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A Unified Approach to Nonlinear Estimation

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Electronics Division

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Engineering Science Operations
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Prepared for SPACE AND MISSILE SYSTEMS ORGANIZATION
AIR FORCE SYSTEMS COMMAND
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FOREWORD

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Approved

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Electronics Division
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Publication of this report does not constitute Air Force approval of the report's findings or conclusions. It is published only for the exchange and stimulation of ideas.

Lt. Ronald P. Fuchs, USAF
Project Officer, SMTTA

ABSTRACT

A unified approach to the dynamic nonlinear estimation problem is presented. General algorithms for filtering, smoothing, and prediction for both continuous and discrete nonlinear systems are derived. The continuous problem is treated first by a direct approach which yields an exact solution. A similar approach is then applied to the more complex discrete estimation problem. The discussion provided points out some subtle, but important features of the main results which should be useful when applying the results to practical problems. To provide insight, motivation and background for the technique used in obtaining the solution of the general nonlinear problem, the special case of linear estimation is discussed in the first two appendices. In a third appendix it is shown that in the continuous measurement limit, the derived discrete and continuous nonlinear filtering algorithms are identical.

CONTENTS

TABLES

I. INTRODUCTION

For nearly two centuries the concept of least squares has served scientists well in the solution of estimation problems. Although the technique is basically linear, it has been used effectively in nonlinear problems by successive application to linearized approximations (differential corrections). In recent years, however, the requirements for increased accuracy and efficiency, particularly in the aerospace industry, have created considerable interest in the development of a purely nonlinear technique. With few exceptions, the least squares concept has been ignored and probability theory has been adopted as the basis for new developments. Although this recent approach has yielded some interesting mathematics, the results reported to date do not give a practical complete approach to nonlinear estimation (see, for example, comments by R. S. Bucy [4]). In any case, with the use of probability theory there is the inherent assumption that a system's dynamics, measurements, and relevant statistics can be modeled perfectly. Clearly this is not generally the case in practical engineering situations. This report will show that in abandoning the least squares approach, many researchers have overlooked a very practical and concise nonlinear estimation technique which is not limited by modeling assumptions.

An important breakthrough in extending least squares directly to nonlinear problems was made by Detchmendy and Sridhar in 1965 [5]. Their results, however, are limited to filtering for continuous nonlinear systems, and even then their derivation is indirect and unnecessarily involved.

In the first part of this report the continuous filtering problem will be solved from a new viewpoint. A similar approach will then be used to derive an appropriate filter for discrete nonlinear systems. Algorithms for smoothing in both continuous and discrete nonlinear systems will also be presented. The discussions provided point out some subtle, but important, features of the overall technique. They should be helpful in application of the main results to practical problems. Three appendices are also included. The first two furnish the necessary background and motivation for the technique used in solving the nonlinear estimation problem. The last one shows that in the continuous measurement limit the derived discrete and continuous nonlinear filtering formulas are identical.

BASIC NOTATIONAL CONVENTIONS

a) Matrices are upper case letters.

b) Vectors are lower case letters.

c) Exceptions to these rules are i, k, n, p, q which are used as integers; t, T which denote time; and J which represents a scalar.

d) Asterisks (*) denote matrix transposition.

e) Vectors are assumed to be columns unless otherwise denoted by an asterisk.

f) Parentheses ( ) are used exclusively for showing functional dependence, while square brackets [ ] are used for grouping terms in equations.

g) The operator $\partial/\partial x$ applied to a scalar (vector, matrix) gives the appropriate gradient (Jacobian, Hessian) evaluated at x.

II. CONTINUOUS STATE ESTIMATION

A. DERIVATION — CONTINUOUS SYSTEMS

Consider the system

$$\dot{x} = f(x, t) + \Gamma(t)w(t) \tag{1}$$

$$z = h(x, t) + v(t) \tag{2}$$

where x is a p-dimensional state vector, z is a q-dimensional vector of measurements, f(x, t) and h(x, t) are appropriately dimensioned vector functions of x and t, and $\Gamma(t)w(t)$ and v(t) are vectors representing unknown disturbances, noise, and general modeling errors, with $\Gamma(t)$ being a known $p \times k$ matrix function of time. It is to be emphasized that w(t) and v(t) are not necessarily random functions. They will, however, be assumed to have a zero time average. Note that a non-zero time average can be estimated as part of the state vector.

The generalized least squares estimation problem can be stated succinctly as follows. Find the y and u which minimize the scalar

$$J = \frac{1}{2}[y(t_0) - \hat{x}_0]^* P_0^{-1}[y(t_0) - \hat{x}_0]$$

$$+ \frac{1}{2}\int_{t_0}^{T}\Big[[z - h(y, t)]^* R^{-1}(t)[z - h(y, t)] + u^* Q^{-1}(t)\, u\Big]\, dt \tag{3}$$

subject to the constraint

$$\dot{y} = f(y, t) + \Gamma(t)u(t) \tag{4}$$

where P_0, R, and Q are arbitrary symmetric positive definite weighting matrices and $\hat{x}_0$ is an a priori guess of the initial state of the system. As stated, the problem is strictly one of smoothing; i.e., the solution at any instant t will be the least squares estimate of the state vector based on the data gathered in the entire interval $t_0 \le t \le T$. The filtering aspects of the problem will become apparent later.

Note from the form of (3) and (4) that the problem can be considered purely as one in control theory; i.e., it can be considered one of finding a control u(t) to force the output of the given system to track an arbitrary signal z(t). This viewpoint is particularly useful for indicating the possible application of the function u(t) to the modeling problem. Namely, since u(t) is the "best control" for driving a system with dynamics f(y, t) to track z(t), it can be considered an estimate of the difference between f(y, t) and the true dynamics of the system generating z(t).

The given problem of (3) and (4) clearly calls for application of the calculus of variations. By a well-known procedure [1, 3, 6] it can be shown that the solution must satisfy the two-point boundary value problem (TPBV)

$$\dot{y} = f(y, t) - \Gamma Q \Gamma^{*} \lambda \tag{5}$$

$$\dot{\lambda} = - \frac{\partial f}{\partial y}^{*} \lambda + \frac{\partial h}{\partial y}^{*} R^{-1}[z - h(y, t)] \tag{6}$$

$$\lambda(t_0) = - P_0^{-1}[y(t_0) - \hat{x}_0] \qquad (\lambda(T) = 0) \tag{7}$$

with

$$u(t) = - Q \Gamma^{*} \lambda(t) \tag{8}$$

where λ is a p-vector of unknown functions (adjoint variables). In general, a direct solution of an arbitrary TPBV problem is quite unlikely. For this case, however, a special approach leads to a very practical method of generating solutions. The procedure used is motivated by the solution of the linear version of this problem for which details are given in Appendix A.

That solution strongly suggests that a substitution of the form

$$y(t) = -P(t)\lambda(t) + \hat{x}(t) \tag{9}$$

where P is arbitrary and $\hat{x}$ is the filtered state estimate, be applied to (5) and (6). The conditions under which (9) satisfies the nonlinear TPBV problem will now be found.

Differentiating (9) with respect to t gives

$$\dot{y} = -\dot{P}\lambda - P\dot{\lambda} + \dot{\hat{x}} \tag{10}$$

which upon application of (5) and (6) becomes

$$f(y,t) - \Gamma Q \Gamma^{*}\lambda = -\dot{P}\lambda + P\frac{\partial f}{\partial y}^{*}\lambda - P\frac{\partial h}{\partial y}^{*}R^{-1}[z - h(y,t)] + \dot{\hat{x}} \tag{11}$$

Now, using (9) and noting that for $\lambda = 0$, $y = \hat{x}$, expand (11) in powers of λ. Then

$$f(\hat{x},t) - \frac{\partial f}{\partial \hat{x}}P\lambda - \Gamma Q \Gamma^{*}\lambda = -\dot{P}\lambda + P\frac{\partial f}{\partial \hat{x}}^{*}\lambda - P\left[\frac{\partial h}{\partial \hat{x}}^{*}R^{-1}[z - h(\hat{x},t)] - \frac{\partial}{\partial \hat{x}}\left(\frac{\partial h}{\partial \hat{x}}^{*}R^{-1}[z - h(\hat{x},t)]\right)P\lambda\right]$$
$$+ \dot{\hat{x}} + \text{higher order terms in } \lambda \tag{12}$$

Equating coefficients of the first two powers of λ to zero gives

$$\dot{\hat{x}} = f(\hat{x},t) + P\frac{\partial h}{\partial \hat{x}}^{*}R^{-1}[z - h(\hat{x},t)] \tag{13}$$

$$\dot{P} = \frac{\partial f}{\partial \hat{x}}P + P\frac{\partial f}{\partial \hat{x}}^{*} + \Gamma Q \Gamma^{*} + P\frac{\partial}{\partial \hat{x}}\left(\frac{\partial h}{\partial \hat{x}}^{*}R^{-1}[z - h(\hat{x},t)]\right)P \tag{14}$$

Assuming continuity of the functions involved, these equations are then necessary and sufficient to ensure the validity of (9) in a neighborhood of $\lambda = 0$. Consequently, by (7) they must hold near $t = T$, the end of the data span. Furthermore, the term "filter" is correct for (13) and (14) since by (9), $\lambda = 0$ gives $y = \hat{x}$, the best estimate of the state vector at T based on data gathered in the interval $t_0 \leq t \leq T$.

To obtain initial conditions for these equations, observe that T can be considered arbitrary, so that if $T \to t_0$, it must follow that

$$\hat{x}(t_0) = \hat{x}_0 \tag{15}$$

$$P(t_0) = P_0 \tag{16}$$

Note also from (9) and (7) that $\hat{x}(T)$ is the missing boundary condition for (5) so that a backward solution of (5)-(6) using

$$\lambda(T) = 0 \qquad \text{and} \qquad y(T) = \hat{x}(T) \tag{17}$$

immediately gives the complete smoothing solution in the interval $t_0 \leq t \leq T$.

These results are summarized in Table I using the standard notation $\hat{x}(t/T)$ for the state estimate at time t based on data up through time T. For completeness the solution to the prediction problem is presented.

B. DISCUSSION — CONTINUOUS SYSTEMS

The basic formulas for filtering (18) and (19) have been derived previously by Detchmendy and Sridhar [5]. Their derivation, however, is incomplete because it does not yield the all-important initial conditions. In addition, their approach is unnecessarily complicated by the introduction of a partial differential equation.

A potential application of least squares estimation, which apparently has been overlooked, is its possible use in the solution of the modeling

Table I. Continuous Nonlinear Estimation

Filtering: $t = T$

$$\frac{d\hat{x}}{dt} = f(\hat{x}, t) + P \frac{\partial h^*}{\partial \hat{x}} R^{-1}[z - h(\hat{x}, t)] \tag{18}$$

$$\frac{dP}{dt} = \frac{\partial f}{\partial \hat{x}} P + P \frac{\partial f^*}{\partial \hat{x}} + \Gamma Q \Gamma^* + P \frac{\partial}{\partial \hat{x}}\left(\frac{\partial h^*}{\partial \hat{x}} R^{-1}[z - h(\hat{x}, t)]\right) P \tag{19}$$

$$\hat{x}(t_0) = \hat{x}_0 \quad \text{and} \quad P(t_0) = P_0 \tag{20}$$

Smoothing: $t < T$

$$\frac{d}{dt}\hat{x}(t/T) = f(\hat{x}(t/T), t) - \Gamma Q \Gamma^* \lambda \tag{21}$$

$$\frac{d\lambda}{dt} = -\frac{\partial f^*}{\partial \hat{x}(t/T)} \lambda + \frac{\partial h^*}{\partial \hat{x}(t/T)} R^{-1}[z - h(\hat{x}(t/T), t)] \tag{22}$$

$$\hat{x}(T/T) = \hat{x}(T) \quad \text{and} \quad \lambda(T) = 0 \tag{23}$$

Prediction: $t > T$

$$\frac{d}{dt}\hat{x}(t/T) = f(\hat{x}(t/T), t) \tag{24}$$

$$\hat{x}(T/T) = \hat{x}(T) \tag{25}$$

problem. At present, errors in modeling system dynamics are often the most prominent cause of inaccuracy in practical estimation problems. In the least squares formulation of the problem, (1)-(4), the general system modeling error is included in the term w. The solution in Table I offers both a filtered and a smoothed estimate of w. From (18) and (8) these are

$$\Gamma\hat{w}(t) = P\frac{\partial h^*}{\partial \hat{x}} R^{-1}[z - h(\hat{x}, t)] \tag{26}$$

and

$$\Gamma\hat{w}(t/T) = -\Gamma Q \Gamma^* \lambda(t) \qquad (t < T) \tag{27}$$

respectively. Note that while the solution to the prediction problem (24) follows from (5)-(6) with $z(t) = h(y, t) = 0$, it may not be the most practical one for modeling problems. That is, if $\Gamma\hat{w}$ is significant, it may be advantageous to extrapolate it in some consistent manner for use in state vector prediction. One approach would be to find an appropriate dynamic model for this error, then augment the given state vector accordingly, and apply the formal prediction algorithm (24).

III. DISCRETE STATE ESTIMATION

A. DERIVATION – DISCRETE SYSTEMS

Although the continuous estimation formulas of Table I can be useful in certain situations, there are many practical estimation problems in which system measurements are not recorded continuously, but rather are known only at specific instants of time. This, coupled with the fact that it is generally desirable to perform analyses digitally, is good reason to seek a discrete analogue of the least squares formulas of Table I. Several attempts at deriving such formulas have been made in the past [11, 12] with no overall success. This is not surprising since the extension is not trivial as the discussion below will show. Although the discrete problem is similar to the previous one, there are sufficient differences to warrant a completely independent treatment. Since the results to be obtained are new, a particularly detailed discussion is provided.

For the discrete problem consider the system

$$x_i = g_{i-1}(x_{i-1}) + \Gamma_{i-1}w_{i-1} \qquad (2 \le i \le n) \qquad (28)$$

$$z_i = h_i(x_i) + v_i \qquad (1 \le i \le n) \qquad (29)$$

where x_i is a p-dimensional state vector, z_i is a q-dimensional vector of measurements, $g_{i-1}(x_{i-1})$ and $h_i(x_i)$ are appropriately dimensioned vector functions, and $\Gamma_{i-1}w_{i-1}$ and v_i are vectors representing unknown disturbances, noise, and general modeling errors, with Γ_{i-1} being a known $p \times k$ matrix function of time. It is to be emphasized that w_{i-1} and v_i are not necessarily random functions. They will, however, be assumed to have a zero time average. Note that a non-zero time average can be estimated as part of the state vector.

In this context the generalized least squares problem is that of finding the sequences $y_1, y_2, \ldots, y_n$ and $u_1, u_2, \ldots, u_{n-1}$ which minimize the scalar

$$J = \frac{1}{2}[y_1 - \hat{x}_0]^* P_0^{-1}[y_1 - \hat{x}_0] + \frac{1}{2}[z_1 - h_1(y_1)]^* R_1^{-1}[z_1 - h_1(y_1)]$$

$$+ \frac{1}{2}\sum_{i=2}^{n}\Big[[z_i - h_i(y_i)]^* R_i^{-1}[z_i - h_i(y_i)] + u_{i-1}^* Q_{i-1}^{-1} u_{i-1}\Big] \tag{30}$$

subject to the constraints

$$y_i = g_{i-1}(y_{i-1}) + \Gamma_{i-1}u_{i-1} \qquad (2 \le i \le n) \tag{31}$$

where P_0, R_i and Q_{i-1} are arbitrary symmetric positive definite weighting matrices, and $\hat{x}_0$ is an a priori guess of the initial state of the system at time t_1. Observe that (28) is meaningless for $n = 1$, and the basic problem reduces to finding the minimum of

$$J_1 = \frac{1}{2}[y_1 - \hat{x}_0]^* P_0^{-1}[y_1 - \hat{x}_0] + \frac{1}{2}[z_1 - h_1(y_1)]^* R_1^{-1}[z_1 - h_1(y_1)] \tag{32}$$

The multistage problem of (30) and (31) clearly falls within the scope of the well-known Lagrange multiplier technique for solving constrained extrema problems. By introducing two dummy constraints, $y_1 = g_0(y_0) + u_0$ and $y_{n+1} = g_n(y_n) + u_n$, a general formulation can be found which conveniently includes the single stage problem (32). In particular, write

$$J' = \frac{1}{2}[y_1 - \hat{x}_0]^* P_0^{-1}[y_1 - \hat{x}_0] + \lambda_{n+1}^*[y_{n+1} - g_n(y_n) - \Gamma_n u_n] + \frac{1}{2}u_0^* Q_0^{-1} u_0$$

$$+ \sum_{i=1}^{n}\Big[\frac{1}{2}[z_i - h_i(y_i)]^* R_i^{-1}[z_i - h_i(y_i)] + \frac{1}{2}u_i^* Q_i^{-1} u_i$$

$$+ \lambda_i^*[y_i - g_{i-1}(y_{i-1}) - \Gamma_{i-1}u_{i-1}]\Big] \tag{33}$$

where $\lambda_1, \lambda_2, \ldots, \lambda_n, \lambda_{n+1}$ are p-vectors of Lagrange multipliers. Now both J and J_1 will be minimized when $\partial J'/\partial y_i = 0$ for $i = 0, 1, 2, \ldots, n+1$; and $\partial J'/\partial u_i = 0$ for $i = 0, 1, 2, \ldots, n$, or, more explicitly, when

$$\lambda_1 = \frac{\partial g_1^*}{\partial y_1} \lambda_2 + \frac{\partial h_1^*}{\partial y_1} R_1^{-1}[z_1 - h_1(y_1)] - P_0^{-1}[y_1 - \hat{x}_0] \tag{34}$$

$$\lambda_i = \frac{\partial g_i^*}{\partial y_i} \lambda_{i+1} + \frac{\partial h_i^*}{\partial y_i} R_i^{-1}[z_i - h_i(y_i)] \qquad (2 \le i \le n) \tag{35}$$

$$y_i = g_{i-1}(y_{i-1}) + \Gamma_{i-1} Q_{i-1} \Gamma_{i-1}^* \lambda_i \qquad (2 \le i \le n) \tag{36}$$

$$\lambda_1 = \lambda_{n+1} = 0 \tag{37}$$

with

$$u_0 = 0 \quad \text{and} \quad u_i = Q_i \Gamma_i^* \lambda_{i+1} \qquad (1 \le i \le n) \tag{38}$$

Note that (34), (37) and (38) show J_1 is minimized for $n = 1$.

Observe that the problem described by (34) through (37) is analogous to the TPBV problem encountered earlier in the continuous estimation problem. Recall in that case that the method of solution was motivated by an approach to the corresponding linear problem. A similar technique can be used here. A solution to the linear discrete estimation problem is presented in Appendix B. That derivation strongly suggests that a substitution of the form

$$y_i = P_i \frac{\partial g_i^*}{\partial \hat{x}_i} \lambda_{i+1} + \hat{x}_i \tag{39}$$

$$y_{i-1} = P_{i-1} \frac{\partial g^*_{i-1}}{\partial \hat{x}_{i-1}} \lambda_i + \hat{x}_{i-1} \tag{40}$$

where P_i and P_{i-1} are arbitrary and $\hat{x}_i$ and $\hat{x}_{i-1}$ are filtered state estimates, be applied to the TPBV problem (34)-(37).

Proceeding along these lines, then, combine (36) and (39) to obtain

$$\Gamma_{i-1} Q_{i-1} \Gamma^*_{i-1} \lambda_i = P_i \frac{\partial g^*_i}{\partial \hat{x}_i} \lambda_{i+1} + \hat{x}_i - g_{i-1}(y_{i-1}) \tag{41}$$

Use (35) to eliminate λ_i, and expand $g_{i-1}(y_{i-1})$ about $\hat{x}_{i-1}$. Then

$$\Gamma_{i-1} Q_{i-1} \Gamma^*_{i-1} \frac{\partial g^*_i}{\partial y_i} \lambda_{i+1} + \Gamma_{i-1} Q_{i-1} \Gamma^*_{i-1} \frac{\partial h^*_i}{\partial y_i} R^{-1}_i [z_i - h_i(y_i)] =$$

$$P_i \frac{\partial g^*_i}{\partial \hat{x}_i} \lambda_{i+1} + \hat{x}_i - g_{i-1}(\hat{x}_{i-1}) - \frac{\partial g_{i-1}}{\partial \hat{x}_{i-1}} [y_{i-1} - \hat{x}_{i-1}] - \cdots \tag{42}$$

At this point the index i is confined to be near n, i.e., $i = n - 1, n$, where n is the last stage of the estimation problem. Other values of i are of no importance in the discussion below. With this in mind, it is assumed that $y_{n-2} - \hat{x}_{n-2}$ and λ_n are "small" in the sense that quadratic and higher order terms in these quantities can be ignored. The implications of this hypothesis will be considered later. Note from (40) that assuming λ_n to be small guarantees that $y_{n-1} - \hat{x}_{n-1}$ will also be small. Thus, for the i of interest, terms nonlinear in $y_{i-1} - \hat{x}_{i-1}$ in (42) can be omitted.

Continuing with the development, then, apply (40) and (35) to (42) to get

$$\Gamma_{i-1}Q_{i-1}\Gamma_{i-1}^{*} \frac{\partial g_i^{*}}{\partial y_i}\lambda_{i+1} + \Gamma_{i-1}Q_{i-1}\Gamma_{i-1}^{*} \frac{\partial h_i^{*}}{\partial y_i} R_i^{-1}[z_i - h_i(y_i)] =$$

$$P_i \frac{\partial g_i^{*}}{\partial \hat{x}_i}\lambda_{i+1} + \hat{x}_i - g_{i-1}(\hat{x}_{i-1})$$

$$- \frac{\partial g_{i-1}}{\partial \hat{x}_{i-1}}\left[P_{i-1} \frac{\partial g_{i-1}^{*}}{\partial \hat{x}_{i-1}}\left[\frac{\partial g_i^{*}}{\partial y_i} \lambda_{i+1} + \frac{\partial h_i^{*}}{\partial y_i} R_i^{-1}[z_i - h_i(y_i)]\right]\right] \quad (43)$$

Now, bearing in mind the fact that y_i is a function of λ_{i+1} by (39), and that λ_{i+1} is assumed to be small, expand both sides of (43) in powers of λ_{i+1} to obtain

$$\Gamma_{i-1}Q_{i-1}\Gamma_{i-1}^{*}\left[\frac{\partial h_i^{*}}{\partial \hat{x}_i} R_i^{-1}[z_i - h_i(\hat{x}_i)] + \frac{\partial}{\partial \hat{x}_i}\left(\frac{\partial h_i^{*}}{\partial \hat{x}_i} R_i^{-1}[z_i - h_i(\hat{x}_i)]\right)P_i \frac{\partial g_i^{*}}{\partial \hat{x}_i}\lambda_{i+1}\right]$$

$$+ \Gamma_{i-1}Q_{i-1}\Gamma_{i-1}^{*} \frac{\partial g_i^{*}}{\partial \hat{x}_i}\lambda_{i+1} = P_i \frac{\partial g_i^{*}}{\partial \hat{x}_i}\lambda_{i+1} + \hat{x}_i - g_{i-1}(\hat{x}_{i-1})$$

$$- \frac{\partial g_{i-1}}{\partial \hat{x}_{i-1}}\left[P_{i-1} \frac{\partial g_{i-1}^{*}}{\partial \hat{x}_{i-1}}\left[\frac{\partial g_i^{*}}{\partial \hat{x}_i}\lambda_{i+1} + \frac{\partial h_i^{*}}{\partial \hat{x}_i} R_i^{-1}[z_i - h_i(\hat{x}_i)]\right.\right.$$

$$\left.\left. + \frac{\partial}{\partial \hat{x}_i}\left(\frac{\partial h_i^{*}}{\partial \hat{x}_i} R_i^{-1}[z_i - h_i(\hat{x}_i)]\right)P_i \frac{\partial g_i^{*}}{\partial \hat{x}_i}\lambda_{i+1}\right]\right]$$

$$(44)$$

For convenience, the matrix is defined as

$$P_{i/i-1} \triangleq \Gamma_{i-1}Q_{i-1}\Gamma_{i-1}^{*} + \frac{\partial g_{i-1}}{\partial \hat{x}_{i-1}} P_{i-1} \frac{\partial g_{i-1}^{*}}{\partial \hat{x}_{i-1}} \quad (45)$$

and the vector as

$$\eta_i(\hat{x}_i) \triangleq \frac{\partial h_i^*}{\partial \hat{x}_i} R_i^{-1} [z_i - h_i(\hat{x}_i)] \tag{46}$$

Then (44) can be expressed more concisely as

$$P_{i/i-1} \frac{\partial g_i^*}{\partial \hat{x}_i} \lambda_{i+1} + P_{i/i-1}\eta_i(\hat{x}_i) + P_{i/i-1} \frac{\partial \eta_i}{\partial \hat{x}_i} P_i \frac{\partial g_i^*}{\partial \hat{x}_i} \lambda_{i+1} = P_i \frac{\partial g_i^*}{\partial \hat{x}_i} \lambda_{i+1} + \hat{x}_i - g_{i-1}(\hat{x}_{i-1}) \tag{47}$$

Let i = n. Then $\lambda_{n+1} = 0$ by (37), and (47) becomes

$$\hat{x}_n = g_{n-1}(\hat{x}_{n-1}) + P_{n/n-1}\eta_n(\hat{x}_n) \tag{48}$$

Now, assume that new data is acquired, i.e., the last stage becomes n + 1. Then, $\lambda_{n+2} = 0$, but $\lambda_{n+1} \neq 0$, so that equating coefficients of λ_{i+1} in (47) for i = n gives

$$P_n = P_{n/n-1} + P_{n/n-1} \frac{\partial \eta_n}{\partial \hat{x}_n} P_n \tag{49}$$

Thus (45) and (46) with i = n, and (48) and (49) are a set of recursion relations for P_n and $\hat{x}_n$. Note that (48) limited from a practical viewpoint because it is implicit in $\hat{x}_n$. Fortunately, the relationship can be solved directly by means of some interesting manipulations. Consider the quantity $\hat{x}_n - \bar{x}_n$ where $\bar{x}_n \triangleq g_{n-1}(\hat{x}_{n-1})$. From (36), (37) and (40)

$$\hat{x}_n - \bar{x}_n = g_{n-1}(y_{n-1}) + \Gamma_{n-1} Q_{n-1} \Gamma^*_{n-1} \lambda_n - g_{n-1}(\hat{x}_{n-1}) \tag{50}$$

Expanding the first term on the right about $\hat{x}_{n-1}$ and applying (40) and (45) yields

$$\hat{x}_n - \bar{x}_n = P_{n/n-1} \lambda_n \tag{51}$$

where λ_n has already been assumed to be small. Thus an expansion of (48) in powers of $\hat{x}_n - \bar{x}_n$ can be truncated after the linear term with no loss of generality. In particular, (48) becomes

$$\hat{x}_n = \bar{x}_n + P_{n/n-1} \left[\eta_n(\bar{x}_n) + \frac{\partial \eta_n}{\partial \bar{x}_n} [\hat{x}_n - \bar{x}_n] \right] \tag{52}$$

or, finally

$$\hat{x}_n = \bar{x}_n + \left[I - P_{n/n-1} \frac{\partial \eta_n}{\partial \bar{x}_n} \right]^{-1} P_{n/n-1} \eta_n(\bar{x}_n) \tag{53}$$

where the indicated matrix inverse is assumed to exist. A practical interpretation of this hypothesis will be given later.

To be consistent with the above result (49) must also be expanded in powers of $\hat{x}_n - \bar{x}_n$. At first it appears that third derivatives of $h_n(\bar{x}_n)$ will be involved in the coefficients of the linear term. Recall, however, that (49) resulted from equating coefficients of λ_{i+1} in (47). Since that quantity itself is assumed to be small, inclusion of the first power of $\hat{x}_n - \bar{x}_n$ would then effectively result in a "second order" term. Thus, (49) immediately reduces to

$$P_n = \left[I - P_{n/n-1} \frac{\partial \eta_n}{\partial \bar{x}_n} \right]^{-1} P_{n/n-1} \tag{54}$$

Note that (53) can now be written as

$$\hat{x}_n = \bar{x}_n + P_n \eta_n(\bar{x}_n) \tag{55}$$

Thus (54), (55) and

$$P_{n/n-1} = \Gamma_{n-1} Q_{n-1} \Gamma^*_{n-1} + \frac{\partial g_{n-1}}{\partial \hat{x}_{n-1}} P_{n-1} \frac{\partial g^*_{n-1}}{\partial \hat{x}_{n-1}} \tag{56}$$

form the desired set of recursion relations for filtering nonlinear systems.

To establish initial conditions for the sequence, let n = 2 and consider the original TPBV problem (34)-(37). From (34) with $\lambda_1 = 0$, it is clear that y_1 is some function of λ_2. Expanding y_1 in powers of λ_2 gives $y_1 = \hat{x}_1 + [\partial y_1 / \partial \lambda_2]\lambda_2 + \ldots$, where $\hat{x}_1$ must be the solution to the single stage estimation problem. Since by hypothesis λ_2 is small, this result shows that $y_1 - \hat{x}_1$ must also be small. Applying these ideas to an expansion of (34) in powers of $y_1 - \hat{x}_1$ yields

$$0 = \frac{\partial g^*_1}{\partial \hat{x}_1} \lambda_2 + \eta_1(\hat{x}_1) + \frac{\partial \eta_1}{\partial \hat{x}_1}[y_1 - \hat{x}_1] - P_0^{-1}[\hat{x}_1 - \hat{x}_0] - P_0^{-1}[y_1 - \hat{x}_1] \tag{57}$$

Since by definition $\hat{x}_1$ satisfies

$$\eta_1(\hat{x}_1) - P_0^{-1}[\hat{x}_1 - \hat{x}_0] = 0 \tag{58}$$

(57) reduces to

$$y_1 = \left[I - P_0 \frac{\partial \eta_1}{\partial \hat{x}_1}\right]^{-1} P_0 \frac{\partial g_1^*}{\partial \hat{x}_1} \lambda_2 + \hat{x}_1 \tag{59}$$

Now assume that $y_1 - \hat{x}_0$ is small and expand $\eta_1(\hat{x}_1)$ and $\partial\eta_1/\partial\hat{x}_1$ about $\hat{x}_0$. Thus (58) and (59) become

$$\hat{x}_1 = \hat{x}_0 + \left[I - P_0 \frac{\partial \eta_1}{\partial \hat{x}_0}\right]^{-1} P_0 \eta_1(\hat{x}_0) \tag{60}$$

$$y_1 = \left[I - P_0 \frac{\partial \eta_1}{\partial \hat{x}_0}\right]^{-1} P_0 \frac{\partial g_1^*}{\partial \hat{x}_1} \lambda_2 + \hat{x}_1 \tag{61}$$

Comparing (61) with (39) shows that

$$P_1 \triangleq \left[I - P_0 \frac{\partial \eta_1}{\partial \hat{x}_0}\right]^{-1} P_0 \tag{62}$$

Thus (60) and (62) are the required initial conditions for the filter. Note that the assumption that $y_1 - \hat{x}_0$ is small is totally consistent with those made earlier in developing the basic recursion relationships.

As in the continuous case, the solution to the smoothing problem follows directly from the results of filtering. That is, $\hat{x}_n$ and $\lambda_{n+1} = 0$ are final conditions for running (35) and (36) backward from $i = n$.

Table II. Discrete Nonlinear Estimation

Filtering: $(i = 1, 2, \ldots)$

$$\bar{x}_1 = \hat{x}_0 \quad \text{and} \quad P_{1/0} = P_0 \tag{63}$$

$$\hat{x}_i = \bar{x}_i + P_i \frac{\partial h_i^*}{\partial \bar{x}_i} R_i^{-1} [z_i - h_i(\bar{x}_i)] \tag{64}$$

$$P_i = \left[I - P_{i/i-1} \frac{\partial}{\partial \bar{x}_i} \left(\frac{\partial h_i^*}{\partial \bar{x}_i} R_i^{-1} [z_i - h_i(\bar{x}_i)] \right) \right]^{-1} P_{i/i-1} \tag{65}$$

$$\bar{x}_{i+1} = g_i(\hat{x}_i) \tag{66}$$

$$P_{i+1/i} = \Gamma_i Q_i \Gamma_i^* + \frac{\partial g_i}{\partial \hat{x}_i} P_i \frac{\partial g_i^*}{\partial \hat{x}_i} \tag{67}$$

Smoothing: $(i = n, n-1, \ldots, 2)$

$$\lambda_{n+1} = 0 \quad \text{and} \quad \hat{x}_{n/n} = \hat{x}_n \tag{68}$$

$$\lambda_i = \frac{\partial g_i^*}{\partial \hat{x}_{i/n}} \lambda_{i+1} + \frac{\partial h_i^*}{\partial \hat{x}_{i/n}} R_i^{-1} [z_i - h_i(\hat{x}_{i/n})] \tag{69}$$

$$\hat{x}_{i-1/n} = g_i^{-1}(\hat{x}_{i/n} - \Gamma_{i-1} Q_{i-1} \Gamma_{i-1}^* \lambda_i) \tag{70}$$

Prediction: $(i = n, n+1, \ldots)$

$$\hat{x}_{n/n} = \hat{x}_n \tag{71}$$

$$\hat{x}_{i+1/n} = g_i(\hat{x}_{i/n}) \tag{72}$$

For the convenience of the reader, the results for the discrete estimation problem are summarized in Table II. The notation $\hat{x}_{i/n}$ is used to indicate an estimate of the state vector at t_i based on data through t_n. For completeness the prediction results are included. It is noted that $\bar{x}_i$ in the filter is itself a prediction which is updated by the measurement z_i to give $\hat{x}_i$.

Observe that the use of (64) and (65) requires inversion of a $p \times p$ matrix at each stage. If $\partial(\partial h_i^* / \partial \bar{x}_i)/\partial \bar{x}_i$ or $z_i - h_i(\bar{x}_i)$ is small (this includes the case when measurements are linear), then

$$\frac{\partial}{\partial \bar{x}_i}\left(\frac{\partial h_i^*}{\partial \bar{x}_i} R_i^{-1} [z_i - h_i(\bar{x}_i)]\right) \approx \frac{\partial h_i^*}{\partial \bar{x}_i} R_i^{-1} \frac{\partial h_i}{\partial \bar{x}_i} \tag{73}$$

and the famous Matrix Inversion Lemma [9] can be applied to (64) and (65) to yield

$$\hat{x}_i = \bar{x}_i + P_{i/i-1} \frac{\partial h_i^*}{\partial \bar{x}_i}\left[\frac{\partial h_i}{\partial \bar{x}_i} P_{i/i-1} \frac{\partial h_i^*}{\partial \bar{x}_i} + R_i\right]^{-1} [z_i - h_i(\bar{x}_i)] \tag{74}$$

$$P_i = P_{i/i-1} - P_{i/i-1} \frac{\partial h_i^*}{\partial \bar{x}_i}\left[\frac{\partial h_i}{\partial \bar{x}_i} P_{i/i-1} \frac{\partial h_i^*}{\partial \bar{x}_i} + R_i\right]^{-1} \frac{\partial h_i}{\partial \bar{x}_i} P_{i/i-1} \tag{75}$$

These formulas require inversion of $q \times q$ matrices. The result (75) can, in turn, be put in the more computationally attractive form

$$K_i = P_{i/i-1} \frac{\partial h_i^*}{\partial \bar{x}_i}\left[\frac{\partial h_i}{\partial \bar{x}_i} P_{i/i-1} \frac{\partial h_i^*}{\partial \bar{x}_i} + R_i\right]^{-1} \tag{76}$$

$$P_i = \left[I - K_i \frac{\partial h_i}{\partial \bar{x}_i}\right] P_{i/i-1} \left[I - K_i \frac{\partial h_i}{\partial \bar{x}_i}\right]^* + K_i R_i K_i^* \tag{77}$$

The consistency of the approach taken here with that used earlier for continuous systems is verified in Appendix C. There it is shown that the limiting case of the discrete filters of Table II as the time between measurements becomes small is indeed the continuous filter of Table I.

B. DISCUSSION – DISCRETE SYSTEMS

1. Fundamental Interpretations

In this section the assumptions made in deriving the formulas of Table II will be discussed in detail along with other items of general practical interest.

First consider the assumption that λ_n and $y_{n-2} - \hat{x}_{n-2}$ are small. From (35) it can be seen that λ_n will be small if $[\partial h_n^* / \partial y_n] R_n^{-1} [z_n - h_n(y_n)]$ is small. This will be true if a least squares fit of the last stage data alone gives results near those of the composite n-stage fit. Thus, small λ_n is essentially a requirement that new data be consistent with previously processed information. Similarly, the assumption of small $y_{n-2} - \hat{x}_{n-2}$ (i.e., closeness of filtering and smoothing near n) is a requirement that the introduction of new data does not significantly change the results. Note by comparison to the derivation of the continuous formulas of Table I, that the above assumptions effectively simulate a "continuity" condition for the discrete problem. Basically then, the restrictions are mathematical in nature and are not necessarily limiting from an engineering point of view. Certainly, if difficulty were to occur in a practical problem, it would be expected at the initial stages of filtering. An approach to be followed in such situations will be suggested in a later section.

In the derivation of (63) - (66) the existence of the indicated matrix inverse was assumed. This can be justified on practical grounds as follows. Let i = 1 and note that $P_{1/0}$ is positive definite. Then

$$\left[I - P_{1/0} \frac{\partial}{\partial \bar{x}_1} \left(\frac{\partial h_1^*}{\partial \bar{x}_1} R_1^{-1} [z_1 - h_1(\bar{x}_1)] \right) \right] = P_{1/0} \left[P_{1/0}^{-1} + \frac{\partial h_1^*}{\partial \bar{x}_1} R_1^{-1} \frac{\partial h_1}{\partial \bar{x}_1} - \frac{\partial}{\partial \bar{x}_1} \left(\frac{\partial h_1^*}{\partial \bar{x}_1} \right) R_1^{-1} [z_1 - h_1(\bar{x}_1)] \right] \tag{78}$$

Now, clearly $[\partial h_1^*/\partial \bar{x}_1] R_1^{-1} [\partial h_1/\partial \bar{x}_1]$ is at least positive semidefinite. The matrix sum on the right of (78) will thus be positive definite unless the quantity $R_1^{-1} [z_1 - h_1(\bar{x}_1)]$ is "large". Such a situation, however, implies that the data z_1 is especially inconsistent with $\bar{x}_1$. It could then be argued that the weighting matrix R_1^{-1} be decreased to limit the effect of z_1 in the resulting estimate. This, in turn, would ensure the positive definiteness of the matrix in question. This reasoning can be extended inductively for all i by observing from (67) that $P_{n/n-1}$ must be positive definite if P_{n-1} has that property (it is implicitly assumed that $[\partial g_{n-1}/\partial \hat{x}_{n-1}]$ is invertibie). Thus, in practical situations where weighting matrices have been sensibly chosen, a matrix singularity should not arise.

An interesting interpretation of $P_{i/i-1}$ and P_i can be obtained by comparing (64) and (65) with (60) and (62). These relations can readily be used to show that $P_{i/i-1}$ is the appropriate "a priori" weighting matrix which transforms the basic problem of (30) into a sequence of single stage problems. That is, minimizing

$$J_i = \frac{1}{2} [y_i - \bar{x}_i]^* P_{i/i-1}^{-1} [y_i - \bar{x}_i] + \frac{1}{2} [z_i - h_i(y_i)]^* R_i^{-1} [z_i - h_i(y_i)] \quad (79)$$

for $i = 2, 3, \ldots, n$ is equivalent to minimizing (30). In this context, it can also be shown that $P_i = \partial(\partial J_i/\partial \bar{x}_i)/\partial \bar{x}_i$.

The formula (79) leads to a heuristic discussion of the behavior of $P_{i/i-1}$. In well behaved filtering problems where large amounts of data are available it seems that there should be a point beyond which new data is redundant. By (79) this could only occur if $P_{i/i-1}$ reached a steady-state value which is "small" compared to R_i. Certainly if $\Gamma_{i-1} Q_{i-1} \Gamma_{i-1}^*$ is large, this could not happen. Note, however, that new data should never be redundant in such cases since Q is chosen to be large when large disturbances are anticipated or when there is little confidence in the system dynamic model. In general, the behavior of the matrix $P_{i/i-1}$ in practical problems should provide some indication as to the adequacy of the filtering process. Note that establishing conditions for convergence of the sequential estimation formulas of Table II is an open problem. It is conjectured that the solution is the key to nonlinear observability.

The modeling problem will not be discussed in detail because the comments made for continuous systems in that regard apply here essentially unchanged. Note that the appropriate filtered and smoothed estimates of the dynamic error are, respectively,

$$\Gamma_{i-1}\hat{w}_{i-1} = P_i \frac{\partial h_i^*}{\partial \bar{x}_i} R_i^{-1} [z_i - h_i(\bar{x}_i)] \tag{80}$$

$$\Gamma_{i-1}\hat{w}_{i-1,n} = \Gamma_{i-1} Q_{i-1} \Gamma_{i-1}^* \lambda_i \qquad (2 \le i \le n) \tag{81}$$

2. Comparison With Differential Corrections

Adequate results have often been obtained in practical nonlinear estimation problems using linear techniques such as the Kalman Filter in conjunction with the method of differential corrections [7,8]. The basic philosophy of this approach can be easily demonstrated and compared to the method proposed here using a single stage estimation problem as an example.

To minimize

$$J = \frac{1}{2} [z - h(y)]^* R^{-1} [z - h(y)] \tag{82}$$

by the method of differential corrections, it is assumed that the minimizing value of y is close to some nominal $\bar{x}$ so that h(y) can be adequately approximated as a linear function of $y - \bar{x}$. That is,

$$J \approx \frac{1}{2}\left[z - h(\bar{x}) - \frac{\partial h}{\partial \bar{x}} [y - \bar{x}]\right]^* R^{-1} \left[z - h(\bar{x}) - \frac{\partial h}{\partial \bar{x}} [y - \bar{x}]\right] \tag{83}$$

Then, $\partial(J)/\partial y = 0$ gives the estimate

$$\hat{x} = \bar{x} + \left[\frac{\partial h^*}{\partial \bar{x}} R^{-1} \frac{\partial h}{\partial \bar{x}}\right]^{-1} \frac{\partial h^*}{\partial \bar{x}} R^{-1} [z - h(\bar{x})] \tag{84}$$

Note that the linearizing approximation is made <u>before</u> minimization is attempted.

The alternative presented here is essentially one of <u>first</u> minimizing and <u>then</u> approximating. Thus, referring to (82),

$$\frac{\partial}{\partial y}(J) = - \frac{\partial h^*}{\partial y} R^{-1}[z - h(y)] = 0 \tag{85}$$

Now assume that $\partial(J)/\partial y$ is a linear function of $y - \bar{x}$ and obtain

$$\hat{x} = \bar{x} + \left[- \frac{\partial}{\partial \bar{x}} \left(\frac{\partial h^*}{\partial \bar{x}} R^{-1}[z - h(\bar{x})]\right)\right]^{-1} \frac{\partial h^*}{\partial \bar{x}} R^{-1}[z - h(\bar{x})] \tag{86}$$

or

$$\hat{x} = \bar{x} + \left[\frac{\partial h^*}{\partial \bar{x}} R^{-1} \frac{\partial h}{\partial \bar{x}} - \frac{\partial}{\partial \bar{x}} \left(\frac{\partial h^*}{\partial \bar{x}}\right) R^{-1}[z - h(\bar{x})]\right]^{-1} \frac{\partial h^*}{\partial \bar{x}} R^{-1}[z - h(\bar{x})] \tag{87}$$

Compare (84) and (87) and note the term missing in (84). The basic result (87) can also be derived by assuming J to be a quadratic function of $y - \bar{x}$ and then minimizing that approximation.

The observations made above apply in principle to the much more complex dynamic estimation problem. Thus, use of the formulas presented in this paper instead of those for the differentially corrected Kalman filter should provide impr ed accuracy and computational efficiency in practical problems.

3. Unusual Cases

The direct use of the basic nonlinear filtering algorithm of (63)-(67) should provide efficient and accurate state estimation for most practical problems.

Some difficulty could arise, however, when the data taken in the initial stages are of a particularly poor quality. The following heuristic "start up" procedure is recommended in such cases.

Apply (63)-(67) for several stages. Use the smoothing algorithm (68)-(70) to obtain λ_1. If $\lambda_1 = 0$, continue filtering from the point it was interrupted. On the other hand, if $\lambda_1 \neq 0$, two choices remain: either continue with filtering for a few more stages and smooth again, or restart the problem using the smoothed result at the first stage for linearization. That is, let [8]

$$P_1 = \left[I - P_{1/0} \frac{\partial}{\partial \hat{x}_{1/n}} \left(\frac{\partial h_1^*}{\partial \hat{x}_{1/n}} R_1^{-1} [z_1 - h_1(\hat{x}_{1/n})] \right) \right]^{-1} P_{1/0} \qquad (88)$$

$$\hat{x}_1 = \hat{x}_{1/n} + P_1 \left[\frac{\partial h_1^*}{\partial \hat{x}_{1/n}} R_1^{-1} [z_1 - h_1(\hat{x}_{1/n})] + P_{1/0}^{-1} [\hat{x}_0 - \hat{x}_{1/n}] \right] \qquad (89)$$

Continue this "looping" procedure until $\lambda_1 = 0$.

Another special case of interest is encountered if large amounts of reliable data are available. In such situations the filtering algorithm of Table II may "converge" sufficiently at some point so that the term $[\partial(\partial h_i^* / \partial \bar{x}_i)/\partial \bar{x}_i] R_i^{-1} [z_i - h_i(\bar{x}_i)]$ can be omitted and the formulas (74), (76), and (77) used in place of (64) and (65). Of course, this would only be advantageous if the dimension of the measurement vector q is less than that of the state vector p.

APPENDIX A. CONTINUOUS LINEAR ESTIMATION

Although the solutions to linear estimation problems are well known[2], they are derived by a special technique here and in Appendix B specifically to provide insight and motivation for the techniques used to solve the corresponding nonlinear estimation problems.

The linear version of the continuous least squares estimation problem is obtained if in (1)-(2) $f(x,t) = F(t)x$ and $h(x,t) = H(t)x$. Then (5)-(7) become

$$\dot{y} = Fy - \Gamma Q \Gamma^* \lambda \tag{A-1}$$

$$\dot{\lambda} = -F^*\lambda + H^* R^{-1}[z - Hy] \tag{A-2}$$

$$\lambda(t_0) = -P_0^{-1}[y(t_0) - \hat{x}_0] \qquad (\lambda(T) = 0) \tag{A-3}$$

By expressing (A-1) and (A-2) in the augmented vector-matrix form

$$\begin{bmatrix} \dot{y} \\ \dot{\lambda} \end{bmatrix} = \begin{bmatrix} F & -\Gamma Q \Gamma^* \\ -H^* R^{-1} H & -F^* \end{bmatrix} \begin{bmatrix} y \\ \lambda \end{bmatrix} + \begin{bmatrix} 0 \\ H^* R^{-1} z \end{bmatrix} \tag{A-4}$$

and partitioning the corresponding transition matrix accordingly, the solution can be expressed as

$$y(t) = \Phi_{11}(t,t_0)y(t_0) + \Phi_{12}(t,t_0)\lambda(t_1) + \int_{t_0}^{t} \Phi_{12}(t,\sigma)H^* R^{-1} z(\sigma)\, d\sigma \tag{A-5}$$

$$\lambda(t) = \Phi_{21}(t,t_0)y(t_0) + \Phi_{22}(t,t_0)\lambda(t_1) + \int_{t_0}^{t} \Phi_{22}(t,\sigma)H^* R^{-1} z(\sigma)\, d\sigma \tag{A-6}$$

This result can be transformed to a much more useful form as follows. Solve (A-6) for $y(t_0)$ and substitute the results in (A-5). Then

$$y(t) = \Phi_{11}(t,t_0)\Phi_{21}^{-1}(t,t_0)\lambda(t) + \left[\Phi_{12}(t,t_0) - \Phi_{11}(t,t_0)\Phi_{21}^{-1}(t,t_0)\Phi_{22}(t,t_0)\right]\lambda(t_0)$$

$$- \Phi_{11}(t,t_0)\Phi_{21}^{-1}(t,t_0)\int_{t_0}^{t} \Phi_{22}(t,\sigma)H^*R^{-1}z(\sigma)d\sigma + \int_{t_0}^{t} \Phi_{12}(t,\sigma)H^*R^{-1}z(\sigma)d\sigma$$

(A-7)

Since (A-7) holds for arbitrary t_0, the solution to equations (A-1) and (A-2) for the boundary condition $\lambda(T) = 0$ can be expressed in the general form

$$y(t) = -P(t)\lambda(t) + \xi(t) \qquad \text{(A-8)}$$

with $P(t)$ and $\xi(t)$ as yet unspecified. Differentiating (A-8) and substituting from (A-1) and (A-2) yields

$$-\dot{\xi} + F\xi - PH^*R^{-1}H\xi + PH^*R^{-1}z = [-\dot{P} + FP + PF^* - PH^*R^{-1}HP + \Gamma Q\Gamma^*]\lambda$$

(A-9)

For this to hold for all λ requires

$$\dot{\xi} = F\xi + PH^*R^{-1}[z - H\xi] \qquad \text{(A-10)}$$

$$\dot{P} = FP + PF^* + \Gamma Q\Gamma^* - PH^*R^{-1}HP \qquad \text{(A-11)}$$

From (A-8) and (A-3) the initial conditions for these equations are seen to be

$$\xi(t_0) = \hat{x}_0 \qquad \text{(A-12)}$$

$$P(t_0) = P_0 \tag{A-13}$$

Further, since $\lambda(T) = 0$, this gives

$$y(T) = \xi(T) \tag{A-14}$$

The proper interpretation of the result is the key to the solution desired. Recall that y is the smoothed estimate in the interval $t_0 \leq t \leq T$. From this it should be clear that $\xi(t)$ for arbitrary t is an estimate of the state vector based on the data gathered from t_0 to t. By definition, then, this is the "filtered" estimate of the state. Note also that for any T, the term $\xi(T)$ is the missing boundary condition on (A-1) so that a backward solution of (A-1)-(A-2) using

$$\lambda(T) = 0 \qquad \text{and} \qquad y(T) = \xi(T) \tag{A-15}$$

immediately gives the complete smoothing solution in the interval $t_0 \leq t \leq T$.

Note that for the standard statistical model where in (1) and (2) w and v are white noise processes with spectral density matrices Q and R, respectively, the filter (A-10)-(A-11) is the Kalman filter. In any case it is the optimal least squares filter.

A useful interpretation of the matrix P can be obtained from the defining relationship (A-8). Differentiating with respect to λ yields

$$\frac{\partial}{\partial \lambda}\left(\hat{x}(t/T) - \hat{x}(t)\right) = -P(t) \tag{A-16}$$

From this it is clear that a "small" P indicates that the difference between the smoothed and filtered estimates is relatively unaffected by changes in λ. This would be expected to occur after large amounts of data have been processed or when the a priori guess of the state vector is especially consistent with the observed data. (Observability is implicitly assumed here.) In such cases, the smoothing algorithm (A-1), (A-2) and (A-15) would be of little value.

On the other hand, if P were "large", (A-16) clearly indicates an improvement in estimating the state from smoothing. In a qualitative sense, then, P is a measure of the effectiveness of filtering where the ultimate standard of "goodness" is the smoothed estimate.

An interesting observation can be made from (A-8). The main motivation for its introduction was to solve the filtering problem. If, instead, the object had been to solve the smoothing (tracking) problem directly, the desired substitution would be

$$\lambda(t) = -S(t)y(t) + \zeta(t) \tag{A-17}$$

This gives "feedback control" directly when substituted in (A-1). Now, comparing (A-17) with (A-8) it is immediately obvious that

$$S = P^{-1} \tag{A-18}$$

and, indeed, the resulting differential equation for S is precisely the equation for P^{-1}. This is, of course, the well-known duality between the optimal control and optimal filtering problems. Note that substitutions of the form (A-17) have been used effectively in solving nonlinear TPBV problems by McReynolds [10].

APPENDIX B. DISCRETE LINEAR ESTIMATION

The linear version of the discrete least squares estimation problem is obtained if in (28)-(29) $g_i(x_i) = \Phi_i x_i$ and $h_i(x_i) = H_i x_i$. In that case (34)-(37) becomes

$$\lambda_1 = \Phi_1^* \lambda_2 + H_1^* R_1^{-1}[z_1 - H_1 y_1] - P_0^{-1}[y_1 - \hat{x}_0] \tag{B-1}$$

$$\lambda_i = \Phi_i^* \lambda_{i+1} + H_i^* R_i^{-1} [z_i - H_i y_i] \tag{B-2}$$

$$y_i = \Phi_{i-1} y_{i-1} + \Gamma_{i-1} Q_{i-1} \Gamma_{i-1}^* \lambda_i \qquad (2 \le i \le n) \tag{B-3}$$

$$\lambda_1 = \lambda_{n+1} = 0 \tag{B-4}$$

Solving (B-1) for x_1 with $\lambda_1 = 0$ immediately gives

$$y_1 = \left[I + P_0 H_1^* R_1^{-1} H_1\right]^{-1} P_0 \left[\Phi_1^* \lambda_2 + P_0^{-1} \hat{x}_0 + H_1 R_1^{-1} z_1\right] \quad . \tag{B-5}$$

Now let

$$P_1 \triangleq \left[I + P_0 H_1^* R_1^{-1} H_1\right]^{-1} P_0 \tag{B-6}$$

and

$$\begin{aligned} \xi_1 &\triangleq \left[I + P_0 H_1^* R_1^{-1} H_1\right]^{-1} P_0 \left[P_0^{-1} \hat{x}_0 + H_1^* R_1^{-1} z_1\right] \\ &= \hat{x}_0 + \left[I + P_0 H_1^* R_1^{-1} H_1\right]^{-1} P_0 H_1^* R_1^{-1} [z_1 - H_1 \hat{x}_0] \end{aligned} \tag{B-7}$$

so that (B-5) can be written as

$$y_1 = P_1 \Phi_1^* \lambda_2 + \xi_1 \tag{B-8}$$

This suggests that in general the change of variable

$$y_i = P_i \Phi_i^* \lambda_{i+1} + \xi_i \qquad (1 \le i \le n) \tag{B-9}$$

(where P_i and ξ_i are arbitrary) be applied to (B-1) and (B-2). Note from (B-4), however, that if (B-9) is to be valid, ξ_i must be the smoothed state vector at the end of an i-stage process. By definition this is the "filtered" state vector and, therefore,

$$\xi_i = \hat{x}_i \tag{B-10}$$

where the notation $\hat{x}_i$ is introduced to specifically indicate a filtered estimate. Note that for $i = n = 1$, ξ_1 has already been defined correctly.

To apply (B-9) to (B-2) and (B-3) write

$$y_{i-1} = P_{i-1}\Phi^*_{i-1}\lambda_i + \hat{x}_{i-1} \tag{B-11}$$

$$y_i = P_i\Phi^*_i\lambda_{i+1} + \hat{x}_i \tag{B-12}$$

Combining this with (B-3) gives

$$\Gamma_{i-1}Q_{i-1}\Gamma^*_{i-1}\lambda_i = P_i\Phi^*_i\lambda_{i+1} + \hat{x}_i - \Phi_{i-1}P_{i-1}\Phi^*_{i-1}\lambda_i - \Phi_{i-1}\hat{x}_{i-1} \tag{B-13}$$

By first applying (B-2) and then (B-12), λ_i and x_i can be eliminated from (B-13) to give

$$\begin{aligned}
&\Gamma_{i-1}Q_{i-1}\Gamma^*_{i-1}\lambda_{i+1} + \Gamma_{i-1}Q_{i-1}\Gamma^*_{i-1}H^*_iR^{-1}_i\left[z_i - H_i\left[P_i\Phi^*_i\lambda_{i+1} + \hat{x}_i\right]\right] \\
&= P_i\Phi^*_i\lambda_{i+1} + \hat{x}_i - \Phi_{i-1}\hat{x}_{i-1} - \Phi_{i-1}P_{i-1}\Phi^*_{i-1}\Bigg[\Phi^*_i\lambda_{i+1} + H^*_iR^{-1}_i \\
&\qquad \times \left[z_i - H_i\left[P_i\Phi^*_i\lambda_{i+1} + \hat{x}_i\right]\right]\Bigg]
\end{aligned} \tag{B-14}$$

For convenience define

$$P_{i/i-1} \triangleq \Gamma_{i-1}Q_{i-1}\Gamma_{i-1}^{*} + \Phi_{i-1}P_{i-1}\Phi_{i-1}^{*} \tag{B-15}$$

Then (B-14) can be simplified to

$$P_{i/i-1}\Phi_i^{*}\lambda_{i+1} + P_{i/i-1}H_i^{*}R_i^{-1}z_i - \hat{x}_i + \Phi_{i-1}\hat{x}_{i-1}$$

$$= P_i\Phi_i^{*}\lambda_{i+1} + P_{i/i-1}H_i^{*}R_i^{-1}H_iP_i\Phi_i^{*}\lambda_{i+1} + P_{i/i-1}H_i^{*}R_i^{-1}H_i\hat{x}_i \tag{B-16}$$

For $i = n$, the last stage of an estimation problem, $\lambda_{i+1} = 0$ by (B-4) so that (B-16) becomes

$$\hat{x}_n = \Phi_{n-1}\hat{x}_{n-1} + \left[I + P_{n/n-1}H_n^{*}R_n^{-1}H_n\right]^{-1} P_{n/n-1}H_n^{*}R_n^{-1}[z_n - H_n\Phi_{n-1}\hat{x}_{n-1}] \tag{B-17}$$

Suppose now that additional data is acquired, i.e., the last stage becomes $n + 1$. Then $\lambda_{n+2} = 0$ but $\lambda_{n+1} \neq 0$, so that equating coefficients of λ_{i+1} in (B-16) for $i = n$ and simplifying gives

$$P_n = \left[I + P_{n/n-1}H_n^{*}R_n^{-1}H_n\right]^{-1} P_{n/n-1} \tag{B-18}$$

Thus the formulas (B-17), (B-18), and (B-15) with the initial conditions (B-6) and (B-7) can be used recursively for least squares filtering of an arbitrary linear process.

Note also that for any n, the term $\hat{x}_n$ is the missing boundary condition on (B-4) so that a backward solution of (B-2)-(B-3) using

$$y_n = \hat{x}_n \qquad \text{and} \qquad \lambda_{n+1} = 0 \tag{B-19}$$

immediately gives the complete smoothing solution.

APPENDIX C. CONTINUOUS MEASUREMENT LIMIT FOR THE DISCRETE NONLINEAR FILTER

It will be shown here that as the time between successive measurements approaches zero, the discrete filtering formulas of Table II approach the continuous ones of Table I.

To begin, some basic relationships must be established between the various quantities involved. Integrating (4) between t_{i-1} and t_i with $\Delta t = t_i - t_{i-1}$ assumed to be small yields

$$y_i \approx y_{i-1} + f(y_{i-1}, t_{i-1})\Delta t + \Gamma(t_{i-1})u(t_{i-1})\Delta t \tag{C-1}$$

Comparing this with (31) immediately gives

$$g_{i-1}(y_{i-1}) = y_{i-1} + f(y_{i-1}, t_{i-1})\Delta t \tag{C-2}$$

$$\Gamma_{i-1}u_{i-1} = \Gamma(t_{i-1})u(t_{i-1})\Delta t \tag{C-3}$$

whereupon

$$\frac{\partial g_{i-1}}{\partial y_{i-1}} = I + \frac{\partial f}{\partial y_{i-1}} \Delta t \tag{C-4}$$

To force the two least squares criteria (3) and (30) to be equivalent, it is clear that

$$R_i^{-1} = R^{-1}(t_i)\Delta t \tag{C-5}$$

$$Q_{i-1}^{-1} = \frac{Q^{-1}(t_i)}{\Delta t} \tag{C-6}$$

where (C-3) was used to establish the last result. Combining (C-4), (C-6), and (67) gives

$$P_{i/i-1} = \Gamma(t_{i-1})Q(t_{i-1})\Gamma^{*}(t_{i-1})\Delta t + P_{i-1} + \frac{\partial f}{\partial y_{i-1}} P_{i-1}\Delta t + P_{i-1} \frac{\partial f^{*}}{\partial y_{i-1}} \Delta t$$

$$+ \frac{\partial f}{\partial y_{i-1}} P_{i-1} \frac{\partial f^{*}}{\partial y_{i-1}} \Delta t^{2} \qquad \text{(C-7)}$$

In order to use this relationship effectively, (65) must be expressed in the form

$$P_i = \left[I - P_{i/i-1} \frac{\partial \eta_i}{\partial \bar{x}_i}\right]^{-1} P_{i/i-1}$$

$$= P_{i/i-1} - P_{i/i-1}P^{-1}_{i/i-1}\left[I - P_{i/i-1} \frac{\partial \eta_i}{\partial \bar{x}_i}\right]\left[I - P_{i/i-1} \frac{\partial \eta_i}{\partial \bar{x}_i}\right]^{-1} P_{i/i-1}$$

$$+ \left[I - P_{i/i-1} \frac{\partial \eta_i}{\partial \bar{x}_i}\right]^{-1} P_{i/i-1}$$

$$= P_{i/i-1} + P_{i/i-1} \frac{\partial \eta_i}{\partial \bar{x}_i}\left[I - P_{i/i-1} \frac{\partial \eta_i}{\partial \bar{x}_i}\right]^{-1} P_{i/i-1} \qquad \text{(C-8)}$$

where the notational abbreviation (46) has been used for convenience. Now use (C-7) to eliminate the first term on the right of (C-8). Then

$$\frac{P_i - P_{i-1}}{\Delta t} = \frac{\partial f}{\partial y_{i-1}} P_{i-1} + P_{i-1} \frac{\partial f^{*}}{\partial y_{i-1}} + \Gamma(t_{i-1})Q(t_{i-1})\Gamma^{*}(t_{i-1})$$

$$+ \frac{\partial f}{\partial y_{i-1}} P_{i-1} \frac{\partial f^{*}}{\partial y_{i-1}} \Delta t + P_{i/i-1}\left[\frac{\partial \eta_i}{\partial \bar{x}_i} \frac{1}{\Delta t}\right]\left[I - P_{i/i-1} \frac{\partial \eta_i}{\partial \bar{x}_i}\right]^{-1} P_{i/i-1}$$

(C-9)

Similarly, applying (C-2) and (66) to (64) gives

$$\frac{\hat{x}_i - \hat{x}_{i-1}}{\Delta t} = f(\hat{x}_{i-1}, t_{i-1}) + P_i \frac{\eta_i(\bar{x}_i)}{\Delta t} \tag{C-10}$$

By applying (C-5) to the explicit forms of $\eta_i(\bar{x}_i)$ and $\partial\eta_i/\partial\bar{x}_i$ and then invoking (C-7) it can readily be shown that as $\Delta t \rightarrow 0$ (C-9) and (C-10) become as desired, respectively,

$$\frac{d\hat{x}}{dt} = f(\hat{x}, t) + P \frac{\partial h^*}{\partial \hat{x}} R^{-1}[z - h(\hat{x}, t)] \tag{C-11}$$

$$\frac{dP}{dt} = \frac{\partial f}{\partial \hat{x}} P + P \frac{\partial f^*}{\partial \hat{x}} + \Gamma Q \Gamma^* + P \frac{\partial}{\partial \hat{x}} \left(\frac{\partial h^*}{\partial \hat{x}} R^{-1}[z - h(\hat{x}, t)] \right) P \tag{C-12}$$

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13 ABSTRACT

A unified approach to the dynamic nonlinear estimation problem is presented. General algorithms for filtering, smoothing, and prediction for both continuous and discrete nonlinear systems are derived. The continuous problem is treated first by a direct approach which yields an exact solution. A similar approach is then applied to the more complex discrete estimation problem. The discussion provided points out some subtle, but important features of the main results which should be useful when applying the results to practical problems. To provide insight, motivation and background for the technique used in obtaining the solution of the general nonlinear problem, the special case of linear estimation is discussed in the first two appendices. In a third appendix it is shown that in the continuous measurement limit, the derived discrete and continuous nonlinear filtering algorithms are identical.

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