i-1}}^{\tau_i} e^{-\alpha t} dt \right] + K(\lambda/\lambda+\alpha)^n}{1 - (\lambda/\lambda+\alpha)^n} \quad (\tau_0 \equiv 0) \\
 & = \frac{K(\lambda/\lambda+\alpha)^n + 1/\alpha \sum_{i=1}^n C(i-1) E[e^{-\alpha \tau_{i-1}} - e^{-\alpha \tau_i}]}{1 - (\lambda/\lambda+\alpha)^n} \\
 & = \frac{K(\lambda/\lambda+\alpha)^n + 1/\alpha \sum_{i=1}^n C(i-1) [(\lambda/\lambda+\alpha)^{i-1} - (\lambda/\lambda+\alpha)^i]}{1 - (\lambda/\lambda+\alpha)^n} .
 \end{aligned}$$

When  $C(i) = iC$ , we have that (19) equals

$$(20) \quad \frac{K(\lambda/\lambda+\alpha)^n + \lambda C/\alpha^2 [1 + (n-1)(\lambda/\lambda+\alpha)^n - n(\lambda/\lambda+\alpha)^{n-1}]}{1 - (\lambda/\lambda+\alpha)^n} .$$

## REFERENCES

- [1] Breiman, L., PROBABILITY, Addison-Wesley, (1968).
- [2] Johns, M. and R. Miller, "Average Renewal Loss Rate," Annals of Mathematical Statistics, Vol. 34, pp. 396-401, (1963).
- [3] Savage, I. R., "Surveillance Problems," Naval Research Logistics Quarterly, Vol. 9, Nos. 3 & 4, pp. 187-210, (1962).

Unclassified

Security Classification

DOCUMENT CONTROL DATA - R & D

(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)

1. ORIGINATING ACTIVITY (Corporate author) University of California, Berkeley		2a. REPORT SECURITY CLASSIFICATION Unclassified	
		2b. GROUP	
3. REPORT TITLE OPTIMAL DISPATCHING OF A POISSON PROCESS			
4. DESCRIPTIVE NOTES (Type of report and, inclusive dates) Research Report			
5. AUTHOR(S) (First name, middle initial, last name) ROSS, Sheldon M.			
6. REPORT DATE November 1968		7a. TOTAL NO. OF PAGES 11	7b. NO. OF REFS 3
8a. CONTRACT OR GRANT NO. Nonr-222(83)		8b. ORIGINATOR'S REPORT NUMBER(S) ORC 68-27	
b. PROJECT NO. NR 047 033			
c. Research Project No.: RR 003 07 01		9b. OTHER REPORT NO(S) (Any other numbers that may be assigned this report)	
d.			
10. DISTRIBUTION STATEMENT This document has been approved for public release and sale; its distribution is unlimited.			
11. SUPPLEMENTARY NOTES Also supported by the Nat'l. Sci. Found. under Grant GP-8695 and the U.S. Army Res. Office-Durham under Contract DA-31-124-ARO-D-331.		12. SPONSORING MILITARY ACTIVITY MATHEMATICAL SCIENCE DIVISION	
13. ABSTRACT SEE ABSTRACT.			

14 KEY WORDS	LINK A		LINK B		LINK C	
	ROLE	WT	ROLE	WT	ROLE	WT
Optimal Dispatching Stopping Time Nonhomogeneous Poisson Process Multiple Dispatching						