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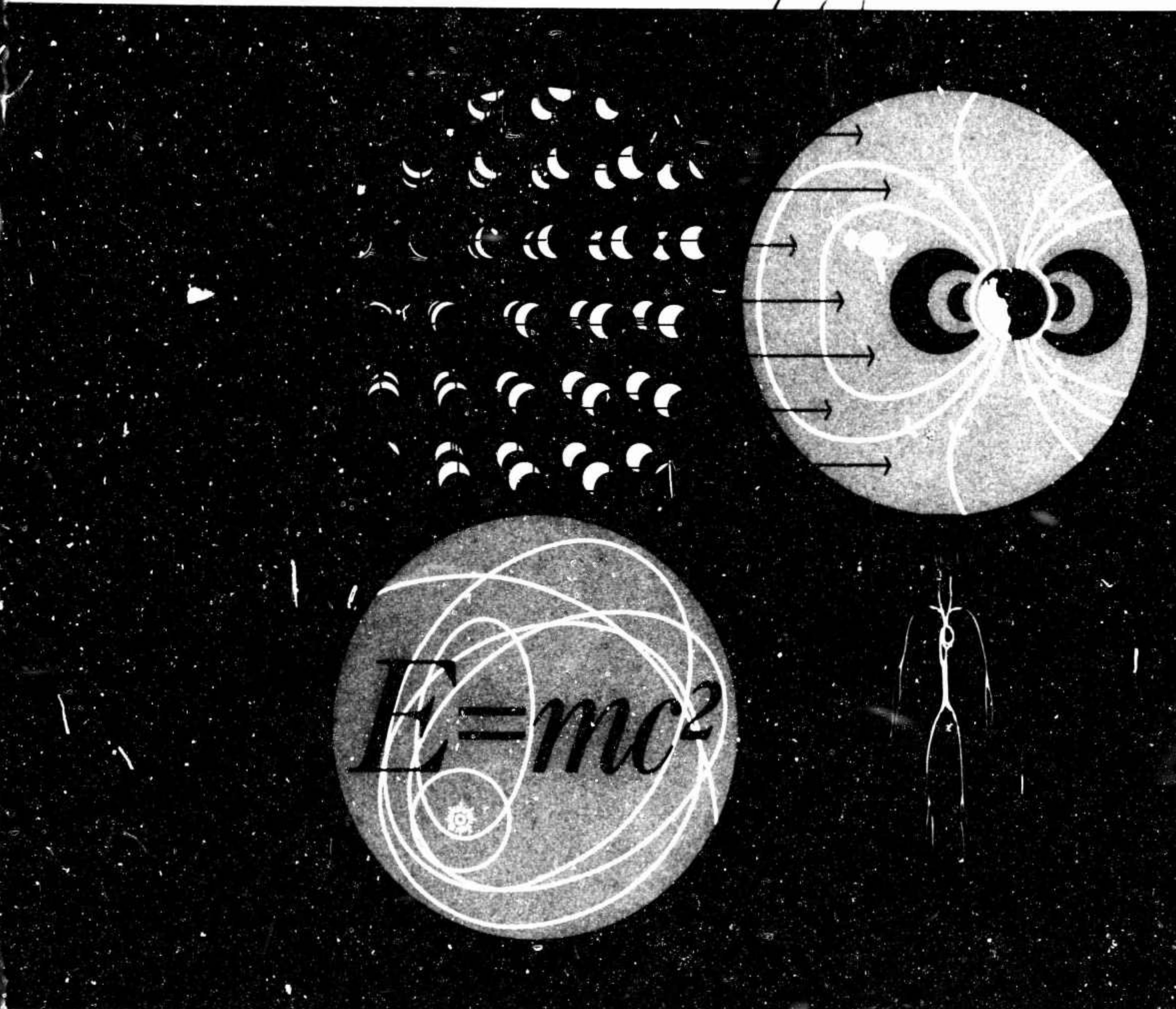

UNIFIED FIELD THEORY

D. PANDRES, JR.

NOVEMBER 1968

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ABSTRACT

A solution is given to the problem of constructing a unified theory of the gravitational and electromagnetic fields. The theory exhibits the following features: (1) It is fully geometrical, the geometry being a special type of four-dimensional Riemannian space-time. (2) It is completely unified, all physical features (i.e., all curvature) being introduced via a quantity which transforms irreducibly under the coordinate transformations of general relativity. (3) The gravitational field and the electromagnetic field are inextricably linked, each being merely a different manifestation of the geometrical structure. (4) Charged particles move according to the Lorentz force law. (5) The energy density is everywhere nonnegative. (6) The field equations insure that the electrical current density is everywhere timelike and that charged particles have extended structures rather than point structures.

CONTENTS

Section 1

INTRODUCTION

The success of Einstein in geometrizing the gravitational field led him and others to search for a way to geometrize the electromagnetic field as well. However, no complete geometrization of electrodynamics was achieved, and it now seems that the search has been abandoned by most physicists. The purpose of this paper is to present a solution to the problem. As we shall see, the key to the problem is that one must specialize the metric of space-time sufficiently so that all curvature is introduced via a quantity which is irreducible under the coordinate transformations of general relativity.

Section 2

SPECIALIZATION OF THE METRIC

We follow Einstein in considering a four-dimensional Riemannian space-time with metric $g_{\mu\nu}$. The signature of the metric is taken to be +2, and we adopt the summation convention that repeated indices are summed from 0 to 3. It is well known that $g_{\mu\nu}$ may be expressed in the form

$$g_{\mu\nu} = g_{ij} A^i_\mu A^j_\nu \tag{1}$$

where g_{ij} is the Lorentz metric diag (−1, 1, 1, 1) and A^i_μ is an orthonormal tetrad of vectors. It must be clearly understood, however, that the superscript i on A^i_μ is not a space-time index. Thus, the symbol A^i_μ must be read as "the μ^{th} component of the i^{th} vector." One advantage of using this tetrad formalism is that $g_{\mu\nu}$ has the same signature as g_{ij} for all A^i_μ. Thus, A^i_μ simply introduces any curvature which may be present in the space described by $g_{\mu\nu}$. If $A^i_{\mu,\nu} = A^i_{\nu,\mu}$, where a comma denotes ordinary partial differentiation, then there exist four functions φ^i such that $A^i_\mu = \varphi^i_{,\mu}$. In this case, Eq 1 merely expresses a coordinate transformation, and the metric $g_{\mu\nu}$ describes a flat space. Thus, physical interest must center upon cases in which at least one of the quantities

$$F^i_{\mu\nu} = A^i_{\nu,\mu} - A^i_{\mu,\nu} \tag{2}$$

is nonvanishing. Now, however, we are faced with a dilemma. One of the most beautiful features of Einstein's gravitational theory is that $g_{\mu\nu}$ transforms irreducibly (Ref 1) under coordinate transformations. Our dilemma is based upon the fact that the quantity $F^i_{\mu\nu}$ (which brings in all curvature) is not, in general, irreducible. Since the superscript i is not a space-time index, it follows that the four quantities $F^0_{\mu\nu}$, $F^1_{\mu\nu}$, $F^2_{\mu\nu}$ and $F^3_{\mu\nu}$ transform separately under the coordinate transformations of general relativity. Irreducibility can be obtained only if these four quantities are scalar multiples of each other. We therefore restrict our attention to cases in which

$$F^i_{\mu\nu} = K^i F_{\mu\nu} \tag{3}$$

where $F_{\mu\nu}$ does not vanish. Next we identify $F_{\mu\nu}$ as the electromagnetic field and hence require that there exist a vector A_μ such that

$$F_{\mu\nu} = A_{\nu,\mu} - A_{\mu,\nu} \tag{4}$$

is nonvanishing. It follows easily from Eqs 2, 3, and 4 that either K^i must be constant, or the invariant $\epsilon^{\alpha\beta\mu\nu}F_{\alpha\beta}F_{\mu\nu}$ must vanish. Since we do not wish to restrict the electromagnetic field in this way, we require that K^i be constant. This implies that A^i_μ is of the form

$$A^i_\mu = \varphi^i_{,\mu} + K^i A_\mu \tag{5}$$

Before proceeding further, we must be certain that we have not specialized to a metric whose determinant g vanishes identically. This may be done in the following way: Assume that g vanishes identically. Then Eq 1 implies that the determinant of A^i_μ vanishes identically. This, in turn, implies that the four vectors A^0_μ, A^1_μ, A^2_μ and A^3_μ are linearly dependent. Hence there must exist four constants C_i such that

$$C_i A^i_\mu = 0 \tag{6}$$

By differentiating Eq 6 with respect to x^ν and subtracting the result from the corresponding equation with μ and ν interchanged,

$$C_i F^i_{\mu\nu} = C_i K^i F_{\mu\nu} = 0 \tag{7}$$

Since $F_{\mu\nu}$ is nonvanishing, we conclude that $C_i K^i$ must vanish. Hence if we substitute Eq 5 into Eq 6, we obtain

$$C_i \varphi^i_{,\mu} + C_i K^i A_\mu = C_i \varphi^i_{,\mu} = 0 \tag{8}$$

But $C_i \varphi^i_{,\mu}$ can vanish only if the determinant of $\varphi^i_{,\mu}$ vanishes. Hence we conclude that g can vanish identically only if the determinant of $\varphi^i_{,\mu}$ vanishes identically.

For the reasons discussed in this section, we restrict our attention to metrics defined by Eq 1, using tetrads of the type given in Eq 5. In summary, then, we take

$$
\begin{aligned}
g_{\mu\nu} &= g_{ij} A^i_\mu A^j_\nu \\
g_{ij} &= \mathrm{diag}(-1, 1, 1, 1) \\
A^i_\mu &= \varphi^i_{,\mu} + K^i A_\mu \\
K^i &= \text{constant} \\
\mathrm{Det}\ \varphi^i_{,\mu} &\neq 0 \\
F_{\mu\nu} &= A_{\nu,\mu} - A_{\mu,\nu} \neq 0
\end{aligned}
\tag{9}
$$

and we identify the antisymmetric tensor $F_{\mu\nu}$ as the electromagnetic field. We shall later specialize further to the case in which $K^i = (K^o, 0, 0, 0)$, but the reason for this choice will be clear only after we have derived field equations.

Section 3

THE LORENTZ FORCE

We define the electrical current density j_μ in the usual way by

$$J_\mu = F_{\mu\ ;\nu}^{\ \nu} \tag{10}$$

where the semicolon denotes covariant differentiation with respect to the Christoffel symbol $\left\{ {\alpha \atop \mu\nu} \right\}$. Next we show that Eq 9 insures that charged particles move according to the Lorentz force law. We begin by defining inverse tetrads A_i^μ through the relations

$$\begin{aligned} A_i^\mu A_\nu^i &= \delta_\nu^\mu \\ A_\mu^i A_j^\mu &= \delta_j^i \end{aligned} \tag{11}$$

and defining the proper charge density ρ associated with j_μ through the usual relation

$$\rho^2 = |j^\mu j_\mu| \tag{12}$$

The velocity field associated with the current is, of course, given by

$$v_\mu = \frac{j_\mu}{\rho} \tag{13}$$

Quite clearly, the acceleration field is given by

$$v_{\mu;\nu} v^\nu = \left(\frac{j_{\mu;\nu}}{\rho} - \frac{j_\mu}{\rho^2} \rho_{,\nu} \right) v^\nu \tag{14}$$

and the quantity $j_{\mu;\nu}$ may be written in the form

$$
\begin{aligned}
j_{\mu;\nu} &= \left(j_\alpha A^\alpha_i A^i_\mu\right)_{;\nu} \\
&= j_{i,\nu} A^i_\mu + j_\alpha A^\alpha_i A^i_{\mu;\nu}
\end{aligned}
\tag{15}
$$

where the scalars j_i defined by

$$
j_i = j_\mu A^\mu_i \tag{16}
$$

are the <u>physical</u> components of the current density (referred to a local Lorentz frame). That the A^μ_i lead to a locally Lorentz frame is clear from Eq 1. Equation 14 may now be written as

$$
v_{\mu;\nu} v^\nu = \frac{j_{i,\nu} v^\nu}{\rho} A^i_\mu - \frac{\rho_{,\nu} v^\nu}{\rho} v_\mu + v_\alpha A^\alpha_i A^i_{\mu;\nu} v^\nu \tag{17}
$$

If we now consider the physical components

$$
a_i = A^\mu_i v_{\mu;\nu} v^\nu \tag{18}
$$

of the acceleration field, we obtain

$$
a_i = \frac{j_{i,\nu} v^\nu}{\rho} - \frac{\rho_{,\nu} v^\nu}{\rho} v_i + v_k A^k_{\mu;\nu} A^\mu_i v^\nu \tag{19}
$$

where the quantities

$$
v_i = v_\mu A^\mu_i \tag{20}
$$

are physical components of the velocity field. Now, the vector A^k_μ being differentiated in forming $A^k_{\mu;\nu}$ is precisely the tetrad through which the metric is defined in Eq 9. By using Eq 9, we easily find that

$$A^k_{\mu;\nu} = A^k_\beta F^\beta_{\mu\nu} \tag{21}$$

where

$$F^\beta_{\mu\nu} = -\frac{1}{2}\left(K^\beta F_{\mu\nu} + K_\mu F^\beta_\nu + K_\nu F^\beta_\mu\right) \tag{22}$$

and

$$K_\mu = K_i A^i_\mu = g_{ij} K^j A^i_\mu \tag{23}$$

Upon substituting Eq 21 into Eq 19, and using Eqs 22 and 23, we obtain

$$\mathcal{a}_i = \frac{j_{i,\nu} v^\nu}{\rho} - \frac{\rho_{,\nu} v^\nu}{\rho} - \left(v^\nu K_\nu\right) F_{ik} v^k \tag{24}$$

where the quantities

$$F_{ik} = A^\mu_i A^\nu_k F_{\mu\nu} \tag{25}$$

represent the physical components of the electromagnetic field. Now, consider the meaning of the quantity $\rho_{,\nu} v^\nu$. It is just the time rate of change of ρ as seen by an observer who is riding along with the charged fluid. Therefore, it vanishes in the important case of incompressible flow. This is true even if there are world tubes within which ρ is large, corresponding to the motion of a charged particle. Consider also what our observer sees if the flow is not incompressible but he is riding along with a stable concentration of charge. (By a "stable concentration" we mean that the charge density is large, and that the flow appears to be incompressible when oscillatory fluctuations in the density are averaged over appropriately small time intervals.) In this case, it is clear that the time-averaged value of $\rho_{,\nu} v^{\nu}$ is small. A similar argument shows that $j_{i,\nu} v^{\nu}$ is small. Moreover, the charge density ρ is large within the world tube of a charged particle. Thus we see that in the case of a charged particle, the first two terms of Eq 24 may be neglected and the equation reduces to

$$a_i = - \left(v^{\nu} K_{\nu}\right) F_{ik} v^k \qquad (26)$$

which is the Lorentz force law. It must be emphasized that we have not proved the existence of charged particles. We have only proved that <u>if</u> charged particles exist, then they must move according to the Lorentz force law. Moreover, it is not clear whether this deficiency can be removed until the present theory is quantized.

Section 4

THE FIELD EQUATIONS

It is clear from Eq 9 that both $g_{\mu\nu}$ and $F_{\mu\nu}$ have geometrical significance. Indeed, $F_{\mu\nu}$ is the quantity which brings all curvature into $g_{\mu\nu}$. We shall therefore form an invariant from $F_{\mu\nu}$ and $g_{\mu\nu}$, by contraction, and we shall use this invariant as a Lagrangian density function in deriving field equations. Moreover, we require that the invariant be only bilinear in $F_{\mu\nu}$, so that it will lead to second-order field equations. The only invariant which satisfies these requirements is $F_{\mu\nu}F^{\mu\nu}$. We therefore derive our field equations from the variation principle

$$\delta \int F_{\mu\nu} F^{\mu\nu} \sqrt{-g}\, dx = 0 \tag{27}$$

where both the vector A_{μ} and the scalars φ^{i} are varied. It is emphasized that, because of Eq 9, the quantity

$$F^{\mu\nu} = g^{\mu\alpha} g^{\nu\beta} F_{\alpha\beta} \tag{28}$$

depends upon A_{μ} through the metric as well as through $F_{\alpha\beta}$, while it depends upon φ^{i} only through the metric. By varying A_{μ}, we obtain

$$j_{\nu} = K^{\mu} E_{\mu\nu} \tag{29}$$

where

$$E_{\mu\nu} = F_{\mu\alpha}F_{\nu}{}^{\alpha} - \frac{1}{4} g_{\mu\nu}F_{\alpha\beta}F^{\alpha\beta} \tag{30}$$

is the usual stress energy tensor for the electromagnetic field. By varying φ^i, we obtain

$$F^{\beta\nu}j_{\nu} = F^{\beta\nu}K^{\mu}E_{\mu\nu} \tag{31}$$

which is automatically satisfied because of Eq 29.

Now, it follows from Eq 29 and the identity (Ref 2)

$$E_{\mu\alpha}E_{\nu}{}^{\alpha} = \frac{1}{4} g_{\mu\nu}E_{\alpha\beta}E^{\alpha\beta} \tag{32}$$

that

$$j^{\nu}j_{\nu} = \frac{1}{4} K_i K^i E_{\alpha\beta}E^{\alpha\beta} \tag{33}$$

But it is well known (Ref 3) that $E_{\alpha\beta}E^{\alpha\beta}$ is nonnegative and vanishes if and only if $F_{\mu\nu}$ is a null field (Ref 4). Since we do not wish to restrict our attention to null fields, we see from Eq 33 that we must choose the constants K^i to be timelike, i.e.,

$$K_i K^i < 0 \tag{34}$$

This is necessary and sufficient to insure that j_{ν} is everywhere timelike. Since we have chosen timelike constants K^i, there is no loss in generality in taking $K^i = (K^o, 0, 0, 0)$ since this form can always be obtained by a Lorentz transformation on the Latin indices, leaving $g_{ij} = \mathrm{diag}(-1, 1, 1, 1)$ unchanged. With this choice of K^i, the Lorentz force law, Eq 26, reduces to

$$a_i = K_o v_o F_{ik} v^k \tag{35}$$

where all indices refer to physical components. It is clear from Eqs 12 and 33 that ρ is nonvanishing if $F_{\mu\nu}$ is not null. This implies that charged particles must have extended structures rather than point structures. At first glance, it also appears to violate the experimental data, since it implies, for example, that there is a nonvanishing charge density where there is a static magnetic field. The difficulty, however, is only apparent, for we shall see in the next section that $1/2(K_o)^2$ must equal Einstein's gravitational constant K. With this value of K_o, the largest magnetic fields as yet produced in the laboratory correspond to charge densities well below the present level of possible detection.

It seems clear that no reliable discussion of charge-to-mass ratio or other features concerning particle structure can be given within the framework of a classical theory. Nevertheless, it is interesting to note that if v_o is unity (corresponding to a charged fluid locally at rest) then the coefficient of $F_{ik}v^k$ in Eq 35 gives a charge-to-mass ratio much smaller than that of any known charged particle. In general, however, v_o is just $(1 - \beta^2)^{-1/2}$, so we can obtain any larger value of charge-to-mass ratio by properly choosing β. This β would then correspond to the velocity of the charged fluid within the particle's world tube (including both its average motion and its internal or oscillatory motion). Then, because of the relativistic law for addition of velocities, a particle's charge-to-mass ratio would become dependent upon particle velocity at sufficiently high energy. Although we cannot predict at what energy this dependence becomes significant, one can construct models such that it becomes significant only at energies much larger than those presently available.

Section 5

THE TOTAL STRESS-ENERGY TENSOR

Following the notation of Synge (Ref 5), we define the Riemann curvature tensor by

$$R^{\alpha}_{\beta\mu\nu} = A^{\alpha}_{i}\left(A^{i}_{\beta;\mu;\nu} - A^{i}_{\beta;\nu;\mu}\right) \tag{36}$$

and the Ricci tensor by $R_{\mu\nu} = R^{\alpha}_{\mu\nu\alpha}$. The Einstein tensor is defined in the usual way by

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \tag{37}$$

where $R = g^{\mu\nu} R_{\mu\nu}$. In this notation, Einstein's equations are

$$G_{\mu\nu} = -KT_{\mu\nu} \tag{38}$$

where K is the Einstein gravitational constant and $T_{\mu\nu}$ is the total stress-energy tensor. Now, it is clear from Eqs 21 and 36 that the Riemann tensor may be expressed in terms of $F^{\beta}_{\mu\nu}$ and its first covariant derivatives. When this is done, we find that the Einstein tensor is given by

$$G_{\mu\nu} = -\frac{1}{2}(K_o)^2 E_{\mu\nu} + M_{\mu\nu} \tag{39}$$

where the "matter term" $M_{\mu\nu}$ is

$$M_{\mu\nu} = K^{\alpha} j_{\alpha} g_{\mu\nu} - \frac{1}{2}\left(K_{\mu} j_{\nu} + K_{\nu} j_{\mu}\right) + \frac{1}{4} K_{\mu} K_{\nu} F_{\alpha\beta} F^{\alpha\beta} + Z_{\mu\nu} \tag{40}$$

with $Z_{\mu\nu}$ defined by

$$Z_{\mu\nu} = \frac{1}{2} U_\mu U_\nu - \frac{1}{2}\left(U_{\mu|\nu} + U_{\nu|\mu}\right)$$

$$+ \frac{1}{4} g_{\mu\nu}\left[U^\alpha U_\alpha + 2(K_o)^2 F_{\alpha\beta} F^{\alpha\beta}\right] \tag{41}$$

and

$$U_\nu = K^\mu F_{\mu\nu} \tag{42}$$

The vertical stroke $|$ in $U_{\mu|\nu}$ denotes covariant differentiation with respect to $L^\alpha_{\mu\nu} = A^\alpha_i A^i_{\mu,\nu}$. When we consider the physical components of U_ν , i.e.,

$$U_i = U_\mu A^\mu_i \tag{43}$$

we see that U_i is of the form

$$U_i = (0, U_1, U_2, U_3) \tag{44}$$

Finally, we consider the physical components of $G_{\mu\nu}$, i.e.,

$$G_{ij} = A^\mu_i A^\nu_j G_{\mu\nu} \tag{45}$$

With the aid of Eq 44 and the fact that

$$A^\mu_{i|\nu} = 0 \tag{46}$$

we find that the time-time component of G_{ij} is just

$$G_{oo} = -\frac{3}{4}(K_o)^2 H^2 \tag{47}$$

From Eqs 38 and 47, we see that the time-time component of the total stress-energy tensor is everywhere nonnegative and vanishes only where all physical components of the magnetic field H vanish. Notice also that the correct value of the gravitational constant may be obtained through suitable choice of the constant K_o.

It is emphasized that Eq 39 has been derived without using the field equations, Eq 29. By using the equations, however, we find that the matter term $M_{\mu\nu}$ defined by Eq 40 may be simplified to

$$M_{\mu\nu} = \left(K^{\alpha} j_{\alpha} g_{\mu\nu}\right) - \left(K_{\mu} j_{\nu} + K_{\nu} j_{\mu}\right) + Q_{\mu\nu} \tag{48}$$

where

$$\begin{aligned} Q_{\mu\nu} &= \frac{1}{2} U_{\mu} U_{\nu} - \frac{1}{2}\left(U_{\mu;\nu} + U_{\nu;\mu}\right) \\ &\quad + \frac{1}{4} g_{\mu\nu}\left[U^{\alpha} U_{\alpha} + 2(K_o)^2 F_{\alpha\beta} F^{\alpha\beta}\right] \end{aligned} \tag{49}$$

and find also that the Riemann curvature invariant R is just

$$R = (K_o)^2\left(\frac{1}{2} E^2 - 3H^2\right) \tag{50}$$

where E is the total electric field referred to physical components. Now, suppose that $g_{\mu\nu}$ describes a flat space. We see from Eqs 47 and 50 that this would imply

$$E = H = F_{\mu\nu} = 0 \tag{51}$$

so we conclude that the space described by $g_{\mu\nu}$ is flat if and only if $F_{\mu\nu}$ vanishes.

Section 6

DISCUSSION

The results of the previous section clearly imply that our space describes a universe containing only gravitation and electromagnetism. If one is optimistic enough, however, one can hope that the short-range forces of high-energy physics might correspond to quantum effects, or to nonlinear electromagnetic and gravitational effects which are important only in regions of intense field strengths. Be that as it may, the need for quantizing the present theory is clear. This is apparent because the timelike physical component of the current given by Eq 29 is either everywhere positive or everywhere negative, depending upon the choice of sign for K_o. Our theory can therefore describe either positively charged particles or their negatively charged antiparticles, but not both simultaneously. This is a feature which is hardly surprising in an unquantized theory.

Section 7

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