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AN EXACT LOWER CONFIDENCE BOUND FOR THE RELIABILITY OF A SERIES SYSTEM

WHERE EACH COMPONENT HAS AN EXPONENTIAL TIME TO

FAILURE DISTRIBUTION

by

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1. Introduction and Summary

Let us consider a complex system consisting of k components. The components are connected in series so that the system fails if at least one component fails. Let T_i represent the time to failure of the ith component. The reliability of the system at time t is given by

$$(1) \qquad R(t) = P[T_1 > t, T_2 > t, \ldots, T_k > t].$$

This implies that T, the time to failure of the system, is given by

$$(2) \qquad T = \min_{i=1,\ldots,k} [T_i] \quad \text{and} \quad R(t) = P[T > t].$$

Let us assume that T_i has an exponential distribution with parameter λ_i so that

$$(3) \qquad P(T_i > t) = e^{-\lambda_i t}, \; i = 1, \ldots, k.$$

Initially, let us assume that $T_1, \ldots, T_k$ are independently distributed. Then from (1) and (2),

$$(4) \qquad R(t) = P[T > t] = e^{-(\lambda_1 + \ldots + \lambda_k)t},$$

so that the time to failure of the system has an exponential distribution with parameter $\lambda_1 + \lambda_2 + \ldots + \lambda_k$. In general, $\lambda_1, \ldots, \lambda_k$ will not be

known. We want to find a lower confidence bound for $R(t)$.

We assume that the system as a whole is either impossible or too expensive to test. Hence, no data is available for T. However, one can test each individual component and observations are available for T_i for each $i = 1,...,k$. We want our confidence bound to be based on these latter observations.

Kraemer [7] has given two methods for obtaining the lower confidence bound of the above reliability. However, her methods do not give an exact lower confidence bound. The method proposed by us will give an exact lower confidence bound. It will also be shown that our method works under certain circumstances even when $T_1,...,T_k$ are not independently distributed.

We have given numerical examples to compare Kraemer's first method with ours. We have also compared our lower confidence bound with that given by maximum-likelihood estimates.

Kraemer's second method is based on a version of Tchebyshev's inequality. It appears to us that in most cases it is inferior to Kraemer's first method. Accordingly, we will not mention it any further.

Lentner and Buehler [8] have used the Lehmann-Scheffé theory to obtain a uniformly most powerful unbiased test for the hypothesis $\lambda_1 + \lambda_2 = \emptyset$. Inverting the test they obtained an exact confidence bound for the reliability of the system. The method has been extended to the case of k components by El Mawaziny [1]. Unfortunately, the methods are complicated and even in the case of $k = 2$ and very small sample sizes one has to solve equations by trial and error to obtain the confidence bounds for $\emptyset$. For larger values of k and/or for moderate sample sizes elaborate computer programming is required. In fact, El Mawaziny and

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Buehler [2] derived an approximate method when there are $k(k \geq 2)$ components because of the difficulty in computation for their exact methods. It will be shown below that our method is very easy to compute and can be easily worked out with a desk calculator. The methods of [8] and [1] are applicable only when the components are independent. As will be shown, our method also works for dependent components.

El Mawaziny and Buehler have used a theorem of R. von Mises to obtain large sample approximations of the methods of [8] and [1]. Their method depends on asymptotic expansions of certain quantities. It turns out that if the first two terms of the expansions are used (and the higher order terms are ignored), the confidence bound of reliability is the same as that given by the maximum-likelihood method. The approximation can be improved by using higher order terms, but the degree of improvement is not known.

2. The Method

To illustrate the method, let us assume that $k = 2$.

Let T_i be independent exponentially distributed random variables with parameters λ_i, $i = 1,2$. Let $T = \min(T_1, T_2)$. Then

$$P(T > t) = P(T_1 > t, \ T_2 > t) = e^{-(\lambda_1 + \lambda_2)t}.$$

Hence, T is exponentially distributed with parameter $(\lambda_1 + \lambda_2)$.

Let us assume that n_i similar components of type i are on test simultaneously. The test is without replacement so that as one component fails it is removed from test and is not replaced. We terminate

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the test as soon as r_i among n_i have failed. This method is known in the literature as type II censoring of r_i out of n_i without replacement.

Let $X_{(1)} < X_{(2)} < \ldots < X_{(r_1)}$ be the order statistics of the times to failure of the components of the first type, and $Y_{(1)} < Y_{(2)} < \ldots < Y_{(r_2)}$ be the same for the components of the second type. For our method we require that $r_1 = r_2$. Let $r = \min(r_1, r_2)$. Hence, we work with termination points r for each component. Kraemer's method or the methods of [1], [2], and [8], however, do not require this restriction. For simplicity, let us assume that $n_1 = n_2 = n$. As will be evident, there is no loss of generality in making this second assumption.

Let

$$D_1 = X_{(1)}.$$

(5)

$$D_j = X_{(j)} - X_{(j-1)}, \quad j = 2,\ldots,r.$$

$$\delta_j = (n-j+1)D_j \quad , \quad j = 1,\ldots,r.$$

Epstein and Sobel [3], and Epstein [4] have shown that $\delta_1,\ldots,\delta_r$ are independently, identically distributed. The distribution is exponential with parameter λ_1. Similarly, let

$$E_1 = Y_{(1)}.$$

(6)

$$E_j = Y_{(j)} - Y_{(j-1)}, \quad j = 2,\ldots,r.$$

$$\tau_j = (n-j+1)E_j \quad , \quad j = 1,\ldots,r.$$

Then $\tau_1,\ldots,\tau_r$ are independently exponentially distributed with parameter

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λ_2. Also, $\tau_1, \ldots, \tau_r$ are independent of $\delta_1, \ldots, \delta_r$.

Let

(7)
$$z_j = \min(\delta_j, \tau_j).$$

Then, as we have already stated, $z_1, \ldots, z_r$ are independently exponentially distributed with parameter $\lambda_1 + \lambda_2$. Let

(8)
$$rV = \sum_{j=1}^{r} z_j.$$

It is well known that $2rV(\lambda_1 + \lambda_2)$ has chi-square distribution with $2r$ degrees of freedom. Let χ^2 denote a random variable distributed as chi-square with $2r$ degrees of freedom. Let $\chi^2_{2r}(\alpha)$ be such that

(9)
$$P[\chi^2 < \chi^2_{2r}(\alpha)] = 1 - \alpha.$$

Hence,

$$P[2rV(\lambda_1 + \lambda_2) < \chi^2_{2r}(\alpha)] = 1 - \alpha$$

or

$$P[\lambda_1 + \lambda_2 < \frac{\chi^2_{2r}(\alpha)}{2rV}] = 1 - \alpha$$

and, since $R(t) = e^{-(\lambda_1 + \lambda_2)t}$ is a decreasing function of $\lambda_1 + \lambda_2$ for fixed t, we get

(10)
$$P[e^{-(\lambda_1 + \lambda_2)t} > \exp\left\{-t \cdot \frac{\chi^2_{2r}(\alpha)}{2rV}\right\}] = 1 - \alpha.$$

This gives an exact $100(1-\alpha)$ per cent lower confidence bound on the reliability of the system at time t when the system consists of two components in series and the time to failure distribution of each component is exponential and the two exponentials are independent.

It is obvious that the above method does not require the assumption

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that $n_1 = n_2 = n$. If n_1 and n_2 are unequal, then in (5) and (6) the definition of δ_j and τ_j will be changed as follows:

$$
(11) \qquad
\left.
\begin{aligned}
\delta_j &= (n_1 - j + 1)D_j \\[2ex]
\tau_j &= (n_2 - j + 1)E_j
\end{aligned}
\right\} \quad j = 1,2,\ldots,r.
$$

The rest of the theory goes through and (7), (8), (9), and (10) hold with the new definitions of δ_j, τ_j, and z_j.

3. Extensions of the Method

A. When the number of components is larger than 2.

Let us consider a system consisting of k components in series. Let T_i be the time to failure of the ith component. Assume that $T_1, T_2, \ldots, T_k$ are independent exponentially distributed random variables having parameters $\lambda_1, \lambda_2, \ldots, \lambda_k$ respectively.

Let $T = \min(T_1, \ldots, T_k)$. Then

$$P(T > t) = P(T_1 > t, \ldots, T_k > t).$$

Since $T_1, T_2, \ldots, T_k$ are independent,

$$P(T > t) = P(T_1 > t) \ldots P(T_k > t)$$
$$= e^{-(\lambda_1 + \ldots + \lambda_k)t}.$$

Hence, T is exponentially distributed with parameter $\lambda_1 + \ldots + \lambda_k \equiv \lambda$, say.

Suppose we have n components of each type on test, and let us perform type II censoring of r out of n without replacement. Let

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our data consist of $X_{(1)}^{(i)} < X_{(2)}^{(i)} < \dots < X_{(r)}^{(i)}$, $i = 1, \dots, k$. As in section 2, for $i = 1, 2, \dots, k$, let

$$D_1^{(i)} = X_1^{(i)}$$

(12)
$$D_j^{(i)} = X_{(j)}^{(i)} - X_{(j-1)}^{(i)}, \quad j = 2, \dots, r,$$

$$\delta_j^{(i)} = (n-j+1)D_j^{(i)}, \quad j = 1, \dots, r,$$

(13)
$$z_j = \min_{i=1,\dots,k} \delta_j^{(i)}, \quad j = 1, \dots, r.$$

$z_1, \dots, z_r$ are independently exponentially distributed with parameter λ. Let

(14)
$$rV = \sum_{i=1}^{r} z_i.$$

Then $2rV\lambda$ has chi-square distribution with $2r$ degrees of freedom, and

(15)
$$P[e^{-\lambda t} > \exp\left\{-t \cdot \frac{\chi_{2r}^2(\alpha)}{2rV}\right\}] = 1-\alpha.$$

This gives us an exact $100(1-\alpha)$ per cent lower confidence bound for the reliability of this system at time t.

If $n_1, \dots, n_k$ are different, then the modification proposed in the last paragraph of section 2 applies.

B. **When $k = 2$ and the distributions are not independent.**

Marshall and Olkin [9] and Harris [6] have considered the following model of a bivariate exponential distribution.

Let us consider 3 independent series of events happening in time. Let us consider the series i, $i = 1,2,3$. Let $U_i^{(n)}$ represent the interval between the nth and the $(n-1)^{th}$ event of type i. We assume that $U_i^{(n)}$

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has exponential distribution with parameter λ_i, independent of n. With an arbitrary time origin let us assume that $U_i^{(1)}$, the time to the first event of type i, has the same distribution as mentioned above. Further, let us assume that for fixed i, the distributions of $U_i^{(n)}$ are independent for all n. Hence, we have three independent Poisson series of events and U_i has distribution $\exp(\lambda_i)$, i = 1,2,3 and U_1, U_2, U_3 are mutually independent.

Let us now consider a system with two components in series. Let us assume that component 1 fails if either U_1 or U_3 happens, and component 2 fails if either U_2 or U_3 happens. The system fails if at least one component fails.

Let T_i, i = 1,2, be the time to failure of the ith component. Then $T_1 = \min(U_1,U_3)$, $T_2 = \min(U_2,U_3)$ and

$$(16) \qquad P[T_1 > s, \ T_2 > t] = P[U_1 > s, \ U_2 > t, \ U_3 > \max(s,t)]$$

$$= \exp[-\lambda_1 s - \lambda_2 t - \lambda_3 \max(s,t)].$$

This defines the distribution of a bivariate exponential distribution. The reliability of the system at time t is

$$(17) \qquad R(t) = P(T_1 > t, \ T_2 > t) = \exp[-(\lambda_1 + \lambda_2 + \lambda_3)t].$$

This shows that the time to failure of the system has exponential distribution with parameter $\lambda_1 + \lambda_2 + \lambda_3$. Also, the marginal distributions of T_1 and T_2 are exponential with parameters $\lambda_1 + \lambda_3$ and $\lambda_2 + \lambda_3$, respectively. T_1 and T_2 are not independently distributed.

Let $T = \min(T_1,T_2)$. Then

$$(18) \qquad P(T > t) = P(T_1 > t, \ T_2 > t) = \exp[-(\lambda_1 + \lambda_2 + \lambda_3)t].$$

(18) is really the same as (17).

Now suppose we have the same data as in section 2. We define δ_j, τ_j, z_j, and rV as in (5), (6), (7), and (8). Then $z_1, \ldots, z_r$ are independently exponentially distributed with parameter $\lambda_1 + \lambda_2 + \lambda_3$. With this modification we get that $2rV(\lambda_1 + \lambda_2 + \lambda_3)$ has chi-square distribution with $2r$ degrees of freedom and, as in (9) and (10), we get

$$(19) \qquad P[e^{-(\lambda_1 + \lambda_2 + \lambda_3)t} > \exp\left\{-t \cdot \frac{x_{2r}^2(\alpha)}{2rV}\right\}] = 1-\alpha.$$

This gives an exact $100(1-\alpha)$ per cent lower confidence bound for $R(t)$. When n_1 and n_2 are different, we proceed as in the last part of section 2.

C. As in case B, but $k > 2$.

The bivariate exponential distribution has been generalized by Marshall and Olkin [9] to multivariate exponential distribution.

Let us consider a system with k components in series. Let S denote the set of vectors $(s_1, \ldots, s_k)$, where each $s_j = 0$ or 1, but $(s_1, \ldots, s_k) \neq (0, \ldots, 0)$. For any vector $s \in S$, $\max(x_i\, s_i)$ is the maximum of the x_i's for which $s_i = 1$. The multivariate exponential distribution is defined as

$$(20) \qquad P(T_1 > x_1, \ldots, T_k > x_k) = \exp[-\sum_{s \in S} \lambda_s \max(x_i\, s_i)].$$

Here $T_1, \ldots, T_k$ denote the time to failure of components $1, \ldots, k$. The marginal distribution of any subset of $T_1, \ldots, T_k$ is again a multivariate exponential distribution and, in particular, the one-dimensional marginal distributions are exponential and are not independent. As in case B, we can generate the distribution in (20) by considering $2^k - 1$ independent

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series of Poisson events and the associated failures of components. For example, for three components we need 7 series. The associated failures are

s	shock	component	parameter
1	1	1	λ_1
2	2	2	λ_2
3	3	3	λ_3
4	4	1,2	λ_4
5	5	1,3	λ_5
6	6	2,3	λ_6
7	7	1,2,3	λ_7

The reliability of the system at time t is given by

$$(21) \qquad R(t) = \exp[- \sum_{s \in S} \lambda_s \cdot t].$$

As in case A, if we have data for the k components and define $\delta_j^{(i)}$, $j = 1,\ldots,r$, $i = 1,\ldots,k$, then $z_j = \min_{i=1,\ldots,k} \delta_j^{(i)}$, $j = 1,\ldots,r$, has exponential distribution with parameter $\lambda \equiv \sum_{s \in S} \lambda_s$ and the method proposed under case A, section 3, is applicable.

D. When type II censoring with replacement.

Let us suppose that n components of a type are on test and we replace each failed component immediately by a new one. We stop as soon as r failures are obtained. This is termed in the literature as type II censoring of r out of n with replacement. Let D_j be defined as in (5). Let us define

$$(22) \qquad \delta_j = nD_j, \quad j = 1,\ldots,r.$$

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Epstein [4] has shown that $\delta_1, \ldots, \delta_r$ are independently exponentially distributed with the same distribution as the parent population. With this new definition of δ_j (and τ_j, $\delta_j^{(i)}$), the methods of sections 2 and 3 go through and we can obtain an exact $100(1-\alpha)$ per cent lower confidence bound for the reliability of the system. As before, the method works when $n_1, \ldots, n_k$ are not all equal.

4. Confidence Bounds Based on Maximum Likelihood Estimates

We will consider case A of section 3 and for illustration we will take $k = 2$.

Let us consider component i and let the order statistics obtained by type II censoring of r out of n without replacement be denoted by

$$X_{(1)}^{(i)} < X_{(2)}^{(i)} < \ldots < X_{(r)}^{(i)}, \quad i = 1,2.$$

Using the transformation given in (5), the likelihood of the sample is proportional to

$$L = \lambda_i^r \exp[-\lambda_i(\delta_1^{(i)} + \ldots + \delta_r^{(i)})], \quad i = 1,2,$$

where $\delta_1^{(i)}, \ldots, \delta_r^{(i)}$ are independently identically distributed as exponential with parameter λ_i. Hence, the maximum likelihood estimate of λ_i is given by

$$(23) \qquad \hat{\lambda}_i = \frac{r}{\sum\limits_{j=1}^{r} \delta_j^{(i)}}, \quad i = 1,2.$$

Let

$$(24) \qquad \delta = \sum_{j=1}^{r} \delta_j^{(1)}, \quad \tau = \sum_{j=1}^{r} \delta_j^{(2)}.$$

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Then δ and τ have gamma distributions with density functions

$$f(\delta) = \frac{\lambda_1^r \delta^{r-1} e^{-\lambda_1 \delta}}{(r-1)!}, \quad \delta \geq 0, \lambda_1 > 0,$$

(25)

$$g(\tau) = \frac{\lambda_2^r \tau^{r-1} e^{-\lambda_2 \tau}}{(r-1)!}, \quad \tau \geq 0, \lambda_2 > 0.$$

Now,

$$\frac{\partial^2 \log L}{\partial \lambda_i^2} = - \frac{r}{\lambda_i^2}, \quad i = 1,2.$$

$\frac{\lambda_i^2}{r}$ is, in a sense, the "asymptotic variance" of $\hat{\lambda}_i$. However, since λ_i is not known, we replace $\frac{\lambda_i^2}{r}$ by $\frac{\hat{\lambda}_i^2}{r}$.

Definition: Let X_n be a random variable depending on n and let X be any other random variable. Then we define

$$X_n \overset{\mathcal{L}}{\to} X \quad \text{as} \quad n \to \infty$$

if, as $n \to \infty$, the cumulative distribution function of X_n tends to the cumulative distribution function of X at the points of continuity of the latter function. Also, we define $X_n \overset{P}{\to} X$ if, given any $\epsilon > 0$,

$$P(|X_n - X| \geq \epsilon) \to 0 \quad \text{as} \quad n \to \infty.$$

We shall prove

Theorem 4.1:

Let $\delta_1, \ldots, \delta_r$ and $\tau_1, \ldots, \tau_r$ be two independent random samples from exponential distributions with parameters λ_1 and λ_2, respectively. Then

$$T_r = \frac{\frac{r}{\delta} + \frac{r}{\tau} - \lambda_1 - \lambda_2}{\sqrt{\frac{r}{\delta^2} + \frac{r}{\tau^2}}} \xrightarrow{\mathcal{L}} N(0,1) \quad \text{as} \quad r \to \infty,$$

where $\delta = \sum\limits_{i=1}^{r} \delta_i^{(1)}$ and $\tau = \sum\limits_{i=1}^{r} \delta_i^{(2)}$.

We will require the following theorem to prove Theorem 4.1:

Theorem 4.2:

Suppose $(x_{1\xi}, \ldots, x_{k\xi}, \xi = 1, \ldots, r)$ is a sample from a k-dimensional distribution with finite means $\{\mu_i\}$ and positive definite covariance matrix $\|\sigma_{ij}\|$, $i, j = 1, \ldots, k$. Let $g(x_1, \ldots, x_k)$ be a function which possesses first derivatives $\frac{\partial g}{\partial x_i} = g_i$, say, $i = 1, \ldots, k$, at all points in some neighborhood of $(\mu_1, \ldots, \mu_k)$, and let $g_i^o = g_i(\mu_1, \ldots, \mu_k)$. Then if at least one of the g_i^o is $\neq 0$,

$$\sqrt{r}[g(\bar{x}_1, \ldots, \bar{x}_k) - g(\mu_1, \ldots, \mu_k)]$$

$$\xrightarrow{\mathcal{L}} N[0, \sum\limits_{i,j=1}^{k} \sigma_{ij} g_i^o g_j^o] \quad \text{as} \quad r \to \infty.$$

Proof: See [14, p. 260, section 9.3.1a].

Proof of Theorem 4.1:

Let $\bar{\delta} = \delta/r$, $\bar{\tau} = \tau/r$. From what we are given,

$$E(\delta_i) = \frac{1}{\lambda_1}, \quad V(\delta_i) = \frac{1}{\lambda_1^2},$$

$$E(\tau_j) = \frac{1}{\lambda_2}, \quad V(\tau_j) = \frac{1}{\lambda_2^2},$$

$$\text{Cov}(\delta_i, \tau_j) = 0; \quad i, j = 1, \ldots, r.$$

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Consider $g(x,y) = \dfrac{\dfrac{1}{x} + \dfrac{1}{y} - \lambda_1 - \lambda_2}{\sqrt{\dfrac{1}{x^2} + \dfrac{1}{y^2}}}$.

Then

$$g_1^o = \frac{\partial g}{\partial x}\bigg|_{x=\frac{1}{\lambda_1}, y=\frac{1}{\lambda_2}} = -\frac{\lambda_1^2}{(\lambda_1^2 + \lambda_2^2)^{1/2}} \neq 0, \pm \infty,$$

and

$$g_2^o = \frac{\partial g}{\partial y}\bigg|_{x=\frac{1}{\lambda_1}, y=\frac{1}{\lambda_2}} = -\frac{\lambda_2^2}{(\lambda_1^2 + \lambda_2^2)^{1/2}} \neq 0, \pm \infty.$$

Hence, by Theorem 4.2, as $r \to \infty$,

$$\sqrt{r}[g(\bar{\delta}, \bar{\tau}) - g(\frac{1}{\lambda_1}, \frac{1}{\lambda_2})]$$

$$\xrightarrow{\mathcal{L}} N[0, \sum_{i,j=1}^{2} \sigma_{ij} g_i^o g_j^o].$$

Now

$$g(\bar{\delta}, \bar{\tau}) = \frac{\dfrac{r}{\delta} + \dfrac{r}{\tau} - \lambda_1 - \lambda_2}{\sqrt{\dfrac{r^2}{\delta^2} + \dfrac{r^2}{\tau^2}}} \, ,$$

$$g(\frac{1}{\lambda_1}, \frac{1}{\lambda_2}) = 0,$$

and

$$\sum_{i,j=1}^{2} \sigma_{ij} g_i^o g_j^o = \frac{1}{\lambda_1^2} \cdot \frac{\lambda_1^4}{(\lambda_1^2 + \lambda_2^2)} + \frac{1}{\lambda_2^2} \cdot \frac{\lambda_2^4}{(\lambda_1^2 + \lambda_2^2)} = 1.$$

Hence,

$$\sqrt{r}[g(\bar{\delta}, \bar{\tau}) - g(\frac{1}{\lambda_1}, \frac{1}{\lambda_2})]$$

$$= \frac{\dfrac{r}{\delta} + \dfrac{r}{\tau} - \lambda_1 - \lambda_2}{\sqrt{\dfrac{r}{\delta^2} + \dfrac{r}{\tau^2}}} \xrightarrow{\mathcal{L}} N(0,1), \quad \text{as } r \to \infty,$$

and the theorem is proved.

Theorem 4.1 gives us an asymptotically exact confidence interval for the reliability of the system. Let k_α be such that

$$\int_{-\infty}^{-k_\alpha} \frac{1}{\sqrt{2\pi}} e^{-1/2 t^2} dt = \alpha.$$

Then

$$P\left[\frac{\frac{r}{\delta} + \frac{r}{\tau} - \lambda_1 - \lambda_2}{\sqrt{\frac{r}{\delta^2} + \frac{r}{\tau^2}}} > - k_\alpha\right] = 1-\alpha,$$

(26) $$P\left[\lambda_1 + \lambda_2 < \frac{r}{\delta} + \frac{r}{\tau} + k_\alpha \sqrt{\frac{r}{\delta^2} + \frac{r}{\tau^2}}\right] = 1-\alpha,$$

(27) $$P\left[e^{-(\lambda_1+\lambda_2)t} > \exp\left\{-t(\frac{r}{\delta} + \frac{r}{\tau} + k_\alpha \sqrt{\frac{r}{\delta^2} + \frac{r}{\tau^2}})\right\}\right] = 1-\alpha.$$

(27) gives the $100(1-\alpha)$ per cent lower confidence bound for the reliability. The bound is exact if $r \to \infty$. The method of this section generalizes when the number of components in series is more than 2.

Since this confidence interval (lower confidence bound, 1) is based on the maximum likelihood estimate, for large samples it is the smallest confidence interval. It is not known how good this bound is for small samples.

We will make come comparison of this method with the method of section 2.

When the number of items on test for each component is different, the remark at the end of section 2 applies.

5. Kraemer's Method [7]

For the case A of section 3, Kraemer's first method is as follows:

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Let $a_i = \dfrac{1}{\lambda_i}$, $i = 1,2,\ldots,k$.

$$\hat{a}_i = [\sum_{j=1}^{r_i} X_{(j)}^{(i)} + (n_i - r_i) X_{(r_i)}^{(i)}]|r_i.$$

$\dfrac{2r_i \hat{a}_i}{a_i}$ is distributed as chi-square with $2r_i$ degrees of freedom.

Let $A = \min\limits_{i=1,2,\ldots,k} (r_i \hat{a}_i)$. Then

$$P[e^{-(\lambda_1 + \ldots + \lambda_k)t} > \exp\left\{-t \frac{\chi^2_{2(r_1 + \ldots + r_k)}(\alpha)}{2A}\right\}] \geq 1-\alpha,$$

where, if U is a chi-square variable with $2(r_1 + \ldots + r_k)$ degrees of freedom, then $\chi^2_{2(r_1 + \ldots + r_k)}(\alpha)$ is defined as

$$P[U < \chi^2_{2(r_1 + \ldots + r_k)}(\alpha)] = 1-\alpha.$$

6. A Comparison of the Methods

We will compare the methods of sections 2, 4, and 5 for a system with two independent components in series. The time to failure of each component is distributed exponentially with parameter λ_i, $i = 1,2$. The comparison will be based on the expected values of the upper confidence bounds for $\lambda_1 + \lambda_2$ under each method. Since the left-hand end point is zero, this gives the expected length of the confidence interval for $\lambda_1 + \lambda_2$. Since the reliability is a (strictly) monotone function of $\lambda_1 + \lambda_2$, this furnishes a valid comparison.

Exact Method of Section 2

$$P[\lambda_1 + \lambda_2 < \frac{\chi^2_{2r}(\alpha)}{2rV}] = 1-\alpha.$$

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Since z_i are independent, identical exponential variables with parameter $\lambda_1 + \lambda_2$, by (8) rV has gamma distribution and

$$(28) \quad E[\frac{\chi^2_{2r}(\alpha)}{2rV}] = \frac{\chi^2_{2r}(\alpha)}{2} E[\frac{1}{\sum\limits_{i=1}^{r} z_i}] = \frac{\chi^2_{2r}(\alpha)}{2} \cdot \frac{\lambda_1 + \lambda_2}{r-1}$$

where

$$E(\frac{1}{\sum\limits_{1}^{r} z_i}) = \int_0^\infty \frac{1}{x} \cdot \frac{(\lambda_1+\lambda_2)^r x^{r-1} e^{-(\lambda_1+\lambda_2)x}}{(r-1)!} dx$$

$$= \frac{\lambda_1 + \lambda_2}{r-1} \int_0^\infty \frac{(\lambda_1+\lambda_2)^{r-1} x^{r-2} e^{-(\lambda_1+\lambda_2)x}}{(r-2)!} dx$$

$$= \frac{\lambda_1 + \lambda_2}{r-1}, \quad \text{provided } r \geq 2.$$

Kraemer's Method of Section 5

$$P[\lambda_1 + \lambda_2 < \frac{\chi^2_{4r}(\alpha)}{2A}] \geq 1-\alpha,$$

where

$$A = \min(r\hat{a}_1, r\hat{a}_2)$$

$$= \min[\frac{1}{2\lambda_1} \cdot 2r\hat{a}_1\lambda_1, \frac{1}{2\lambda_2} \cdot 2r\hat{a}_2\lambda_2].$$

Since $2r\hat{a}_i\lambda_i$ are distributed as χ^2 with $2r$ degrees of freedom and are independent for $i = 1,2$,

$$P(A > x) = P(2r\hat{a}_1\lambda_1 > 2\lambda_1 x) \, P(2r\hat{a}_2\lambda_2 > 2\lambda_2 x)$$

$$= [\int_{\lambda_1 x}^\infty \frac{1}{(r-1)!} u^{r-1}e^{-u}du][\int_{\lambda_2 x}^\infty \frac{1}{(r-1)!} v^{r-1}e^{-v}dv].$$

Hence, the probability density function of A is given by

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$$f(x) = \frac{\lambda_1^r x^{r-1} e^{-\lambda_1 x}}{(r-1)!} \int_x^\infty \frac{\lambda_2^r v^{r-1} e^{-\lambda_2 v}}{(r-1)!} \, dv$$

$$+ \frac{\lambda_2^r x^{r-1} e^{-\lambda_2 x}}{(r-1)!} \int_x^\infty \frac{\lambda_1^r v^{r-1} e^{-\lambda_1 v}}{(r-1)!} \, dv, \quad x \geq 0.$$

Integrating by parts,

$$f(x) = \frac{\lambda_1^r x^{r-1} e^{-\lambda_1 x}}{(r-1)!} \sum_{s=0}^{r-1} e^{-\lambda_2 x} \frac{(\lambda_2 x)^s}{s!}$$

$$+ \frac{\lambda_2^r x^{r-1} e^{-\lambda_2 x}}{(r-1)!} \sum_{s=0}^{r-1} e^{-\lambda_1 x} \frac{(\lambda_1 x)^s}{s!} .$$

Hence,

(29) $$E(\tfrac{1}{A}) = \int_0^\infty \frac{1}{x} f(x) dx = \sum_{s=0}^{r-1} \frac{(r+s-2)!}{(r-1)! s!} \frac{\lambda_1^r \lambda_2^s + \lambda_2^r \lambda_1^s}{(\lambda_1 + \lambda_2)^{r+s-1}}, \quad \text{for} \quad r \geq 2,$$

and

(30) $$E(\frac{\chi_{4r}^2(\alpha)}{2A}) = \frac{\chi_{4r}^2(\alpha)}{2} \sum_{s=0}^{r-1} \frac{(r+s-2)!}{(r-1)! s!} \frac{\lambda_1^r \lambda_2^s + \lambda_2^r \lambda_1^s}{(\lambda_1 + \lambda_2)^{r+s-1}}, \quad \text{provided} \quad r \geq 2.$$

Asymptotic Method of Section 4

$$P[\lambda_1 + \lambda_2 < \frac{r}{\delta} + \frac{r}{\tau} + k_\alpha \sqrt{\frac{r}{\delta^2} + \frac{r}{\tau^2}}] = 1 - \alpha.$$

Using (25),

(31) $$E(\frac{r}{\delta} + \frac{r}{\tau}) = \frac{r}{r-1} (\lambda_1 + \lambda_2).$$

Substituting $\delta = \rho \cos \Theta$, $\tau = \rho \sin \Theta$, and using (25), we get

$$E[\sqrt{\frac{r}{\delta^2} + \frac{r}{\tau^2}}] = E[\frac{\sqrt{r} \sqrt{\delta^2 + \tau^2}}{\delta \tau}]$$

$$= \frac{\sqrt{r} (\lambda_1 \lambda_2)^r (2r-2)!}{(r-1)! (r-1)!} \int_0^{\pi/2} \frac{(\cos \Theta \sin \Theta)^{r-2} d\Theta}{(\lambda_1 \cos \Theta + \lambda_2 \sin \Theta)^{2r-1}},$$

$$\text{provided} \quad r \geq 2.$$

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(Put $t = \tan \frac{\theta}{2}$)

$$= \frac{\sqrt{r}\,(\lambda_1\lambda_2)^r(2r-2)!}{(r-1)!\,(r-1)!} \cdot \frac{2^{r-1}}{\lambda_1^{2r-1}} \int_0^1 \frac{t^{r-2}(1-t^2)^{r-2}(1+t^2)^2 dt}{(a+t)^{2r-1}(b-t)^{2r-1}}$$

where

$$a = \sqrt{1 + (\frac{\lambda_2}{\lambda_1})^2} - \frac{\lambda_2}{\lambda_1} > 0,$$

$$b = \sqrt{1 + (\frac{\lambda_2}{\lambda_1})^2} + \frac{\lambda_2}{\lambda_1} > 1.$$

We will break the integrand into partial fractions:

$$\frac{t^{r-2}(1-t^2)^{r-2}(1+t^2)^2}{(a+t)^{2r-1}(b-t)^{2r-1}} = \sum_{i=1}^{2r-1} \frac{A_i}{(a+t)^i} + \sum_{j=1}^{2r-1} \frac{B_j}{(b-t)^j},$$

or

$$\sum_{s=0}^{r-2} \binom{r-2}{s} (-1)^s [t^{r+2s-2} + 2t^{r+2s} + t^{r+2s+2}]$$

$$= \sum_{t=1}^{2r-1} A_i(a+t)^{2r-1-i}(b-t)^{2r-1}$$

$$+ \sum_{j=1}^{2r-1} B_j(b-t)^{2r-1-j}(a+t)^{2r-1},$$

where A's and B's are independent of t.

We have $4r-2$ unknowns. We will differentiate n times $(n = 0,1,\ldots,2r-2)$ with respect to t and put $t = -a$ and $t = b$. This will give us two systems of $2r-1$ equations each. Each system is triangular and we can easily solve sequentially for $A_{2r-1}, B_{2r-1}, \ldots, A_1, B_1$.

For the nth differentiation the general pair of equations is

$$\sum_{s=0}^{r-2} \binom{r-2}{s} (-1)^s \left[\binom{r+2s-2}{n}(-a)^{r+2s-2-n}\right.$$

$$\left. + 2\binom{r+2s}{n}(-a)^{r+2s-n} + \binom{r+2s+2}{n}(-a)^{r+2s+2-n}\right]$$

$$= \sum_{k=0}^{n} A_{2r-1-k}\binom{2r-1}{n-k}(a+b)^{2r-1-(n-k)}(-1)^{n-k}$$

and

$$\sum_{s=0}^{r-2} \binom{r-2}{s}(-1)^s\left[\binom{r+2s-2}{n}b^{r+2s-2-n}\right.$$

$$\left. + 2\binom{r+2s}{n}b^{r+2s-n} + \binom{r+2s+2}{n}b^{r+2s+2-n}\right]$$

$$= \sum_{k=0}^{n} B_{2r-1-k}\binom{2r-1}{n-k}(a+b)^{2r-1-(n-k)}(-1)^k$$

with the proviso that $\binom{p}{q} = 0$ if $p < q$. After solving for A's and B's, we get, using (31),

(32)

$$E[\frac{r}{\delta} + \frac{r}{\tau} + k_\alpha \sqrt{\frac{r}{\delta^2} + \frac{r}{\tau^2}}]$$

$$= \frac{r}{r-1}(\lambda_1 + \lambda_2) + k_\alpha \frac{\sqrt{r}\,(\lambda_1\lambda_2)^r(2r-2)!\,2^{r-1}}{(r-1)!\,(r-1)!\,\lambda_1^{2r-1}} \times$$

$$[\sum_{i=2}^{2r-1} \frac{A_i}{i-1}\left(\frac{1}{a^{i-1}} - \frac{1}{(a+1)^{i-1}}\right) + A_1 \log\left(\frac{a+1}{a}\right)$$

$$+ \sum_{j=2}^{2r-1} \frac{B_j}{j-1}\left(\frac{1}{(b-1)^{j-1}} - \frac{1}{b^{j-1}}\right) + B_1 \log\left(\frac{b}{b-1}\right)],$$

for $r \geq 2$.

We will now make some numerical comparisons between (28), (30), and (32).

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Numerical Comparisons

The following table gives some numerical values of E (upper 95 per cent confidence bound for $\lambda_1 + \lambda_2$) based on (28), (30), and (32). In (30) we have used tables of cumulative binomial distribution as follows: Let

$$B(n,k,p) = \sum_{s=k}^{n} \binom{n}{s} p^s (1-p)^{n-s}.$$

Then

$$\sum_{s=0}^{r-1} \frac{(r+s-2)!}{(r-1)!s!} \frac{\lambda_1^r \lambda_2^s + \lambda_2^r \lambda_1^s}{(\lambda_1+\lambda_2)^{r+s-1}} \; , \; r \geq 2$$

$$= \frac{\lambda_1}{r-1} \sum_{s=0}^{r-1} \binom{r+s-2}{r-2} \left(\frac{\lambda_1}{\lambda_1+\lambda_2}\right)^{r-1} \left(\frac{\lambda_2}{\lambda_1+\lambda_2}\right)^s$$

$$+ \frac{\lambda_2}{r-1} \sum_{s=0}^{r-1} \binom{r+s-2}{r-2} \left(\frac{\lambda_2}{\lambda_1+\lambda_2}\right)^{r-1} \left(\frac{\lambda_1}{\lambda_1+\lambda_2}\right)^s$$

$$= \frac{\lambda_1}{r-1} \sum_{s=r-1}^{2r-2} \binom{2r-2}{s} \left(\frac{\lambda_1}{\lambda_1+\lambda_2}\right)^s \left(\frac{\lambda_2}{\lambda_1+\lambda_2}\right)^{2r-2-s}$$

$$+ \frac{\lambda_2}{r-1} \sum_{s=r-1}^{2r-2} \binom{2r-2}{s} \left(\frac{\lambda_2}{\lambda_1+\lambda_2}\right)^s \left(\frac{\lambda_1}{\lambda_1+\lambda_2}\right)^{2r-2-s}$$

$$= \frac{\lambda_1}{r-1} B(2r-2, r-1, \frac{\lambda_1}{\lambda_1+\lambda_2})$$

$$+ \frac{\lambda_2}{r-1} \left\{ 1 - B(2r-2, r, \frac{\lambda_1}{\lambda_1+\lambda_2}) \right\} .$$

In (32), instead of using the partial fraction expansion we have performed numerical integration of

$$\int_0^{\pi/2} \frac{(\cos\theta \sin\theta)^{r-2} d\theta}{(\lambda_1 \cos\theta + \lambda_2 \sin\theta)^{2r-1}} .$$

We have used Simpson's rule after subdividing the interval $[0,\pi/2]$ into ten equal parts.

λ_1	λ_2	r	E (exact)	E (Kraemer)	E (asymptotic)
2	2.5	5	10.30	11.37	8.72
		10	7.85	8.40	6.90
		20	6.60	6.98	*
		30	6.14	6.48	*
1	5	5	13.73	19.66	12.20
		10	10.47	15.49	10.47
		20	8.80	13.41	*
		30	8.18	12.64	*
λ	λ	5	4.58λ	5.00λ	3.87λ
		10	3.49λ	3.67λ	3.06λ
		20	2.94λ	3.03λ	*
		30	2.73λ	2.79λ	*

* For larger values of r the error involved in approximate integration becomes large and hence these values have not been given.

As has been pointed out before, the above table also gives a valid comparison between the confidence intervals of $R(t)$ as given by the different methods.

In the range of α that we are interested in (e.g., $1-\alpha = .90$, .95, etc), it is expected that the exact method will give a shorter confidence interval than the Kraemer method. Since Kraemer's method is an

approximate one, in general, for λ_1 and λ_2 widely different (as in the second set of examples), the exact method is expected to be significantly superior to Kraemer's. This happens because the error of approximation in her method exceeds the precision obtained by using a larger number of degrees of freedom in the chi-square distribution. When λ_1 and λ_2 are close (as in the first set of examples), sometimes Kraemer's method may give a better confidence interval than the exact method; but the difference between the two methods is not expected to be large. For very small values of α (e.g., $1-\alpha = .99$, $.999$, etc.), there is some numerical evidence that sometimes Kraemer's method is slightly better than the exact method when $\lambda_1 = \lambda_2$. It may be pointed out that for $\lambda_1 = \lambda_2$, the difference between the expected value of the upper confidence bound for $\lambda_1 + \lambda_2$ given by Kraemer's method, and the expected value as given by the exact method, goes to zero as r goes to infinity.

Kraemer's method has one advantage over the exact method. The former is applicable even when the termination points (r_i) for the components are different. For the exact method some data have to be ignored to equalize the termination points.

For large r it is expected that the asymptotic method will be better than the exact method. Without a knowledge of the rate of convergence of the exact and asymptotic methods, it is not possible to assert to any degree of precision how the two methods will compare. However, it is expected that the exact method will compare favorably with the asymptotic method for large r.

Since the methods of [1] and [8] are based on inverting uniformly most powerful unbiased tests, they may give better lower confidence bounds

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than those given by the method proposed in this paper. This is especially true for small values of r. For large values of r the confidence bounds given in [1] and [8] will tend to the bounds given by the method of [2], and will be the same as those given by the asymptotic method. As we have already pointed out, the methods of [1] and [8] are computationally complicated even for the case of two components, and work only when the components are independent. In fact, the computational difficulties motivated the authors to get (approximately) asymptotic results in their subsequent joint paper [2]. These three methods, however, do not require the same termination points for every component.

When $\lambda_2 \to 0$, the exact method and the asymptotic method proposed in this paper can be easily compared. If $\lambda_2 \to 0$, then we are reduced to the case of a single component with exponential time to failure distribution. In this case, $\tau_i = \infty$ for all i, and

$$z_i = \min(\delta_i, \tau_i) = \delta_i$$

$$rV = \sum_{i=1}^{r} \delta_i.$$

(10) reduces to

$$P[e^{-\lambda_1 t} > \exp(-t \cdot \frac{\chi^2_{2r}(\alpha)}{2rV})] = 1-\alpha ,$$

the well-known result about a single component.

Let $\lambda_2 \to 0$ in (27). Then

$$P[e^{-\lambda_1 t} > \exp\left\{-t(\frac{r}{\delta} + k_\alpha \frac{\sqrt{r}}{\delta})\right\}] = 1-\alpha.$$

Following Halperin [5],

$$\sqrt{r}(R(t) - \hat{R}(t))\sigma_u \xrightarrow{\mathcal{L}} N(0,1).$$

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where

$$\sigma(u) = -\frac{1}{R(t)\log R(t)},$$

$$\hat{R}(t) = e^{-\hat{\lambda}_1 t},$$

$$\hat{\lambda}_1 = \frac{r}{\delta}.$$

If we replace σ_u by $\hat{\sigma}_u = -\dfrac{1}{\hat{R}(t)\log \hat{R}(t)}$, then the lower confidence

bound for $R(t)$ becomes

$$\exp(-t \cdot \frac{r}{\delta})\left\{1 - \frac{k_\alpha t \sqrt{r}}{\delta}\right\}.$$

By the weak law of large numbers (see [10, p. 92]), $\dfrac{\delta}{r} \overset{P}{\to} \dfrac{1}{\lambda_1}$ as

$r \to \infty$. Then $\dfrac{r}{\delta} \overset{P}{\to} \lambda_1$ and

$$\frac{\sqrt{r}}{\delta} = \frac{1}{\sqrt{r}} \cdot \frac{r}{\delta} \overset{P}{\to} 0.$$

Hence, asymptotically $\exp(-\dfrac{k_\alpha t \sqrt{r}}{\delta})$ and $1 - \dfrac{k_\alpha t \sqrt{r}}{\delta}$ are the same.

This checks our asymptotic result for a single component against the

well-known result as given above.

Finally, it is easily shown that the exact lower confidence bound

converges in probability to the true reliability as r approaches infinity

From (10) we note that the lower confidence bound for $R(t)$ is

$\exp\left\{-t \cdot \dfrac{\chi^2_{2r}(\alpha)}{2rV}\right\}$. By (18), V is the arithmetic average of independent

identically distributed random variables. Hence, by the weak law of

large numbers,

$$V \overset{P}{\to} \frac{1}{\lambda_1+\lambda_2} \quad \text{as} \quad r \to \infty.$$

Also,

$$\frac{\chi^2_{2r}(\alpha)}{2r} \to 1 \quad \text{as} \quad r \to \infty.$$

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Hence, as $r \to \infty$,

$$\exp\left\{-t \cdot \frac{\chi_{2r}^2(\alpha)}{2rV}\right\} \xrightarrow{P} e^{-(\lambda_1+\lambda_2)t} = R(t).$$

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13. ABSTRACT

Consider a complex system consisting of several components in series. The system as a whole is either impossible or too expensive to test. The only data available are the times to failure of the components when tested individually. Based on these data an exact lower confidence bound for the reliability of the system is obtained. Initially, the method is developed for the case when the components are independent. Later the method has been extended to the case when the components are dependent and have multivariate exponential time to failure distribution. The sampling method considered is type II censoring of r out of n with or without replacement. An asymptotically smallest confidence interval has also been developed. Kraemer's approximate method and the two proposed methods have been compared. It is also shown that the exact method proposed in this report is computationally much simpler than the exact methods proposed by Lentner, Buehler, and El Mawaziny.

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