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ANALYSIS OF THE POLARIZATION SCATTERING MATRIX

W. T. Payne

DECEMBER 1968

Work Performed for

ADVANCED RESEARCH PROJECTS AGENCY
Contract Administered by
DEVELOPMENT ENGINEERING DIVISION
DIRECTORATE OF PLANNING AND TECHNOLOGY
ELECTRONIC SYSTEMS DIVISION
AIR FORCE SYSTEMS COMMAND
UNITED STATES AIR FORCE
L. G. Hanscom Field, Bedford, Massachusetts

Sponsored by
Advanced Research Projects Agency
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Project 8051
Prepared by
THE MITRE CORPORATION
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FOREWORD

The work reported in this document was performed by The MITRE Corporation, Bedford, Massachusetts, for Advanced Research Projects Agency; the contract, AF 19(628)-5165, was monitored by the Directorate of Planning and Technology, Electronic Systems Division, Air Force Systems Command.

REVIEW AND APPROVAL

Publication of this technical report does not constitute Air Force approval of the report's findings or conclusions. It is published only for the exchange and stimulation of ideas.

A. P. TRUNFIO
Project Officer
Development Engineering Division
Directorate of Planning & Technology

ABSTRACT

It is shown how, by carrying out a sequence of three coordinate axis rotations in Poincaré space, one can calculate the principal basis and the real, nonnegative eigenvalues of any symmetric polarization scattering matrix. Then the two eigenvalues and the three Eulerian angles of the principal axes in Poincaré space constitute a complete set of pure scatterer parameters. A scatterer classification scheme based on these parameters is constructed, with the help of a geometrical representation, in Poincaré space, of polarization transformations. The procedure is applied to several simple scattering configurations. Some conclusions are reached concerning the scatterer geometry in various cases.

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1.0 INTRODUCTION

Efforts to make the polarization scattering matrix more useful for scatterer identification have usually been directed toward extracting from it a set of pure object parameters, i.e. quantities determined solely by the scattering object and independent of what polarizations are transmitted and of what polarization basis is used. If such a set of parameters can be found, then one would hope that they would yield information about the geometry of the scattering object, or at least that they could be used as a basis for classifying scattering objects into a number of characteristic types for purposes of recognition. There have been at least two different approaches to the problem of finding pure object parameters. One of these involves the null polarizations of the scattering matrix[1,2]; but the possibilities appear to be quite limited. The other seeks to find the eigenpolarizations and eigenvalues of the scattering matrix. It has long been known how to do this for an object having bilateral symmetry with respect to a plane containing the line of sight, by means of a simple axis rotation about the line of sight[3]. Recently a procedure was given by Bickel[4] for diagonalizing any general symmetric scattering matrix by means of two successive orthogonal changes of basis, one of which is a rotation about the line of sight, and the other an ellipticity change. The eigenvalues that Bickel obtains for the scattering matrix are in general complex. The real parameters of these and of the basis transformations constitute a complete set of pure object parameters.

The present paper is to some extent equivalent to Bickel's work[4], but goes beyond it in several respects. By means of a third change of basis, the scattering matrix is reduced to real, nonnegative diagonal form; and this makes it possible to take over some concepts from tensor analysis, particularly the concept of principal basis and the concept of degeneracy, which turns out to be quite useful in setting up an object classification scheme and interpreting it. Also the present paper, unlike Bickel's, devotes a good deal of attention to the geometrical representation of polarization transformations, this representation being expressed in terms of rotations, reflections, and other transformations in Poincaré space, i.e. the three dimensional space in which the Poincaré sphere[5] is embedded. The geometrical representation is quite helpful in constructing the object classification scheme, as well as in giving insight into the analytical operations.

The basis finally arrived at by means of the three basis changes mentioned above (i.e. the basis in which the scattering matrix is in real nonnegative diagonal form) will be called the principal basis of the scattering matrix, in analogy with the principal axis system of a tensor. An explicit procedure will be given (Section 5) for calculating the principal basis of any symmetric scattering matrix. This procedure yields three real parameters specifying the principal basis: they are just the Eulerian angles, in Poincaré space, of the principal basis axis system relative to the fundamental Poincaré axes (which will be defined in Section 2). The three Eulerian angles of the principal basis and the two real eigenvalues of the scattering matrix make up the set of pure object parameters that will be used in this paper. It will be shown how their values are related to geometry in Poincaré space (Section 6) and also to the object geometry, and furthermore how the ranges of their values lead naturally to a classification scheme for all scattering surfaces satisfying the reciprocity condition. The procedure will be tried out on a number of simple examples (Section 8).

It is assumed that a monochromatic, monostatic radar system is used, and that the same two orthogonal polarizations are received as are transmitted. Only actual scattering effects are considered; effects due to the propagation medium (such as Faraday rotation) are supposed to have been already removed from the scattering matrix. Absolute phase will not be kept track of; a change in the absolute phase of the complex electric field components or of the scattering matrix will not be considered to have any significance.

2.0 REPRESENTATION OF POLARIZATIONS AND BASES IN POINCARÉ SPACE

We shall make considerable use of the Poincaré sphere, laid out in the usual way[5], as shown in Figure 1. We shall use not only the spherical surface but also the entire three dimensional space in which the Poincaré sphere is embedded. This will be called Poincaré space. It is not the same as physical space.

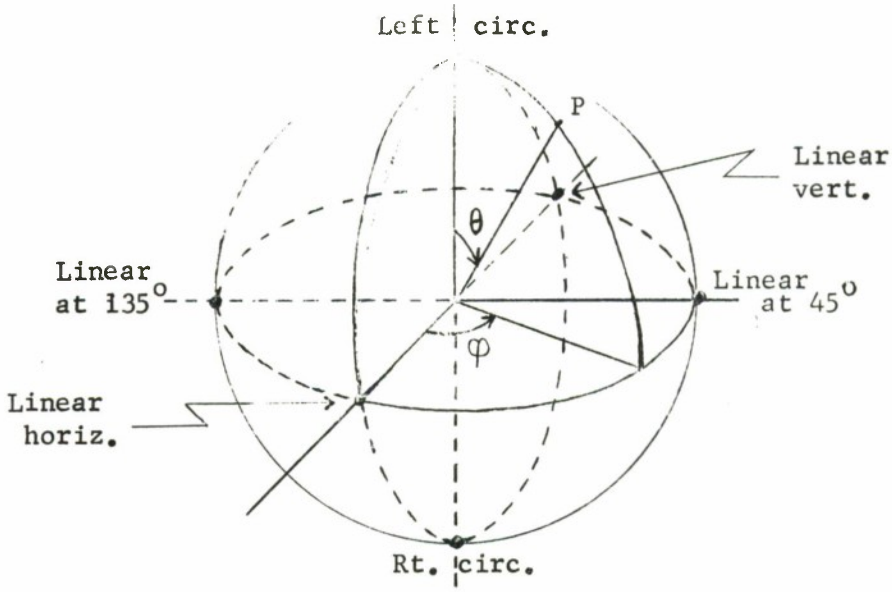

Figure 1 Poincaré sphere

The coordinate axes shown in Figure 1 (i.e. the three axes whose positive senses represent linear horizontal polarization, linear at 45° to horizontal, and left circular) will be called the fundamental Poincaré axes. The angles θ and φ will be used to denote the polar and azimuthal angles of a given direction (OP) in Poincaré space, referred to the fundamental Poincaré axes. Thus a given polarization can always be specified by the values of its θ and its φ. The angle θ is related to the ellipticity angle α by the equation,

$$\theta = \frac{\pi}{2} - 2\alpha$$

as seen by comparing Figure 1 with Reference 5, Figure 6(b). The angle φ is equal to double the azimuthal angle, in physical space, of the polarization ellipse major axis, measured from the positive horizontal direction.

Besides the fundamental Poincaré axes of Figure 1, it will be necessary to use other axis systems having the same origin but differently oriented. Such a system will be denoted by P_1, P_2, P_3. Its orientation in Poincaré space shall correspond to the polarization basis that is being used, as illustrated in Figure 2. To specify a polarization basis, two items must be decided on: first, the two orthogonal basis polarizations that are to be used, and second, a convention establishing what the phase relationship between these two polarizations is to be. Let us take the coordinate axis orientation to be related to the basis as follows. First, the positive and negative P_3 axes shall be in the directions (in Poincaré space) of the two orthogonal basis polarizations; then the complex components of a polarization in the + P_3 direction will be proportional to (1, 0), and those of a polarization in the - P_3 direction will be proportional to (0, 1). Second, the phase convention and the direction of the P_1 axis shall be such that a polarization in the + P_1 direction will always have components proportional to (1, 1), and one in the $-P_1$ direction will have components proportional to (1, -1). Then for polarizations in the $\pm P_2$ directions, the complex components come out to be proportional to (1, -i) and to (1, i) respectively. A careful study shows that all these rules are consistent. Figure 2 shows the axis orientations for two commonly used bases.

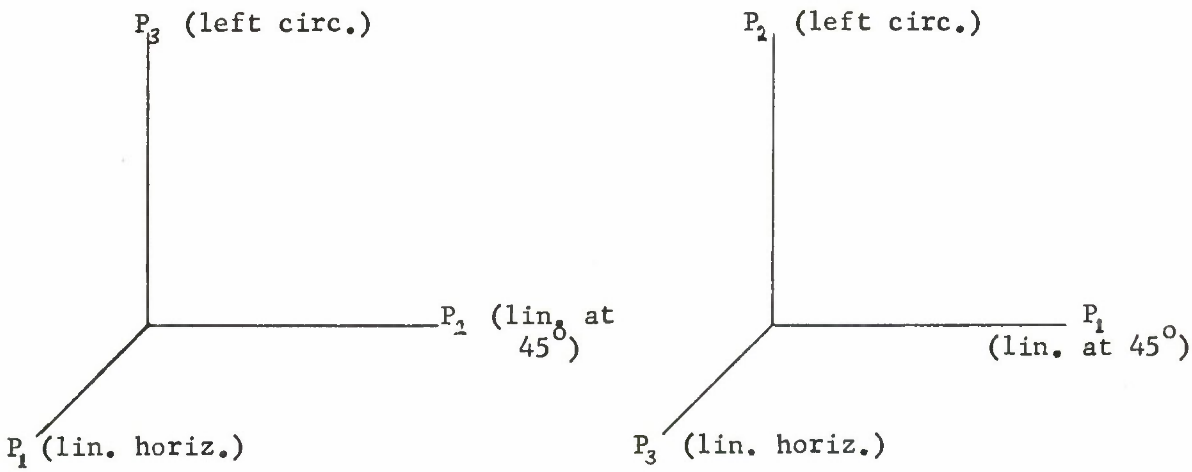

Figure 2 (a) A circular basis. (b) A linear basis

It should be emphasized that the angles θ and φ for a given polarization state will always be referred to the fundamental Poincaré axes of Figure 1; whereas the values of the complex electric components of a given polarization state will depend not only on the polarization state but also on the choice of basis used.

Let us now restrict our attention to the circular basis of Figure 2 (a). In this basis the simplest polarizations, with amplitudes normalized to unity, have components as follows, to within an arbitrary phase factor, according to the conventions already adopted.

$$\left.\begin{array}{ll} \text{Left circular, } (1,\ 0), & \text{Right circular, } (0,\ 1) \\ \text{Linear Horizontal, } \frac{1}{\sqrt{2}}\ (1,\ 1), & \text{Linear Vertical, } \frac{1}{\sqrt{2}}\ (1,\ -1) \\ \text{Linear at } 45^{\circ}, \frac{1}{\sqrt{2}}\ (1,\ -i), & \text{Linear at } 135^{\circ}, \frac{1}{\sqrt{2}}\ (1,\ i) \end{array}\right\} \qquad (1)$$

For a general polarization state specified by spherical angles θ and φ, the complex electric field components, referred to the basis of Figure 2 (a), are proportional to

$$\cos\frac{1}{2}\theta\ e^{i\varphi/2} \text{ and } \sin\frac{1}{2}\theta\ e^{-i\varphi/2} \qquad (2)$$

as follows from the theory of the Poincaré sphere[1,5]. The quantities (2) are to be multiplied by the real amplitude $|E|$ of the electric field, and they can also be multiplied by an arbitrary phasor $e^{\frac{1}{2}i\psi}$, which expresses the absolute phase. Thus,

$$\left.\begin{aligned} E_1 &= |E|\ \cos\frac{1}{2}\theta\ e^{i\ (\varphi + \psi)/2} \\ E_2 &= |E|\ \sin\frac{1}{2}\theta\ e^{i\ (\psi - \varphi)/2} \end{aligned}\right\}, \qquad (3)$$

where the subscripts 1 and 2 refer to the left circular and right circular components respectively. The quantities (3) are just the components of a spinor[6] in Poincaré space, having Eulerian angles (θ, φ, ψ). Therefore all the machinery of spinor theory can be taken over and used on the present problem. For example, if the coordinate axes in Poincaré space are rotated, the components of a spinor undergo a linear transformation with a unitary matrix U. For an axis rotation about the P_3 axis through an angle φ_r,

$$U = \begin{pmatrix} e^{-i\varphi_r/2} & 0 \\ 0 & e^{i\varphi_r/2} \end{pmatrix} \tag{4}$$

For an axis rotation about the P_2 axis through an angle θ_r,

$$U = \begin{pmatrix} \cos\frac{1}{2}\theta_r & \sin\frac{1}{2}\theta_r \\ -\sin\frac{1}{2}\theta_r & \cos\frac{1}{2}\theta_r \end{pmatrix} \tag{5}$$

For derivations see Reference 6, but note that the matrices (4) and (5) are the inverses of those given there. This is because the rotations being considered here are coordinate axis rotations, whereas those in Reference 6 were rotations of spinors with the coordinate axes held fixed.

The matrices (4) and (5) can also be derived from the familiar real orthogonal transformations of solid analytic geometry, without explicit use of spinor theory[7].

It has been seen in the foregoing that, in Poincaré space, any direction from the origin specifies a particular polarization state. Now what does distance from the origin signify? If the complex electric components (3) of a polarized wave are given, then the distance from the origin to the point of Poincaré space that is specified by the components (3) is equal to

$$\frac{1}{2}\left(E_1 E_1^{*} + E_2 E_2^{*}\right) , \tag{6}$$

from spinor theory[6]. But the expression (6) is proportional to the power density of the electromagnetic wave. Thus, distance from the origin in Poincaré space measures power density. Poincaré spheres of different radii represent electromagnetic waves of different power densities. A transformation of Poincaré space that distorts the family of Poincaré spheres into a family of nonspherical surfaces corresponds to a target that changes the power densities of differently polarized waves by different factors. Transformations with this property will be encountered in Section 6.

It is also possible to give the significance of the rectangular coordinates P_1, P_2, and P_3, of a point in Poincaré space. These are related to the complex electric components (3) by the equation[6],

$$\left.\begin{aligned} P_1 &= \frac{1}{2}\,(E_1\,E_2^{*} + E_2\,E_1^{*}) \\ P_2 &= -\frac{1}{2}\,i\,(E_1\,E_2^{*} - E_2\,E_1^{*}) \\ P_3 &= \frac{1}{2}\,(E_1\,E_1^{*} - E_2\,E_2^{*}) \end{aligned}\right\} \qquad (7)$$

These quadratic expressions, together with (6), are the (monochromatic) Stokes parameters[8] of the electromagnetic wave.

3.0 TRANSFORMATION OF POLARIZATIONS BY A SCATTERER

It is usually assumed that, on reflection from a scatterer, the complex electric components undergo an ordinary linear transformation, just as would be the case for transmission across a boundary without change in the propagation direction. The matrix of the linear transformation is the scattering matrix for the particular object. Now the use of an ordinary linear transformation for a reflection leads to difficulties, because it implies that an object whose scattering matrix is the identity matrix would leave every polarization completely unchanged, whereas it is known that every real object causes changes in at least some polarizations. For example, a large plane surface normal to the line of sight changes a right circular or right elliptic polarization into the corresponding left-handed polarization and vice versa; in other words, it reverses the phase difference between the horizontal and vertical complex electric components, as judged by the observer. Similarly, a phase difference reversal of some kind turns out to occur on reflection from all other objects, and this fact ought to be expressed in some way in the transformation equations, if these equations are to give an accurate account of what actually occurs.

The following formulation is designed to satisfy this requirement; and, in addition, it leads to a complete geometrical representation, in Poincaré space, of the effects of the object on the complex electric components, as will be shown in Section 6. We assume that, on reflection, the complex electric components undergo not an ordinary linear transformation, but rather a linear conjugate transformation of the form

$$\begin{aligned} E_{r1} &= S_{11} E_{t1}^{*} + S_{12} E_{t2}^{*} \\ E_{r2} &= S_{21} E_{t1}^{*} + S_{22} E_{t2}^{*} \end{aligned} \quad , \qquad (8)$$

where the subscripts "t" and "r" stand for "transmitted" and "reflected" and the subscripts 1 and 2 designate the components in whatever basis is being used. The conjugation of E_{t1} and E_{t2} on the right side of (8) takes account of the phase difference reversal mentioned in the preceding paragraph. The matrix $\begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix}$ is the scattering matrix.

With this formulation, it is necessary to adopt two conventions. First, the numerical specification of the different polarization states shall be exactly the same for the reflected wave as for the transmitted wave. For example, if the basis of Fig. 2(a) is used, the complex components of both waves shall have their values determined according to formulas (1). Second, the transformation of the complex electric components under a change of basis shall be exactly the same for the reflected wave as for the transmitted wave and shall follow the spinor rules based on the transformations (4) and (5).

Eqs. (8) can, of course, be forced into the form of an ordinary linear transformation by changing the terminology, e.g. by defining the components of the reflected field to be the complex conjugates of the quantities that we are calling E_{r1} and E_{r2}. Then a corresponding change must be made in the definition of the scattering matrix elements; and also the two conventions of the preceding paragraph must be suitably altered. Using a scheme equivalent to this, Graves[9] succeeded in deriving the correct transformation rule for the behavior of the scattering matrix under a change of basis, i.e. the congruent transformation (18). However, Graves' treatment is confusing.

It should be noted that, in ordinary practical radar work, it makes no difference whether one uses the transformation (8) or an ordinary linear transformation without conjugation, because ordinarily the basis used is selected to match the polarizations actually transmitted and received, so that E_{t1} and E_{t2} would always be proportional either to (1, 0) or to (0, 1), and hence would not be changed by conjugation.

The distinctive feature of Equations (8) is the conjugation; and in Graves' theory[9] also a conjugation is implicit, namely in the relationship between the transformations of the incident and reflected components under a change of basis. The mathematical reason why a conjugation inevitably turns up, in one place or another, is that the reflection of the radio waves by the object in physical space induces a reflection of the points of Poincaré space in one of the diametral planes (as will be shown), and this must be expressed in some way in the theory. Now in dealing with complex components a reflection is expressed by conjugation; thus, in expressions (2), conjugation reverses the sense of the angle φ, and this constitutes a reflection of all points in the P_1 P_2 plane of Figure 2 (a). A reflection in any other diametral plane can be accomplished by a combination of this reflection and a suitably chosen rotation.

As an illustration of the above statement that the reflection of the radio waves in physical space induces a reflection of the points of Poincaré space, consider again the large plane scattering surface normal to the line of sight. Aside from absolute phase, this object surface returns all linear polarizations unchanged, and all circular and elliptic polarizations are returned unchanged except for a reversal of rotational sense. But this is exactly the same as saying that all the points in the Poincaré space of Figure 1 have gotten reflected in the equatorial plane, or plane of linear polarizations.

Following are some elementary scattering matrices, in the basis of Figure 2 (a).

Large plane normal to line of sight $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ (9)

Large right angle corner with edge along P_1 axis $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ (10)

Horizontal straight wire $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ (11)

Straight wire at angle λ to horizontal $\begin{pmatrix} e^{2i\lambda} & 1 \\ 1 & e^{-2i\lambda} \end{pmatrix}$ (12)

Each of these can be verified by substituting it into Equation (8) and taking for E_{t1} and E_{t2} the numerical values, given in formulas (1), for each of the different polarizations in turn; it will be found that empirically correct results will be obtained. Note, however, that if an ordinary linear transformation without conjugation is used, incorrect conclusions will be reached for some polarizations. For example, if the scattering matrix (9) for a large plane normal to the line of sight is applied to each of the sets of components (1), without conjugation, one arrives at the incorrect conclusion that the 45° linear and the 135° linear are interchanged.

4.0 TRANSFORMATION OF SCATTERING MATRIX WITH CHANGE OF BASIS

A change of basis consists of a rotation of the coordinate axes in Poincaré space, as already noted in Section 2. It was also noted there that such a rotation transforms the complex electric components by a unitary transformation; let its matrix be called U. Let primes refer to the new basis. Then in matrix notation,

$$E_r = S\, E_t^{*} \text{ (in old basis)} \tag{13}$$

$$\text{and } E_r' = S'\, E_t'^{*} \text{ (in new basis)} \tag{14}$$

Also,

$$E_t' = UE_t \text{ and } E_r' = UE_r \, . \tag{15}$$

From Equations (14) and (15), it follows that

$$UE_r = S' \; (UE_t)^{*} \; . \tag{16}$$

From Equation (13),

$$UE_r = USE_t^*$$

$$= US(U^*)^{-1} U^* E_t^*$$

$$= US(U^*)^{-1} (UE_t)^* \quad . \qquad (17)$$

From Equations (16) and (17),

$$S' = US(U^*)^{-1} \quad ,$$

and therefore,

$$S' = USU^T \quad , \qquad (18)$$

where U^T is the transpose of U and is equal, since U is unitary, to $(U^{-1})^*$.

Equation (18) is the desired transformation for the scattering matrix. It is called a congruent transformation. As already mentioned at the beginning of Section 3, this transformation has been derived by Graves[9] in a different way.

The congruent transformation (18) will now be applied to the problem of calculating the principal basis of any symmetric scattering matrix.

5.0 CALCULATION OF PRINCIPAL BASIS AND EIGENVALUES OF THE SCATTERING MATRIX

This will be done by means of three rotations of the coordinate axes in Poincaré space, starting from the circular basis of Figure 2 (a). The coordinate axes after the first, second, and third rotations will be denoted by single, double, and triple primes respectively. The three rotations will be, respectively, a rotation about the P_3 axis through an angle φ_r, one about the $P_2{}'$ axis through an angle θ_r, and finally one about the $P_3{}''$ axis through an angle ψ_r. The three angles φ_r, θ_r, ψ_r, are yet to be determined.

5.1 Rotation about P_3 through φ_r

Let the original scattering matrix be written in polar form:

$$\begin{pmatrix} ke^{i\kappa} & \ell e^{i\lambda} \\ \ell e^{i\lambda} & me^{i\mu} \end{pmatrix}$$

Then from formulas (18) and (4),

$$\begin{pmatrix} S_{11}{}' & S_{12}{}' \\ S_{21}{}' & S_{22}{}' \end{pmatrix} = \begin{pmatrix} e^{-i\varphi_r/2} & 0 \\ 0 & e^{i\varphi_r/2} \end{pmatrix} \begin{pmatrix} ke^{i\kappa} & \ell e^{i\lambda} \\ \ell e^{i\lambda} & me^{i\mu} \end{pmatrix} \begin{pmatrix} e^{-i\varphi_r/2} & 0 \\ 0 & e^{i\varphi_r/2} \end{pmatrix}$$

$$= \begin{pmatrix} ke^{i(\kappa-\varphi_r)} & \ell e^{i\lambda} \\ \ell e^{i\lambda} & me^{i(\mu+\varphi_r)} \end{pmatrix} . \tag{19}$$

For reasons that will appear in the next rotation, it is necessary to determine φ_r so that the quantity $(S_{11}' - S_{22}')/S_{12}'$ will be real. This requires that

$$\mathrm{Im}\left[\left(ke^{i(\kappa-\varphi_r)} - me^{i(\mu+\varphi_r)}\right)/\ell e^{i\lambda}\right] = 0,$$

or:

$$k \sin(\kappa-\lambda) \cos\varphi_r - k\cos(\kappa-\lambda)\sin\varphi_r$$

$$= m\sin(\mu-\lambda)\cos\varphi_r + m\cos(\mu-\lambda)\sin\varphi_r \quad ,$$

or:

$$\tan\varphi_r = \frac{k\sin(\kappa-\lambda) - m\sin(\mu-\lambda)}{k\cos(\kappa-\lambda) + m\cos(\mu-\lambda)} \quad ,$$

or:

$$\tan\varphi_r = \frac{\mathrm{Im}[(ke^{i\kappa} - me^{i\mu})/e^{i\lambda}]}{\mathrm{Re}[(ke^{i\kappa} + me^{i\mu})/e^{i\lambda}]} \quad ,$$

or:

$$\tan\varphi_r = \frac{\mathrm{Im}[(S_{11} - S_{22})/S_{12}]}{\mathrm{Re}[(S_{11} + S_{22})/S_{12}]} \quad . \tag{20}$$

This to be solved for φ_r and then S_{11}' and S_{22}' can be calculated by Equation (19). Note that $S_{12}' = S_{12}$. There is no loss of generality in restricting φ_r to the interval $0 \leqq \varphi_r \leqq 180^o$, and this should be done to avoid ambiguity.

5.2 Rotation about P_2' through θ_r

The effect of this rotation on the scattering matrix S' is calculated by using formulas (18) and (5), and it can be shown that the transformed scattering matrix S" will be in diagonal form if the rotation angle θ_r is suitably chosen. However, it is more efficient to write the congruent transformation (18) the other way round, i.e. to show S' equal to a congruent transformation of S" (which is assumed to be diagonal), the transforming matrix being the inverse of (5). Thus,

$$\begin{pmatrix} S_{11}' & S_{12}' \\ S_{21}' & S_{22}' \end{pmatrix} = \begin{pmatrix} \cos\frac{1}{2}\theta_r & -\sin\frac{1}{2}\theta_r \\ \sin\frac{1}{2}\theta_r & \cos\frac{1}{2}\theta_r \end{pmatrix} \begin{pmatrix} S_{11}'' & 0 \\ 0 & S_{22}'' \end{pmatrix} \begin{pmatrix} \cos\frac{1}{2}\theta_r & \sin\frac{1}{2}\theta_r \\ -\sin\frac{1}{2}\theta_r & \cos\frac{1}{2}\theta_r \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} (S_{11}''+S_{22}'') + (S_{11}''-S_{22}'')\cos\theta_r & (S_{11}''-S_{22}'')\sin\theta_r \\ (S_{11}''-S_{22}'')\sin\theta_r & (S_{11}''+S_{22}'')-(S_{11}''-S_{22}'')\cos\theta_r \end{pmatrix}.$$

Therefore

$$S_{11}' + S_{22}' = S_{11}'' + S_{22}'' \tag{21}$$

$$S_{11}' - S_{22}' = (S_{11}'' - S_{22}'')\cos\theta_r \tag{22}$$

$$S_{12}' = \frac{1}{2}(S_{11}'' - S_{22}'')\sin\theta_r \quad . \tag{23}$$

From Equations (22) and (23), it follows that

$$\cot \theta_r = (S_{11}' - S_{22}')/2S_{12}' \quad . \tag{24}$$

This is the condition that θ_r must satisfy in order for S" to be diagonal. Equation (24) can be satisfied by a real θ_r because the right side of Equation (24) is guaranteed to be real by Equation (20). Equation (24) is to be solved for θ_r, with $0 \leqq \theta_r \leqq 180^\circ$; and then S_{11}'' and S_{22}'' can be calculated as follows.

From Equations (21) and (23),

$$S_{11}'' = \frac{1}{2} \; (S_{11}' + S_{22}') + (S_{12}'/\sin \theta_r) \tag{25}$$

$$S_{22}'' = \frac{1}{2} \; (S_{11}' + S_{22}') - (S_{12}'/\sin \theta_r) \quad . \tag{26}$$

But $\sin \theta_r$ can be expressed in terms of S_{11}', S_{22}', and S_{12}' by using Equation (24), and thus Equations (25) and (26) can be written in the form,

$$S_{11}'' = \frac{1}{2} (S_{11}' + S_{22}') + S_{12}' \left[1 + \left(\frac{S_{11}'-S_{22}'}{2S_{12}'}\right)^2 \right]^{\frac{1}{2}} \tag{27}$$

$$S_{22}' = \frac{1}{2} (S_{11}' + S_{22}') - S_{12}' \left[1 + \left(\frac{S_{11}'-S_{22}'}{2S_{12}'}\right)^2 \right]^{\frac{1}{2}} \tag{28}$$

5.3. Rotation about P_3'' through ψr

The scattering matrix S'' is diagonal, with (in general) complex diagonal elements. Let these be $S_{11}'' = ae^{i\alpha}$ and $S_{22}'' = be^{i\beta}$, with a, b, α, and β all real and nonnegative. Then by formulas (18) and (4),

$$S''' = \begin{pmatrix} e^{-i\psi_r/2} & 0 \\ 0 & e^{i\psi_r/2} \end{pmatrix} \begin{pmatrix} ae^{i\alpha} & 0 \\ 0 & be^{i\beta} \end{pmatrix} \begin{pmatrix} e^{-i\psi_r/2} & 0 \\ 0 & e^{i\psi_r/2} \end{pmatrix} \qquad (29)$$

Let us choose

$$\psi r = \frac{1}{2} (\alpha - \beta) \ . \qquad (30)$$

Then Equation (29) simplifies to

$$S''' = \begin{pmatrix} ae^{i(\alpha + \beta)/2} & 0 \\ 0 & be^{i(\alpha + \beta)/2} \end{pmatrix} \ . \qquad (31)$$

This is the simplest attainable form for the scattering matrix in any basis. If the absolute phase is discarded in (31), then

$$S''' = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \ , \qquad (32)$$

and is all real, with a and b both $\geqq 0$.

At this point, it is appropriate to state the mathematical relationship between the basis changes just given and those given by Bickel[4], already cited in Section 1. Bickel's work starts from the linear basis of Figure 2(b). His congruent transformation [Ref. 4, Eq. (13)] is the opposite way round from the one used here [Equation (18)]; to get it the same way round, his two basis change matrices [Reference 4, formulas (12) and (20)] would have to be replaced by their transposes, i.e. by the matrices (still in Bickel's notation),

$$\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \text{ and } \begin{pmatrix} \cos\alpha & j\sin\alpha \\ j\sin\alpha & \cos\alpha \end{pmatrix}. \tag{33}$$

By formula (5), the first of these represents a rotation through 2θ about Bickel's P_2 axis; and the second, according to spinor theory[6], represents a rotation through 2α about Bickel's P_1 axis. But from Figure 2, it is seen that the P_2 and the P_1 axes of the linear basis of Figure 2 (b) correspond to the P_3 and the P_2 axes, respectively, of the circular basis of Figure 2 (a), which is the basis used in this paper. Therefore Bickel's two basis changes, transformed to our present basis, would be identical, after appropriate changes in notation, with basis changes described by the matrices (4) and (5).

6.0 GEOMETRY OF POLARIZATION TRANSFORMATIONS IN POINCARÉ SPACE

From Section 5 it is seen that the principal axis system P_1''' P_2''' P_3''' of the scattering matrix has Eulerian angles θ_r, φ_r, ψ_r, referred to the fundamental Poincaré axis system of Figure 1. Specifically, θ_r and φ_r are the polar and azimuthal angles of the P_3''' axis referred to the fundamental Poincaré axes, and ψ_r is the azimuthal angle of the P_1''' P_2''' P_3''' system about the P_3''' axis. The Eulerian angles of the principal axes are shown in Figure 3.

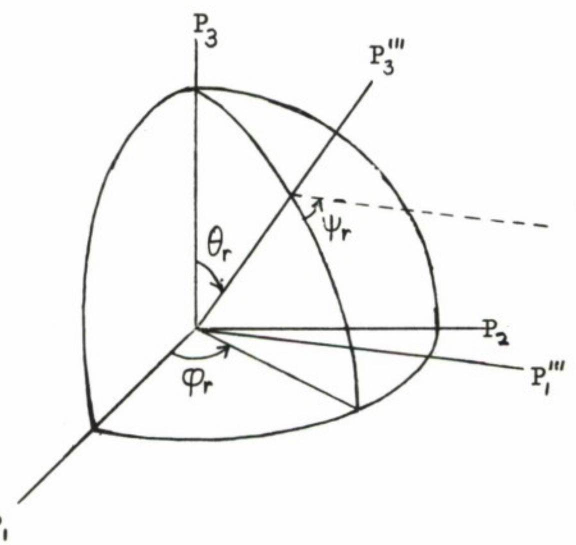

Figure 3. Eulerian angles of the principal axes (P_2''' axis not shown).

From Equation (32), the equations for the polarization transformation in the principal basis are

$$\left.\begin{aligned} E_{r1}''' &= a\, E_{t1}'''^{*} \\ E_{r2}''' &= \qquad\qquad b\, E_{t2}'''^{*} \end{aligned}\right\} . \tag{34}$$

This transformation consists of two operations, which are commutative, viz. (1) conjugation and (2) multiplication by a and b respectively. A geometrical interpretation of these two operations will now be given.

(1) Conjugation. From formulas (2) and Figure 2 (a), conjugation in the unprimed coordinate system causes all points of the Poincaré space to be reflected in the $P_1\ P_3$ plane. Analogously, in the $P_1'''\ P_2'''\ P_3'''$ system conjugation causes all points to be reflected in the $P_1'''\ P_3'''$ plane.

(2) Multiplication of the components by a and b respectively. This is a hermitian transformation of Poincaré space. If $a \neq b$ and if a and b are both $\neq 0$, then according to spinor theory[10] it transforms the family of Poincaré spheres of different sizes into a family of confocal, equieccentric prolate spheroids all having their major axes along the P_3''' axis, as shown in Figure 4.

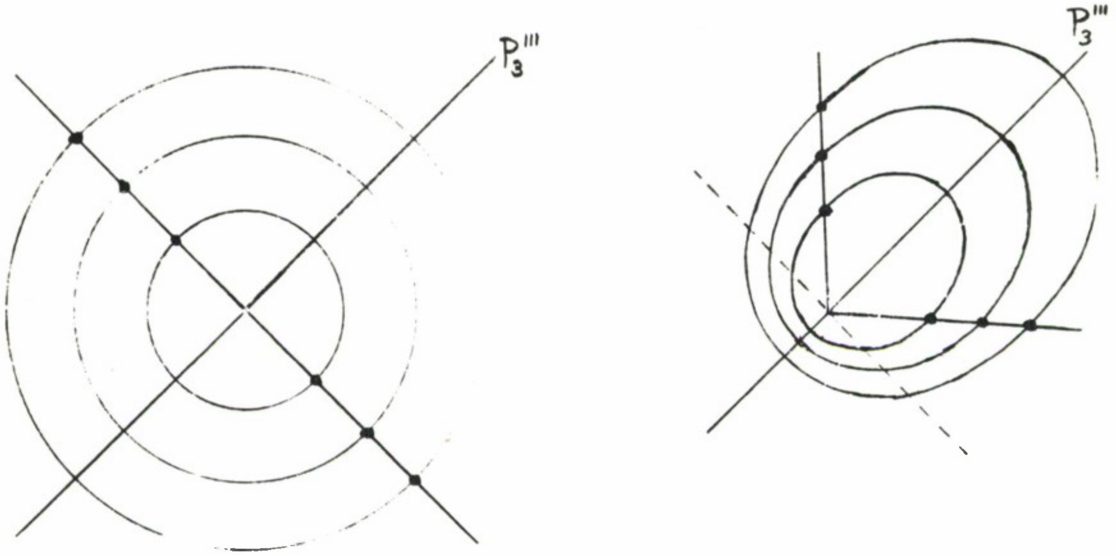

Figure 4. Hermitian transformation of Poincaré space

Thus the two orthogonal polarizations in the $\pm P_3'''$ directions are changed in magnitude only; these are the eigenpolarizations of the scattering matrix. All other polarizations get their directions changed.

From the above, it is seen that the total effect in Poincaré space of the polarization transformation (8) is to reflect all the points of Poincaré space in a certain diametral plane (the $P_1''' P_3'''$ plane) and to cause a hermitian transformation in Poincaré space whose axis (the P_3''' axis) lies in the plane of the reflection. The angles θ_r, φ_r, ψ_r specify the reflection plane and the hermitian axis relative to the fundamental Poincaré axis system, and the quantities a and b are the eigenvalues of the hermitian transformation. It is evident from this purely geometrical interpretation of the polarization transformation, that the quantities θ_r, φ_r, ψ_r, a, b are independent of what polarizations are transmitted and received, and also independent of what basis is used. They are pure scatterer parameters.

The above statements need to be modified if a or b vanishes, in which case the scattering matrix is singular, or if a = b, in which case the scattering matrix can be called degenerate, in analogy with the terminology of tensor theory. These two special cases will now be discussed.

Singular Scattering Matrix. The determinant is zero; this is true even before transformation since the determinant is invariant to the congruent transformations used here. The hermitian transformation maps the entire Poincaré space onto the positive P_3''' axis or the negative P_3''' axis, the prolate spheroids shrinking to line segments along the $\pm P_3'''$ axis.

Degenerate Scattering Matrix. In this case, the hermitian transformation degenerates to a multiplication of all polarization amplitudes by one and the same positive number; and the overall transformation consists of this together with a reflection of Poincaré space in a diametral plane. It is geometrically evident that the eigenpolarizations (i.e. those whose directions in Poincaré space are not changed) are all the polarizations lying in the P_1''' P_3''' plane; thus there is a single infinity of them. However, in general only one orthogonal pair of these eigenpolarizations can be linear, since the P_1''' P_3''' plane can intersect the plane of linear polarizations only in a single line. (Exception: the case where the P_1''' P_3''' plane coincides with the plane of linears; for this case $\psi_r = 90^\circ$).

Let us now return to the non-singular non-degenerate case. It is convenient to divide this into two sub-cases, according as $\theta_r = 90^\circ$ or $\theta_r \neq 90^\circ$. If $\theta_r = 90^\circ$, the two eigenpolarizations are linear polarizations, and therefore, according to the geometrical interpretation given earlier in this section, the transformation (34) or (8) treats the upper and lower hemispheres of the Poincaré sphere symmetrically; thus the object will respond equally to left circular and right circular polarizations. If $\theta_r \neq 0$, the response is greater for one circular than for the other. An object of this type can be said to have "helicity", according to Huynen.[11] Quantitatively, the helicity is probably best defined as $(a - b) \cos \theta_r$, for if this is positive the response is greater to left circular, if negative to right circular, and if zero, there is no preference. A degenerate object shows no preference, as seen from the geometric interpretation.

An example of a nondegenerate scatterer with zero helicity ($\theta_r = 90^o$) is any elongated convex surface having bilateral symmetry relative to a plane containing the line of sight, e.g. a surface of revolution. This case has been widely treated. However, it is not certain that bilateral symmetry is a necessary condition for the helicity to vanish because the different parts of the scatterer might conceivably be arranged unsymmetrically, but in such a way that the effects of the dissymmetries would cancel out.

An example of an object with non-zero helicity ($\theta_r \neq 90^o$) is a twisted configuration in which the twist can be recognized by circularly polarized waves as either left handed or right handed.

7.0 REDUCTION OF SCATTERING MATRIX WHEN GIVEN IN AN ARBITRARY BASIS

In Section 5, it was shown how to calculate the five scatterer parameters θ_r, φ_r, ψ_r, a, b, when the initial basis was that shown in Figure 2 (a). How are the calculations to be made if the scattering matrix is given in a different basis, such as that of Figure 2 (b)?

A direct procedure can surely be worked out; but probably it is nearly as simple to proceed indirectly, by converting the scattering matrix from the given basis to the basis of Figure 2 (a), and then going through the procedure of Section 5. To make the conversion, it is only necessary to subject the given scattering matrix to the congruent transformation (18), using for U the matrix of the rotation that rotates the given basis axes into the axes of Figure 2 (a).

To illustrate this conversion, suppose that the scattering matrix is given in the basis of Figure 2 (b). The axis rotation that rotates the axes of Figure 2 (b) into those of Figure 2 (a) can be accomplished by a rotation of the axes about P_3 through -90°, followed by a rotation about P_2' through -90°. The matrices of these two rotations can be found from formulas (4) and (5). Their product (with the first one on the right) comes out to be

$$\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix} \tag{35}$$

after dropping an absolute phase factor $(\sqrt{i}\,)$. Therefore the conversion is accomplished by subjecting the given scattering matrix to a congruent transformation with the matrix (35).

8.0 APPLICATIONS TO SOME SIMPLE PROBLEMS

The classification scheme to be used has been indicated to some extent in Section 6. The full scheme is as follows:

(A) Singular objects ($a = 0$ or $b = 0$).

(B) Degenerate objects ($a = b$).

(1) Isotropic about line of sight ($\psi_r = 90^{\circ}$).

(2) Anisotropic about line of sight ($\psi_r \neq 90^{\circ}$).

(C) Non-singular, non-degenerate objects.

$(a \neq b,\ a \neq 0,\ b \neq 0)$.

(1) Objects with zero helicity ($\theta_r = 90^\circ$).

(2) Objects with non-zero helicity ($\theta_r \neq 90^\circ$).

The angle φ_r does not appear in the above. It always measures the azimuth of the object about the line of sight.

It is of interest to compare the above scheme with Kennaugh's classification[1,2], which is based on null polarizations and consists of four types of objects, viz. linear, isotropic, symmetrical, and all others. Kennaugh's linear objects are those singular scatterers [Class (A) above] whose eigenpolarizations are linear. His isotropic objects are the same as Class (B1) above. His symmetrical scatterers appear to be closely related to Class (C1).

A number of simple examples will now be given and the five object parameters will be calculated for each.

(A) Singular Object. Straight wire at angle γ to horizontal and perpendicular to line of sight. The normalized scattering matrix [see formula (12)] is

$$\frac{1}{\sqrt{2}}\begin{pmatrix} e^{2i\gamma} & 1 \\ 1 & e^{-2i\gamma} \end{pmatrix} \quad \text{or} \quad \frac{1}{\sqrt{2}}\begin{pmatrix} 1+\cos 2\gamma & \sin 2\gamma \\ \sin 2\gamma & 1-\cos 2\gamma \end{pmatrix} \tag{36}$$

in the bases of Figure 2 (a) and (b) respectively. The procedure of Section 5 yields the values,

$$\varphi_r = 2\gamma,\ \theta_r = 90^\circ,\ \psi_r = \text{arb.},\ a = 1,\ b = 0. \tag{37}$$

The single eigenpolarization is linear, along the line of the wire. The angle ψ_r is indeterminate, as would be expected.

(B1) Degenerate object, isotropic about line of sight.

Large plane perpendicular to line of sight. The scattering matrix [see formula (9)] is

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{38}$$

in the two bases. The parameters come out to be φ_r = arb., $\theta_r = 90^\circ$, $\psi_r = 90^\circ$, $a = 1$, $b = 1$. (39)

The eigenpolarizations are the linears. The geometrical transformation of Poincaré space is a reflection in the plane of the linear polarizations.

(B2) Degenerate object, anisotropic about line of sight.

First Case. Large right-angle corner, with edge perpendicular to line of sight and horizontal. The scattering matrix, in the basis of Figure 2 (a), is

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{40}$$

The parameters are

$$\varphi_r = 0,\ \theta_r = \text{arb.},\ \psi_r = 0,\ a = 1,\ b = 1. \tag{41}$$

The geometrical transformation of Poincaré space is a reflection in the $P_1\ P_3$ plane of Figure 2 (a). The eigenpolarizations are all the polarizations in that plane. The linear eigenpolarizations are the horizontal and the vertical.

If the right-angle corner is rotated about the line of sight so that its edge is at angle γ to the horizontal, then $\varphi_r = 2\gamma$, and the other parameters are unchanged.

Second Case. Two equal straight wires, both perpendicular to the line of sight and separated (along the line of sight) by distance d. Let the front wire be horizontal and the rear wire vertical. Assume simple superposition of returns, no multiple scattering.

The normalized scattering matrix in the circular basis of Figure 2 (a) is

$$\frac{1}{2} e^{2ikd} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} , \tag{42}$$

the first term being the scattering matrix of the horizontal wire, with phase advanced by 2 kd. On adding the two terms of (42) and shifting absolute phase, one gets for the scattering matrix.

$$\begin{pmatrix} i \sin kd & \cos kd \\ \cos kd & i \sin kd \end{pmatrix} . \tag{43}$$

The procedure of Section 5 yields

$$\varphi_r = 0, \ \theta_r = 90^\circ, \ \psi_r = kd \pm 90^\circ, \ a = 1, \ b = 1. \tag{44}$$

The geometrical transformation of Poincaré space is a reflection in the plane that contains the linear horizontal-vertical axis and makes a dihedral angle equal to ψ_r with the plane of the linear horizontal-vertical axis and the circular axis. The eigenpolarizations are all polarizations in the plane of reflection. The linear eigenpolarizations are the horizontal and vertical.

Three special cases are of interest.

$$\left.\begin{array}{llll} \text{If} & d = 0 & \text{then} & \psi_r = 90^\circ , \\ & d = \pi/2k, & & \psi_r = 0, \\ & 0<d< (\pi/2k), & & 90^\circ > \psi_r > 0 . \end{array}\right\} \quad (45)$$

Thus, for d = 0 the parameters are identical with those of a large plane perpendicular to the line of sight; the configuration is recognized as isotropic. For $d = \pi/2k$, the object is indistinguishable from a right angle corner if the system is monochromatic. (However, it could be distinguished by varying the frequency).

(C1) Non-singular non-degenerate object with zero helicity.

Two equal coplanar straight wires, both perpendicular to line of sight, one horizontal and the other at 45° to the horizontal. Again assume no multiple scattering. The (unnormalized) scattering matrix in the basis of Figure 2 (a) is

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} + \begin{pmatrix} i & 1 \\ 1 & -i \end{pmatrix} = \begin{pmatrix} 1+i & 2 \\ 2 & 1-i \end{pmatrix} . \quad (46)$$

The parameters are calculated to be

$$\phi_r = 45^\circ,\ \theta_r = 90^\circ,\ \psi_r = 90^\circ ,$$

$$a = 2 + \sqrt{2} = 3.41,\ b = 2 - \sqrt{2} = .59. \quad (47)$$

Since $\phi_r = 45^\circ$, the a - axis is at 22.5° to the horizontal and the b-axis is at 112.5° to the horizontal, in physical space, as in Figure 5.

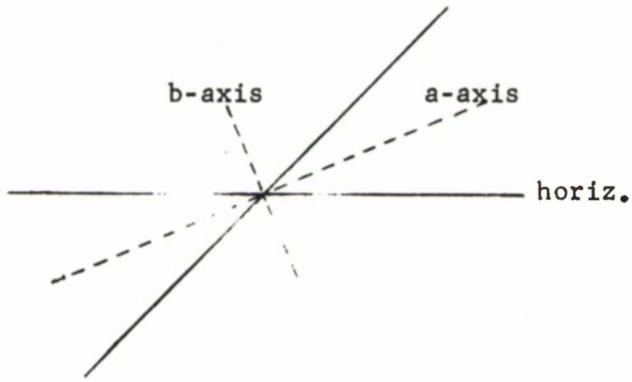

Figure 5. Principal axes (in physical space) for case (C1).

The two eigenpolarizations are linear, and are along these axes. The plane of reflection in Poincaré space is the plane of linears.

(C2) Non-singular non-degenerate object with non-zero helicity.

Two equal straight wires, both perpendicular to line of sight but separated in range by $\lambda/8$, with the front wire horizontal and the rear wire at 45° to horizontal. Then

$$\text{S.M. for rear wire} = \begin{pmatrix} i & 1 \\ 1 & -i \end{pmatrix}$$

$$\text{S.M. for front wire} = \begin{pmatrix} i & i \\ i & i \end{pmatrix} \text{(with phase advance of } 90^{\circ}\text{)}.$$

$$\text{Total S.M.} = \begin{pmatrix} 2i & 1+i \\ 1+i & 0 \end{pmatrix} .$$

The parameters are found to be

$$\varpi_r = 45^{\circ},\ \varphi_r = \cot^{-1}(1/\sqrt{2}) = 54.7^{\circ},\ \psi_r = 90^{\circ}\ ,$$

$$a = \sqrt{1.5} + \sqrt{.5} = 1.94, \quad b = \sqrt{1.5} - \sqrt{.5} = .52. \tag{48}$$

The helicity [$= (a-b) \cos \theta_r$] is positive, corresponding to the left-handedness of the configuration. The two eigenpolarizations are both elliptic polarizations. The two linears that come closest to being eigenpolarizations are those at 22.5° and at 112.5°, as one would expect.

9.0 SOME CONCLUSIONS ON SCATTERER GEOMETRY

A. Singular Object. The only singular object considered in Section 8 was one having a linear eigenpolarization. It is not certain whether a singular object can be realized having a circular or elliptic eigenpolarization.

B. Degenerate Object. In the second example of case (B2), Section 8, any arbitrarily assigned values of φ_r, Ψ_r, and a can be realized by suitable choices of the azimuth, the separation, and the lengths of the two wires. Therefore, at a fixed frequency, the polarization effects of any given degenerate object can be duplicated by this model. [Note that, for the degenerate case, a = b, and θ_r can always be taken equal to 90°].

C. Non-singular, non-degenerate object. If the helicity is zero, the surface is recognized as elongated. The long and short axes, whose directions (in physical space) are parallel to the (linear) eigenpolarizations, have their azimuths about the line of sight determined by φ_r. The effective lengths of these axes are measured by the eigenvalues a and b. These "lengths" are, of course, electrical dimensions, and are not directly related to the actual physical dimensions. There appears to be no useful information contained in the angle Ψ_r.

Objects with non-vanishing helicity have elliptic eigenpolarizations, but are also recognized as elongated, with φ_r, a, and b related to the object axes as before. (This would not be true if the eigenpolarizations were the two circulars, a case that is probably not realizable). Also the object has a twist, as already seen in Section 7.

All the above conclusions are based on the assumption of monochromatic operation. If the frequency were changed, the values of the parameters would also change, in general; and probably more could be learned about the scatterer.

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13. ABSTRACT

It is shown how, by carrying out a sequence of three coordinate axis rotations in Poincaré space, one can calculate the principal basis and the real, nonnegative eigenvalues of any symmetric polarization scattering matrix. Then the two eigenvalues and the three Eulerian angles of the principal axes in Poincaré space constitute a complete set of pure scatterer parameters. A scatterer classification scheme based on these parameters is constructed, with the help of a geometrical representation, in Poincaré space, of polarization transformations. The procedure is applied to several simple scattering configurations. Some conclusions are reached concerning the scatterer geometry in various cases.

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