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123

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SEVER INSTITUTE OF TECHNOLOGY

PURSUIT-EVASION DIFFERENTIAL GAMES

BY

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Prepared under the direction of Professor J. Zaborszky

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ABSTRACT

PURSUIT-EVASION DIFFERENTIAL GAMES

by John B. Berger

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Differential game theory is applied to several classes of pursuit-evasion problems. For these differential games the dynamics of the participants are described by linear nonstationary differential equations.

One class of differential games that was formulated and studied is the differential game, where the evader has to out maneuver a pursuer, if it is to strike the target that the pursuer is defending. This differential game will be called the differential endgame.

The differential endgame's payoff functional is the square of the terminal engagement miss distance weighted against the difference of the participants' control energies, spent during their respective flight times. The evader's target constraints are the position coordinates of the target and the evader's kinetic energy as it strikes the target.

The necessary and sufficient conditions for the existence of a saddle point, and the participants' control algorithms are determined for this differential endgame.

For this type of differential endgame when the intercept and target times, and the pursuer's initial position and velocity vectors, constrained in magnitude, are unknown, the relationships that determine these parameters are derived.

For a class of differential games, which results when the evader's target constraints and postengagement flight time are not considered, it is shown how the relationships that determine the intercept time and the pursuer's initial state are used in determining when the pursuer is launched.

Another class of differential games, formulated and studied in this dissertation, is the one where an additional pursuer is cooperating with the primary pursuer that is trying to intercept the evader. Here the payoff functional, which is constrained by the inner product of the terminal miss vector between the cooperating pursuer and the evader, is the square of the terminal miss distance between the primary pursuer and the evader, weighted against the difference of their control energies. For this differential game the two point boundary value problem is derived and solved.

TABLE OF CONTENTS

TABLE OF CONTENTS
(continued)

TABLE OF CONTENTS
(continued)

TABLE OF CONTENTS
(continued)

LIST OF TABLES

LIST OF FIGURES

PURSUIT-EVASION DIFFERENTIAL GAMES

1. INTRODUCTION

1.1 HISTORICAL BACKGROUND

In 1954 Isaacs (1, 2)* studied the pursuit problem by trying to determine the optimal path for a pursuer in order to intercept maneuverable targets. Isaacs' formal and hueristic approach, which was similar to Bellman's dynamic programming method, initiated the study of differential games.

In 1957 Berkovitz and Fleming (3) applied the calculus of variations technique to a class of differential games. Later Berkovitz (4) gave a rigorous treatment of a wider class of differential games, based upon the calculus of variations. Here Berkovitz obtained the necessary conditions that must exist along a path resulting from the use of optimal strategies of the two adversaries, by relating the differential game problem to a Bolza problem with differential inequalities added as side constraints.

*The numbers in parentheses indicate references in the Bibliography.

Berkovitz also developed a sufficiency theory which in principle verifies the existence of a saddle point.

Kelendzheridze (5) studied the problem of two maneuverable adversaries to which he determined the minimax time it takes for the pursuer to capture an evader. Here he used the minimax analogue of the Maximum Principle to determine the necessary conditions for optimality.

Ho, Bryson, and Baron (6,7), assuming the existence of a saddle point, used variational techniques to derive conditions for capture and optimality of a linear class of differential games. These conditions depend on the authors' definition of the relative controllability matrix. Baron (8), assuming the existence of a saddle point, derived conditions for capture and optimality for a linear class of differential games where the adversaries have limited energy resources or where the magnitudes of the control forces are limited. Baron also derived necessary and sufficient conditions for a class of nonlinear differential games. Gahzhiev (9) solved a similar version of the problem that was studied by Ho et al (6,7) but without the relative controllability condition.

More recently Meschler (10) and Chattopadhyay (11) have formulated pursuit-evasion problems where the objectives of the evader are not only to avoid interception of the pursuer but also to strike the target, the pursuer is defending.

Meschler has formulated his pursuit-evasion problem as a differential game. Its payoff is the square of the terminal miss distance. The dynamics of the participants are represented by linear time-invariant differential equations and the participants' control force components have specified magnitude constraints. Here the extremum of this differential game is determined analytically by dynamic programming. A serious drawback of this differential game is that it was optimized with respect to one component of the participants' control vectors. Meschler's work was published after the solutions for this dissertation were obtained. In the pursuit-evasion problem studied by Chattopadhyay, although it contains the notion of the pursuer defending the evader's target, it is not a differential game because the trajectory of the evader is predetermined.

1.2 SCOPE OF INVESTIGATION

One of the objectives of this dissertation is to apply differential game theory to the endgame problem. The endgame problem is defined as the terminal flight stage of an offensive missile that tries to penetrate its target by out maneuvering a maneuverable pursuer. This class of differential games will be called the differential endgame.

For this differential endgame the payoff functional is the square of the terminal engagement miss distance, weighted against the difference of the control energies, spent by the participants during their respective flight times.

The dynamics of the participants are described by linear time-varying differential equations. The evader's target constraints are the position coordinates of the target and the evader's target speed which is a measure of the evader's terminal kinetic energy.

For this differential endgame the necessary and sufficient conditions for the existence of a saddle point are derived by the calculus of variations method. From these necessary conditions control algorithms are developed for the participants.

Meschler (10) is the only one who studied the endgame problem in terms of differential game theory. His work was published after the solution to the differential endgame problem proposed in Chapter 3 was obtained. In Meschler's differential endgame the payoff functional is the square of the terminal engagement's miss distance. The dynamics of the participants are defined by linear time-invariant differential equations. The target constraint is a target zone. Here the minimax value of the differential game is determined with respect to one component of the participants' control vectors. This component of the participants' control vectors is constrained in magnitude. This differential endgame is solved by dynamic programming.

In all previous classes of pursuit-evasion differential games important parameters such as the intercept time, target time, and the pursuer's initial position and velocity vectors were assummed to be some known values.

In Chapter 4 these parameters are considered unknown and they are determined via differential game theory. For these differential games the pursuer's unknown position and velocity vectors have constrained magnitudes. Here the necessary conditions needed to determine these optimal parameters are derived for these classes of differential games. Also in Chapter 4 it is shown how the differential game with its optimized parameters is used to determine when the pursuer is launched.

Finally, in Chapter 5 a class of differential games involving two pursuers, trying to intercept an evader, is formulated. The two point boundary value problem, which determines the value of this type of differential game, is derived by the calculus of variations method. The solution of this two point boundary value problem is determined.

2. DIFFERENTIAL GAMES

2.1 PARTICIPANTS' LINEAR NONSTATIONARY SYSTEMS

Both participants of the differential games studied in this dissertation have linear, nonstationary, continuous systems of the following form:

$$\dot{x}_p = F_p(t)x_p + G_p(t)\,u + n_p \tag{2.1.1}$$

$$\dot{x}_e = F_e(t)x_e + G_e(t)\,v + n_e \tag{2.1.2}$$

where x_p and x_e are n-vectors describing the state of the pursuer and evader respectively; u and v are m-vectors, representing the control vectors of the pursuer and evader respectively; n_p and n_e are n-vectors, representing any disturbance acting upon the pursuer and evader; $F_p(t)$ and $F_e(t)$ are nxn matrices, continuous in t; and $G_p(t)$ and $G_e(t)$ are nxm matrices, continuous in t.

The participants' state vectors are determined by solving the set of differential equations, describing their systems,

$$x_p(t)=\Phi_p(t,t_0)x_p(t_0)+\int_{t_0}^{t}\Phi_p(t,\tau)[G_p(\tau)u(\tau)+n_p(\tau)]d\tau \tag{2.1.3}$$

$$x_e(t)=\Phi_e(t,t_0)x_e(t_0)+\int_{t_0}^{t}\Phi_e(t,\tau)[G_e(\tau)v(\tau)+n_e(\tau)]d\tau \tag{2.1.4}$$

where $\Phi_p(t,t_0)$ and $\Phi_e(t,t_0)$, the state transition matrices, are the solutions of the following set of differential equations:

$$\dot{\Phi}_k(t,t_0) = F_k(t)\Phi_k(t,t_0); \quad k=p,e \tag{2.1.5}$$

subject to the initial conditions

$$\Phi_k(t_0,t_0) = I \tag{2.1.6}$$

Since the state transition matrix is used to derive the optimal solution of the differential games that are formulated in this dissertation, it is appropriate to point out the following properties of the state transition matrix. Proof of these statements is in (12).

1) By definition

$$\Phi(t,t) = I \tag{2.1.7}$$

2) The group property of the state transition matrix is

$$\Phi(t_2,t_0) = \Phi(t_2,t_1)\Phi(t_1,t_0) \tag{2.1.8}$$

3) The inverse of the state transition matrix is

$$\Phi^{-1}(t,\tau) = \Phi(\tau,t) \tag{2.1.9}$$

In physical terms the participants' state vectors represent their position and velocity components. The position vector for the participants is defined as

$$x_{k1} = Ax_k;\ k=p,e \tag{2.1.10}$$

where the 2mxm matrix A is partitioned into the mxm identity and null matrices

$$A = [I \vdots 0] \tag{2.1.11}$$

The velocity vector for the participants is defined as

$$x_{k2} = Qx_k;\ k=p,e \tag{2.1.12}$$

where the 2mxm matrix, Q, is partitioned into the mxm null and identity matrices

$$Q = [0 \vdots I] \tag{2.1.13}$$

The participants' position, velocity, and control vectors are considered to be three dimensional vectors for the differential games studied in this dissertation.

2.2 GENERAL DEFINITION OF A DIFFERENTIAL GAME

The basic differential game problem is nonrigorously condensed from Berkowitz (4) as follows:

For the payoff functional

$$J = \Psi(x(T),T) + \int_{t_o}^{T} L(x,u,v,t)dt \tag{2.2.1}$$

and the participants' optimal strategies, u^* and v^*, determine $W(x_0,t_0)$, the value of J, such that*

$$W(x_0,t_0) = \underset{u\in U}{\text{Min}}\underset{v\in V}{\text{Max}}\ J \tag{2.2.2}$$

subject to the constraints

$$\dot{x} = F(x,u,v,t) \tag{2.2.3}$$

$$x(t_0) = x_0 \tag{2.2.4}$$

and

$$u\in U(t),\ v\in V(t) \tag{2.2.5}$$

Here $x(t)$, which is defined as the state of the game, is composed of the pursuer's and the evader's state vectors, and u and v are the control vectors of the pursuer and evader, respectively. T is the fixed termination time of the game, and the game's fixed time interval is $[t_0,T]$. $\Psi(x(T),T)$ is some terminal nonlinear function of the state variables of the game, and $L(u,v,x,t)$ is some nonlinear penalty functional of the control energy spent by both players.

Now a saddle point for the differential game is defined as the pair (u^*,v^*) satisfying the relation

$$J(u^*,v) \leq J(u^*,v^*) \leq J(u,v^*) \tag{2.2.6}$$

*It is assumed that $\underset{u\in U}{\text{Min}}\underset{v\in V}{\text{Max}}\ J = \underset{v\in V}{\text{Max}}\underset{u\in U}{\text{Min}}\ J$

for arbitrary $u \in U$, $v \in V$. If and only if (2.2.6) is satisfied, u^* and v^* are optimal strategies and $J(u^*,v^*)=W(x_0,t_0)$.

2.3 FEEDBACK CONTROL LAWS

There are two types of control strategies, one is open loop control and the other is closed loop control. The open loop controls are admissible controls which determine the saddle point for (2.2.1) subject to (2.2.3-2.2.5) The open loop controls are optimum for a particular initial state and its corresponding optimal path.

$$u^* = h_1(x_0,t_0,t) \qquad (2.3.1)$$

$$v^* = h_2(x_0,t_0,t) \qquad (2.3.2)$$

Closed loop optimal controls are optimum for any initial state and any deviation from the optimal nominal trajectories along these optimal nominal trajectories

$$u^*=k_1(x_0,t_0,x,t) \qquad (2.3.3)$$

$$v^*=k_2(x_0,t_0,x,t) \qquad (2.3.4)$$

Although optimal closed loop controls, which are determined by solving the Hamilton-Jacobi equation, are more desirable than the optimal open loop controls, the optimal open loop controls are determined because it is easier to solve the two point boundary value problem than the Hamilton-Jacobi equation. The optimal open loop controls can approximate closed loop control by instantaneously and continuously computing optimal open loop controls from updated measurements on the present state of the game.

These optimal open loop strategies are derived by the calculus of variations method.

2.4 NECESSARY AND SUFFICIENT CONDITIONS FOR THE DIFFERENTIAL GAME

The calculus of variations technique is used to derive both the necessary conditions which must be satisfied if a saddle point for the differential game exists, and the set of sufficiency conditions which determine the saddle point. From the necessary conditions the optimal open loop strategies are derived. The differential game's value and the participants' optimal control strategies are determined when

$$\underset{u\in U}{\text{Min}}\ \underset{v\in V}{\text{Max}}\ J_c = \underset{u\in U}{\text{Min}}\ \underset{v\in V}{\text{Max}} \left\{ \Psi(x(T),T) + \int_{t_0}^{T} [L(x,u,v,t) + \lambda' F(x,u,v,t) - \lambda' \dot{x}]dt \right\} \qquad (2.4.1)$$

Now defining the Hamiltonian as

$$H(x,\lambda,u,v,t) = L(x,u,v,t) + \lambda' F(x,u,v,t) \qquad (2.4.2)$$

one can rewrite the minimax operation (2.4.1) as

$$\underset{u\in U}{\text{Min}}\ \underset{v\in V}{\text{Max}}\ J_c = \underset{u\in U}{\text{Min}}\ \underset{v\in V}{\text{Max}} \left\{ \Psi(x(T),T) + \int_{t_0}^{T} (H(x,\lambda,u,v,t) - \lambda' \dot{x})dt \right\} (2.4.3)$$

where t_0, T, and $x(t_0)$ are fixed, and x(T) is free.

The variation of J is written in the following form

$$\Delta J_c = \delta J_c + \delta^2 J_c \qquad (2.4.4)$$

where δJ_c is the first order variation and $\delta^2 J_c$ is the second order variation.

The necessary conditions that must be satisfied over the time interval $[t_0,T]$ if J_c has a saddle point for the strategies $v = v^*$ and $u = u^*$ are:

1) The Euler-Lagrange equations and the boundary conditions must be satisfied such that δJ is zero.
2) The analogous Legendre-Clebsch conditions satisfy

$$J_{uu} \geqq 0 \tag{2.4.5}$$

$$J_{vv} \leqq 0 \tag{2.4.6}$$

3) Nonexistence of a conjugate point for the accessory minimax problem.

If J has a saddle point for the control strategies $u = u^*$ and $v = v^*$ then the following conditions are sufficient if they are satisfied simultaneously:

1) u^* and v^* satisfy the Euler-Lagrange equations and their boundary conditions.
2) Along u^* and v^* (2.4.5) and (2.4.6) are satisfied over the interval $[t_0,T]$.
3) No conjugate points exist over the interval $[t_0,T]$ for the accessory minimax problem.

2.4.1 Determination of the Euler-Lagrange Equations and their Boundary Conditions

The variation of the constrained payoff functional, J_c, excluding all terms higher than second order is

$$\Delta J_c = \Psi_x \delta x(T) + \delta x(T)' \Psi_{xx} \delta x(T) - \lambda(T)' \delta x(T) +$$

$$\int_{t_0}^{T} \left((H_x + \dot{\lambda}')\delta x + H_u \delta_u + H_v \delta_v \right) dt +$$

$$1/2 \int_{t_0}^{T} [\delta x' \vdots \delta u' \vdots \delta v'] \begin{bmatrix} H_{xx} H_{xu} H_{xv} \\ H_{ux} H_{uu} H_{uv} \\ H_{vx} H_{vu} H_{vv} \end{bmatrix} \begin{bmatrix} \delta x \\ \delta u \\ \delta v \end{bmatrix} dt \tag{2.4.7}$$

Requiring the first order variation to vanish on an optimal trajectory and control leads to the Euler-Lagrange equations and their boundary conditions from which the optimal strategies of the pursuer and evader can be determined. The first order variation is set to zero by equating the coefficients of the variationals equal to zero. This leads to the Euler-Lagrange equations

$$H_x + \dot{\lambda}' = 0 \tag{2.4.8}$$

$$H_u = 0 \tag{2.4.9}$$

$$H_v = 0 \tag{2.4.10}$$

The set of terminal boundary conditions for the state vector $x(t)$ and costate vector $\lambda(t)$ are:

$$\Psi_x - \lambda(T) = 0 \tag{2.4.11}$$

$$x(t_0) = x_0 \tag{2.4.12}$$

Equations (2.4.8) and (2.2.3) are the differential equations representing the unknown nth order state vector $x(t)$ and the unknown nth order costate vector $\lambda(t)$. Equation (2.4.11) represents the terminal boundary conditions of the costate vector, and equation (2.4.12) is the initial boundary condition of the state vector. The solution of equations (2.4.8-2.4.12) and (2.2.3) determines $x(t)$ and $\lambda(t)$. Knowing $x(t)$ and $\lambda(t)$, the participants' optimal open loop strategies are determined by (2.4.9,2.4.10)

2.4.2 Determination of the Analogous Legendre-Clebsch Conditions

The Legendre conditions that must be satisfied if a saddle point exists are

$$J_{uu} = H_{uu} \geqq 0 \tag{2.4.13}$$

$$J_{vv} = H_{vv} \leqq 0 \tag{2.4.14}$$

These are a direct analogy to the two-sided calculus extremum problem.

2.4.3 Conjugate Point Problem

One of the sufficiency conditions that form the set of sufficiency conditions is the nonexistence of a conjugate point along the optimal path. The following definition of a conjugate point for the differential game is similar to the one for the one-sided optimization problem (13).

Definition 1: The point $\tilde{a}(\neq a)$ is said to be conjugate to the point a, if the Euler-Lagrange equations for the differential game have a solution which vanishes for $t=a$ and $t=\tilde{a}$, but is not identically zero. Another definition of a conjugate point which can be used to formulate a procedure for determining the existence of a conjugate point is Definition 2 : The point $t=\tilde{a}$ is said to be conjugate to the point $t=a$ with respect to the payoff functional of the differential game if it is conjugate to $t=a$ with respect to its second order variation.

2.4.4 Procedure for Determining the Existence of a Conjugate Point

If the necessary conditions exist such that the first variation vanishes, the total variation of the payoff functional (2.2.1) is reduced to the following

$$\Delta J_c = \delta x(T)'\Psi_{xx}\delta x(T) + \int_{t_0}^{T} [\delta x' \vdots \delta u' \vdots \delta v'] \begin{bmatrix} H_{xx} & H_{xu} & H_{xv} \\ H_{ux} & H_{uu} & H_{uv} \\ H_{vx} & H_{vu} & H_{vv} \end{bmatrix} \begin{bmatrix} \delta x \\ \delta u \\ \delta v \end{bmatrix} dt \tag{2.4.15}$$

subject to the constraints

$$\delta \dot{x} = F_x \delta x + F_u \delta u + F_v \delta v \tag{2.4.16}$$

$$\delta x(t_0) = 0 \tag{2.4.17}$$

Now according to Definition 2 a test is devised to determine whether conjugate points exist for the quadratic functional (2.4.15) subject to the constraints defined by (2.4.16) and (2.4.17). This procedure is called the accessory minimax problem. This is analogous to the one-sided accessory minimum(maximum) problem (14).

2.4.4 Accessory Minimax Problem

Adjoining the differential constraint of equation (2.4.16) to the quadratic second order variational

$$\Delta J_C = \delta x(T)' \Psi_{xx} \delta x(T) + \int_{t_0}^{T} \Big\{ [\delta x' \vdots \delta u' \vdots \delta v'] \begin{bmatrix} H_{xx} & H_{xu} & H_{xv} \\ H_{uv} & H_{uu} & H_{uv} \\ H_{vx} & H_{vu} & H_{vv} \end{bmatrix} \begin{bmatrix} \delta x \\ \delta u \\ \delta v \end{bmatrix} +$$

$$\delta \lambda' (F_x \delta x + F_u \delta u + F_v \delta v - \delta \dot{x}) \Big\} dt \tag{2.4.18}$$

and redefining

$$\delta x = y \tag{2.4.19}$$

$$\delta u = \eta \tag{2.4.20}$$

$$\delta v = \nu \tag{2.4.21}$$

$$\delta \lambda = \mu \tag{2.4.22}$$

the Hamiltonian for (2.4.18) is

$$H(\mu, \eta, \nu, y, t) = [\, y' \vdots \eta' \vdots \nu'] \begin{bmatrix} H_{xx} & H_{xu} & H_{xv} \\ H_{uv} & H_{uu} & H_{uv} \\ H_{vx} & H_{vu} & H_{vv} \end{bmatrix} \begin{bmatrix} y \\ \eta \\ \nu \end{bmatrix} +$$

$$\mu' (F_x y + F_u \eta + F_v \nu) \tag{2.4.23}$$

and

$$\psi(y(T),T) = \delta x(T)'\Psi_{xx}\delta x(T) \qquad (2.4.24)$$

The second order variation can be written as

$$\Delta J_c = \psi(y(T),T) + \int_{t_0}^{T}\left(H(\mu,\eta,\nu,y,t) - \mu'\dot{y}\right)dt \qquad (2.4.25)$$

Now the necessary conditions for an extremum of ΔJ_c are the Euler-Lagrange equations

$$\dot{y} = F_x y + F_u \eta + F_v \nu \qquad (2.4.26)$$

$$H_\mu + \dot{\mu} = 0 \qquad (2.4.27)$$

$$H_\eta = 0 \qquad (2.4.28)$$

$$H_\nu = 0 \qquad (2.4.29)$$

subject to the boundary conditions

$$y(t_0) = 0 \qquad (2.4.30)$$

$$\mu(T) = \psi_y(y(T),T) \qquad (2.4.31)$$

It is possible to devise the following test for the existence of conjugate points. From (2.4.31) one sees that there are 2n unknowns and n equations. Assuming that (2.4.31) is linearly independent, one can in principle solve n unknowns in terms of the n unknowns which are free. The free unknowns are labeled as $y_i(T)$. $Y(T)$, which is composed of the column vectors $y_i(T)$, is defined as

$$Y(T) = \left[y_1(T)\cdots y_n(T)\right] = \begin{bmatrix} 1 & & & 0 \\ & \ddots & & \\ & & \ddots & \\ 0 & & & 1 \end{bmatrix} \qquad (2.4.32)$$

Now corresponding to these choices of $y_i(T)$ vectors, one can determine the $\mu_i(T)$ vectors from (2.4.31) and form the matrix:

$$\Lambda(T) = [\mu_1(T)\cdots\mu_n(T)] \qquad (2.4.33)$$

After obtaining matrix solution of the Euler-Lagrange equations (2.4.26-2.4.29) with boundary conditions $Y(T)$ and $\Lambda(T)$, if $Y(t)$ becomes singular at some time t during the interval $[t_0,T]$, then a conjugate point of $Y(t)$ exists on the interval $[t_0,T]$.

2.5 SUMMARY

The purposes of this chapter are:

1) To define the dynamics of the participants for the type of differential games studied in this dissertation.
2) To illustrate the general concept of the differential game and the techniques for determining the necessary and sufficient conditions for the differential game.

3. DIFFERENTIAL ENDGAME

3.1 INTRODUCTION

The endgame problem is one where an offensive type missile has to out maneuver an antimissile if it is to strike its target. It is the purpose of this chapter to place the endgame problem within the framework of differential game theory. This particular type of differential game will be called the "differential endgame". The necessary and sufficient conditions for the existence of the differential endgame's saddle point are determined.

3.2 FORMULATION OF THE DIFFERENTIAL ENDGAME

The special class of differential endgame to be studied is as follows:

For the payoff functional

$$J=\frac{a^2}{2}[x_p(T_1)-x_e(T_1)]'A'A[x_p(T_1)-x_e(T_1)] + 1/2 \int_{t_0}^{T_1} u'R_p(t)u\,dt - 1/2 \int_{t_0}^{T_2} v'R_e(t)v\,dt \tag{3.2.1}$$

and the participants' optimal strategies, u^* and v^*, determine $W(x_p(t_0), x_e(t_0), t_0)$, the value of the game, such that

$$W(x_p(t_0),x_e(t_0),t_0) = \underset{u\in U}{\text{Min}}\ \underset{v\in V}{\text{Max}}\ J \tag{3.2.2}$$

subject to the constraints

$$\dot{x}_p=F_p(t)x_p+G_p(t)u+n_p \tag{3.2.3}$$

$$\dot{x}_e=F_e(t)x_e+G_e(t)v+n_e \tag{3.2.4}$$

$$x_p(t_0) = x_{p0} \tag{3.2.5}$$

$$x_e(t_0) = x_{e0} \qquad (3.2.6)$$

$$x_{e1}(T_2) = 0 \qquad (3.2.7)$$

$$x_{e2}(T_2)'x_{e2}(T_2) = V_T^2 \qquad (3.2.8)$$

and

$$u, v \in R^3. \qquad (3.2.9)$$

where the state vectors, x_p and x_e, represent the position and velocity components of the pursuer and evader; the control vectors, u and v, represent the components of the pursuer's and evader's acceleration commands; n_p and n_e represent any disturbance vectors such as the earth's gravitational field; R^3 is the three dimensional open Euclidean space; the 6x6 matrices, $F_p(t)$ and $F_e(t)$, and the 6x3 matrices, $G_p(t)$ and $G_e(t)$, are continuous in time; $R_p(t)$ and $R_e(t)$ are 3x3 positive definite matrices, continuous in time; a^2 is a weighting factor.

The differential endgame considered in this chapter has a finite duration of (T_2-t_0), t_0 being the fixed commencement time of the game, and T_2 being the evader's fixed target time. The differential endgame has a finite engagement interval (T_1-t_0), T_1 being the fixed terminal engagement or intercept time. The postengagement time interval of the differential endgame is (T_2-T_1).

For the differential endgame the payoff functional proposed by (3.2.1) is the engagement's final miss distance, squared, weighted against the difference of the control energies spent by the participants. The pursuer's control energy is spent over the engagement interval and the evader's control energy is spent over the duration of the game. When interception occurs the evader's postengagement trajectory is the optimal path it would have if it were not destroyed.

Both participants have linear nonstationary dynamics, defined by the differential constraints (3.2.3, 3.2.4).

In order to facilitate the application of the method of Lagrangian multipliers the evader's differential constraint over the postengagement interval is converted to an integral constraint

$$x_e(T_2)=\Phi_e(T_2,T_1)x_e(T_1)+$$

$$\int_{T_1}^{T_2}\Phi_e(T_2,t)[G_e(t)v(t)+n_e(t)]dt; \quad T_1< t \leq T_2 \qquad (3.2.10)$$

where $\Phi_e(T_2,t)$ is the evader's state transition matrix and $x_e(T_1)$ and $x_e(T_2)$ are the evader's state vectors at the intercept and target times respectively.

The evader's target constraints are expressed by (3.2.7, 3.2.8). Equation (3.2.7) defines $x_{e1}(T_2)$, the evader's target position vector, which is the origin of the differential endgame's coordinate system. By specifying the inner product of the evader's target velocity vector (3.2.8) represents a measure of the evader's kinetic energy as it strikes the target.

In conclusion, the interpretation of this differential endgame is that at some fixed time, T_1, the pursuer tries to intercept an evader, which is attempting to penetrate the pursuer's defense in order to strike its target at some fixed time T_2. Both participants have limited energy sources. An open-loop version of this endgame problem is considered since the optimal control forces of the participants are considered only as functions of time.

Variational calculus as applied to one-sided optimal control problems (14, 15) is applied to the differential endgame problem as follows. Vector Lagrangian multipliers λ_p and λ_e are introduced in order to adjoin the differential constraints (3.2.3) and (3.2.4) to the payoff functional (3.2.1) over the engagement interval. Also the vector Lagrangian multiplier μ adjoins over the postengagement interval the integral constraint (3.2.10) to (3.2.1), and the scalar Lagrangian multiplier η adjoins the evader's target constraint (3.2.8) to (3.2.1). In terms of the differential endgame's constrained payoff functional, the differential endgame is mathematically expressed as

$$\begin{aligned}\underset{u\in U}{\text{Min}}\ \underset{v\in V}{\text{Max}}\ J_c = \underset{u\in U}{\text{Min}}\ \underset{v\in V}{\text{Max}} \Big\{ & \frac{a^2}{2}[x_p(T_1)-x_e(T_1)]'A'A[x_p(T_1)-x_e(T_1)] \\ & + \int_{t_0}^{T_1} [1/2u'R_p(t)u-1/2v'R_e(t)v+\lambda_p'(F_p(t)x_p+G_p(t)u+n_p-\dot{x}_p) \\ & + \lambda_e'(F_e(t)x_e+G_e(t)v+n_e-\dot{x}_e]dt+\eta/2(x_{e2}(T_2)'x_{e2}(T_2)-V_T^2) \\ & + \mu'[x_e(T_2)-\Phi_e(T_2,T_1)x_e(T_1)] \\ & - \int_{T_1}^{T_2} [1/2v'R_e(t)v+\mu'\Phi_e(T_2,t)(G_e(t)v+n_e)]dt \Big\} \end{aligned} \qquad (3.2.11)$$

3.3 DETERMINATION OF THE EULER-LAGRANGE EQUATIONS FOR THE DIFFERENTIAL ENDGAME

Applying variations, δu and δv, about a particular pair of controls, u and v, the total variation of the differential endgame's constrained payoff functional J_c is

$$\begin{aligned}
\Delta J_c &= [a^2(x_p(T_1)-x_e(T_1))'A'A-\lambda_p'(T_1)]\delta x_p(T_1) \\
&+[a^2(x_e(T_1)-x_p(T_1))A'A-\lambda_e'(T_1)-\mu'\Phi_e(T_2,T_1)]\delta x_e(T_1) \\
&+[\mu'Q'+\eta x_{e2}(T_2)']\delta x_{e2}(T_2) \\
&+\int_{t_0}^{T_1}[(u'R_p(t)+\lambda_p'G_p(t))\delta u+(-v'R_e(t)+\lambda_e'G_e(t))\delta v \\
&+(\dot{\lambda}_p'+\lambda_p'F_p(t))\delta x_p+(\dot{\lambda}_e'+\lambda_e'F_e(t))\delta x_e]dt \\
&-\int_{T_1}^{T_2}[v'R_e(t)+\mu'\Phi_e(T_2,t)G_e(t)]\delta v\, dt \\
&+a^2/2[\delta x_p(T_1)-\delta x_e(T_1)]'A'A[\delta x_p(T_1)-\delta x_e(T_1)] \\
&+\eta/2\delta x_{e2}(T_2)'\delta x_{e2}(T_2) \\
&+1/2\int_{t_0}^{T_1}[\delta u' : \delta v']\begin{bmatrix} R_p(t) & 0 \\ 0 & -R_e(t)\end{bmatrix}\begin{bmatrix}\delta u \\ \delta v\end{bmatrix}dt-1/2\int_{T_1}^{T_2}\delta v'R_e(t)\delta v\, dt \qquad (3.3.1)
\end{aligned}$$

From ΔJ_c the necessary and sufficient conditions for the existence of a saddle point for the differential endgame's payoff functional are determined. Of primary interest are the necessary conditions which result in the determination of the Euler-Lagrange equations and their associated boundary conditions. These necessary conditions are derived by requiring the first order variations of ΔJ_c to vanish. Table 1 summarizes these necessary conditions.

Table 1

Differential Endgame's Necessary Conditions Required for δJ to Vanish

Variational	Coefficients of Variationals Equated to Zero	
δu	$u' R_p(t)+\lambda_p' G_p(t)=0 ; \quad t_0 \leq t \leq T_1$	(3.3.2)
δv	$-v' R_e(t)+\lambda_e' G_e(t)=0 ; \quad t_0 \leq t \leq T_1$	(3.3.3)
	$v' R_e(t) + \mu' \Phi_e(T_2,t) G_e(t)=0; \quad T_1 < t \leq T_2$	(3.3.4)
δx_p	$\dot{\lambda}_p' + \lambda_p' F_p(t)=0; \quad t_0 \leqq t \leqq T_1$	(3.3.5)
δx_e	$\dot{\lambda}_e' + \lambda_e' F_e(t)=0; \quad t_0 \leqq t \leqq T_1$	(3.3.6)
$\delta x_p(T_1)$	$\lambda_p'(T_1)-a^2[x_p(T_1)-x_e(T_1)]'A'A=0$	(3.3.7)
$\delta x_e(T_1)$	$\lambda_e'(T_1)+a^2[x_p(T_1)-x_e(T_1)]'A'A+\mu'\Phi_e(T_2,T_1)=0$	(3.3.8)
$\delta x_e(T_2)$	$\eta x_{e2}(T_2)+\mu_2=0$*	(3.3.9)

*The (3x1) partitioned vector, μ_2, is obtained by pre-multiplying the (6x1) vector, μ, by the matrix Q.

The complete set of Euler-Lagrange equations is formed by combining the differential constraints of the participants (3.2.3) and (3.2.4) with the necessary conditions (3.3.2-3.3.6). The boundary conditions for the unknown variables of the Euler-Lagrange equations are formed by combining the known boundary conditions(3.2.5-3.2.8) with the necessary conditions (3.3.7-3.3.9). Therefore the Euler-Lagrange equations are:

$$\begin{bmatrix}\dot{x}_p\\ \dot{x}_e\\ \dot{\lambda}_p\\ \dot{\lambda}_e\end{bmatrix}=\begin{bmatrix}F_p(t) & 0 & -G_p(t)R_p(t)^{-1}G_p'(t) & 0\\ 0 & F_e(t) & 0 & G_e(t)R_e^{-1}(t)G_e'(t)\\ 0 & 0 & -F_p'(t) & 0\\ 0 & 0 & 0 & -F_e'(t)\end{bmatrix}\begin{bmatrix}x_p\\ x_e\\ \lambda_p\\ \lambda_e\end{bmatrix}+\begin{bmatrix}n_p\\ n_e\\ 0\\ 0\end{bmatrix}$$

for $t_0 \leqq t \leqq T_1$ (3.3.10)

$\dot{x}_e=F_e(t)x_e-G_e(t)R_e^{-1}(t)G_e'(t)\Phi_e'(T_2,t)\mu+n_e$; for

$T_1 < t \leqq T_2$ (3.3.11)

The boundary conditions for the state vectors, x_p and x_e, the costate vectors, λ_p and λ_e, in terms of the constant costate vector μ and the scalar Lagrangian multiplier η are:

$$\begin{bmatrix}x_p(t_0)\\ \hline x_e(t_0)\end{bmatrix}=\begin{bmatrix}x_{p0}\\ \hline x_{e0}\end{bmatrix} \quad (3.3.12)$$

$$x_{e1}(T_2) = 0 \quad (3.3.13)$$

$$\eta x_{e2}(T_2) = -\mu_2 \quad (3.3.14)$$

$$\lambda_p(T_1) = a^2A'A[x_p(T_1)-x_e(T_1)] \quad (3.3.15)$$

$$\lambda_e(T_1)+a^2A'A[x_p(T_1)-x_e(T_1)]+\Phi_e'(T_2,T_1)\,\mu=0 \quad (3.3.16)$$

$$x_{e2}(T_2)'x_{e2}(T_2) = V_T^2 \quad (3.3.17)$$

The simultaneous solution of these linear differential equations and their nonlinear algebraic set of boundary equations yields the costate vectors λ_p, λ_e, and μ, and the scalar Lagrangian multiplier η. With the determination of these Lagrangian multipliers the optimal strategies for the participants and the differential endgame's value can be determined.

3.4 SOLUTION OF THE EULER-LAGRANGE EQUATIONS

The solution for the costate vectors of the Euler-Lagrange equations (3.3.10) defined over the engagement interval $[t_0, T_1]$ is

$$\lambda_k = \Phi_k'(T_1,t)\lambda_k(T_1); \; k = p,e \quad (3.4.1)$$

where $\Phi_k'(T_1,t)$ are the transition matrices for the costate vectors, λ_p and λ_e, and $\lambda_p(T_1)$ and $\lambda_e(T_1)$ are the unknown costate vectors' corner conditions. The participants' state vectors at T_1 in terms of the corner condition vectors are:

$$\begin{bmatrix} x_p(T_1) \\ x_e(T_1) \end{bmatrix} = \begin{bmatrix} \Phi_p(T_1,t_0) & 0 \\ 0 & \Phi_e(T_1,t_0) \end{bmatrix} \begin{bmatrix} x_p(t_0) \\ x_e(t_0) \end{bmatrix} +$$

$$\begin{bmatrix} -M_p(T_1,t_0) & 0 \\ 0 & M_e(T_1,t_0) \end{bmatrix} \begin{bmatrix} \lambda_p(T_1) \\ \lambda_e(T_1) \end{bmatrix} + \begin{bmatrix} k_p(T_1,t_0) \\ k_e(T_1,t_0) \end{bmatrix} \quad (3.4.2)$$

where the controllability matrices are defined as

$$M_k(T_1,t_0) = \int_{t_0}^{T_1} \Phi_k(T_1,t)G_k(t)R_k^{-1}(t)G_k'(t)\Phi_k'(T_1,t)dt; \; k=p,e \quad (3.4.3)$$

and the 6x1 column vectors $k_k(T_1,t_0)$ due to the disturbance vectors n_k are

$$k_k(T_1,t_0) = \int_{t_0}^{T_1} \Phi_k(T_1,t)n_k dt; \qquad k=p,e \qquad (3.4.4)$$

The solution of the Euler-Lagrange equation (3.3.11) defined at $t=T_2$ in terms of the unknown costate vector μ and the evaders unknown terminal engagement state $x_e(T_1)$ is

$$x_e(T_2)=\Phi_e(T_2,T_1)x_e(T_1)-M_e(T_2,T_1)\mu+k_e(T_2,T_1) \qquad (3.4.5)$$

where the 6x6 controllability matrix is defined as

$$M_e(T_2,T_1)= \int_{T_1}^{T_2} \Phi_e(T_2,t)G_e(t)R_e(t)^{-1}G_e'(t)\Phi_e'(T_2,t)dt \qquad (3.4.6)$$

and the 6x1 column vector $k_e(T_2,T_1)$ due to the disturbance vector n_e is

$$k_e(T_2,T_1)= \int_{T_1}^{T_2} \Phi_e(T_2,t)n_e dt \qquad (3.4.7)$$

Therefore, by solving for the unknown corner condition vectors, $\lambda_p(T_1)$ and $\lambda_e(T_1)$, and the costate vector μ, the solution of the Euler-Lagrange equations are obtained.

3.4.1 Determination of the Costate Vectors' Corner Conditions

With the use of the evader's boundary condition

$$x_e(T_2)=\Phi_e(T_2,T_1)x_e(T_1)-M_e(T_2,T_1)\mu+k_e(T_2,T_1)=\begin{bmatrix} 0 \\ - - \\ -1/\eta\ \mu_2 \end{bmatrix} \qquad (3.4.8)$$

The costate vector μ is expressed in terms of the evader's unknown terminal engagement state $x_e(T_1)$ and the unknown Lagrangian multiplier η as:

$$\mu=[M_e(T_2,T_1)-1/\eta Q'Q]^{-1}\{\Phi_e(T_2,T_1)x_e(T_1)+k_e(T_2,T_1)\} \qquad (3.4.9)$$

Substituting for $x_e(T_1)$ μ becomes, in terms of the evader's corner condition costate vector $\lambda_e(T_1)$ and the scalar Lagrangian multiplier η:

$$\mu=[M_e(T_2,T_1)-1/\eta Q'Q]^{-1}\{\Phi_e(T_2,t_0)x_e(t_0)+$$
$$\Phi_e(T_2,T_1)M_e(T_1,t_0)\lambda_e(T_1)+\Phi_e(T_2,T_1)k_e(T_1,t_0)+k_e(T_2,T_1)\} \qquad (3.4.10)$$

Substituting the unknown vectors- $x_e(T_1)$, $x_p(T_1)$ and μ given by (3.4.2) and (3.4.10) into the corner condition equations (3.3.15, 3.3.16) the unknown variables in the corner condition equations are reduced to $\lambda_p(T_1)$, $\lambda_e(T_1)$ and η. The corner conditions become

$$\left[\begin{array}{c|c} U+a^2A'AM_p(T_1,t_0) & a^2A'AM_e(T_1,t_0) \\ \hline -a^2A'AM_p(T_1,t_0) & U-[a^2A'A-K(\eta)]M_e(T_1,t_0) \end{array}\right]\left[\begin{array}{c} \lambda_p(T_1) \\ \hline \lambda_e(T_1) \end{array}\right] =$$

$$\left[\begin{array}{c|c} a^2A'A & -a^2A'A \\ \hline -a^2A'A & a^2A'A-K(\eta) \end{array}\right]\left\{\left[\begin{array}{cc} \Phi_p(T_1,t_0) & 0 \\ 0 & \Phi_e(T_1,t_0) \end{array}\right]\left[\begin{array}{c} x_p(t_0) \\ \hline x_e(t_0) \end{array}\right]\right.$$
$$\left.+\left[\begin{array}{c} k_p(T_1,t_0) \\ \hline k_e(T_1,t_0) \end{array}\right]\right\}-\left[\begin{array}{c} 0 \\ \hline K(\eta)\Phi_e(T_1,T_2)k_e(T_2,T_1) \end{array}\right] \qquad (3.4.11)$$

where the matrix $K(\eta)$ is defined as

$$K(\eta)=\Phi'_e(T_2,T_1)\ [M_e(T_2,T_1)-1/\eta Q'Q]^{-1}\Phi_e(T_2,T_1) \qquad (3.4.12)$$

In shorthand form (3.4.11) is rewritten as

$$B(\eta)\lambda(T_1)= C(\eta)x(t_0)+\ell(\eta) \qquad (3.4.13)$$

The corner condition costate vectors $\lambda_p(T_1)$ and $\lambda_e(T_1)$ are determined as functions of the unknown parameter η as

$$\lambda_e(T_1)=[0\,|\,U]B(\eta)^{-1}(C(\eta)x(t_0)+\ell(\eta)) \qquad (3.4.14)$$

$$\lambda_p(T_1)=[U\,|\,0]B(\eta)^{-1}(C(\eta)x(t_0)+\ell(\eta)) \qquad (3.4.15)$$

where the order of the null and identity matrices are 6x6 for the three dimensional coordinate system.

Substituting (3.4.12) and (3.4.14) into (3.4.10) the costate vector μ is expressed in terms of the parameter η as

$$\mu=\Phi_e'(T_1,T_2)\; K(\eta)[M_e(T_1,t_0)[0\,|\,U]B(\eta)^{-1}(C(\eta)x(t_0)+\ell(\eta))$$
$$+\Phi_e(T_1,t_0)x_e(t_0)+k_e(T_1,t_0)+\Phi_e(T_1,T_2)k_e(T_2,T_1)] \qquad (3.4.16)$$

The parameter η is determined by substituting the evader's target boundary condition (3.4.8) into the evader's kinetic energy target constraint (3.2.8).

$$x_{e2}(T_2)'x_{e2}(T_2)=1/\eta^2\mu'Q'Q\mu=V_T^2 \qquad (3.4.17)$$

By substituting μ (3.4.16) into (3.4.17) the evader's target kinetic energy constraint (3.4.17) is expressed in terms of the scalar multiplier η.

$$1/\eta^2\mu'Q'Q\mu-V_T^2 = \sum_{i=0}^{L} p_i\eta^i = 0 \qquad (3.4.18)$$

The value for η is determined by extracting the roots of (3.4.18).

Equation (3.4.18) represents an exact polynomial. L depends on the weighting matrices, $R_p(t)$ and $R_e(t)$, and the matrices, $F_k(t)$, $G_k(t)$; k=p,e, defining the dynamics of the participants.

In the Appendix 8.1.5 it is shown that when

$$F_k = \begin{bmatrix} 0 & U \\ 0 & 0 \end{bmatrix};\; k=p,e \qquad (3.4.19)$$

$$G_k = \begin{bmatrix} 0 \\ U \end{bmatrix} \tag{3.4.20}$$

$$R_k(t) = r_k[U]; \quad k = p, e \tag{3.4.21}$$

the degree of the polynomial (3.4.18) expressing the evader's kinetic energy constraint is six.

3.4.2 Determination of the Control Algorithms for the Pursuer and the Evader

From the necessary conditions (3.3.2, 3.3.3) requiring δJ_c to vanish the open loop optimal strategies for both the pursuer and the evader over the engagement time interval are:

$$v^* = R_e^{-1}(t) G_e'(t)\, \Phi_e'(T_1, t)\, \lambda_e(T_1) \tag{3.4.22}$$

$$u^* = -R_p^{-1}(t) G_p'(t)\, \Phi_p'(T_1, t)\, \lambda_p(T_1) \tag{3.4.23}$$

Substituting the known corner condition vectors (3.4.14) and (3.4.15) into (3.4.22, 3.4.23), the control algorithms for the participants over the engagement interval are;

$$v^* = R_e^{-1}(t) G_e'(t)\, \Phi_e'(T_1, t)\, [0 \vdots U] B(\eta)^{-1} \big(C(\eta) x(t_0) + \ell(\eta) \big) \tag{3.4.24}$$

$$u^* = -R_p^{-1}(t) G_p'(t)\, \Phi_p'(T_1, t)\, [U \vdots 0] B(\eta)^{-1} \big(C(\eta) x(t_0) + \ell(\eta) \big) \tag{3.4.25}$$

These time-varying optimal strategies are functions of the initial state and initial parameters of the endgame.

The evader's optimal strategy over the postengagement interval is derived from one of the necessary conditions (3.3.4), requiring δJ_c to vanish. This strategy is expressed as

$$v^* = -R_e(t)^{-1} G_e'(t) \Phi_e'(T_2, t) \mu \tag{3.4.26}$$

Substituting for μ, given by (3.4.9), (3.4.26) becomes

$$v^{*} = -R_e(t)^{-1}G_e'(t)\bar{\Phi}_e'(T_2,t)\Phi_e'(T_1,T_2)K(\eta)\times$$
$$[x_e(T_1)+\bar{\Phi}_e(T_1,T_2)k_e(T_2,T_1)] \qquad (3.4.27)$$

where the evader's state vector $x_e(T_1)$ is considered as the evader's initial state vector to be measured during the postengagement interval. The scalar multiplier η is determined by substituting the vector μ, defined by (3.4.9) into the evader's terminal kinetic energy constraint (3.4.17).

These strategies do not take into account errors such as noise from radar measurements and the approximations of the exact dynamics of the participants. The uncertainties due to these errors during the endgame's duration can be reduced by continuously measuring the initial state of the participants and updating the initial parameters of the differential endgame.

3.5 NECESSARY AND SUFFICIENT CONDITIONS FOR THE DIFFERENTIAL ENDGAME

The necessary conditions that must be satisfied over the differential endgame's time interval, $[t_0,T_2]$, if the payoff functional, J, defined by (3.2.1) has a saddle point for the strategies, $u=u^{*}$ and $v=v^{*}$, are:

1) The Euler-Lagrange equations and their boundary conditions must be satisfied when the first order variation of ΔJ_c is equated to zero. This has been previously accomplished in Section 3.2.

2) An analogous Legendre-Clebsch condition for the saddle point must be satisfied over the time interval of the differential endgame.

3) Nonexistence of a conjugate point for the accessory minimax problem over the time duration of the differential endgame must be shown.

If J has a saddle point for the control strategies, $u = u^*$ and $v = v^*$, then the three previous necessary conditions are sufficient if they are satisfied simultaneously.

3.5.1 Analogous Legendre-Clebsch Conditions for the Differential Endgame

With reference to the variation of the constrained payoff functional (3.3.1), when the first order variation δJ_c vanishes, ΔJ_c becomes

$$\delta^2 J_c = a^2/2[\delta x_p(T_1)-\delta x_e(T_1)]'A'A[\delta x_p(T_1)-\delta x_e(T_1)]+$$
$$\eta/2\,\delta x_{e2}(T_2)'\delta x_{e2}(T_2)+1/2\int_{t_0}^{T_1}\delta u' R_p(t)\delta u dt$$
$$-1/2\int_{t_0}^{T_2}\delta v' R_e(t)\delta v dt \qquad (3.5.1)$$

Another necessary condition for J to have a saddle point with respect to the control strategies (u, v) is that the second order variational terms with respect to δu must be nonnegative and the second order variational terms with respect to δv must be nonpositive. These conditions are satisfied if

$$R_p(t) \geq 0 \tag{3.5.2}$$

$$R_e(t) \geq 0 \tag{3.5.3}$$

These conditions are analogous to the one-sided optimization problem (13).

3.5.2 Conjugate Point Problem

The final necessary condition for the existence of a saddle point (u^*,v^*) is the nonexistence of a conjugate point for the accessory minimax problem of the differential endgame.

3.5.2.1 Accessory Minimax Problem

The accessory minimax problem is $\underset{\delta u}{\text{Min}}\ \underset{\delta v}{\text{Max}}\ \delta^2 J_c =$

$$\underset{\delta u}{\text{Min}}\ \underset{\delta v}{\text{Max}}\Big\{a^2/2[\delta x_p(T_1)-\delta x_e(T_1)]'A'A[\delta x_p(T_1)-\delta x_e(T_1)]$$

$$+\eta/2\delta x_e(T_2)'Q'Q\delta x_e(T_2)$$

$$+\ 1/2\int_{t_0}^{T_1}[\delta u' \vdots \delta v']\begin{bmatrix} R_p(t) & 0 \\ 0 & -R_e(t)\end{bmatrix}\begin{bmatrix}\delta u \\ \delta v\end{bmatrix}dt$$

$$-1/2\int_{T_1}^{T_2}\delta v' R_e(t)\delta v dt\Big\} \tag{3.5.5}$$

subject to the constraints

$$\delta\dot{x}_p = F_p(t)\delta x_p + G_p(t)\delta u \tag{3.5.6}$$

$$\delta\dot{x}_e = F_e(t)\delta x_e + G_e(t)\delta v;\ t_0 \leqq t \leqq T_1 \tag{3.5.7}$$

$$\delta x_e(T_2) = \Phi_e(T_2,T_1)\delta x_e(T_1) + \int_{T_1}^{T_2}\Phi_e(T_2,t)G_e(t)\delta v;$$

$$T_1 < t \leq T_2 \tag{3.5.8}$$

$$\begin{bmatrix}\delta x_p(t_0) \\ \hdashline \delta x_e(t_0)\end{bmatrix} = 0 \tag{3.5.9}$$

$$\delta x_{e1}(T_2) = 0 \tag{3.5.10}$$

$$x_e(T_2)Q'Q\delta x_e(T_2) = 0 \tag{3.5.11}$$

where the payoff functional for the accessory minimax problem is the second order variation of the differential endgame's constrained payoff functional (3.2.11); the differential constraints (3.5.6, 3.5.7) and the integral constraint (3.5.8) represent the perturbated dynamics of the participants, due to the variationals δu and δv; the boundary condition constraints (3.5.9-3.5.11) represent the perturbated boundary and target constraints due to the variationals δu and δv.

If the solution of the Euler-Lagrange equations that results from $\underset{\delta u}{\text{Min}}\ \underset{\delta v}{\text{Max}}\,\delta^2 J_c$ does not vanish, then the non-existence of a conjugate point is assured.

Adjoining to $\delta^2 J_c$, the differential constraints, (3.5.6, 3.5.7) by $\delta\lambda_p$ and $\delta\lambda_e$, the integral constraint (3.5.8) by $\delta\mu$, and the target constraint (3.5.11) by $\delta\eta$, the accessory minimax problem becomes

$$\underset{\delta u}{\text{Min}}\ \underset{\delta v}{\text{Max}}\Big\{ a^2/2[\delta x_p(T_1)-\delta x_e(T_1)]'A'A[\delta x_p(T_1)-\delta x_e(T_1)]$$

$$+\eta/2\,\delta x_e(T_2)'Q'Q\delta x_e(T_2)+\delta\eta x_e(T_2)'Q'Q\delta x_e(T_2)$$

$$+\int_{t_0}^{T_1}\Big(1/2\delta u'R_p(t)\delta u-1/2\delta v'R_e(t)\delta v+\delta\lambda_p'(F_p(t)\delta x +G_p(t)\delta u-\delta\dot{x}_p)$$

$$+\delta\lambda_e'(F_e(t)\delta x_e+G_e(t)\delta v-\delta\dot{x}_e\Big)dt+\delta\mu'(\delta x_e(T_2)-\Phi_e(T_2,T_1)\delta x_e(T_1))$$

$$-\int_{T_1}^{T_2}\Big(1/2\delta v'R_e(t)\delta v+\Phi_e(T_2,t)G_e(t)\delta v\Big)dt\Big\} \tag{3.5.12}$$

Applying variational calculus to (3.5.12) the variation of $\delta^2 J$ is

$$\Delta(\delta^2 J_c) = \delta(\delta^2 J_c) + \ . \ . \ . \ + \qquad (3.5.13)$$

where $\delta(\delta^2 J_c)$ is the first order variation of $\delta^2 J_c$. A necessary condition required for the existence of a saddle point for $\delta^2 J_c$ is

$$\delta(\delta^2 J_c) = 0 \qquad (3.5.14)$$

The necessary conditions satisfying (3.5.14) are in Table 2.

3.5.2.2 Solution of the Accessory Minimax Problem

Before proceeding to establish the test for the existence of a conjugate point it is necessary to eliminate the unknown costate vector $\delta\mu$ from the corner condition equations (3.5.22) and (3.5.23). With the aid of the necessary condition (3.5.19) defining the optimal control δv over the interval $(T_1, T_2]$, and the integral constraint (3.5.8) the boundary condition for $\delta x_e(t)$ is written as

$$\delta x_e(T_2) = -\begin{bmatrix} 0 \\ \hline \delta\eta/\eta\, x_{e2}(T_2) \end{bmatrix} - \begin{bmatrix} 0 \\ \hline 1/\eta\, \delta\mu_2 \end{bmatrix} = \Phi_e(T_2, T_1)\delta x_e(T_1)$$

$$-M_e(T_2, T_1)\delta\mu \qquad (3.5.15)$$

Solving for $\delta\mu$

$$\delta\mu = [M_e(T_2, T_1) - 1/\eta Q'Q]^{-1}\Big\{\Phi_e(T_2, T_1)\delta x_e(T_1)$$

$$+\delta\eta \begin{bmatrix} 0 \\ \hline 1/\eta\, x_{e2}(T_2) \end{bmatrix}\Big\} \qquad (3.5.16)$$

Substituting $\delta\mu$ in the corner condition equations (3.5.22-3.5.23) the corner conditions become;

Table 2

Necessary Conditions for $\delta(\delta^2 J)$ to Vanish

Variational	Coefficients of Variationals Equated to Zero	
$\delta(\delta u)$	$R_p(t)\delta u+G_p'(t)\delta\lambda_p=0$	(3.5.17)
$\delta(\delta v)$	$-R_e(t)\delta v+G_e'(t)\delta\lambda_e=0;\ t_0\leqq t\leqq T_1$	(3.5.18)
	$-R_e(t)\delta v-G_e'(t)\ \tilde{\Phi}_e'(T_2,t)\ \delta\mu=0;\ T_1<t\leqq T_2$	(3.5.19)
$\delta(\delta x_p)$	$\delta\dot{\lambda}_p'+\delta\lambda_p'F_p(t)=0$	(3.5.20)
$\delta(\delta x_e)$	$\delta\dot{\lambda}_e'+\delta\lambda_e'F_e(t)=0$	(3.5.21)
$\delta(\delta x_p(T_1))$	$-\delta\lambda_p'(T_1)+a^2[\delta x_p(T_1)-\delta x_e(T_1)]'A'A=0$	(3.5.22)
$\delta(\delta x_e(T_1))$	$-\delta\lambda_e(T_1)'-a^2[\delta x_p(T_1)-\delta x_e(T_1)]'A'A$	
	$-\delta\mu'\tilde{\Phi}_e(T_2,T_1)=0$	(3.5.23)
$\delta(\delta x_{e2}(T_2))$	$\eta\delta x_{e2}(T_2)+\delta\eta x_{e2}(T_2)+\delta\mu_2=0$	(3.5.24)

$$\delta\lambda(T_1) = \left[\begin{array}{c|c} a^2A'A & -a^2A'A \\ \hline -a^2A'A & a^2A'A - \Phi_e'(T_2,T_1)[M_e(T_2,T_1) - 1/\eta\, Q'Q]^{-1}\Phi_e(T_2,T_1) \end{array}\right]\delta x(T_1)$$

$$- \left[\begin{array}{c} 0 \\ \hline \delta\eta\,\Phi_e'(T_2,T_1)[M_e(T_2,T_1) - 1/\eta\, Q'Q]^{-1}\left[\begin{array}{c} 0 \\ \hline 1/\eta\, x_{e2}(T_2) \end{array}\right] \end{array}\right] \qquad (3.5.25)$$

where $\delta\lambda(T_1)$ and $\delta x(T_1)$ are defined as

$$\delta\lambda(T_1) = \begin{bmatrix} \delta\lambda_p(T_1) \\ \delta\lambda_e(T_1) \end{bmatrix} \qquad (3.5.26)$$

$$\delta x(T_1) = \begin{bmatrix} \delta x_p(T_1) \\ \delta x_e(T_1) \end{bmatrix} \qquad (3.5.27)$$

From the corner condition equations (3.5.25) one can solve for the unknown corner condition of the costate vectors, $\delta\lambda_p(T_1)$ and $\delta\lambda_e(T_1)$, and the constant costate vector $\delta\mu$ in terms of the unknown state vectors, $\delta x_p(T_1)$ and $\delta x_e(T_1)$. Now specifying the free unknowns $\delta x_k(T_1)$; $k = p, e$, as m_i, the system of the free unknowns, $X(T_1)$ is

$$X(T_1) = [m_1 \; . \; . \; . \; m_n] = \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix} \qquad (3.5.28)$$

The system of the corner condition costate vectors $\Lambda(T_1)$ is

$$\Lambda(T_1) = \begin{bmatrix} a^2A'A & -a^2A'A \\ -a^2A'A & [a^2A'A - K(\eta)] \end{bmatrix} + C \qquad (3.5.29)$$

where the matrix C is composed of the number of column vectors equal to the number of free unknown column vectors m_i. These column vectors which are partitioned into two 6x1 column vectors are defined as:

$$c_i = \begin{bmatrix} 0 \\ \hline \delta\eta_i \bar{\Phi}_e'(T_2,T_1)[M_e(T_2,T_1)-1/\eta\, Q'Q]^{-1} \begin{bmatrix} 0 \\ 1/\eta\, x_{e2}(T_2) \end{bmatrix} \end{bmatrix} \quad (3.5.30)$$

Substituting the m_i vectors into (3.5.16) the system of costate vectors composed of $\delta\mu_i$ becomes

$$M = \bar{\Phi}_e'(T_1,T_2)[K(\eta)+C] \quad (3.5.31)$$

Now for the accessory minimax problem, the system of Euler-Lagrange equations over the engagement time interval $[t_0,T_1]$ is

$$\begin{bmatrix} \dot{X} \\ \\ \dot{\Lambda} \\ \end{bmatrix} = \begin{bmatrix} F_p(t) & 0 & -G_p(t)R_p(t)^{-1}G_p'(t) & 0 \\ 0 & F_e(t) & 0 & G_e(t)R_e^{-1}(t)G_e'(t) \\ 0 & 0 & -F_p'(t) & 0 \\ 0 & 0 & 0 & -F_e'(t) \end{bmatrix} \begin{bmatrix} X \\ \\ \Lambda \\ \end{bmatrix} \quad (3.5.32)$$

subject to the boundary conditions $X(T_1)$ and $\Lambda(T_1)$, defined by (3.5.28) and (3.5.29). The solution of the Euler-Lagrange equations over the engagement time interval is

$$\begin{bmatrix} X(t) \\ \\ \Lambda(t) \\ \end{bmatrix} = \begin{bmatrix} \bar{\Phi}_p(t,T_1) & 0 & 0 & 0 \\ 0 & \bar{\Phi}_e(t,T_1) & 0 & 0 \\ 0 & 0 & \bar{\Phi}_p'(t,T_1) & 0 \\ 0 & 0 & 0 & \bar{\Phi}_e'(t,T_1) \end{bmatrix} \times$$

$$\begin{bmatrix} U & 0 & M_p(T_1,t_0) & 0 \\ 0 & U & 0 & -M_e(T_1,t_0) \\ 0 & 0 & U & 0 \\ 0 & 0 & 0 & U \end{bmatrix} \begin{bmatrix} X(T_1) \\ \\ \Lambda(T_1) \\ \end{bmatrix} \quad (3.5.33)$$

Substituting the corner condition equations (3.5.28, 3.5.29) into (3.5.33)

$$X(t) = \begin{bmatrix} \Phi_p(t,T_1) & 0 \\ 0 & \Phi_e(t,T_1) \end{bmatrix} \times$$

$$\left(U + \begin{bmatrix} M_p(T_1,t_0) & 0 \\ 0 & -M_e(T_1,t_0) \end{bmatrix} \left\{ \begin{bmatrix} a^2A'A & -a^2A'A \\ -a^2A'A & a^2A'A-K(\eta) \end{bmatrix} + C \right\} \right);$$

$$t_0 \leq t \leq T_1 \qquad (3.5.34)$$

As shown by (3.5.30) the matrix C is a function of the unknown scalar Lagrangian multipliers $\delta\eta_i$. $\delta\eta_i$ is determined as follows: substituting $\delta x_e(T_2)$ obtained from the necessary condition (3.5.24) into the target constraint (3.5.11) one obtains

$$\delta\eta x_{e2}(T_2)'x_{e2}(T_2) = -x_{e2}(T_2)'\delta\mu_2 \qquad (3.5.35)$$

Substituting for $\delta\mu_2$, (3.5.35) becomes

$$\delta\eta x_{e2}(T_2)'x_{e2}(T_2) = -x_{e2}(T_2)'Q\Phi_e'(T_1,T_2)K(\eta)\delta x_e(T_1)$$
$$- \delta\eta/\eta Q\Phi_e'(T_1,T_2)K(\eta)\Phi_e(T_1,T_2)x_e(T_2) \qquad (3.5.36)$$

Solving for $\delta\eta$ and using the terminal kinetic energy constraint

$$\delta\eta = \frac{-x_{e2}(T_2)'Q\Phi_e'(T_1,T_2)K(\eta)\delta x_e(T_1)}{V_T^2+1/\eta Q\Phi_e'(T_1,T_2)K(\eta)\Phi_e(T_1,T_2)x_e(T_2)} \qquad (3.5.37)$$

Now for each selection of the column vector m_i

$$\delta\eta_i = \frac{-x_{e2}(T_2)'Q\Phi_e'(T_1,T_2)K(\eta)m_i}{V_T^2+1/\eta Q\Phi_e'(T_1,T_2)K(\eta)\Phi_e(T_1,T_2)x_e(T_2)} \qquad (3.5.38)$$

Over the postengagement interval X (t) is defined as

$$X_e(t) = \Phi_e(T_2,t) - M_e(t,T_1)M \qquad (3.5.39)$$

3.5.2.3 Conditions for Nonexistence of a Conjugate Point

The solution of the Euler-Lagrange equations X(t) for the accessory minimax problem does not vanish, and consequently no conjugate points exist, if

$$\det\left\{ U + \begin{bmatrix} M_p(T_1,t_0) & 0 \\ 0 & -M_e(T_1,t_0) \end{bmatrix} \left(\begin{bmatrix} a^2A'A & -a^2A'A \\ -a^2A'A & a^2A'A-K(\eta) \end{bmatrix} + C \right) \right\}$$

$$\neq 0, \quad t_0 \leqq t \leqq T_1 \tag{3.5.40}$$

$$\det\{ \Phi_e(T_2,t)+M_e(t,T_1)M\} \neq 0, \quad T_1 < t \leqq T_2 \tag{3.5.41}$$

3.6 EXAMPLE OF A DIFFERENTIAL ENDGAME

Here the framework of the differential endgame is defined by specifying the dynamics of the pursuer and evader and their weighting matrices, $R_p(t)$ and $R_e(t)$. For this particular differential endgame, which was programmed on the digital computer, the dynamics of the participants both have the identical form:

$$\begin{bmatrix} \dot{x}_{k1} \\ \dot{x}_{k2} \end{bmatrix} = \begin{bmatrix} 0 & U \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_{k1} \\ x_{k2} \end{bmatrix} + \begin{bmatrix} 0 \\ U \end{bmatrix} u_k(t) + \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 32 \end{bmatrix}; \quad k=p,e \tag{3.6.1}$$

$$\begin{bmatrix} x_{k1}(t_0) \\ \hline x_{ik}(t_0) \end{bmatrix} = \begin{bmatrix} x_{k10} \\ x_{k20} \end{bmatrix}; \quad k=p,e \tag{3.6.2}$$

The partitioned (3x1) state vectors, x_{k1} and x_{k2}, represent the position and velocity vectors of the participants respectively. The components of the (6x1) disturbance vectors, n_p and n_e, represent the acceleration of gravity. The weighting matrices are diagonal ones of the form

$$R_k = r_k\,[U];\; k = p,e \qquad (3.6.3)$$

3.6.1 Control Algorithms for the Participants

The control algorithms for the participants of this endgame are specifically formulated as follows. For the pursuer the components of the (3x1) control force are

$$u_j(t) = -T_1\lambda_{p1j}(T_1)/r_{pj} + (\lambda_{p1j}(T_1)/r_{pj})t;\; t_0 \leqq t \leqq T_1 \qquad (3.6.4)$$

For the evader the control components are

$$v_j(t) = \begin{cases} (T_1\lambda_{e1j}(T_1) + \lambda_{e2j}(T_1))/r_{ej} - (\lambda_{e1j}(T_1)/r_{ej})t; & t_0 \leqq t \leqq T_1 \\ -(T_2\mu_{1j} + \mu_{2j})/r_{ej} + (\mu_{2j}/r_{ej})t; & T_1 < t \leqq T_2 \end{cases}$$

$$j = 1,3 \qquad (3.6.5)$$

The components of the corner condition costate vectors, $\lambda_p(T_1)$ and $\lambda_e(T_1)$, and of the costate vector μ which are defined by (3.4.14) and (3.4.15) are explicitly formulated in the Appendix 8.1.

3.6.2 Determination of the Lagrangian Multiplier η.

With reference to the boundary condition (3.4.17) the target kinetic energy constraint is expressed in terms of the (3x1) partitioned costate vector μ_2. In Appendix 8.1.5 the components of this partitioned costate vector are rational fractional polynomials in terms of Lagrangian multiplier η. Thus the scalar product of μ_2 with itself yields a sixth order polynomial function of η.

Because of the degree of this polynomial is greater than one, it appears that the existence of multiple values of η would cause great difficulty in determining the true saddle point for the differential endgame. But, fortunately this polynomial generates one root which can be used for the determination of a saddle point. In the numerous cases of differential endgames that were simulated on the digital computer four of the roots of the η dependent polynomial were always complex. Of the two real roots, there results the evader's target velocities which are of the same magnitude but of opposite direction. Thus from a physical viewpoint one of the real roots is meaningless for it assumes that the evader can reverse its direction during the postengagement period.

3.6.3 Differential Endgame Simulated on the Digital Computer

For this particular differential endgame the initial state of the participants are

$$x_p(t_0) = \begin{bmatrix} 50000.(\text{ft}) \\ 1000. \\ 70000. \\ 7000.(\text{ft/sec}) \\ -100. \\ 50. \end{bmatrix}; \quad x_e(t_0) = \begin{bmatrix} 150000.(\text{ft}) \\ 5000. \\ 90000. \\ -10000.(\text{ft/sec}) \\ 100. \\ -200. \end{bmatrix} \tag{3.6.6}$$

The other parameters of the endgame are: the terminal engagement time, T_1, is 6.4 seconds; the target time, T_2, is 20.7 seconds; the evader's target speed is 9000 ft/sec; the weighting coefficient, a^2, is 1; the evader's weighting matrix, $R_e(t)$ is 60[U] and the pursuer's weighting matrix $R_p(t)$ is r_p[U]; $1 \leqq r_p \leqq 120$. For these weighting matrices the participants' controllability matrices satisfy the following relationship

$$M_e(T_1, t_0) = (r_p/r_e)M_p(T_1, t_0) \tag{3.6.7}$$

where (r_p/r_e) is called the evader's controllability factor. The effect of the controllability factor upon the terminal engagement miss distance and the participants' maximum control accelerations are studied.

Figure 1 shows how the terminal miss distance increases with an increase in the evader's controllability factor. From Figure 1 one can relate the evader's controllability factor to the intercept capability of the pursuer's warhead. For this differential endgame the evader can be intercepted regardless of the evader's controllability factor if a nuclear warhead is used (16).

Figure 2 illustrates how the participants' maximum control accelerations vary with respect to the evader's controllability factor for this particular differential endgame. The pursuer's maximum control acceleration occurs at the commencement of the differential endgame. The evader's maximum control acceleration occurs at the terminal engagement time. From Figure 2 the participants' control capability can be related to the structural design of the participants' airframes.

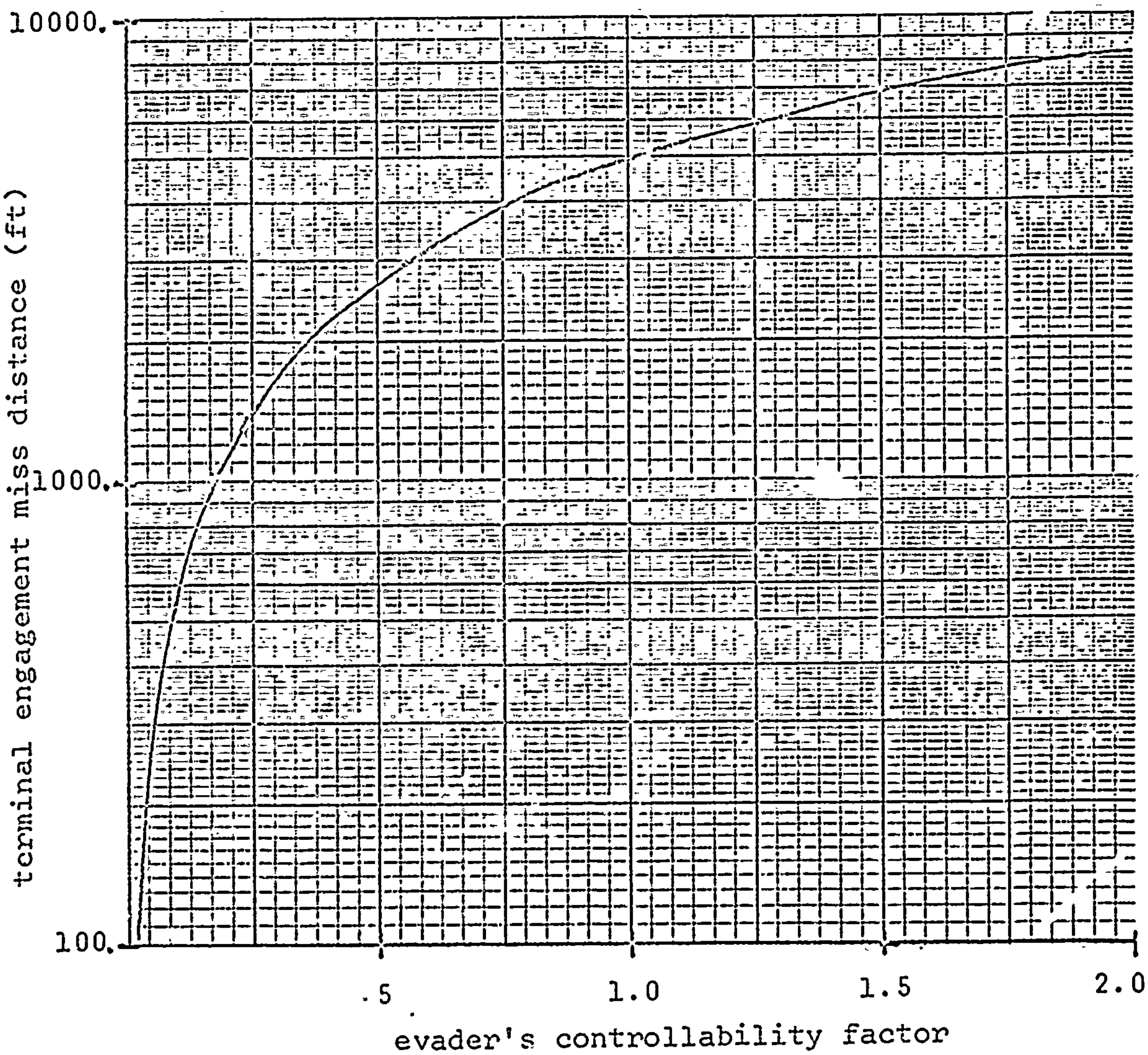

Figure 1. Evader's Controllability Factor Versus Terminal Engagement Miss Distance

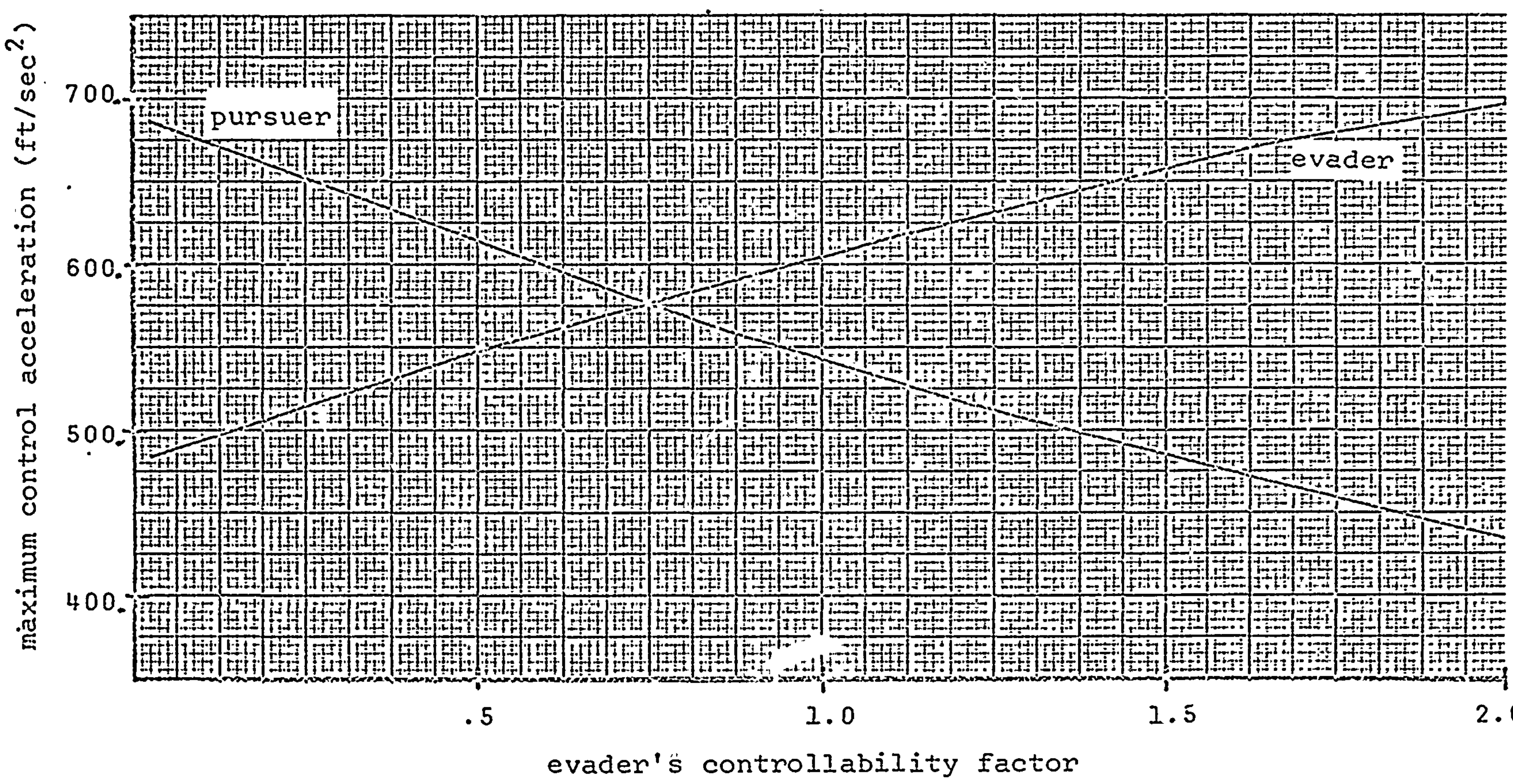

Figure 2. Participants' Maximum Control Acceleration Versus Evader's Controllability Factor

3.7 SUMMARY OF THE CHAPTER

The principal contributions contained in this chapter are:

1. Formulation of the terminal endgame between an offensive missile and its pursuer within the framework of differential game theory.
2. Determination of the necessary and the sufficient conditions for the existence of a saddle point for the differential endgame problem.
3. Determination of optimal open loop strategies for the participants of the differential endgame.
4. Through an example it is illustrated how the differential endgame can be used in determining certain design specifications such as structural capability and the pursuer's warhead.

4. PARAMETER OPTIMIZATION FOR DIFFERENTIAL GAMES

4.1 INTRODUCTION

For the differential endgame studied in Chapter 3 the pursuer's initial state vector, the intercept time, and the target time are fixed values. Now it is proposed to study the differential endgame whose intercept time, target time, and pursuer's initial state vector are unknown. Here the relationships that determine these unknown parameters are derived.

For the class of differential games studied by Ho et al (6), the pursuer's initial state vector and terminal engagement time are fixed values. This class of differential games is formulated where the pursuer's initial state vector and the terminal engagement time are unknown. The relationships that determine these unknown parameters are derived.

For the differential game it is shown how the determination of the intercept time and the pursuer's initial state via differential game theory is used to determine when the pursuer is launched.

4.2 PARAMETER OPTIMIZATION FOR THE DIFFERENTIAL ENDGAME

For the participants' optimal strategies, u^* and v^*, and the differential endgame's payoff functional

$$J = a^2/2[x_p(T_1)-x_e(T_1)]'A'A[x_p(T_1)-x_e(T_1)]$$

$$+1/2\int_{t_0}^{T_1} u'R_p(t)u\,dt-1/2\int_{t_0}^{T_2} v'R_e(t)v\,dt \qquad (4.2.1)$$

subject to the constraints

$$\dot{x}_p = F_p(t)x_p + G_p(t)u + n_p \qquad (4.2.2)$$

$$\dot{x}_e = F_e(t)x_e + G_e(t)v + n_e \qquad (4.2.3)$$

$$x_e(t_0) = x_{e0} \qquad (4.2.4)$$

$$x_{p1}(t_0)'x_{p1}(t_0) = R_0^2 \qquad (4.2.5)$$

$$x_{p2}(t_0)'x_{p2}(t_0) = V_0^2 \qquad (4.2.6)$$

$$x_{e1}(T_2) = 0 \qquad (4.2.7)$$

$$x_{e2}(T_2)'x_{e2}(T_2) = V_T^2 \qquad (4.2.8)$$

$$u, v \in R^3 \qquad (4.2.9)$$

determine the differential endgame's parameters, T_1, T_2, and $x_p(t_0)$ such that $W(x_p(t_0), x_e(t_0), t_0)$, the value of the differential endgame, is determined by

$$W(x_p(t_0), x_e(t_0), t_0) = \underset{u \in U}{\text{Min}}\ \underset{v \in V}{\text{Max}}\ J \qquad (4.2.10)$$

subject to the constraints (4.2.2-4.2.9).

The state vectors, x_p and x_e, represent the position and velocity components of the pursuer and evader; the control vectors, u and v, represent the components of the pursuer's and evader's acceleration commands; n_p and n_e represent any disturbance vector such as the earth's gravitational field; R^3 is the 3-dimensional open Euclidean space; the 6x6 matrices, $F_p(t)$ and $F_e(t)$, and the 6x3 matrices, $G_p(t)$ and $G_e(t)$, are continuous in time; $R_p(t)$ and $R_e(t)$ are 3x3 positive definite matrices, continuous in time; a^2 is a weighting factor.

The differential endgame has an unknown duration (T_2-t_0), t_0 being the fixed commencement time of the game, and T_2 being the evader's unknown target time. The participants have an unknown engagement interval (T_1-t_0), T_1 being the unknown terminal engagement or intercept time. The unknown postengagement interval is (T_2-T_1).

For the differential endgame the payoff functional is the engagement's final miss distance, squared, weighted against the difference between the pursuer's control energy, spent during the engagement interval and the evader's control energy, spent during the duration of the endgame.

Both participants have linear nonstationary dynamics, defined by the differential constraints (4.2.2, 4.2.3). Over the postengagement interval the evader's differential constraint (4.2.3) is converted into the integral constraint

$$x_e(T_2)=\Phi_e(T_2,T_1)x_e(T_1)+\int_{T_1}^{T_2}\Phi_e(T_2,t)[G_e(t)v(t)+n_e]\,dt$$

$$T_1 < t \leqq T_2 \tag{4.2.11}$$

where $\Phi_e(T_2,t)$ is the evader's state transition matrix, and $x_e(T_2)$ and $x_e(T_1)$ are the evader's unknown state vectors at T_2 and T_1.

The evader's target constraints are its target position (4.2.7) and its kinetic energy as it strikes the target (4.2.8). The pursuer's unknown initial state vector $x_p(t_0)$ is composed of its initial position vector $x_{p1}(t_0)$ and velocity vector $x_{p2}(t_0)$. The square of the magnitudes of these vectors are constrained by (4.2.5,4.2.6). These magnitudes represent the distance and speed of the pursuer at the initiation of the differential endgame. Vector Lagrangian multipliers, λ_p and λ_e, are introduced in order to adjoin the differential constraints (4.2.2, 4.2.3) to the payoff functional (4.2.1). The vector μ adjoins the integral constraint (4.2.11) to the payoff functional (4.2.1). The scalar multipliers η, η_r, and η_v adjoin the evader's kinetic energy constraint (4.2.8) and the inner product constraints of the pursuer's initial position and velocity vectors (4.2.5, 4.2.6) to the payoff functional (4.2.1).

The differential endgame is expressed as

$$
\begin{aligned}
\underset{u\in U\, v\in V}{\mathrm{MinMax}} J_c = \underset{u\in U\, v\in V}{\mathrm{MinMax}} \Big\{ & a^2/2[x_p(T_1)-x_e(T_1)]'A'A[x_p(T_1)-x_e(T_1)] \\
& + \int_{t_0}^{T_1} [1/2u'R_p(t)u-1/2v'R_e(t)v+\lambda_p'(F_p(t)x_p+G_p(t)u+n_p-\dot{x}_p) \\
& +\lambda_e'(F_e(t)x_e+G_e(t)v+n_e-\dot{x}_e)]dt \\
& +\eta_r/2(x_{p1}(t_0)'x_{p1}(t_0)-R_0^2)+\eta_v/2(x_{p2}(t_0)'x_{p2}(t_0)-V_0^2) \\
& +\eta/2(x_{e2}(T_2)'x_{e2}(T_2)-V_T^2)+\mu'[x_e(T_2)-\Phi_e(T_2,T_1)x_e(T_1)] \\
& - \int_{T_1}^{T_2} (1/2v'R_e(t)v+\mu'\Phi_e(T_2,t)[G_e(t)v+n_e])dt \Big\} \qquad (4.2.12)
\end{aligned}
$$

4.2.1 Derivation of the Relationships that Determine the Intercept Time, Target Time and the Pursuer's Initial State

Applying variations, δu and δv, about a particular pair of controls, u and v, the first order variation of the differential endgame's constrained payoff functional (4.2.12) is

$$\delta J_c = [a^2(x_p(T_1)-x_e(T_1))'A'A-\lambda_p'(T_1)]\delta x_p(T_1)$$
$$+[a^2(x_e(T_1)-x_p(T_1))'A'A-\lambda_e'(T_1)-\mu'\Phi_e(T_2,T_1)]\delta x_e(T_1)$$
$$+[\mu_2'+\eta x_{e2}'(T_2)]\,\delta x_e(T_2)$$
$$+[\eta_r x_{p1}'(t_0)+\lambda_{p1}'(t_0)]\delta x_{p1}(t_0)+[\eta_v x_{p2}'(t_0)+\lambda_{p2}'(t_0)]\delta x_{p2}(t_0)$$
$$+[1/2u(T_1)'R_p(T_1)u\ (T_1)+a^2[x_p(T_1)-x_e(T_1)]'A'A[\dot{x}_p(T_1)-\dot{x}_e(T_1)]\delta T_1$$
$$+[\eta x_{e2}(T_2)'\dot{x}_{e2}(T_2)-1/2v(T_2)'R_e(T_2)v(T_2)]\delta T_2$$
$$+\int_{t_0}^{T_1}[(u'R_p(t)+\lambda_p'G_p(T))\delta u+(-v'R_e(t)+\lambda_e'G_e(t))\delta v$$
$$+(\dot{\lambda}_p'+\lambda_p'F_p(t))\delta x_p+(\dot{\lambda}_e'+\lambda_e'F_e(t))\delta x_e]dt$$
$$-\int_{T_1}^{T_2}[v'R_e(t)+\mu'\Phi_e(T_2,t)G_e(t)]\delta v\,dt \qquad (4.2.13)$$

The relationships that determine the intercept time T_1, target time T_2, and the pursuer's initial state vector $x_p(t_0)$ such that

$$W(x_p(t_0),x_e(t_0),t_0) = \underset{u\in U}{\text{Min}}\ \underset{v\in V}{\text{Max}} J_c \qquad (4.2.14)$$

are the Euler-Lagrange equations and their boundary conditions. The Euler-Lagrange equations and their boundary conditions are determined by the differential endgame's constraints (4.2.2-4.2.8) and the necessary conditions that result by requiring δJ to vanish. Table 3 summarizes the necessary conditions required for δJ to vanish.

Table 3

Necessary Conditions Required for δJ to Vanish

Variational	Coefficients of Variationals Equated to Zero	
δu	$u'R_p(t)+\lambda_p'G_p(t)=0;\ t_0 \leq t \leq T_1$	(4.2.15)
δv	$-v'R_e(t)+\lambda_e'G_e(t)=0;\ t_0 \leq t \leq T_1$	(4.2.16)
	$v'R_e(t$[illegible]$\Phi_e(T_2,t)G_e(t)=0;\ T_1 \leq t \leq T_2$	(4.2.17)
δx_p	$\dot{\lambda}_p'+$ [illegible] $)=0;\ t_0 \leq t \leq T_1$	(4.2.18)
δx_e	$\dot{\lambda}_e'+\lambda_e' F_e(t)=0;\ t_0 \leq t \leq T_1$	(4.2.19)
$\delta x_p(t_0)$	$\begin{bmatrix} x_{p1}(t_0) \\ x_{p2}(t_0) \end{bmatrix} = -\begin{bmatrix} 1/\eta_r U & 0 \\ 0 & 1/\eta_v U \end{bmatrix} \begin{bmatrix} \lambda_{p1}(t_0) \\ \lambda_{p2}(t_0) \end{bmatrix}$	
	$= -G(\eta_r,\eta_v)\lambda_p(t_0)$	(4.2.20)
$\delta x_p(T_1)$	$\lambda_p'(T_1)-a^2[x_p(T_1)-x_e(T_1)]'A'A=0$	(4.2.21)
$\delta x_e(T_1)$	$\lambda_e'(T_1)+a^2[x_p(T_1)-x_e(T_1)]'A'A$	
	$+\mu'\Phi_e(T_2,T_1)=0$	(4.2.22)
$\delta x_{e2}(T_2)$	$\eta x_{e2}(T_2)+\mu_2=0$	(4.2.23)
δT_1	$1/2u(T_1)'R_p(T_1)u(T_1)$	
	$+a^2[x_p(T_1)-x_e(T_1)]'A'A[\dot{x}_p(T_1)-\dot{x}_e(T_1)]=0$	(4.2.24)
δT_2	$\eta x_{e2}(T_2)'\dot{x}_{e2}(T_2)-1/2v(T_2)'R_e(T_2)v(T_2)=0$	(4.2.25)

The Euler-Lagrange equations are

$$\begin{bmatrix} \dot{x}_p \\ \dot{x}_e \\ \dot{\lambda}_p \\ \dot{\lambda}_e \end{bmatrix} = \begin{bmatrix} F_p(t) & 0 & -G_p(t)R_p^{-1}(t)G_p'(t) & 0 \\ 0 & F_e(t) & 0 & G_e(t)R_e^{-1}(t)G_e'(t) \\ 0 & 0 & -F_p'(t) & 0 \\ 0 & 0 & 0 & -F_e'(t) \end{bmatrix} \begin{bmatrix} x_p \\ x_e \\ \lambda_p \\ \lambda_e \end{bmatrix} + \begin{bmatrix} n_p \\ n_e \\ 0 \\ 0 \end{bmatrix}$$

for $t_0 \leqq t \leqq T_1$ (4.2.26)

and

$\dot{x}_e = F_e(t)x_e - G_e(t)R_e^{-1}(t)G_e'(t)\Phi_e'(T_2,t)\mu + n_e$

$T_1 < t \leqq T_2$ (4.2.27)

The boundary conditions are

$x_e(t_0)=x_{e0}$ (4.2.28)

$x_{p1}(t_0)=-1/\eta_r\lambda_{p1}(t_0)$ (4.2.29)

$x_{p2}(t_0)=-1/\eta_v\lambda_{p2}(t_0)$ (4.2.30)

$x_{p1}'(t_0)x_{p1}(t_0)=R_0^2$ (4.2.31)

$x_{p2}'(t_0)x_{p2}(t_0)=V_0^2$ (4.2.32)

$x_{e1}(T_2)=0$ (4.2.33)

$\eta x_{e2}(T_2)+\mu_2= 0$ (4.2.34)

$x_{e2}(T_2)'x_{e2}(T_2)=V_T^2$ (4.2.35)

$\lambda_p(T_1)=a^2A'A[x_p(T_1)-x_e(T_1)]$ (4.2.36)

$\lambda_e(T_1)=a^2A'A[x_e(T_1)-x_p(T_1)]-\Phi_e'(T_2,T_1)\mu=0$ (4.2.37)

$1/2u(T_1)'R_p(T_1)u(T_1)$

$+a^2[x_p(T_1)-x_e(T_1)]'A'A[\dot{x}_p(T_1)-\dot{x}_e(T_1)]=0$ (4.2.38)

$\eta x_{e2}(T_2)'\dot{x}_{e2}(T_2)-1/2v(T_2)'R_e(T_2)v(T_2)=0$ (4.2.39)

The solution of the Euler-Lagrange equations (4.2.26) defined at T_1 in terms of the participants' unknown boundary states, $x_p(T_1)$ and $x_e(T_1)$, and their unknown corner condition vectors, $\lambda_p(T_1)$, $\lambda_e(T_1)$ are

$$\lambda_k(T_1)=\Phi_k'(t_0,T_1)\lambda_k(t_0);\quad k=p,e \tag{4.2.40}$$

$$x_p(T_1)=\Phi_p(T_1,t_0)x_p(t_0)-M_p(T_1,t_0)\lambda_p(T_1)+k_p(T_1,t_0) \tag{4.2.41}$$

$$x_e(T_1)=\Phi_e(T_1,t_0)x_e(t_0)+M_e(T_1,t_0)\lambda_e(T_1)+k_e(T_1,t_0) \tag{4.2.42}$$

where $\Phi_k(T_1,t_0)$ and $\Phi_k'(t_0,T_1)$; $k=p,e$ are the transition matrices for the state and costate vectors; the controllability matrices are defined as

$$M_k(T_1,t_0)=\int_{t_0}^{T_1}\Phi_k(T_1,t)G_k(t)R_k^{-1}(t)G_k'(t)\Phi_k'(T_1,t)dt;$$

$$k=p,e \tag{4.2.43}$$

and the column vectors $k_k(T_1,t_0)$; $k=p,e$ due to the disturbance vectors n_k are

$$k_k(T_1,t_0)=\int_{t_0}^{T}\Phi_k(T_1,t)n_k dt;\qquad k=p,e \tag{4.2.44}$$

The solution of the Euler-Lagrange equation (4.2.27) defined at T_2 in terms of the evader's unknown state $x_e(T_1)$ and costate vector μ is

$$x_e(T_2)=\Phi_e(T_2,T_1)x_e(T_1)-M_e(T_2,T_1)\mu+k_e(T_2,T_1) \tag{4.2.45}$$

where the controllability matrix is defined as

$$M_e(T_2,T_1)=\int_{T_1}^{T_2}\Phi_e(T_2,t)G_e(t)R_e^{-1}(t)G_e'(t)\Phi_e'(T_2,t)dt \tag{4.2.46}$$

and the column vector $k_e(T_2,T_1)$ due to the disturbance vector n_e is

$$k_e(T_2,T_1)=\int_{T_1}^{T_2}\Phi_e(T_2,t)n_e dt \tag{4.2.47}$$

The relationships that determine the Lagrangian multipliers, $\lambda_p(T_1)$, $\lambda_e(T_1)$ and μ, in terms of the unknown parameters, T_1, T_2 and $x_p(t_0)$, were derived in Section 3.4. They are expressed as

$$\begin{bmatrix} U+a^2A'AM_p(T_1,t_0) & a^2A'AM_e(T_1,t_0) \\ -a^2A'AM_p(T_1,t_0) & U-[a^2A'A-K(\eta)]M_e(T_1,t_0) \end{bmatrix} \begin{bmatrix} \lambda_p(T_1) \\ \lambda_e(T_1) \end{bmatrix} =$$

$$\begin{bmatrix} a^2A'A & -a^2A'A \\ -a^2A'A & a^2A'A-K(\eta) \end{bmatrix} \left\{ \begin{bmatrix} \Phi_p(T_1,t_0) & 0 \\ 0 & \Phi_e(T_1,t_0) \end{bmatrix} \begin{bmatrix} x_p(t_0) \\ x_e(t_0) \end{bmatrix} \right.$$

$$\left. + \begin{bmatrix} k_p(T_1,t_0) \\ k_e(T_1,t_0) \end{bmatrix} \right\} - \begin{bmatrix} 0 \\ K(\eta)\Phi_e(T_1,T_2)k_e(T_2,T_1) \end{bmatrix} \tag{4.2.48}$$

$$\mu=[M_e(T_2,T_1)-1/\eta Q'Q]^{-1}\{\Phi_e(T_2,t_0)x_e(t_0)+\Phi_e(T_2,T_1)M_e(T_1,t_0)\lambda_e(T_1)$$
$$+\Phi_e(T_2,T_1)k_e(T_1,t_0)+k_e(T_2,T_1)\} \tag{4.2.49}$$

$$1/\eta^2\mu'Q'Q\mu-V_T^2=\sum_{i=0}^{L} p_i\eta^i= 0 \tag{4.2.50}$$

The parameters, T_1, T_2 and $x_p(T_0)$, are determined when (4.2.48-4.2.50) and the following set of boundary conditions are solved simultaneously:

$$x_p(t_0)=-G(\eta_r,\eta_v)\lambda_p(t_0) \tag{4.2.51}$$

$$1/2u(T_1)'R_p(T_1)u(T_1)$$
$$+a^2[x_p(T_1)-x_e(T_1)]'A'A[\dot{x}_p(T_1)-\dot{x}_e(T_1)]=0 \tag{4.2.52}$$

$$\eta x_{e2}(T_2)'\dot{x}_{e2}(T_2)-1/2v(T_2)'R_e(T_2)v(T_2)=0 \tag{4.2.53}$$

$$u(T_1)=-R_p^{-1}(T_1)G_p'(T_1)\lambda_p(T_1) \tag{4.2.54}$$

$$v(T_1)=R_e^{-1}(T_1)G_e'(T_1)\lambda_e(T_1) \tag{4.2.55}$$

$$v(T_2)=-R_e^{-1}(T_2)G_e'(T_2)\mu \tag{4.2.56}$$

$$\lambda_p(t_0)'G(\eta_r,\eta_v)A'AG(\eta_r,\eta_v)\lambda_p(t_0)=R_0^2 \tag{4.2.57}$$

$$\lambda_p(t_0)'G(\eta_r,\eta_v)Q'QG(\eta_r,\eta_v)\lambda_p(t_0)=V_0^2 \tag{4.2.58}$$

$$\dot{x}_p(T_1)=F_p(T_1)x_p(T_1)+G_p(T_1)u(T_1)+n_p(T_1) \tag{4.2.59}$$

$$\dot{x}_e(T_1)=F_e(T_1)x_e(T_1)+G_e(T_1)v(T_1)+n_e(T_1) \tag{4.2.60}$$

$$\dot{x}_e(T_2)=F_e(T_2)x_e(T_2)+G_e(T_2)v(T_2)+n_e(T_2) \tag{4.2.61}$$

$$\lambda_p(t_0)=\Phi_p{}'(T_1,t_0)\lambda_p(T_1) \tag{4.2.62}$$

$$x_e(T_1)=\Phi_e(T_1,t_0)x_e(t_0)+M_e(T_1,t_0)\lambda_e(T_1)+k_e(T_1,t_0) \tag{4.2.63}$$

$$x_p(T_1)=\Phi_p(T_1,t_0)x_p(t_0)-M_p(T_1,t_0)\lambda_p(T_1)+k_p(T_1,t_0) \tag{4.2.64}$$

$$\dot{x}_e(T_2)=\Phi_e(T_2,T_1)x_e(T_1)-M_e(T_2,T_1)\mu+k_e(T_2,T_1) \tag{4.2.65}$$

The boundary conditions (4.2.51-4.2.56) are the necessary conditions in Table 3. The boundary conditions (4.2.57, 4.2.58) are the inner product constraints of the pursuer's initial position and velocity vectors. The boundary conditions (4.2.59 - 4.2.61) are the participants' differential constraints defined at times, T_1 and T_2. The boundary conditions (4.2.62-4.2.66) are the solutions of the Euler-Lagrange equations in terms of the unknown boundary vectors of the state and costate vectors.

4.2.2 Example of the Differential Endgame with Unknown Intercept and Target Times

Presented here is a differential endgame whose intercept time, T_1, and target time, T_2, are unknown. The relationships that determine the boundary conditions for the Euler-Lagrange equations and the time parameters, T_1 and T_2, are (4.2.48 - 4.2.50), (4.2.52-4.2.56), (4.2.59 - 4.2.61) and (4.2.63-4.2.65). The dynamics of both participants have the form

$$\begin{bmatrix}\dot{x}_{k1}\\ \dot{x}_{k2}\end{bmatrix}=\begin{bmatrix}0 & U\\ 0 & 0\end{bmatrix}\begin{bmatrix}x_{k1}\\ x_{k2}\end{bmatrix}+\begin{bmatrix}0\\ U\end{bmatrix}u_k+\begin{bmatrix}0\\ \vdots\\ 0\\ 32\end{bmatrix} \quad ; \; k=p,e \tag{4.2.66}$$

$$x_{p0} = \begin{bmatrix} 50{,}000.\,ft \\ 1{,}000. \\ 70{,}000. \\ 7{,}000.\,ft/sec \\ -100. \\ 50. \end{bmatrix} ; x_{e0} = \begin{bmatrix} 150{,}000.\,ft \\ 5{,}000. \\ 90{,}000. \\ -10{,}000.\,ft/sec \\ 100. \\ -200. \end{bmatrix} \tag{4.2.67}$$

The other parameters of the differential endgame are:

$$a^2 = 1 \tag{4.2.68}$$

$$R_p(t) = 90[U] \tag{4.2.69}$$

$$R_e(t) = 60[U] \tag{4.2.70}$$

$$7000 \leqq \|V_T\|\,ft/sec \leqq 10{,}000. \tag{4.2.71}$$

The procedure for solving the differential endgame problem when its intercept time T_1 and target time T_2 are unknown is shown in Appendix 8.1.5. For this differential endgame the effect of the evader's target speed upon the intercept time, target time, value, and the terminal engagement miss distance are analyzed. Figure 3 shows the variation of target and intercept times versus the variation of evader's target speed.

Figure 4 shows how the participants' control energies, spent over the duration of their flight times, and the differential endgame's value vary with the evader's target speed. These curves are normalized with respect to the results that occur when the target speed is -10,000 ft/sec. For the target speed of -10,000 ft/sec the pursuer's and evader's control energies are 1.39×10^7 and 8.32×10^7 units, and the value is -5.42×10^7 units.

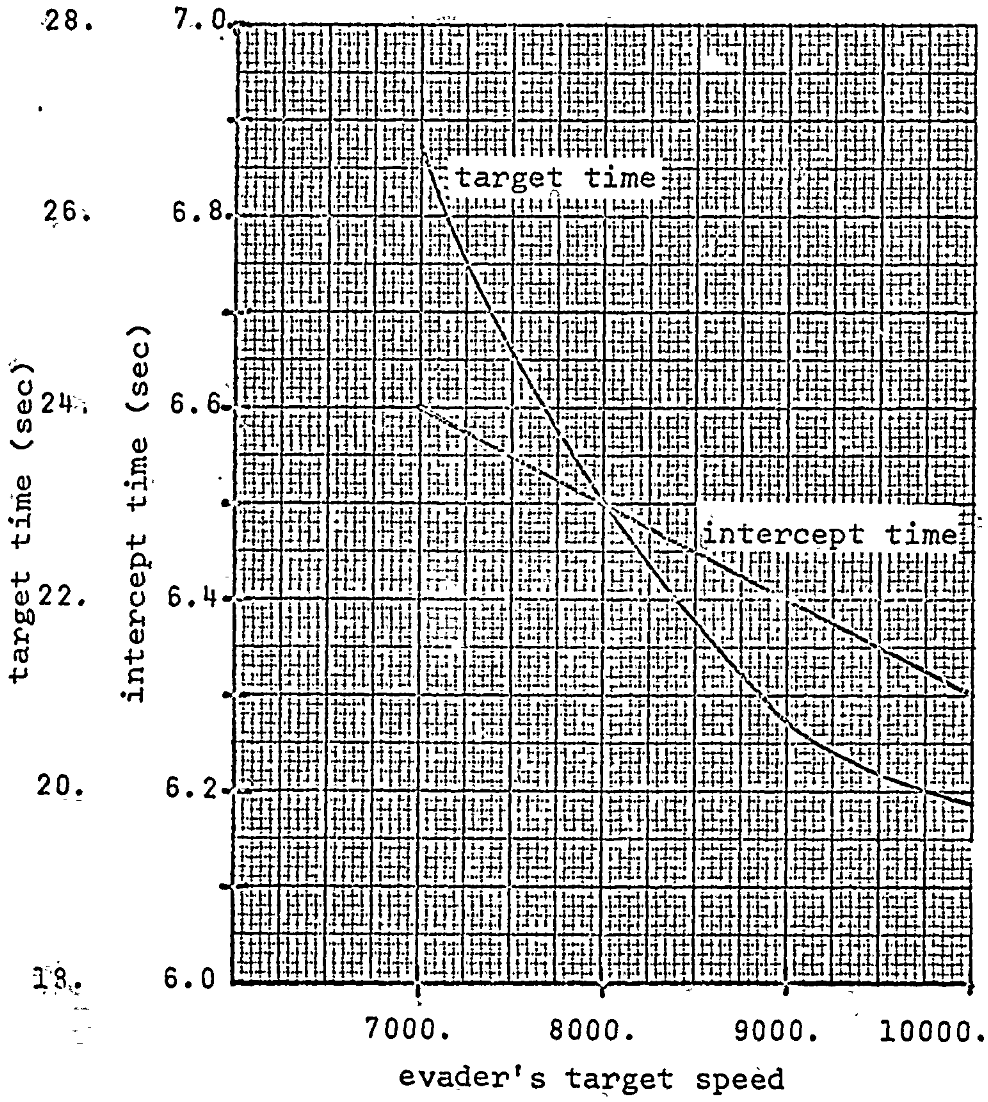

Figure 3. Intercept and Target Times Versus Evader's Target Speed

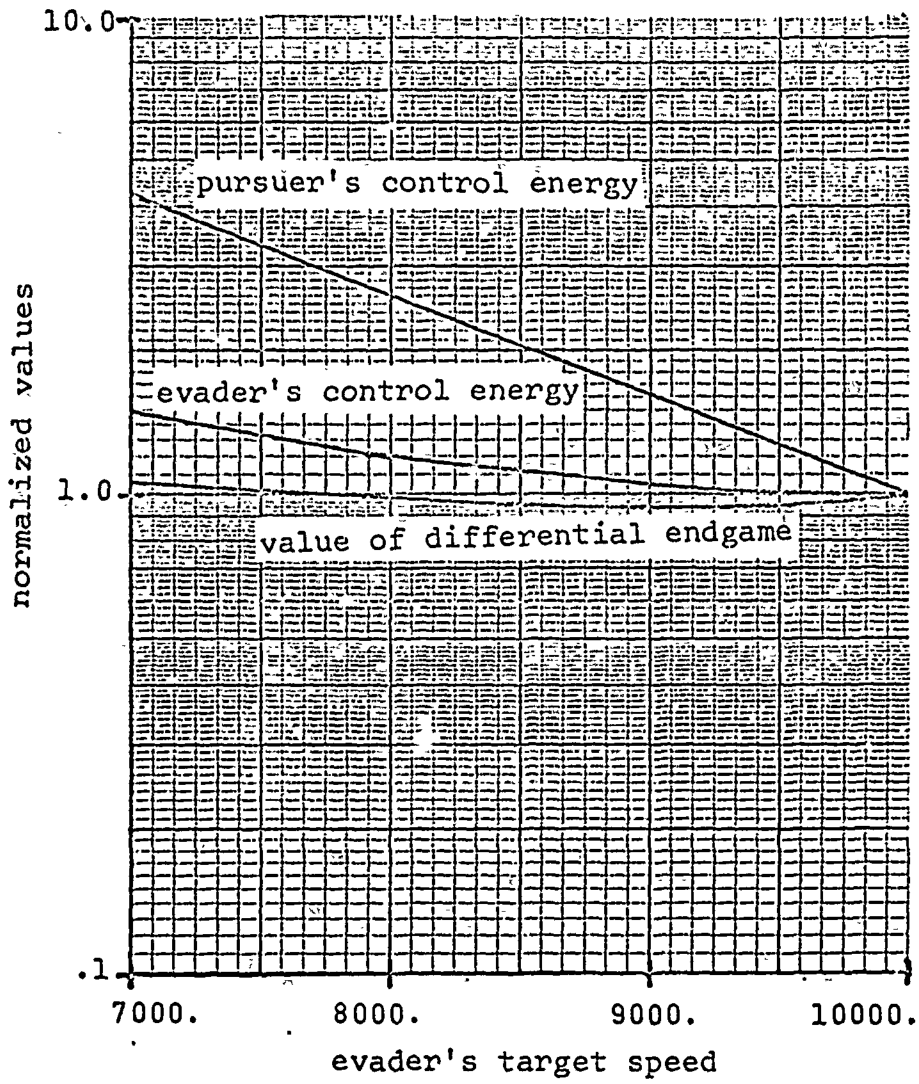

Figure 4. Participants' Control Energies and the Differential Endgame's Value Versus Evader's Target Speed

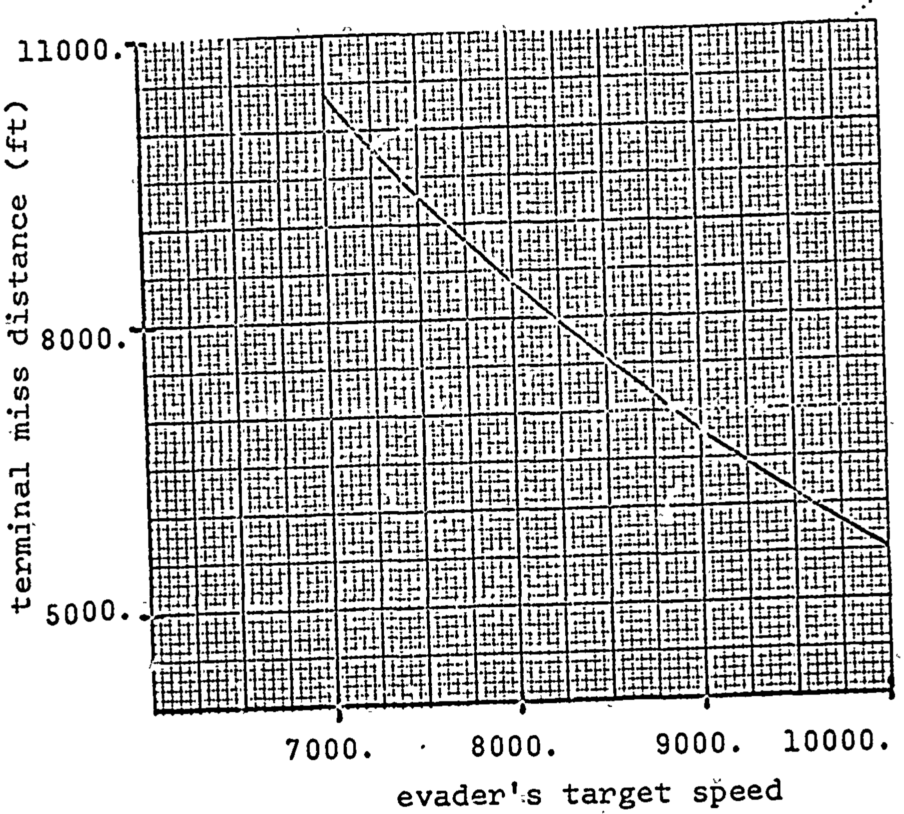

Figure 5. Terminal Miss Distance Versus Evader's Target Speed

Figure 5 illustrates how the terminal miss distance varies with the evader's terminal speed. Here the variation of target speed has slight effect upon the intercept time and the value of the differential game.

Figures 4 and 5 point out that when the evader is required to spend its control energy in reducing its kinetic energy, part of this energy is used in avoiding interception. This conclusion is supported by the fact that both the pursuer's control energy and the terminal engagement miss distance increase as the target's terminal speed decreases. The fact that increase of the pursuer's control energy partially eliminates the effect of the increase of the evader's control energy and the terminal miss distance, explains the slight variation of the value as the evader's target speed is reduce.

4.3 PARAMETER OPTIMIZATION FOR THE DIFFERENTIAL GAME

For the participants' optimal strategies, u^* and v^*, and the differential game's payoff functional

$$J=a^2/2[x_p(T)-x_e(T)]'A'A[x_p(T)-x_e(T)]$$

$$+1/2 \int_{t_0}^{T} [u'R_p(t)u-v'R_e(t)v]dt \qquad (4.3.1)$$

subject to the constraints

$$\dot{x}_p=F_p(t)x_p+G_p(t)u+n_p \qquad (4.3.2)$$

$$\dot{x}_e=F_e(t)x_e+G_e(t)v+n_e \qquad (4.3.3)$$

$$x_e(t_0) = x_{e0} \qquad (4.3.4)$$

$$x_{p1}(t_0)'x_{p1}(t_0)=R_0^2 \qquad (4.3.5)$$

$$x_{p2}(t_0)'x_{p2}(t_0)=V_0^2 \qquad (4.3.6)$$

$u, v \in R^3$. (4.3.7)

determine the pursuer's initial state vector $x_p(t_0)$ and the intercept time T, if it exists, such that $W(x_p(t_0), x_e(t_0), t_0)$, the value of the differential game, is determined by

$$W(x_p(t_0), x_e(t_0), t_0) = \underset{u \in U}{\text{Min}} \; \underset{v \in V}{\text{Max}} J \tag{4.3.8}$$

subject to the constraints defined by (4.3.2-4.3.7).

The state vectors, x_p and x_e, represent the position and velocity components of the pursuer and evader; the control vectors, u and v, represent the components of the participants' acceleration commands; n_p and n_e represent any disturbance vector such as the earth's gravitational field; R^3 is the 3-dimensional open Euclidean space; the 6x6 matrices, $F_p(t)$ and $F_e(t)$, and the 6x3 matrices, $G_p(t)$ and $G_e(t)$, are continuous in time; $R_p(t)$ and $R_e(t)$ are 3x3 positive definite matrices, continuous in time; a^2 is a weighting factor.

The differential game has an unknown duration $(T-t_0)$, t_0 being the fixed commencement time of the game and T being the unknown terminal engagement time.

The differential game's payoff functional (4.3.1) is the engagement's final miss distance, squared, weighted against the difference between the participants' control energies. Both participants have linear nonstationary dynamics, defined by the differential constraints (4.3.2, 4.3.3). The pursuer's initial state vector is composed of its unknown position and velocity vectors, $x_{p1}(t_0)$ and

$x_{p2}(t_0)$. The square of the magnitudes of these vectors are constrained by (4.3.5,4.3.6). These magnitudes represent the distance and speed of the pursuer at the initiation of the differential game.

Vector Lagrangian multipliers, λ_p and λ_e, are introduced in order to adjoin the differential constraints (4.3.2,4.3.3) to the payoff functional (4.3.1). The scalar multipliers, η_r and η_v, adjoin the pursuer's quadratic magnitude constraints of its initial position and velocity vectors (4.3.5,4.3.6) to the payoff functional (4.3.1). The differential game is expressed as

$$\begin{aligned}\underset{u\in U}{\text{Min}}\ \underset{v\in V}{\text{Max}} J_c = \underset{u\in U}{\text{Min}}\ \underset{v\in V}{\text{Max}} \Big\{ & a^2/2[x_p(T)-x_e(T)]A'A[x_p(t)-x_e(T)] \\ & + \eta_r/2(x_{p1}(t_0)'x_{p1}(t_0)-R_0^2) + \eta_v/2(x_{p2}(t_0)'x_{p2}(t_0)-V_0^2) \\ & + \int_{t_0}^{T} [1/2u'R_p(t)u - v'R_e(t)v + \lambda_p'(F_p(t)x_p + G_p(t)u + n_p - \dot{x}_p) \\ & + \lambda_e'(F_e(t)x_e + G_e(t)v + n_e - \dot{x}_e)]dt \Big\} \end{aligned} \tag{4.3.9}$$

4.3.1 Derivation of the Relationships that Determine the Intercept Time and the Pursuer's Initial State

Applying variations, δu and δv, about a particular pair of controls, u and v, the first order variation of the differential game's constrained payoff functional (4.3.9) is

$$\delta J_c = [a^2(x_p(T)-x_e(T))'A'A-\lambda_p'(T)]\delta x_p(T)$$
$$+[a^2(x_e(T)-x_p(T))'A'A-\lambda_e'(T)]\delta x_e(T)$$
$$+[\eta_r x_{p1}'(t_0)+\lambda_{p1}'(t_0)]\delta x_{p1}(t_0)+[\eta_v x_{p2}'(t_0)+\lambda_{p2}'(t_0)]\delta x_{p2}(t_0)$$
$$+1/2[u(T)'R_p(T)u(T)-v(T)'R_e(T)v(T)+$$
$$2a^2(x_p(T)-x_e(T))'A'A(\dot{x}_p(T)-\dot{x}_e(T))]\delta T$$
$$+\int_{t_0}^{T}[(u'R_p(t)+\lambda_p'G_p(t))\delta u+(-v'R_e(t)+\lambda_e'G_e(t))\delta v$$
$$+(\dot{\lambda}_p'+\lambda_p'F_p(t))\delta x_p+(\dot{\lambda}_e'+\lambda_e'F_e(t))\delta x_e]dt \qquad (4.3.10)$$

The relationships that determine the terminal engagement time, T, and the pursuer's initial state vector, $x_p(t_0)$, such that

$$W(x_p(t_0),x_e(t_0),t_0)=\underset{u\in U}{\text{Min}}\ \underset{v\in V}{\text{Max}} J_c \qquad (4.3.11)$$

are the Euler-Lagrange equations and their boundary conditions.

The Euler-Lagrange equations and their boundary conditions are determined by the differential game's constraints (4.3.2-4.3.6) and the necessary conditions that result by requiring δJ_c to vanish. Table 4 summarizes the necessary conditions required for δJ_c to vanish.

The Euler-Lagrange equations are

$$\begin{bmatrix}\dot{x}_p\\ \dot{x}_e\\ \dot{\lambda}_p\\ \dot{\lambda}_e\end{bmatrix}=\begin{bmatrix}F_p(t) & 0 & -G_p(t)R_p^{-1}(t)G_p'(t) & 0\\ 0 & F_e(t) & 0 & G_e(t)R_e^{-1}(t)G_e'(t)\\ 0 & 0 & -F_p'(t) & 0\\ 0 & 0 & 0 & -F_e'(t)\end{bmatrix}\begin{bmatrix}x_p\\ x_e\\ \lambda_p\\ \lambda_e\end{bmatrix}+\begin{bmatrix}n_p\\ n_e\\ 0\\ 0\end{bmatrix} \qquad (4.3.12)$$

Table 4

Differential Game's Necessary
Conditions Required for δJ_c to Vanish

Variational	Coefficients of Variationals Equated to Zero	
δu	$u'R_p(t)+\lambda_p'G_p(t)=0$	(4.3.13)
δv	$-v'R_e(t)+\lambda_e'G_e(t)=0$	(4.3.14)
δx_p	$\dot{\lambda}_p'+\lambda_p'F_p(t)=0$	(4.3.15)
δx_e	$\dot{\lambda}_e'+\lambda_e'F_e(t)=0$	(4.3.16)
$\delta x_p(t_0)$	$\begin{bmatrix} x_{p1}(t_0) \\ x_{p2}(t_0) \end{bmatrix} + \begin{bmatrix} 1/\eta_r U & 0 \\ 0 & 1/\eta_v U \end{bmatrix} \begin{bmatrix} \lambda_{p1}(t_0) \\ \lambda_{p2}(t_0) \end{bmatrix} = 0$	(4.3.17)
$\delta x_p(T)$	$a^2[x_p(T)-x_e(T)]'A'A-\lambda_p(T)=0$	(4.3.18)
$\delta x_e(T)$	$a^2[x_e(T)-x_p(T)]'A'A-\lambda_e(T)=0$	(4.3.19)
δT	$1/2[u(T)'R_p(T)u(T)-v(T)'R_e(T)v(T)]$ $+a^2[x_p(T)-x_e(T)]'A'A[\dot{x}_p(T)-\dot{x}_e(T)]=0$	(4.3.20)

The boundary conditions are:

$$x_e(t_0)=x_{e0} \tag{4.3.21}$$

$$x_{p1}(t_0)+1/\eta_r\lambda_{p1}(t_0)=0 \tag{4.3.22}$$

$$x_{p2}(t_0)+1/\eta_v\lambda_{p2}(t_0)=0 \tag{4.3.23}$$

$$x_{p1}(t_0)'x_{p1}(t_0)=R_0^{\ 2} \tag{4.3.24}$$

$$x_{p2}(t_0)'x_{p2}(t_0)=V_0^{\ 2} \tag{4.3.25}$$

$$\lambda_p(T)=a^2A'A[x_p(T)-x_e(T)] \tag{4.3.26}$$

$$\lambda_e(T)=-a^2A'A[x_p(T)-x_e(T)] \tag{4.3.27}$$

$$1/2[u(T)'R_p(T)u(T)-v(T)'R_e(T)v(T)]$$
$$+a^2[x_p(T)-x_e(T)]'A'A[\dot{x}_p(T_1)-\dot{x}_e(T_1)]=0 \tag{4.3.28}$$

The solution of the Euler-Lagrange equations, defined at T in terms of the state and costate vectors' boundary conditions are:

$$\lambda_k(t_0)=\Phi_k'(T,t_0)\lambda_k(T);\ k=p,e \tag{4.3.29}$$

$$x_p(T)=\Phi_p(T,t_0)x_p(t_0)-M_p(T,t_0)\lambda_p(T)+k_p(T,t_0) \tag{4.3.30}$$

$$x_e(T)=\Phi_e(T,t_0)x_e(t_0)+M_e(T,t_0)\lambda_e(T)+k_e(T,t_0) \tag{4.3.31}$$

where $\Phi_k(T,t_0)$ and $\Phi_k'(T,t_0)$; k=p,e are the transition matrices for the state and costate vectors; the controllability matrices are defined as

$$M_k(T,t_0)=\int_{t_0}^{T}\Phi_k(T,t)G_k(t)R_k^{-1}(t)G_k'(t)\Phi_k'(T,t)dt;\ k=p,e \tag{4.3.32}$$

and the column vectors $k_k(T,t_0)$; k=p,e due to the disturbance vectors n_k are:

$$k_k(T,t_0)=\int_{t_0}^{T}\Phi_k(T,t)n_k dt;\qquad k=p,e \tag{4.3.33}$$

Substituting $\lambda_p(t_0)$ (4.3.29) into the boundary conditions (4.3.22,4.3.23) the pursuer's initial state vector is expressed in terms, $\lambda_p(T)$, as

$$x_p(t_0)=-G(\eta_r,\eta_v)\Phi_p'(T,t_0)\lambda_p(T) \qquad (4.3.34)$$

where

$$G(\eta_r,\eta_v)=\begin{bmatrix}1/\eta_r U & 0\\ 0 & 1/\eta_v U\end{bmatrix} \qquad (4.3.35)$$

With the use of (4.3.34) $x_p(T)$ is defined as

$$x_p(T) = -[\Phi_p(T,t_0)G(\eta_r,\eta_v)\Phi_p'(T,t_0)+M_p(T,t_0)]\lambda_p(T)+k_p(T,t_0) \qquad (4.3.36)$$

Substituting for $x_e(T)$ and $x_p(T)$, (4.3.31) and (4.3.36), into the boundary conditions (4.3.26,4.3.27), the boundary condition in terms of $\lambda_p(T)$, $\lambda_e(T)$, η_r and η_v are:

$$\left[\begin{array}{c:c} U+a^2A'A[M_p(T,t_0)+\Phi_p(T,t_0)G(\eta_r,\eta_v)\Phi_p'(T,t_0)] & a^2A'AM_e(T,t_0)\\ -a^2A'A[M_p(T,t_0)+\Phi_p(T,t_0)G(\eta_r,\eta_v)\Phi_p'(T,t_0)] & U-a^2A'AM_e(T,t_0)\end{array}\right]\times$$

$$\begin{bmatrix}\lambda_p(T)\\ \lambda_e(T)\end{bmatrix}=\begin{bmatrix}0 & -a^2A'A\Phi_e(T,t_0)\\ 0 & a^2A'A\Phi_e(T,t_0)\end{bmatrix}\begin{bmatrix}0\\ x_e(t_0)\end{bmatrix}+\begin{bmatrix}a^2A'A(k_p-k_e)\\ a^2A'A(k_e-k_p)\end{bmatrix} \qquad (4.3.37)$$

The parameters, T_1 and $x_p(t_0)$, are determined when (4.3.37) and the following set of boundary conditions are solved simultaneously:

$$x_p(t_0)=-G(\eta_r,\eta_v)\lambda_p(t_0) \qquad (4.3.38)$$

$$1/2u(T)'R_p(T)u(T)-1/2v(T)'R_e(T)v(T)$$
$$+a^2[x_p(T)-x_e(T)]'A'A[\dot{x}_p(T)-\dot{x}_e(T)]=0 \qquad (4.3.39)$$

$$u(T)=-R_p^{-1}(T)G_p'(T)\lambda_p(T) \qquad (4.3.40)$$

$$v(T)=R_e^{-1}(T)G_e(T)\lambda_e(T) \qquad (4.3.41)$$

$$\lambda_p(t_0)'G'(\eta_r,\eta_v)A'AG(\eta_r,\eta_v)\lambda_p(t_0)=R_0^2 \qquad (4.3.42)$$

$$\lambda_p(t_0)'G'(\eta_r,\eta_v)Q'QG(\eta_r,\eta_v)\lambda_p(t_0)=V_0^2 \qquad (4.3.43)$$

$$\dot{x}_p(T)=F_p(T)x_p(T)+G_p(T)u(T)+n_p(T) \qquad (4.3.44)$$

$$\dot{x}_e(T)=F_e(T)x_e(T)+G_e(T)v(T)+n_e(T) \qquad (4.3.45)$$

$$\lambda_p(t_0)=\Phi_p'(T,t_0)\lambda_p(T) \qquad (4.3.46)$$

$$x_e(T)=\Phi_e(T,t_0)x_e(t_0)+M_e(T,t_0)\lambda_e(T)+k_e(T,t_0) \qquad (4.3.47$$

$$x_p(T)=\Phi_p(T,t_0)x_p(t_0)-M_p(T,t_0)\lambda_p(T)+k_p(T,t_0) \qquad (4.3.48)$$

The boundary conditions (4.3.38-4.3.41) are the necessary conditions in Table 4. The boundary conditions (4.3.42, 4.3.43) are the inner product constraints of the pursuer's initial position and velocity vectors. The boundary conditions (4.3.44,4.3.45) are the participants' differential constraints defined at time, T . The boundary conditions (4.3.46-4.3.48) are the solutions of the Euler-Lagrange equations in terms of the unknown boundary vectors of the state and costate vectors.

4.3.2 Launch Logic Example for the Differential Game with the Pursuer's Initial State and Intercept Time Unknown

Examples of differential games are presented where the intercept time and pursuer's initial position and velocity vectors are unknown. The initial position and velocity vectors are constrained by the square of their magnitudes.

For these differential games the participants' dynamics are defined in the X-Y plane of the cartesian coordinate system as

$$\begin{bmatrix} \dot{x}_{k1} \\ \dot{x}_{k2} \end{bmatrix} = \begin{bmatrix} 0 & U \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_{k1} \\ x_{k2} \end{bmatrix} + \begin{bmatrix} 0 \\ U \end{bmatrix} u_k + \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 32 \end{bmatrix} \quad k=p,e \qquad (4.3.49)$$

the weighting matrices are defined as

$R_p(t)=r_p[U]=90[U]$ (4.3.50)

$R_e(t)=r_e[U]=60[U]$ (4.3.51)

The other unknown parameters are the weighting coefficient, $a^2=1$; the pursuer's initial distance, $R_0=125{,}000$ ft; and the pursuer's initial speed, $V_0=10{,}000$ ft/sec. The evader's initial state is varied as a function of launch delay, Δt_0. Launch delay is the time the pursuer delays in achieving its initial state with respect to its earliest possible launch time. The evader's trajectory is illustrated in Figure 6.

The results for the differential game are determined by solving for the boundary costate vectors, $\lambda_p(T)$ and $\lambda_e(T)$. These boundary conditions are defined in Appendix 8.2 in terms of the unknown intercept time for the parameters specified by (4.3.49- 4.3.51). The computational scheme used in determining the intercept time is shown in the Appendix 8.2.1.

In this example it is desirous to choose the pursuer's launch delay and initial state such that the terminal miss distance is minimum and also that the pursuer's terminal distance from the origin is greater than 155,000 ft. Figure 7 depicts the intercept time and the pursuer's terminal distance versus launch delay. Figures 8 and 9 show how the terminal miss distance and the pursuer's estimated initial state vary with respect to the pursuer's launch time delay.

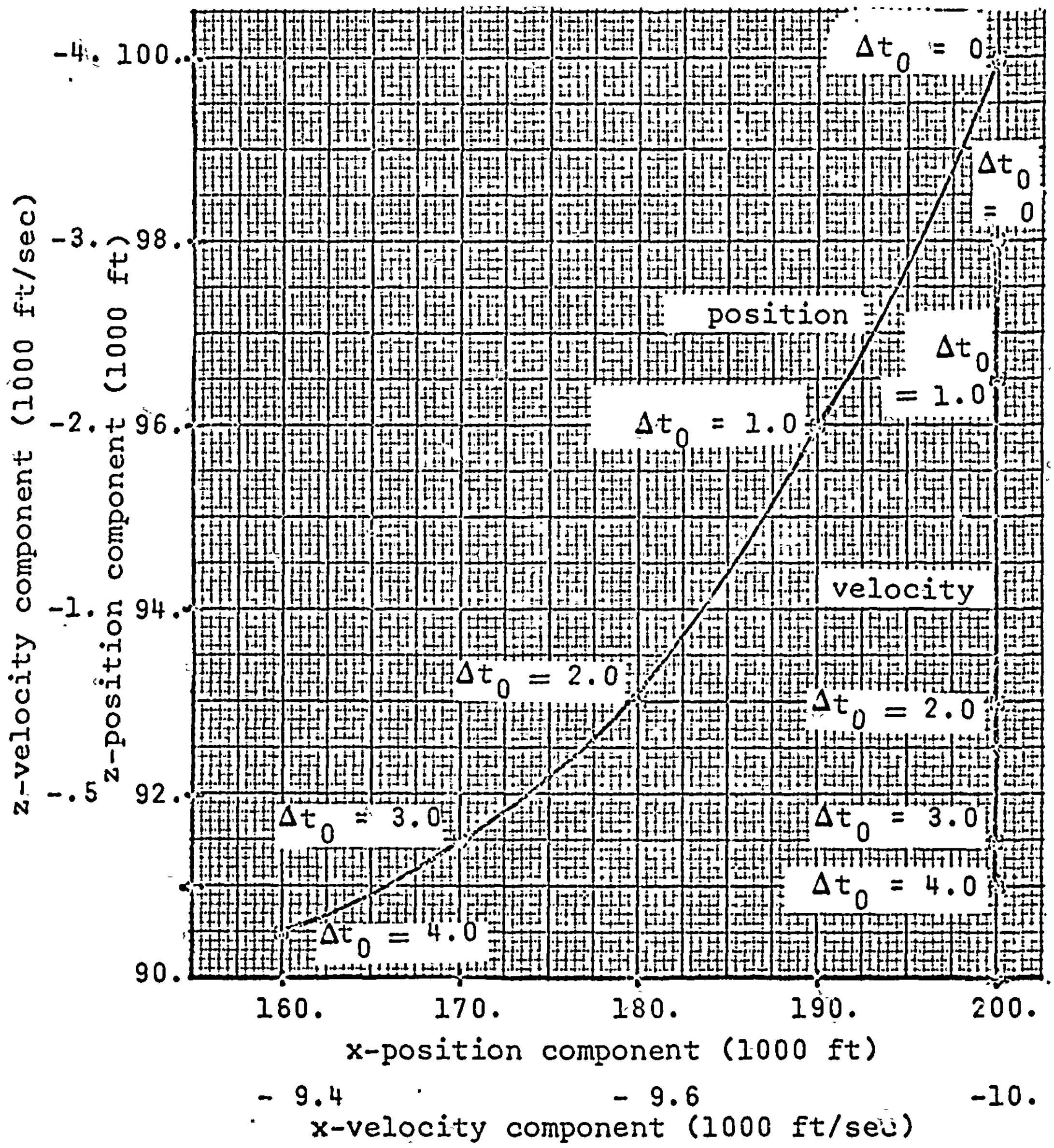

Figure 6. Evader's Estimated Trajectory of Its Initial State as a Function of the Pursuer's Launch Delay Time

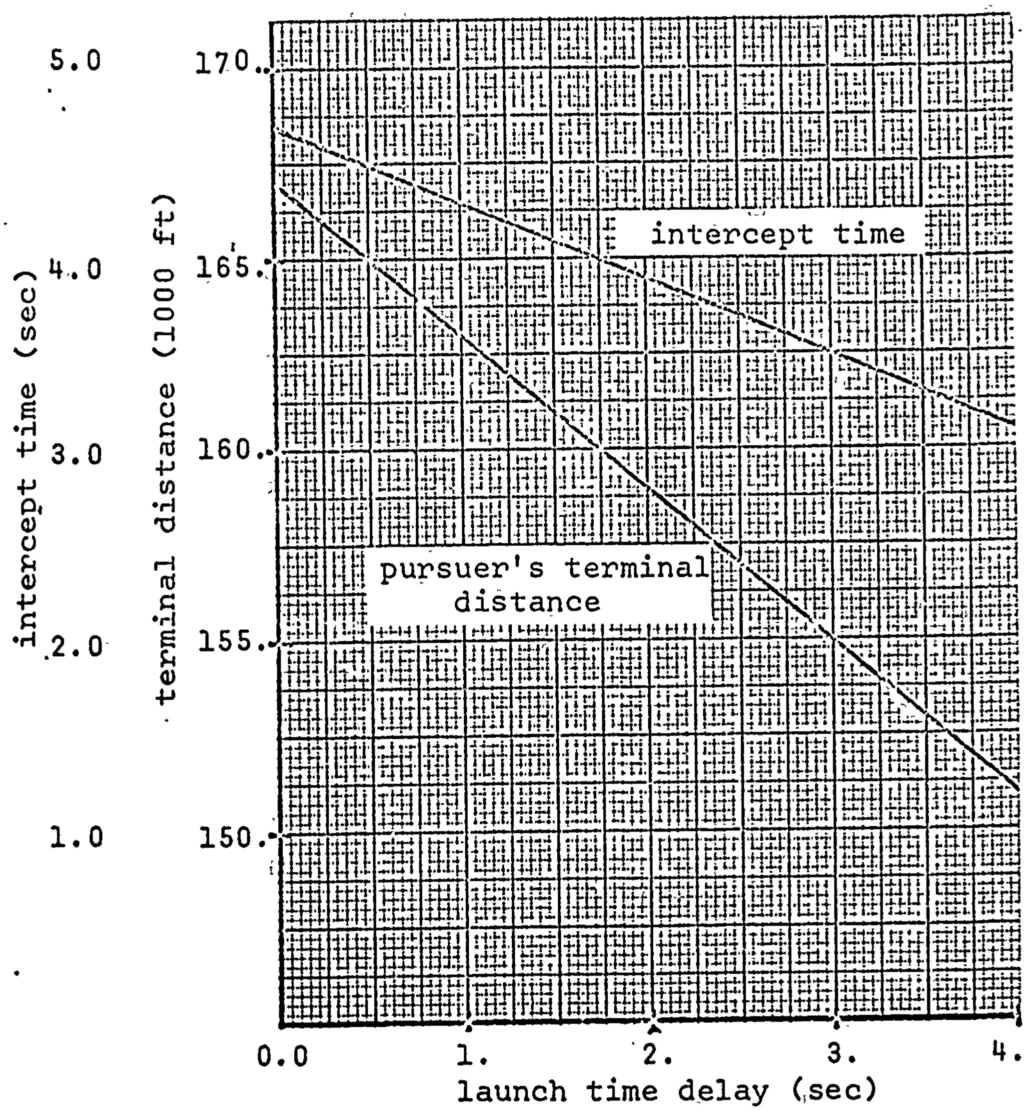

Figure 7. Intercept Time and the Pursuer's Terminal Distance Versus Launch Time Delay

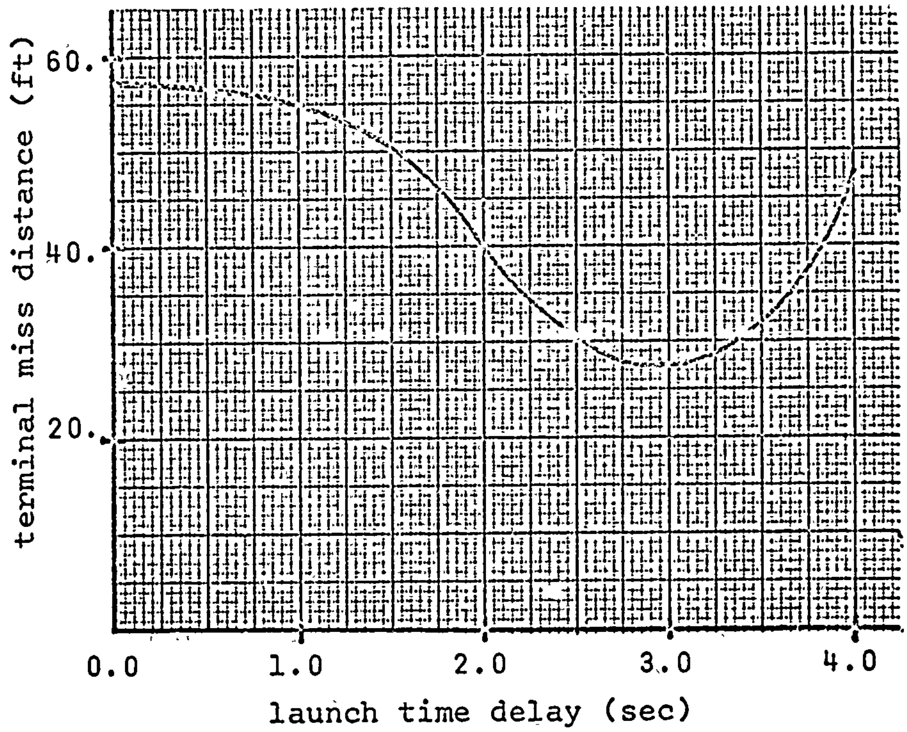

Figure 8. Terminal Miss Distance Versus Launch Time Delay

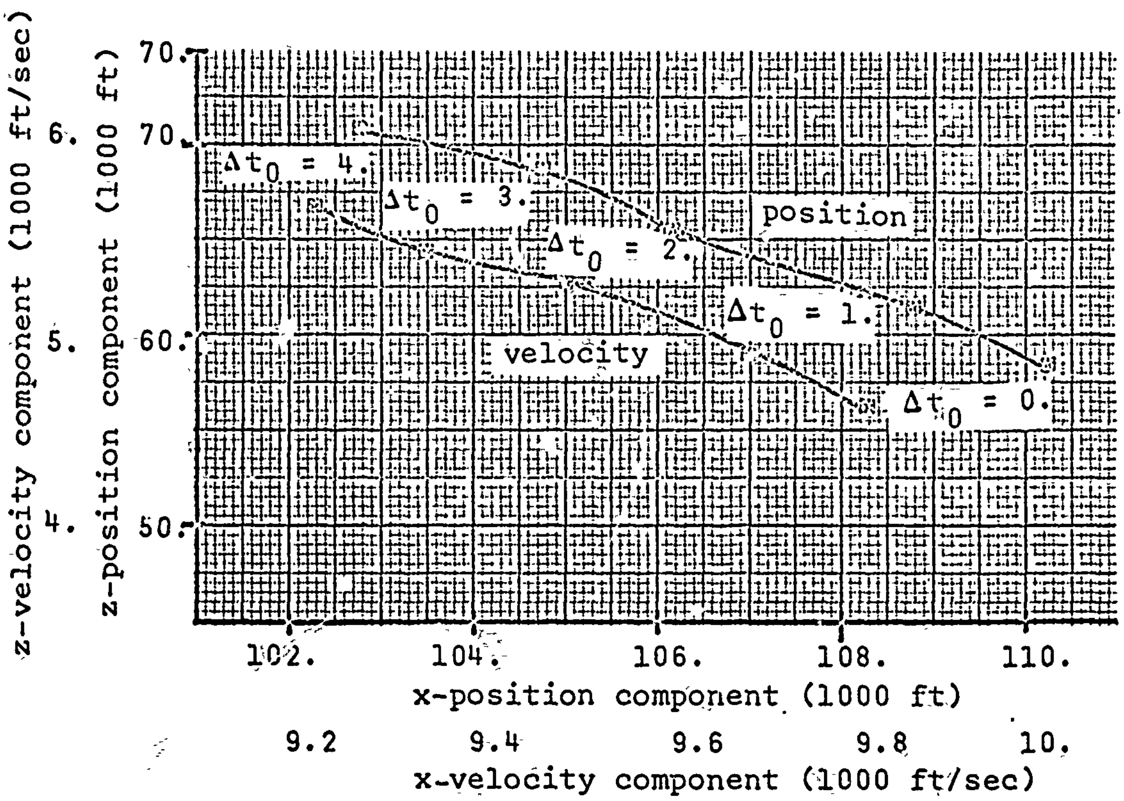

Figure 9. Pursuer's Optimal Initial State
Versus Launch Time Delay

From Figures 7 through 9 it is shown that if the pursuer delays its launch by three seconds, it obtains the following initial state:

$$x_p(t_0) = \begin{bmatrix} 106185.\text{ft} \\ 65950. \\ 8495.\text{ft/sec} \\ 5276. \end{bmatrix} \qquad (4.3.52)$$

The intercept distance for the differential game will be 28 ft.

4.4 SUMMARY OF THE CHAPTER

The principal contributions contained in this chapter are:

(1) Formulation of the differential endgame where the target time, intercept time, and the pursuer's initial position and velocity vectors are unknown. The inner products of the pursuer's initial position and velocity vectors are constrained. The relationships that determine the intercept time, target time, and the pursuer's initial position and velocity vectors are derived.

(2) Formulation of a differential game where the intercept time and pursuer's initial position and velocity vectors are unknown. The inner products of the pursuer's initial position and velocity vectors are constrained. The relationships that determine the intercept time and the pursuer's initial position and velocity vectors are derived.

(3) Application of the differential game in the determination when the pursuer should begin the engagement with the evader.

5. DIFFERENTIAL GAME WITH AN ADDITIONAL PURSUER

5.1 INTRODUCTION

The differential game proposed here is one where an additional pursuer is cooperating with the primary pursuer that is trying to intercept the evader. The objective of this cooperating pursuer is to help the primary pursuer intercept the evader or to increase the evader's cost, if it is able to avoid interception. For this class of differential games the two point boundary value problem is derived from the necessary conditions required for the differential game's payoff functional to have a saddle point. From the solution of the two point boundary value problem the value of the differential game is determined.

5.2 FORMULATION OF THE DIFFERENTIAL GAME WITH AN ADDITIONAL PURSUER

The special class of differential games formulated in this chapter is as follows:

If

$$\|x_{cp1}(T)-x_{e1}(T)\| < \|z_m\| < \|z(T)\| \qquad (5.2.1)$$

then for the primary pursuer's and evader's optimal strategies, u^* and v^*, the differential game's payoff functional

$$J=a^2/2[x_p(T)-x_e(T)]'A'A[x_p(T)-x_e(T)]$$

$$+1/2\int_{t_0}^{T}[u'R_p(t)u-v'R_e(t)v]dt \qquad (5.2.2)$$

subject to the constraints

$$\dot{x}_p = F_p(t)x_p + G_p(t)u + n_p \qquad (5.2.3)$$

$$\dot{x}_e + F_e(t)x_e + G_e(t)v + n_e \qquad (5.2.4)$$

$$x_p(t_0) = x_{p0} \qquad (5.2.5)$$

$$x_e(t_0) = x_{e0} \qquad (5.2.6)$$

$$u, v \in R^3 \qquad (5.2.7)$$

$$[x_{cp}(T) - x_e(T)]'A'A[x_{cp}(T) - x_e(T)] = \|z_m\|^2 \qquad (5.2.8)$$

determine the differential game's value,

$W(x_p(t_0), x_e(t_0), t_0)$, such that

$$W(x_p(t_0), x_e(t_0), t_0) = \underset{u \in U}{\text{Min}}\ \underset{v \in V}{\text{Max}}\ J \qquad (5.2.9)$$

subject to the constraints (5.2.3-5.2.8).

The state vectors, x_p and x_e, represent the position and velocity components of the primary pursuer and the evader respectively, and x_{cp} is the state vector of the cooperating pursuer; the control vectors, u and v, represent the components of the primary pursuer's and evader's acceleration commands; n_p and n_e represent any distrubance vectors, such as the earth's gravitational field; R^3 is the 3-dimensional open Euclidean space; the 6x6 matrices, $F_p(t)$ and $F_e(t)$, and the 6x3 matrices, $G_p(t)$ and $G_e(t)$, are continuous in time; $R_p(t)$ and $R_e(t)$ are 3x3 positive definite weighting matrices, continuous in t; and a^2 is a weighting factor. The differential game has a finite duration of $(T-t_0)$ seconds, T being the fixed intercept time.

The differential game's payoff functional (5.2.2) is the terminal miss distance, squared, weighted against the difference between the primary pursuer's and evader's control energies, spent during the engagement interval

(t_0,T). Both the primary pursuer and the evader have linear nonstationary dynamics, defined by the differential constraints (5.2.3,5.2.4)

$\|Z_m\|$ is the minimum intercept distance allowed between the cooperating pursuer and the evader, if the evader is to avoid interception by the cooperating pursuer. If the inequ\lity condition defined by (5.2.1) exists, then $\|x_{cp1}(T)-x_{e1}(T)\|$, the terminal miss distance between the cooperating pursuer and the evader, is quadratically constrained by (5.2.8).

$\|z(T)\|$ is the terminal miss distance that results from determining $W^*(x_p(t_0),x_e(t_0),t_0)$ the value of the differential game without the cooperating pursuer by

$$W^*(x_p(t_0),x_e(t_0),t_0)=\underset{u\in U}{\mathrm{Min}}\ \underset{v\in V}{\mathrm{Max}}\ J \qquad (5.2.10)$$

subject to the constraints (5.2.3-5.2.7). J is the payoff functional, defined by (5.2.2). This differential game has been studied in reference (6).

5.2.1 Terminal Miss Distance Constraint between the Cooperating Pursuer and the Evader

Before playing the differential game defined by (5.2.1-5.2.8) the evader's terminal miss distance constraint with the cooperating evader (5.2.8) is determined in terms of the evader's free terminal state $x_e(T)$. The cooperating pu.suer's control strategy is determined by

$$\underset{u_{cp}}{\mathrm{Min}}\left\{ a_{cp}^2/2[x_{cp}(T)-x_e(T)]'A'A[x_{cp}(T)-x_e(T)] \right.$$

$$\left. +1/2\int_{t_0}^{T} u_{cp}'R_{cp}(t)u_{cp}dt\right\} \qquad (5.2.11)$$

subject to the constraints:

$$\dot{x}_{cp}=F_{cp}(t)x_{cp}+G_{cp}(t)u_{cp}+n_{cp} \tag{5.2.12}$$

$$x_{cp}(t_0)=x_{cp0} \tag{5.2.13}$$

$$u_{cp}\in R^3 \tag{5.2.14}$$

where the state vector x_{cp} represents the position and velocity components of the cooperating pursuer; $x_e(T)$, the evader's unknown terminal state, represents the cooperating pursuer's desired terminal state; u_{cp} represents the components of the cooperating pursuer's acceleration commands; n_{cp} represents the earth's gravitational field; R^3 is the 3-dimensional Euclidean space; the 6x6 matrix $F_{cp}(t)$ and the 6x3 matrix $G_{cp}(t)$ are continuous in time; a^2_{cp} is a weighting factor and $R_{cp}(t)$ is a 3x3 positive definite weighting matrix, continuous in t.

The performance criterion to be minimized is the square of the norm of the error of the cooperating pursuer's terminal position with respect to its desired terminal position, weighted against the control energy, spent by the cooperating pursuer. The dynamics of the pursuer are defined by (5.2.12).

The terminal miss vector that results from minimizing the performance criterion (5.2.11) is derived in Appendix 8.3 and is expressed as

$$[x_{cp1}(T)-x_{e1}(T)]=A[x_{cp}(T)-x_e(T)]=$$

$$A[U+a^2_{cp}M_{cp}(T,t_0)A'A]^{-1}[\Phi_{cp}(T,t_0)x_{cp0}+k_{cp}(T,t_0)-x_e(T)] \tag{5.2.15}$$

where $\Phi_{cp}(T,t_0)$ is the transition matrix for the state vector x_{cp}. $M_{cp}(T,t_0)$, the controllability matrix, is defined as

$$M_{cp}(T,t_0)=\int_{t_0}^{T}\Phi_{cp}(T,t)G_{cp}(t)R_{cp}(t)^{-1}G'_{cp}(t)\Phi'_{cp}(T,t)dt \quad (5.2.16)$$

and the column vector $k_{cp}(T,t_0)$ due to the disturbance vector n_{cp} is

$$k_{cp}(T,t_0)=\int_{t_0}^{T}\Phi_{cp}(T,t)n_{cp}dt \quad (5.2.17)$$

In shorthand form the terminal miss distance between the evader and cooperating pursuer (5.2.15) is

$$[x_{cp1}(T)-x_{e1}(T)]=K_1[\Phi_{cp}(T,t_0)x_{cp}(t_0)+k_{cp}(T,t_0)-x_e(T)] \quad (5.2.18)$$

where

$$K_1=A[U+a_{cp}^2M_{cp}(T,t_0)A'A]^{-1} \quad (5.2.19)$$

5.3 DETERMINATION OF THE TWO POINT BOUNDARY VALUE PROBLEM FOR THE DIFFERENTIAL GAME WITH AN ADDITIONAL PURSUER

Vector Lagrangian multipliers, λ_p and λ_e, are introduced in order to adjoin the differential constraints (5.2.3,5.2.4) to the payoff functional (5.2.2). The scalar multiplier, γ, adjoins the quadratic constraint of the evader's terminal miss distance with the cooperating pursuer (5.2.8) to the payoff functional (5.2.2).

The differential game is expressed as

$$\underset{u\in U}{\text{Min}}\ \underset{v\in V}{\text{Max}}\ J_c = \underset{u\in U}{\text{Min}}\ \underset{v\in V}{\text{Max}}\Big\{[x_p(T)-x_e(T)]'A'A[x_p(T)-x_e(T)]$$

$$+\gamma/2[(x_{cp0}'\Phi_{cp}'(T,t_0)+k_{cp}'(T,t_0)-x_e'(T))K_1'\times$$

$$K_1(\Phi_{cp}(T,t_0)x_{cp0}+k_{cp}(T,t_0)-x_e(T))-\|z_m\|^2]$$

$$+\int_{t_0}^{T}[1/2u'R_p(t)u-1/2v'R_e(t)v+\lambda_p'(F_p(t)x_p+G_p(t)u+n_p-\dot{x}_p)$$

$$+\lambda_e'(F_e(t)x_e+G_e(t)v+n_e-\dot{x}_e)]dt\Big\} \qquad (5.3.1)$$

Applying variations, δu and δv, about a particular pair of controls, u and v, the first order variation of the differential game's constrained payoff functional is

$$\delta J_c=(-\lambda_p'(T)+a^2[x_p(T)-x_e(T)]'A'A)\delta x_p(T)$$

$$+(-\lambda_e'(T)-a^2[x_p(T)-x_e(T)]'A'A$$

$$-\gamma(x_{cp0}'\Phi_{cp}'(T,t_0)+k_{cp}'(T,t_0)-x_e'(T))K_1'K_1\ \delta x_e(T)$$

$$+\int_{t_0}^{T}[(u'R_p(t)+\lambda_p'G_p(t))\delta u+(-v'R_e(t)+\lambda_e'G_e(t))\delta v$$

$$+(\dot{\lambda}_p'+\lambda_p'F_p(t))\delta x_p+(\dot{\lambda}_e'+\lambda_e'F_e(t))\delta x_e]dt \qquad (5.3.2)$$

Necessary conditions required for the game's saddle point are the coefficients of the variationals equated to zero.

The two point boundary value problem is the set of Euler-Lagrange equations and their boundary conditions. The Euler-Lagrange equations and their boundary conditions are determined by the differential game's constraints (5.2.3-5.2.6) and (5.2.8) and the necessary conditions that result by requiring δJ to vanish. Table 5 summarizes the necessary conditions for δJ to vanish.

Table 5

Necessary Conditions for δJ to Vanish

Variationals	Coefficients of Variationals Equated to Zero	
δu	$u'R_p(t)+\lambda_p{}'G_p(t)=0$	(5.3.3)
δv	$-v'R_e(t)+\lambda_e{}'G_e(t)=0$	(5.3.4)
δx_p	$\dot{\lambda}_p{}'+\lambda_p{}'F_p(t)=0$	(5.3.5)
δx_e	$\dot{\lambda}_e{}'+\lambda_e{}'F_e(t)=0$	(5.3.6)
$\delta x_p(T)$	$a^2[x_p(T)-x_e(T)]'A'A-\lambda_p(T)=0$	(5.3.7)
$\delta x_e(T)$	$\gamma x_e(T)'K_1{}'K_1-a^2[x_p(T)-x_e(T)]'A'A-\lambda_e{}'(T)$	
	$=\gamma[x_{cp}{}'(t_0)\Phi_{cp}{}'(T,t_0)+k_{cp}{}'(T,t_0)]K_1{}'K_1=0$	(5.3.8)

The Euler-Lagrange equations and their boundary conditions are

$$\begin{bmatrix} \dot{x}_p \\ \dot{x}_e \\ \dot{\lambda}_p \\ \dot{\lambda}_e \end{bmatrix} = \begin{bmatrix} F_p(t) & 0 & -G_p(t)R_p^{-1}(t)G_p'(t) & 0 \\ 0 & F_e(t) & 0 & G_e(t)R_e^{-1}(t)G_e'(t) \\ 0 & 0 & -F_p'(t) & 0 \\ 0 & 0 & 0 & -F_e'(t) \end{bmatrix} \begin{bmatrix} x_p \\ x_e \\ \lambda_p \\ \lambda_e \end{bmatrix} + \begin{bmatrix} h_p \\ h_e \\ 0 \\ 0 \end{bmatrix} \quad (5.3.9)$$

$$x_p(t_0)=x_{p0} \quad (5.3.10)$$

$$x_e(t_0)=x_{e0} \quad (5.3.11)$$

$$\lambda_p(T)=a^2A'A[x_p(T)-x_e(T)] \quad (5.3.12)$$

$$\gamma x_e(T)'K_1'K_1-a^2[x_p(T)-x_e(T)]'A'A-\lambda_e'(T)$$

$$-\gamma[x_{cp}'(t_0)\Phi_{cp}'(T,t_0)+k_{cp}'(T,t_0)]K_1'K_1=0 \quad (5.3.13)$$

$$[x_{cp}(T)-x_e(T)]'A'A[x_{cp}(T)-x_e(T)]= \|z_m\|^2 \quad (5.3.14)$$

The solution of the two point boundary value problem (5.3.9-5.3.14) determines the optimal strategies of the participants, the terminal miss distance, and the value of this differential game.

5.4 SOLUTION OF THE TWO POINT BOUNDARY VALUE PROBLEM

The solution of the Euler-Lagrange equations (5.3.9) defined at T in terms of the primary pursuer's and evader's unknown state vectors, $x_p(T)$ and $x_e(T)$, and their unknown costate vectors, $\lambda_p(T)$ and $\lambda_e(T)$, are:

$$\lambda_k(T)=\Phi_k'(t_0,T)\lambda_k(t_0); \quad k=p, \quad (5.4.1)$$

$$x_p(T)=\Phi_p(T,t_0)x_p(T_0)-M_p(T,t_0)\lambda_p(T)+k_p(T,t_0) \quad (5.4.2)$$

$$x_e(T)=\Phi_e(T,t_0)x_e(t_0)+M_e(T,t_0)\lambda_e(T)+k_e(T,t_0) \quad (5.4.3)$$

where $\Phi_k(T,t_0)$ and $\Phi_k'(t_0,T)$; $k=p,e$ are the transition matrices for the state and costate vectors; the controllability matrices are defined as

$$M_k(T,t_0)=\int_{t_0}^{T}\Phi_k(T,t)G_k(t)R_k(t)^{-1}G_k'(t)\Phi_k'(T,t)dt;\quad k=p,e \tag{5.4.4}$$

and the column vectors $k_k(T,t_0)$; $k=p,e$ due to the disturbance vectors $\bar{n}_k$ are

$$k_k(T,t_0)=\int_{t_0}^{T}\Phi_k(T,t)n_k dt;\qquad k=p,e \tag{5.4.5}$$

The two point boundary value problem is solved when the multiplier vectors, $\lambda_p(T)$ and $\lambda_e(T)$, and the scalar multiplier, γ, are determined. Substituting (5.4.2,5.4.3) into the boundary conditions (5.3.7,5.3.8) these boundary conditions are expressed in terms of the multiplier vectors, $\lambda_p(T)$ and $\lambda_e(T)$, and the scalar multiplier, γ.

$$\begin{bmatrix} U+a^2A'AM_p(T,t_0) & a^2A'AM_e(T,t_0) \\ -a^2A'AM_p(T,t_0) & U-[a^2A'A+\gamma K_1'K_1]M_e(T_1,t_0) \end{bmatrix}\begin{bmatrix}\lambda_p(T)\\ \lambda_e(T)\end{bmatrix}$$

$$=\begin{bmatrix} a^2A'A\Phi_p(T,t_0) & -a^2A'A\Phi_e(T,t_0) \\ -a^2A'A\Phi_p(T,t_0) & [a^2A'A+\gamma K_1'K_1]\Phi_e(T,t_0) \end{bmatrix}\begin{bmatrix}x_p(t_0)\\ x_e(t_0)\end{bmatrix}$$

$$+\begin{bmatrix} a^2A'A & -a^2A'A \\ -a^2A'A & a^2A'A+\gamma K_1'K_1 \end{bmatrix}\begin{bmatrix}k_p(T,t_0)\\ k_e(T,t_0)\end{bmatrix}-\gamma\begin{bmatrix} 0 & 0 \\ K_1'K_1\Phi_{cp}(T,t_0) & K_1'K_1 \end{bmatrix}\begin{bmatrix}x_{cp}(t_0)\\ k_{cp}(T,t_0)\end{bmatrix}$$

(5.4.6)

The terminal miss distance constraint between the evader and the cooperating pursuer is expressed in terms of $\lambda_e(T)$ as

$$[x_{cp1}(T)-x_{e1}(T)]'[x_{cp1}(T)-x_{e1}(T)]=$$

$$[K_2'-\lambda_e'(T)M_e'(T,t_0)]K_1'K_1[K_2-M_e(T,t_0)\lambda_e(T)]=\|z_m\|^2 \tag{5.4.7}$$

where

$$K_2=\Phi_{cp}(T,t_0)x_{cp0}+k_{cp}(T,t_0)-\Phi_e(T,t_0)x_{e0}-k_e(T,t_0) \qquad (5.4.8)$$

The boundary costate vectors, $\lambda_p(T)$ and $\lambda_e(T)$, and the scalar multiplier, γ, are determined by solving the simultaneous equations (5.4.6,5.4.7). With $\lambda_p(T)$ and $\lambda_e(T)$ known, the solution of the two point boundary value problem is determined. Knowing the solution of the two point boundary value problem, it is determined whether or not the primary pursuer intercepts the evader.

5.5 EXAMPLE OF A DIFFERENTIAL GAME WITH A COOPERATING PURSUER

For this differential game the dynamics of all participants are defined as

$$\begin{bmatrix}\dot{x}_{k1}\\ \dot{x}_{k2}\end{bmatrix}=\begin{bmatrix}0 & U\\ 0 & 0\end{bmatrix}\begin{bmatrix}x_{k1}\\ x_{k2}\end{bmatrix}+\begin{bmatrix}0\\ U\end{bmatrix}u_k+\begin{bmatrix}0\\ \vdots\\ 0\\ 32\end{bmatrix};\ k=p,cp,e \qquad (5.5.1)$$

the weighting matrices are

$$R_k(t)=r_k[U]=90[U];\ k=p,cp \qquad (5.5.2)$$

$$R_e(t)=r_e[U]=60[U] \qquad (5.5.3)$$

the weighting coefficients are

$$a_{cp}^2=a^2=1 \qquad (5.5.4)$$

The initial state for all participants is

$$x_{p0}=\begin{bmatrix}103600.\text{ft}\\ 0\\ 75200.\\ 8079.\text{ft/sec}\\ 0\\ 5894.\end{bmatrix};x_{cp0}=\begin{bmatrix}99800.\text{ft}\\ 0\\ 75200.\\ 8079.\text{ft/sec}\\ 0\\ 5894.\end{bmatrix};x_{e0}=\begin{bmatrix}150000.\text{ft}\\ 0\\ 90000.\text{ft}\\ -10000.\text{ft/sec}\\ 0\\ -100.\end{bmatrix} \qquad (5.5.5)$$

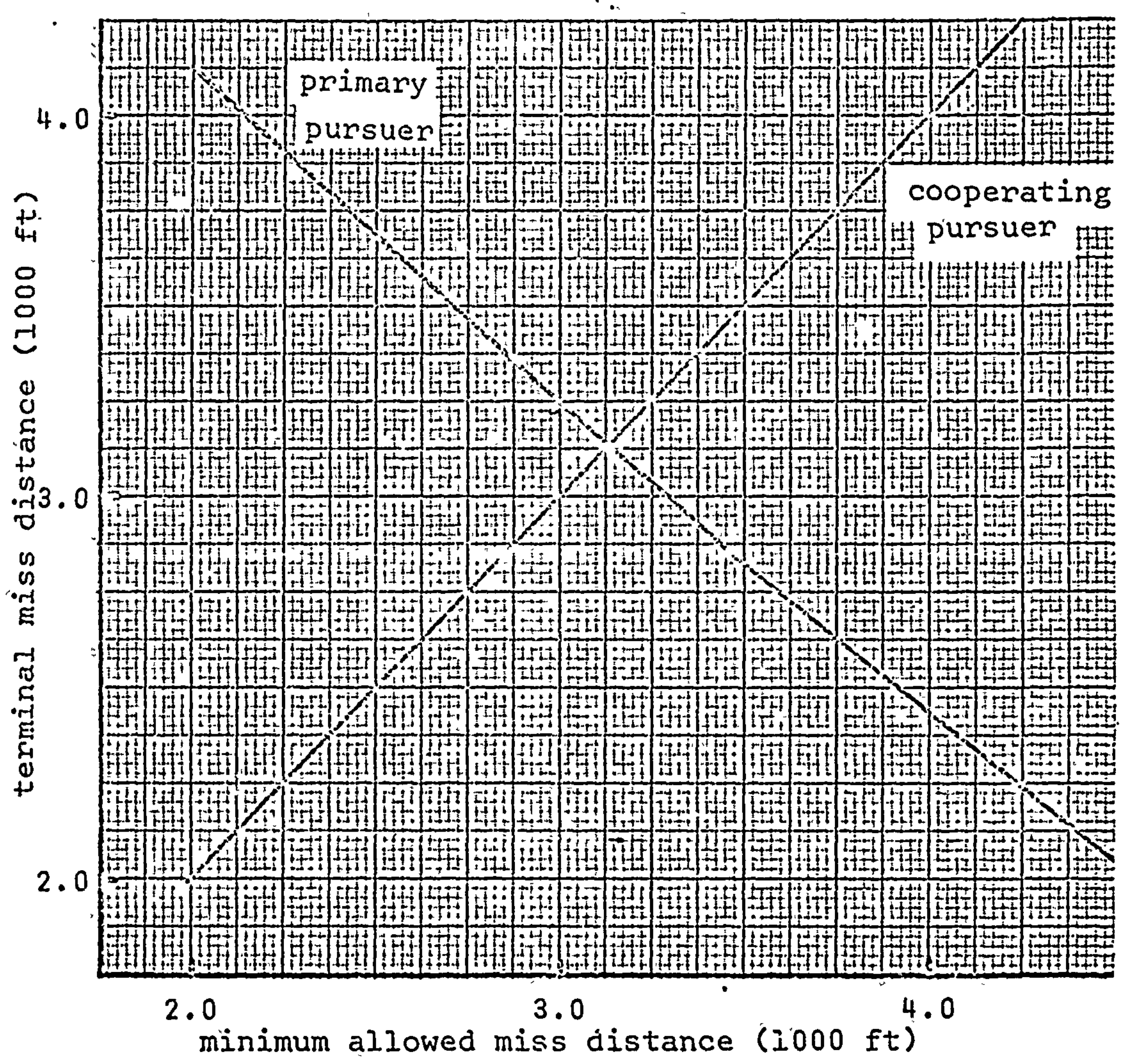

Figure 10. Terminal Miss Distance between the Pursuers and the Evader Versus Minimum Allowed Miss Distance

The results for this differential game depend on the determination of the Lagrangian multipliers, $\lambda_p(T)$, $\lambda_e(T)$ and γ. These Lagrangian multipliers are defined in Appendix 8.4 in terms of the parameters specified by (5.5.1-5.5.4).

For this particular differential game $\|Z(T)\| = 4744$ ft while $\|x_{cp1}(T)-x_{e1}(T)\| = 1357$ ft. The problem is to determine the minimum value of $\|Z_m\|$ such that

$$1357\text{ft} \leq \|Z_m\| \leq 4744 \text{ ft} \tag{5.5.6}$$

and

$$\|x_{p1}(T)-x_{e1}(T)\| \geq \|Z_m\| \tag{5.5.7}$$

Figure 10 illustrates how $\|x_{p1}(T)-x_{e1}(T)\|$ varies with $\|Z_m\|$. From Figure 10 it is seen that the value of $\|Z_m\| = 3125$ ft satisfies (5.5.6,5.5.7)

5.6 SUMMARY

The principal contribution contained in this chapter is the formulation of a class of differential games where an additional pursuer is cooperating with the primary pursuer that is trying to intercept the evader.

6. CONCLUSION

It is evident that differential game theory is an effective tool for analyzing pursuit-evasion problems. Here it is attempted to apply differential game theory to two types of pursuit-evasion problems, the endgame problem and the differential game with a cooperating pursuer. Also, it is shown how differential game theory is used to determine the differential game parameters such as intercept time, pursuer's initial state, and, in the case of the differential endgame, the target time.

Further areas of research resulting from this dissertation could be:

(1) A differential endgame with an additional pursuer, cooperating with the primary pursuer that is trying to intercept the evader.

(2) Development of a differential game theory that can be applied to a differential game that has more than three participants.

7. ACKNOWLEDGMENT

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8. APPENDICES

APPENDIX 8.1

Determination of the Corner Condition Costate Vectors and the Postengagement Costate Vector in Terms of the Specific Differential Endgame Parameters

Here the corner condition vectors, $\lambda_p(T_1)$ and $\lambda_e(T_1)$, and the post engagement costate vector μ, as defined by (3.4.11) and (3.4.16), are expressed in terms of the specified parameters of the differential endgame. These parameters include the initial state of the game, target constraints, participants' dynamics and the diagonal weighting matrices as specified in Section 3.6.

8.1.1 Determination of $B(\eta)^{-1}$

With reference to the participants' specified dynamics (3.6.1) and the diagonal weighting matrices (3.6.3), the state transition matrices and the controllability matrices are defined in terms of partitioned diagonal matrices as

$$\Phi_k(T_1,t_0) = \begin{bmatrix} U & k\tau U \\ 0 & U \end{bmatrix} \; ; \; k = p, e \qquad (8.1.1)$$

$$\Phi_e(T_2, T_1) = \begin{bmatrix} U & \tau U \\ 0 & U \end{bmatrix} \qquad (8.1.2)$$

$$M_k(T_1, t_0) = \begin{bmatrix} [(k\tau)^3/3r_{kj}] & [(k\tau)^2/2r_{kj}] \\ [(k\tau)^2/2r_{kj}] & [k\tau/r_{kj}] \end{bmatrix}^{*} ; k=p,e \qquad (8.1.3)$$

$$M_e(T_2,T_1) = \begin{bmatrix} [\tau^3/3r_{ej}] & [\tau^2/2r_{ej}] \\ [\tau^2/2r_{ej}] & [\tau/r_{ej}] \end{bmatrix} \qquad (8.1.4)$$

*The elements of the partitioned matrices represent the general diagonal term.

where the engagement interval is defined as

$$(T_1 - t_0) = k\tau \tag{8.1.5}$$

and the post engagement interval is defined as

$$(T_2 - T_1) = \tau \tag{8.1.6}$$

The factor k must be chosen such that

$$\tau + k\tau = (T_2 - t_0) \tag{8.1.7}$$

In terms of the evader's state transition matrix and controllability matrix over the post engagement interval the matrix $K(\eta)$ defined by (3.4.12) is expressed in terms of partitioned diagonal matrices as

$$K(\eta) = \begin{bmatrix} \left[\dfrac{12r_{ej}(\tau - r_{ej}/\eta)}{\tau^3(\tau - 4r_{ej}/\eta)}\right] & \left[\dfrac{6r_{ej}(\tau - 2r_{ej}/\eta)}{\tau^2(\tau - 4r_{ej}/\eta)}\right] \\ \left[\dfrac{6r_{ej}(\tau - 2r_{ej}/\eta)}{\tau^2(\tau - 4r_{ej}/\eta)}\right] & \left[\dfrac{4r_{ej}(\tau - 3r_{ej}/\eta)}{\tau(\tau - 4r_{ej}/\eta)}\right] \end{bmatrix} \tag{8.1.8}$$

The matrix $B(\eta)$ defined by (3.4.11) is partitioned into four (6x6) matrices which, in turn, are partitioned into four (3x3) diagonal matrices. Therefore, the four (6x6) partition matrices are

$$B_{11} = U + a^2A'A\ M_p(T_1, t_0) = \begin{bmatrix} [1 + a^2(k\tau)^3/3r_{pj}] & [a^2(k\tau)^2/2r_{pj}] \\ [0] & [U] \end{bmatrix} \tag{8.1.9}$$

$$B_{12} = a^2A'A\ M_e(T_1, t_0) = \begin{bmatrix} [a^2(k\tau)^3/3r_{ej}] & [a^2(k\tau)^2/2r_{ej}] \\ [0] & [0] \end{bmatrix} \tag{8.1.10}$$

$$B_{21} = -a^2A'A\ M_p(T_1, t_0) = \begin{bmatrix} [-a^2(k\tau)^3/3r_{pj}] & [-a^2(k\tau)^2/2r_{pj}] \\ [0] & [0] \end{bmatrix} \tag{8.1.11}$$

and

$$B_{22}(\eta) = U - [a^2A'A - K(\eta)]M_e(t,t_0) \tag{8.1.12}$$

$$= \begin{bmatrix} (B_{22}(\eta))_{11} & (B_{22}(\eta))_{12} \\ (B_{22}(\eta))_{21} & (B_{22}(\eta))_{22} \end{bmatrix} \tag{8.1.13}$$

where the following (3x3) diagonal matrices are defined as

$$(B_{22}(\eta))_{11} = 1 - \frac{a^2k^3r^3}{3r_{ej}} + \frac{k^2(4k+3)r - 2k^2(2k+3)r_{ej}/\eta}{r - 4r_{ej}/\eta} \tag{8.1.14}$$

$$(B_{22}(\eta))_{12} = \frac{-a^2(kr)^2}{2r_{ej}} + \frac{6k(k+1)r - 6kr_{ej}(k+2)/\eta}{r(r - 4r_{ej}/\eta)} \tag{8.1.15}$$

$$(B_{22}(\eta))_{21} = \frac{2k^2(k+1)r^2 - 2k^2(2k+3)r_{ej}r/\eta}{(r - 4r_{ej}/\eta)} \tag{8.1.16}$$

$$(B_{22}(\eta))_{22} = 1 + \frac{k(3k+4)r - 6k(k+2)r_{ej}/\eta}{r - 4r_{ej}/\eta} \tag{8.1.17}$$

Now in terms of the (6x6) partitioned matrices

$$B(\eta)^{-1} = \left[\begin{array}{c|c} (B_{11} - B_{12}B_{22}(\eta)B_{21})^{-1} & -(B_{11} - B_{12}B_{22}(\eta)B_{21})^{-1}B_{12}B_{22} \\ -(B_{22}(\eta) - B_{21}B_{11}B_{12})^{-1}B_{21}B_{11} & (B_{22}(\eta) - B_{21}B_{11}B_{12})^{-1} \end{array}\right] \tag{8.1.18}$$

$B(\eta)^{-1}$ is partitioned into (3x3) diagonal matrices as follows. The general diagonal terms of these partitioned matrices are defined as

$$B^{-1}_{ik}(\eta) = \frac{\alpha_{ik}(j) + \beta_{ik}(j)/\eta}{\Delta(j)} \;;\; i = 1, 4;\; k = 1, 4 \tag{8.1.19}$$

where $\Delta(j)$ and $\alpha_{ik}(j) + \beta_{ik}(j)/\eta$ are expressed as

$$\Delta(j) = \Delta_1(j) + \frac{\Delta_2(j)}{\eta} = \left\{3r_{pj}r_{ej}(k+1)^4 r + a^2k^3(k+1)\,[r_{ej}(k+1)^3 - r_{pj}]r^4\right\}$$
$$+\left\{-12r_{pj}r_{ej}^2(k+1)^3 + r_{ej}a^2k^3[(4+3k)r_{pj} - 4r_{ej}(k+1)^3]r^3\right\}/\eta \tag{8.1.20}$$

$$\alpha_{11}(j)+\beta_{11}(j)/\eta = 3r_{pj}r_{ej}[(k+1)^4 r - a^2k^3(k+1)r^4/3r_{ej}]$$

$$+3r_{pj}r_{ej}[-4(k+1)^3 r_{ej}+a^2k^3(4+3k)r^3/3]/\eta \qquad (8.1.21)$$

$$\alpha_{12}(j) + \beta_{12}(j)/\eta = -1.5r_{ej}a^2k^2(k+1)^4 r^3$$

$$+6r_{ej}{}^2a^2k^2(k+1)^3 r^2/\eta \qquad (8.1.22)$$

$$\alpha_{13}(j)+\beta_{13}(j)/\eta = -r_{pj}a^2k^3(k+1)r^4$$

$$+r_{pj}r_{ej}a^2k^3(3k+4)r^3/\eta \qquad (8.1.23)$$

$$\alpha_{14}(j) + \beta_{14}(j)/\eta = 1.5r_{pj}a^2k^2(k^2-1)r^3$$

$$+3r_{pj}r_{ej}a^2k^2(2-k^2)r^2/\eta \qquad (8.1.24)$$

$$\alpha_{2k}(j) + \beta_{2k}(j)/\eta = \begin{cases} 0; & k = 1,\ 3,\ 4 \\ \Delta; & k = 2 \end{cases} \qquad (8.1.25)$$

$$\alpha_{31}(j) + \frac{\beta_{31}(j)}{\eta}$$

$$= r_{ej}a^2k^3(3k^2+4k+1)r^4 - a^2k^3(6k^2+12k+4)r_{ej}{}^2 r^3/\eta \qquad (8.1.26)$$

$$\alpha_{32}(j)+\frac{\beta_{32}(j)}{\eta} = 1.5r_{ej}a^2k^2(3k^2+4k+1)r^3 - \frac{3a^2k^2(6k^2+12k+4)}{2\eta} r_{ej}{}^2 r^2 \qquad (8.1.27)$$

$$\alpha_{33}(j)+\beta_{33}(j)/\eta = [3r_{pj}r_{ej}(3k^2+4k+1)r + a^2k^3r_{ej}(3k^2+4k+1)r^4]$$

$$-[3r_{pj}r_{ej}{}^2(6k^2+12k+4)+a^2k^3r_{ej}{}^2(6k^2+12k+4)r^3]/\eta \qquad (8.1.28)$$

$$\alpha_{34}(j)+\frac{\beta_{34}(j)}{\eta} = \left\{-18r_{pj}r_{ej}k(k+1)+[1.5a^2k^2r_{pj}-6r_{ej}a^2k^2(k+1)]r^3\right\}$$

$$+\left\{18r_{ej}{}^2r_{pj}k(k+2)/r + [6r_{ej}{}^2a^2k^4(k+2)-6r_{ej}r_{pj}a^2k^2]r^2\right\}/\eta \qquad (8.1.29)$$

$$\alpha_{41}(j)+\beta_{41}(j)/\eta = -2a^2k^5(k+1)r_{ej}r^5+2a^2k^5(2k+3)r_{ej}{}^2 r^4/\eta \qquad (8.1.30)$$

$$\alpha_{42}(j)+\beta_{42}(j)/\eta=-3r_{ej}a^2k^4(k+1)r^4+3a^2k^4(2k+3)r_{ej}{}^2r^3/\eta \quad (8.1.31)$$

$$\alpha_{43}(j)+\beta_{43}(j)/\eta=[-6r_{pj}r_{ej}k^2(k+1)r^2-2r_{ej}a^2k^5(k+1)r^5]$$

$$+[6r_{pj}r_{ej}{}^2k^2(2k+3)r+2r_{ej}{}^2a^2k^5(2k+3)r^4]/\eta \quad (8.1.32)$$

$$\alpha_{44}(j)+\frac{\beta_{44}(j)}{\eta}=\{3r_{pj}r_{ej}(1+3k^2+4k^3)r+$$

$$a^2k^3[r_{ej}-r_{pj}+k^2(4k+3)r_{ej}]r^4\}$$

$$-\{6r_{pj}r_{ej}{}^2(2+3k^2+2k^3)$$

$$+2r_{ej}a^2k^3[2r_{ej}-2r_{pj}+r_{ej}k^2(2k+3)]r^3\}/\eta \quad (8.1.33)$$

8.1.2 Determination of the Corner Condition Costate Vectors

The solution for the corner condition costate vectors derived in Section 3.4 is:

$$\lambda(T_1) = B(\eta)^{-1}\{C(\eta)\,x(t_0) + \ell(\eta)\} \quad (8.1.34)$$

With the aid of (3.4.11) and (8.1.1-8.1.4) the column vector, $C(\eta)\,x(t_0) + \ell(\eta)$, is partitioned into 3x1 column vectors:

$$C(\eta)\,x(t_0) + \ell(\eta) = \begin{bmatrix} \dfrac{c_{k1}(j) + d_{k1}(j)/\eta}{r^4-4r_{ej}r^3/\eta} \\ \vdots \\ \dfrac{c_{k1}(j)+d_{k1}(j)/\eta}{r^4-4r_{ej}r^3/\eta} \end{bmatrix} \quad (8.1.35)$$

whose general terms $c_{k1}(j)+d_{k1}(j)$; $k = 1, 4$ are expressed as

$$c_{k1}(j)+d_{k1}(j)/\eta=[r^4-4r_{ej}r^3/\eta]\,(a^2[x_{p1j}(t_0)-x_{e1j}(t_0)]+$$

$$a^2kr[x_{p2j}(t_0)-x_{e2j}(t_0)]) \quad (8.1.36)$$

$$c_{21}(j) + d_{21}(j)/\eta = 0 \tag{8.1.37}$$

$$c_{31}(j) + \frac{d_{31}(j)}{\eta} =$$

$$(\{-a^2\tau^4[x_{p1j}(t_0)+k\tau x_{p2j}(t_0)]+[a^2\tau^4-12r_{ej}\tau]x_{e1j}(t_0)$$

$$+[a^2k\tau^5-6r_{ej}(2k+1)\tau^2]x_{e2j}(t_0)-6(j-1)(j-2)r_{ej}k(k+1)\tau^3 g\}$$

$$+\{4r_{ej}a^2\tau^3[x_{p1j}(t_0)+k\tau x_{p2j}(t_0)]+[12r_{ej}^2-4r_{ej}a^2\tau^3]x_{e1j}(t_0)$$

$$+[12r_{ej}^2(k+1)\tau-4r_{ej}a^2k\tau^4]x_{e2j}(t_0)$$

$$+6(j-1)(j-2)r_{ej}^2(k+1)^2\tau^2 g\}/\eta \tag{8.1.38}$$

and

$$c_{41}(j)+d_{41}(j)/\eta = \{-6r_{ej}\tau^2 x_{e1j}(t_0)-2r_{ej}(3k+2)\tau^3 x_{e2j}(t_0)$$

$$-(j-1)(j-2)r_{ej}(k+1)(3k+1)\tau^4 g\}+\{12r_{ej}^2\tau x_{e1j}(t_0)$$

$$+12r_{ej}^2(k+1)\tau^2 x_{e2j}(t_0)+6r_{ej}^2(k+1)^2\tau^3 g(j-1)(j-2)/\eta\} \tag{8.1.39}$$

Now carrying out the matrix operation defined by (8.1.34) the corner condition costate vectors partitioned into (3x1) column vectors are expressed in terms of η :

$$\lambda_i(T_1)=\Big\{\sum_{K=1}^{4} \alpha_{ik}(j)c_{K1}(j)+\Big(\sum_{K=1}^{4}[\alpha_{ik}(j)d_{K1}(j)+\beta_{ik}(j)c_{K1}(j)]\Big)1/\eta$$

$$+\Big(\sum_{K=1}^{4}\beta_{ik}(j)d_{K1}(j)\Big)\ (1/\eta)^2\Big\}/\Big\{\tau^4\Delta_1(j)$$

$$+[\tau^4\Delta_2(j)-4r_{ej}\tau^3\Delta_1(j)]1/\eta-[4r_{ej}\tau^3\Delta_2(j)](1/\eta)^2\Big\};\ i=1,\ 4 \tag{8.1.40}$$

The j index indicates the components of $\lambda_i(T_1)$.

where
$$\begin{bmatrix} \lambda_{p1}(T_1) \\ \lambda_{p2}(T_1) \\ \lambda_{e1}(T_1) \\ \lambda_{e2}(T_1) \end{bmatrix} = \begin{bmatrix} \lambda_1(T_1) \\ \lambda_2(T_1) \\ \lambda_3(T_1) \\ \lambda_4(T_1) \end{bmatrix} \qquad (8.1.41)$$

8.1.3 Determination of the Evader's Costate Vector During the Postengagement Interval

With reference to (3.4.16) the components of the (3x1) partitioned costate vectors μ_1 and μ_2 are defined in terms of the components of the corner condition costate vectors as

$$\begin{aligned}\mu_1(j) = \{&12r_{ej}(\tau-r_{ej}/\eta)(x_{e1j}(t_0)+(k+1)\tau x_{e2j}(t_0) \\ &+[k^2(2k+3)\tau^3\lambda_{e1j}(T_1)+3k\tau^2(k+2)\lambda_{e2j}(T_1)]/6r_{ej} \\ &+(j-1)(j-2)(k+1)^2\tau^2 g/2)-6r_{ej}\tau^2(x_{e2j}(t_0) \\ &+[k^2\tau^2\lambda_{e1j}(T_1)+2k\tau\lambda_{e2j}(T_1)]/2r_{ej}+(j-1)(j-2)(k+1)\tau g)\} \\ &/[\tau^4-4r_{ej}\tau^3/\eta]\end{aligned} \qquad (8.1.42)$$

and

$$\begin{aligned}\mu_2(j)=\{&-6r_{ej}(x_{e1j}(t_0)+(k+1)\tau x_{e2j}(t_0)+[k^2(2k+3)\tau^3\lambda_{e1j}(T_1) \\ &+3k\tau^2(k+2)\lambda_{e2j}(T_1)]/6r_{ej}+(j-1)(j-2)(k+1)^2\tau^2 g/2) \\ &+4r_{ej}\tau(x_{e2j}(t_0)+[k^2\tau^2\lambda_{e1j}(T_1)+2k\tau\lambda_{e2j}(T_1)]/2r_{ej} \\ &+(j-1)(j-2)(k+1)\tau g)\}/[\tau^2-4r_{ej}\tau/\eta]\end{aligned} \qquad (8.1.43)$$

8.1.4 Generation of the Target Kinetic Energy Constraint as a Polynomial in Terms of η

With reference to (3.4.17) the target kinetic energy constraint is

$$(1/\eta)^2 \ \mu_2' \ \mu_2 = V_T^2 \tag{8.1.44}$$

Since

$$R_p = r_p[U] \tag{8.1.45}$$

and

$$R_e = r_e[U] \tag{8.1.46}$$

The coefficients of the $(1/\eta)$ terms in the denominators of the corner condition costate vectors $\Delta(j)$ become invariant with respect to the j index

$$\Delta(j) = \Delta(1); \quad j \neq 1 \tag{8.1.47}$$

If the components of $\lambda_e(T_1)$ are substituted in $\mu_2(j)$, (8.1.43) $\mu_2(j)$ has the form

$$\mu_2(j) = \frac{a_0(j)+a_1(j)/\eta+a_2(j)/\eta^2}{b_0+b_1/\eta+b_2/\eta^2+b_3/\eta^3} \tag{8.1.48}$$

Now substituting the components of $\mu_2(j)$ into (8.1.44) the resultant rational fractional polynomial in terms of $1/\eta$ is

$$\frac{1}{(b_0+b_1/\eta+b_2/\eta^2+b_3/\eta^3)^2}\sum_{j=1}^{3} 1/\eta^2 (a_0(j)+a_1(j)/\eta+a_2(j)/\eta^2)^2 = V_T^2 \tag{8.1.49}$$

Now by multiplying (8.1.49) by $(b_0\eta^3+b_1\eta^2+b_2\eta+b_3)^2$ one obtains a polynomial of the form

$$\sum_{i=1}^{6} p_i(j)\eta^i = 0 \tag{8.1.50}$$

8.1.5 Procedure for Solving the Euler-Lagrange Equations when the Intercept and Target Times Are Unknown.

With reference to (3.4.1,3.4.2) and (3.4.5) the solution of the differential endgame's Euler-Lagrange equations depends on the determination of the vector multipliers, $\lambda_p(T_1)$, $\lambda_e(T_1)$ and μ, when the intercept and target times are known. The relationships, derived in Chapter 3, that determine $\lambda_p(T_1)$, $\lambda_e(T_1)$ and μ are:

$$B(\eta)\begin{bmatrix}\lambda_p(T_1)\\ \lambda_e(T_1)\end{bmatrix} = C(\eta)\begin{bmatrix}x_p(t_0)\\ x_e(t_0)\end{bmatrix} + \ell(\eta) \qquad (8.1.51)$$

$$\mu = \Phi_e'(T_1,T_2)K(\eta)[M_e(T_1,t_0)[0\vdots U]B(\eta)^{-1}(C(\eta)\begin{bmatrix}x_p(t_0)\\ x_e(t_0)\end{bmatrix}+\ell(\eta))$$

$$+\Phi_e(T_1,t_0)x_e(t_0)+k_e(T_1,t_0)+\Phi_e(T_1,T_2)k_e(T_2,T_1)] \qquad (8.1.52)$$

$$1/\eta^2\mu'Q'Q\mu = V_T^2 \qquad (8.1.53)$$

When the intercept and target times are unknown, the additional boundary conditions needed to determine the intercept time T_1, and the target time T_2 are:

$$u(T_1) = -R_p^{-1}(T_1)G_p'(T_1)\lambda_p(T_1) \qquad (8.1.54)$$

$$v(T_1) = R_e^{-1}(T_1)G_e'(T_1)\lambda_e(T_1) \qquad (8.1.55)$$

$$v(T_2) = -R_e^{-1}(T_2)G_e'(T_2)\mu \qquad (8.1.56)$$

$$\dot{x}_p(T_1) = F_p(T_1)x_p(T_1)+G_p(T_1)u(T_1)+n_p(T_1) \qquad (8.1.57)$$

$$\dot{x}_e(T_1) = F_e(T_1)x_e(T_1)+G_e(T_1)v(T_1)+n_e(T_1) \qquad (8.1.58)$$

$$\dot{x}_e(T_2) = F_e(T_2)x_e(T_2)+G_e(T_2)v(T_2)+n_e(T_2) \qquad (8.1.59)$$

$$x_e(T_1) = \Phi_e(T_1,t_0)x_e(t_0)+M_e(T_1,t_0)\lambda_e(T_1)+k_e(T_1,t_0) \qquad (8.1.60)$$

$$x_p(T_1)=\Phi_p(T_1,t_0)x_p(t_0)-M_p(T_1,t_0)\lambda_p(T_1)+k_p(T_1,t_0) \tag{8.1.61}$$

$$x_e(T_2)=\Phi_e(T_2,T_1)x_e(T_1)-M_e(T_2,T_1)\mu+k_e(T_2,T_1) \tag{8.1.62}$$

$$1/2u(T_1)'R_p(T_1)u(T_1)+a^2[x_p(T_1)-x_e(T_1)]'A'A[\dot{x}_p(T_1)-\dot{x}_e(T_1)]=0 \tag{8.1.63}$$

$$\eta x_e(T_2)'Q'Q\dot{x}_e(T_2)-1/2v(T_2)'R_e(T_2)v(T_2)=0 \tag{8.1.64}$$

The procedure for determining the multiplier vectors, $\lambda_p(T_1)$, $\lambda_e(T_1)$ and μ, and the intercept and target times is as follows:

1) Choose a combination of discrete values for the intercept and target times

$$T_1=T_{1i}+n\Delta T_1; \quad n=1, \ldots .\ell \tag{8.1.65}$$

$$T_2=T_{2i}+n\Delta T_2; \quad n = 1,\ldots .k \tag{8.1.66}$$

where T_{1i} and T_{2i} are the initial intercept and target times; ΔT_1 and ΔT_2 are the incremental changes of the intercept and target times.

2) For any set of T_1 and T_2 values, $\lambda_p(T_1)$, $\lambda_e(T_1)$ and μ are determined by solving (8.1.51-8.1.53).

3) With the solution of (8.1.51-8.1.53) the boundary conditions defined by (8.1.54-8.1.62) are determined.

4) Substituting the boundary conditions determined by (8.1.54-8.1.62) into (8.1.63,8.1.64) it is determined whether (8.1.63,8.1.64) are satisfied simultaneously.

5) If (8.1.63,8.1.64) are not satisfied simultaneously, then T_1 and T_2 are updated. This procedure is repeated for all combinations of T_1 and T_2 values until (8.1.63-8.1.64) are satisfied.

APPENDIX 8.2

Determination of the Boundary Condition Costate Vectors in Terms of the Terminal Engagement Time and the Specific Differential Game Parameters

The solution of the following equations, derived in Section 4.3, determines the boundary costate vectors, $\lambda_p(T)$ and $\lambda_e(T)$, in terms of the terminal engagement time T.

$$\begin{bmatrix}\lambda_p(T)\\ \lambda_e(T)\end{bmatrix} = \left[\begin{array}{c|c} U+a^2A'A[M_p(T,t_0)+\Phi_p(T,t_0)G(\eta_r,\eta_v)\Phi_p'(T,t_0)] & a^2A'AM_e(T,t_0) \\ -a^2A'A[M_p(T,t_0)+\Phi_p(T,t_0)G(\eta_r,\eta_v)\Phi_p'(T,t_0)] & U-a^2A'AM_e(T,t_0)\end{array}\right]^{-1}$$

$$\times\left(\begin{bmatrix}0 & -a^2A'A\Phi_e(T,t_0)\\ 0 & a^2A'A\Phi_e(T,t_0)\end{bmatrix}\begin{bmatrix}0\\ x_e(t_0)\end{bmatrix} + \begin{bmatrix}a^2A'A(k_p-k_e)\\ a^2A'A(k_e-k_p)\end{bmatrix}\right) \quad (8.2.1)$$

$$\lambda_p'(T)\Phi_p(T,t_0)G'(\eta_r,\eta_v)A'AG(\eta_r,\eta_v)\Phi_p'(T,t_0)\lambda_p(T)=R_0^2 \quad (8.2.2)$$

$$\lambda_p'(T)\Phi_p(T,t_0)G'(\eta_r,\eta_v)Q'QG(\eta_r,\eta_v)\Phi_p'(T,t_0)\lambda_p(T)=V_0^2 \quad (8.2.3)$$

When the dynamics for both participants are specified as

$$\begin{bmatrix}\dot{x}_{k1}\\ \dot{x}_{k2}\end{bmatrix} = \begin{bmatrix}0 & U\\ 0 & 0\end{bmatrix}\begin{bmatrix}x_{k1}\\ x_{k2}\end{bmatrix} + \begin{bmatrix}0\\ U\end{bmatrix}u_k + \begin{bmatrix}0\\ \vdots\\ 0\\ g\end{bmatrix} \quad ; \; k=p,e \quad (8.2.4)$$

and the weighting matrices are specified as

$$R_k(t)=r_k[U]; \; k=p,e \quad (8.2.5)$$

the transition matrices for the participants become:

$$\Phi_k(T,t_0) = \begin{bmatrix}U & (T-t_0)U\\ 0 & U\end{bmatrix}; \quad k=p,e \quad (8.2.6)$$

and the controllability matrices for both participants become:

$$M_k(T,t_0) = \begin{bmatrix} \frac{(T-t_0)^3}{3r_k} U & \frac{(T-t_0)^2}{2r_k} U \\ \frac{(T-t_0)^2}{2r_k} U & \frac{(T-t_0)}{r_k} U \end{bmatrix}; \; k=p,e \tag{8.2.7}$$

Substituting $\Phi_k(T,t_0)$ and $M_k(T,t_0)$; $k=p,e$ into (8.2.1,8.2.2) these relationships are reduced to

$$\eta_r^{\,2} = 1/R_0^{\,2} \sum_{i=1}^{3} \lambda_{p1i}(T\ )^2 \tag{8.2.8}$$

$$\eta_v = R_0(T-t_0)\eta_r/V_0 \tag{8.2.9}$$

and the components of the costate vectors boundary conditions become:

$$\lambda_{e1i}(T\ )=a^2[x_{e1i}(t_0)+(T\ -t_0)x_{e2i}(t_0)]/[1+a^2(T\ -t_0)^3\times (1/3r_p - 1/3r_e)+a^2(1+(T\ -t_0)V_0/R_0)/\eta_r]; \quad i=1,3 \tag{8.2.10}$$

$$\lambda_{p1i}(T\ ) = \frac{a_0(i)}{b_0(i)+b_1(i)/\eta_r}\ ; \; i=1,3 \tag{8.2.11}$$

$$\lambda_{p2i}(T\ ) = \lambda_{e2i}(T_1) = 0; \; i=1,3 \tag{8.2.12}$$

where

$$a_0(i)=-a^2[x_{e1i}(t_0)+(T-t_0)x_{e2i}(t_0)]\left[1+\frac{a^2(T\ -t_0)^3}{3r_e-a^2(T\ -t_0)^3}\right] \tag{8.2.13}$$

$$b_0(i)=1+\frac{a^2(T\ -t_0)^3}{3r_p} + \frac{a^4(T\ -t_0)^6}{3r_p(3r_e-a^2(T-t_0)^3)} \tag{8.2.14}$$

$$b_1(i)=[1+\frac{(T\ -t_0)V_0}{R_0}]\ [a^2+\frac{a^4(T\ -t_0)^3}{3r_e-a^2(T\ -t_0)^3}] \tag{8.2.15}$$

By substituting (8.2.11) into (8.2.8) there results a second order polynomial in terms of η_r. The roots of this polynomial determine two initial state vectors which are symmetrical to each other with respect to the origin. One root is superfluous from a physical viewpoint.

8.2.1 Procedure for Solving the Differential Game's Euler-Lagrange Equations when the Terminal Engagement Time Is Unknown

With reference to (4.3.29-4.3.31) the solution of the differential game's Euler-Lagrange equations depends on the determination of the vector multipliers, $\lambda_p(T)$ and $\lambda_e(T)$. The relationships that determine $\lambda_p(T)$ and $\lambda_e(T)$ in terms of the terminal engagement time are (8.2.1-8.2.3).

When the terminal engagement time is unknown, the additional boundary conditions needed to determine the engagement time are:

$$u(T)=-R_p^{-1}(T)G_p'(T)\lambda_p(T) \tag{8.2.16}$$

$$v(T)=R_e^{-1}(T)G_e'(T)\lambda_e(T) \tag{8.2.17}$$

$$x_e(T)=\Phi_e(T,t_0)x_e(t_0)+M_e(T,t_0)\lambda_e(T)+k_e(T,t_0) \tag{8.2.18}$$

$$x_p(T)=\Phi_p(T,t_0)x_p(t_0)-M_p(T,t_0)\lambda_p(T)+k_p(T,t_0) \tag{8.2.19}$$

$$\dot{x}_e(T)=F_e(T)x_e(T)+G_e(T)v(T)+n_e(T) \tag{8.2.20}$$

$$\dot{x}_p(T)=F_p(T)x_p(T)+G_p(T)u(T)+n_p(T) \tag{8.2.21}$$

$$1/2[u(T)'R_p(T)u(T)-v(T)'R_e(T)v(T)]$$
$$+a^2[x_p(T)-x_e(T)]'A'A[\dot{x}_p(T)-\dot{x}_e(T)]=0 \tag{8.2.22}$$

The procedure for determining the boundary costate vectors and the terminal engagement time is as follows:

1) Choose a discrete value for the terminal engagement time:

$$T = T_i + n\Delta T; \; n = 1, \ldots k \tag{8.2.23}$$

where T_i is the initial terminal engagement time and ΔT is the incremental change in the terminal engagement time.

2) For a discrete value of T the boundary costate vectors are determined by solving (8.2.1-8.2.3).

3) With the determination of the boundary costate vectors the boundary conditions determined by (8.2.16-8.2.21) are determined.

4) Substituting the boundary conditions determined by (8.2.16-8.2.21) into (8.2.22) it is determined whether (8.2.22) is satisfied.

5) If (8.2.22) is not satisfied, then T is updated.

APPENDIX 8.3.

Determination of the Terminal Miss Distance Vector between the Cooperating Pursuer and the Evader in Terms of the Evader's State Vector

The terminal miss distance between the evader and the cooperating pursuer is determined by

$$\underset{u_{cp}}{\mathrm{Min}}\ J=\underset{u_{cp}}{\mathrm{Min}}\Big\{a_{cp}^2/2[x_{cp}(T)-x_e(T)]'A'A[x_{cp}(T)-x_e(T)]$$

$$+1/2\int_{t_0}^{T} u'_{cp}R_{cp}(t)u_{cp}dt\Big\} \qquad (8.3.1)$$

subject to the constraints

$$\dot{x}_{cp}=F_{cp}(t)x_{cp}+G_{cp}(t)u_{cp}+n_{cp} \qquad (8.3.2)$$

$$x_{cp}(t_0)=x_{cp0} \qquad (8.3.3)$$

where the state vector x_{cp} represents the position and velocity vectors of the cooperating pursuer; $Ax_e(T)$, the evader's terminal engagement position, is the cooperating pursuer's desired terminal position; the control vector u_{cp} is the cooperating pursuer's acceleration command ; n_{cp} represents any disturbance vector such as the earth's gravitational field; the 6x6 matrix $F_{cp}(t)$ and the 6x3 matrix $G_{cp}(t)$ are continuous in time; $R_{cp}(t)$ is a 3x3 positive definite matrix continuous in time; a_{cp}^2 is a weighting factor; T is the fixed terminal time.

The cooperating pursuer's performance criterion J is the square of the error norm between the cooperating pursuer's position and its desired position, weighted against the energy spent by the cooperating pursuer. The dynamics of the cooperating pursuer are defined by (8.3.2).

Adjoining the cooperating pursuer's differential constraint (8.3.2) to the performance criterion by the vector multiplier λ_{cp}, the minimization of the performance criterion is expressed as

$$\underset{u_{cp}}{\mathrm{Min}} J_c = \underset{u_{cp}}{\mathrm{Min}} \Big\{ a_{cp}{}^2/2[x_{cp}(T)-x_e(T)]'A'A[x_{cp}(T)-x_e(T)]$$

$$+ \int_{t_0}^{T} [1/2u_{cp}'R_{cp}(t)u_{cp}+\lambda_{cp}'(F_{cp}(t)x_{cp}+G_{cp}(t)u_{cp}+n_{cp}-\dot{\hat{x}}_{cp})]dt \Big\} \qquad (8.3.4)$$

The cooperating pursuer's terminal state is determined by solving the Euler-Lagrange equations that result from minimizing J_c. By the method of calculus of variations the conditions necessary for an extremum is that the first order variation of J_c vanishes

$$\delta J=[a_{cp}^2[x_{cp}(T)-x_e(T)]'A'A-\lambda_{cp}'(T)]\delta x_{cp}(T)$$

$$+ \int_{t_0}^{T} [(u_{cp}'R_{cp}(t)+\lambda_{cp}'G_{cp}(t))\delta u_{cp}$$

$$+(\dot{\lambda}_{cp}'+\lambda_{cp}'F_{cp}(t))\delta x_{cp}]dt=0 \qquad (8.3.5)$$

The Euler-Lagrange equations which result from the differential constraint (8.3.2) and the necessary conditions resulting from equating the coefficients of the first order variationals to zero are:

$$\begin{bmatrix} \dot{x}_{cp} \\ \dot{\lambda}_{cp} \end{bmatrix} = \begin{bmatrix} F_{cp}(t) & -G_{cp}(t)R_{cp}^{-1}(t)G'_{cp}(t) \\ 0 & -F'_{cp}(t) \end{bmatrix} \begin{bmatrix} x_{cp} \\ \lambda_{cp} \end{bmatrix} + \begin{bmatrix} n_{cp} \\ 0 \end{bmatrix} \quad (8.3.6)$$

subject to the boundary conditions

$$x_{cp}(t_0)=x_{cp0} \quad (8.3.7)$$

$$\lambda_{cp}(T)=a_{cp}^2A'A[x_{cp}(T)-x_e(T)] \quad (8.3.8)$$

The solution for x_{cp} defined at T is

$$x_{cp}(T)=\Phi_{cp}(T,t_0)x_{cp0}-M_{cp}(T,t_0)\lambda_{cp}(T)+k_{cp}(T,t_0) \quad (8.3.9)$$

where $\Phi_{cp}(T,t_0)$ is the transition matrix; $M_{cp}(T,t_0)$, the controllability matrix, is defined as

$$M_{cp}(T,t_0)= \int_{t_0}^{T}\Phi_{cp}(T,t)G_{cp}(t)R_{cp}^{-1}(t)G'_{cp}(t)\Phi'_{cp}(T,t)dt \quad (8.3.10)$$

and the column vector $k_{cp}(T,t_0)$ due to the disturbance vector n_{cp} is

$$k_{cp}(T,t_0)= \int_{t_0}^{T}\Phi_{cp}(T,t)n_{cp}dt \quad (8.3.11)$$

Substituting for $\lambda_{cp}(T)$, defined by the boundary condition, (8.3.8) into (8.3.9)

$$x_{cp}(T)=\Phi_{cp}(T,t_0)x_{cp0}-a_{cp}^2M_{cp}(T,t_0)A'A[x_{cp}(T)-x_e(T)]$$
$$+k_{cp}(T,t_0) \quad (8.3.12)$$

Subtracting $x_e(T)$ from both sides of (8.3.12) and solving for the error vector $[x_{cp}(T)-x_e(T)]$

$$[x_{cp}(T)-x_e(T)]=[U+a_{cp}^2M_{cp}(T,t_0)A'A]^{-1}\times$$
$$[\Phi_{cp}(T,t_0)x_{cp0}+k_{cp}(T,t_0)-x_e(T)] \quad (8.3.13)$$

The terminal miss distance vector between the cooperating pursuer and the evader is obtained by premultiplying $[x_{cp}(T)-x_e(T)]$ by the 3x6 matrix A.

APPENDIX 8.4

Determination of the Boundary Costate Vectors in Terms of the Specific Parameters for the Differential Game with the Cooperating Pursuer

The solution of the following equations derived in Section 5.4 determines the boundary costate vectors, $\lambda_p(T)$ and $\lambda_e(T)$.

$$\begin{bmatrix} U+a^2A'AM_p(T,t_0) & a^2A'AM_e(T,t_0) \\ -a^2A'AM_p(T,t_0) & U-[a^2A'A+\gamma K_1'K_1]M_e(T,t_0) \end{bmatrix} \begin{bmatrix} \lambda_p(T) \\ \lambda_e(T) \end{bmatrix}$$

$$= \begin{bmatrix} a^2A'A\Phi_p(T,t_0) & -a^2A'A\Phi_e(T,t_0) \\ -a^2A'A\Phi_p(T,t_0) & [a^2A'A+\gamma K_1'K_1]\Phi_e(T,t_0) \end{bmatrix} \begin{bmatrix} x_e(t_0) \\ x_p(t_0) \end{bmatrix}$$

$$+ \begin{bmatrix} a^2A'A & -a^2A'A \\ -a^2A'A & a^2A'A+\gamma K_1'K_1 \end{bmatrix} \begin{bmatrix} k_p(T,t_0) \\ k_e(T,t_0) \end{bmatrix} -\gamma \begin{bmatrix} 0 & 0 \\ K_1'K_1\Phi_{cp}(T,t_0) & K_1'K_1 \end{bmatrix} \times$$

$$\begin{bmatrix} x_{cp}(t_0) \\ k_{cp}(T,t_0) \end{bmatrix} \tag{8.4.1}$$

$$[K_2' - \lambda_e'(T)M_e'(T,t_0)]K_1'K_1[K_2 - M_e(T,t_0)\lambda_e(T)]$$

$$= \|z_m\|^2 \tag{8.4.2}$$

where

$$K_1 = A[U+a_{cp}^2M_{cp}(T,t_0)A'A]^{-1} \tag{8.4.3}$$

and

$$K_2 = \Phi_{cp}(T,t_0)x_{cp0}+k_{cp}(T,t_0)-\Phi_e(T,t_0)x_{e0}-k_e(T,t_0) \tag{8.4.4}$$

When the dynamics for the participants are specified as

$$\begin{bmatrix} \dot{x}_{k1} \\ \dot{x}_{k2} \end{bmatrix} = \begin{bmatrix} 0 & U \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_{k1} \\ x_{k2} \end{bmatrix} + \begin{bmatrix} 0 \\ U \end{bmatrix} u_k + \begin{bmatrix} 0 \\ \vdots \\ 0 \\ g \end{bmatrix}; \quad k=p,cp,e \tag{8.4.5}$$

and the weighting matrices are specified as

$$R_k(t)=r_k[U];\ k=p,cp,e \tag{8.4.6}$$

the transition matrices for the participants become

$$\Phi_k(T,t_0)=\begin{bmatrix} U & (T-t_0)U \\ 0 & U \end{bmatrix};\ k=p,cp,e \tag{8.4.7}$$

and the controllability matrices for the participants become

$$M_k(T,t_0)=\begin{bmatrix} \frac{(T-t_0)^3}{3r_k}U & \frac{(T-t_0)^2}{2r_k}U \\ \frac{(T-t_0)^2}{2r_k}U & \frac{(T-t_0)}{r_k}U \end{bmatrix};\ k=p,cp,e \tag{8.4.8}$$

Substituting $\Phi_k(T,t_0)$ and $M_k(T,t_0)$; $k=p,cp,e$ into (8.4.1-8.4.4) the boundary costate vectors are expressed in terms of the unknown scalar Lagrangian multiplier, γ, as

$$\lambda_{pj}(T) = \frac{a_{pj}+b_{pj}/\gamma}{c_{p1}+c_{p2}/\gamma};\ j=1,2,3 \tag{8.4.9}$$

$$\lambda_{pj}(T) = 0;\ j=4,5,6 \tag{8.4.10}$$

$$\lambda_{ej}(T) = \frac{a_{ej}+b_{ej}/\gamma}{c_{e1}+c_{e2}/\gamma};\ j=1,2,3 \tag{8.4.11}$$

$$\lambda_{ej}(T) = 0;\ j=4,5,6 \tag{8.4.12}$$

where

$$a_{pj}=a^2[x_{pj}(t_0)-x_{ej}(t_0)+(T-t_0)(x_{p\overline{j+3}}(t_0)-x_{e\overline{j+3}}(t_0)];\quad j=1,2,3 \tag{8.4.13}$$

$$b_{pj}=a_{pj}\left[\frac{3r_{cp}}{3r_{cp}+a^2(T-t_0)^3}\right]^2\times\left[\frac{(T-t_0)^3}{3r_e}\right]+\left[\frac{a^2(T-t_0)^3}{3r_e}\right]\times \left[\frac{3r_{cp}}{3r_{cp}+a^2(T-t_0)^3}\right]^2\times[x_{ej}(t_0)-x_{cpj}(t_0)+(T-t_0)(x_{e\overline{j+3}}(t_0)-x_{cp\overline{j+3}}(t_0))]\quad j=1,2,3 \tag{8.4.14}$$

$$c_{p1}=1+a^2(T-t_0)^3\times[1/r_p-1/r_e]/3 \tag{8.4.15}$$

$$c_{p2}=\left[\frac{3r_{cp}}{3r_{cp}+a^2(T-t_0)^3}\right]^2\times\left[\frac{3r_p+a^2(T-t_0)^3}{3r_p}\right]\times\frac{(T-t_0)^3}{3r_e} \tag{8.4.16}$$

$$a_{ej}=a_{pj}\left[\frac{a^2(T-t_0)^3}{3r_p}\Big/\frac{3r_p+a^2(T-t_0)^3}{3r_p}-1\right]\ ;\ j=1,2,3 \tag{8.4.17}$$

$$b_{ej}=\left[\frac{3r_{cp}}{3r_{cp}+a^2(T-t_0)^3}\right]^2\times[x_{cpj}(t_0)-x_{ej}(t_0)+(T-t_0)\times$$
$$(x_{c\overline{pj+3}}(t_0)-x_{\overline{ej+3}}(t_0))];\ j=1,2,3 \tag{8.4.18}$$

$$c_{e1}=\frac{3r_e-a^2(T-t_0)^3}{3r_e}+\frac{3r_p}{3r_p+a^2(T-t_0)^3}\times\frac{a^4(T-t_0)^6}{9r_er_p} \tag{8.4.19}$$

$$c_{e2}=\left[\frac{3r_{cp}}{3r_{cp}+a^2(T-t_0)^3}\right]^2\times\frac{(T-t_0)^3}{3r_e} \tag{8.4.20}$$

The unknown scalar multiplier γ is determined by substituting $\lambda_e(T)$ into (8.4.2) and solving for γ. For the dynamics and weighting matrices specified by (8.4.5,8.4.6) the evader's quadratic terminal miss distance constraint with the cooperating pursuer is exactly expressed in terms of γ as a second order polynomial. The root which yields the minimum change of control energy for the evader due to the cooperating pursuer is sought. The minimum change of control energy is determined by comparing the evader's control energy for the differential game with the cooperating pursure to the evader's control energy for the game played without the cooperating pursuer (6).

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13. ABSTRACT

Differential game theory is applied to several classes of pursuit-evasion problems. For these differential games the dynamics of the participants are described by linear nonstationary differential equations.

One class of differential games that was formulated and studied is the differential game, where the evader has to out maneuver a pursuer, if it is to strike the target that the pursuer is definding. This differential game will be called the differential endgame.

The differential endgame's payoff functional is the square of the terminal engagement miss distance weighted against the difference of the participants' control energies, spent during their respective flight times. The evader's target constraints are the position coordinates of the target and the evader's kinetic energy as it strikes the target.

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