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**ACOUSTIC-GRAVITY WAVES FROM AN
ENERGY SOURCE AT THE GROUND IN
AN ISOTHERMAL ATMOSPHERE**

J. D. Cole and C. Greifinger

PREPARED FOR:
ADVANCED RESEARCH PROJECTS AGENCY
AND THE UNITED STATES AIR FORCE

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PREFACE

This report is part of RAND's continuing interest in the geophysical effects produced by nuclear explosions. It is a continuation of the work in RM-5738 on some aspects of the acoustic effects of such explosions. The work was sponsored by the United States Air Force and the Advanced Research Projects Agency.

SUMMARY

The pressure pulse generated in an isothermal atmosphere by an energy source at the ground is calculated from an integral representation of the pressure field previously derived by the authors. The shape of the signal is shown, as a function of time, at several distances from the source for a fixed altitude, and at several altitudes at a fixed lateral distance. The first signal to arrive at any location is a high-frequency acoustic wave, followed by a low-frequency acoustic-gravity wave. The onset of the latter is marked by a sharp front, or caustic. At any instant (after the arrival of the caustic), there are three principal frequency components at any location, the lowest of which becomes dominant as time progresses. It is shown how the qualitative features of the flow, as well as the exact location of the caustic, can be obtained from kinematic theory.

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I. INTRODUCTION

A great deal of attention has been devoted to the subject of acoustic-gravity waves in the earth's atmosphere. Historically, interest in this area was first aroused by the worldwide detection of atmospheric disturbances of natural origin. This interest has been renewed, in recent years, by the detection, in widely spaced regions, of atmospheric waves generated by nuclear explosions. One aspect of such waves, and the one which underlies this study, is the existence, at ionospheric heights, of outward traveling disturbances produced by explosions near the ground. For a summary of the experimental and theoretical work pertinent to this particular aspect of the problem, the reader is referred to Row (1967), where an extensive bibliography is also to be found.

Despite its well-known shortcomings, the isothermal atmosphere, because of its tractability, has received a great deal of theoretical attention. It has been remarked often in the literature that, although caution must be exercised in comparing experimental observations with the predictions of so simple a model, useful insight into the propagation of atmospheric disturbances is nevertheless provided thereby. One of the principal results of the work presented here, viz., the existence of a front (caustic) bounding the spatial domain of the low frequency part of the disturbance, could be considered a case in point. It may reasonably be expected that this is a property which the real atmosphere shares with the isothermal atmosphere, with the shape and location of the caustic depending, of course, on the actual density variation of the atmosphere.

Any theoretical treatment of the atmospheric waves generated by a nuclear explosion obviously requires some simplifying assumptions about the source. (For a discussion of this aspect of the problem, see Pierce, 1968.) In this paper, the interest is in the dispersive waves far from the source, with periods much larger than the characteristic period of the source. For this purpose, an instantaneous point source seems adequate. In particular, a point energy source is assumed, since that seems more appropriate to the initial conditions represented by a nuclear explosion than the point mass and point impulse sources which have appeared in other treatments. Some characteristics of acoustic-gravity waves depend only on the properties of the medium, and not on its excitation. However, other properties, such as the spectrum at a given location, obviously do depend on the source. Such properties also depend on the boundary conditions, for which reason the presence of the ground is taken explicitly into account in this work.

In the following, two alternative approaches to the problem discussed above will be presented. The first is a formulation as an initial value problem, with the appropriate boundary conditions, from which the characteristics of the disturbance can be calculated by the method of stationary phase. Details of this method have been presented elsewhere (Cole and Greifinger, 1968), and will be briefly reviewed here for the sake of completeness. It will then be shown how the main properties of the dispersive waves in the asymptotic field can be obtained from the dispersion relation and the kinematic formulation of Whitham (1961). Thus, in addition to providing useful insight into

the propagation of atmospheric disturbances, the isothermal atmosphere also furnishes an illuminating example of the relationship between these two alternative methods.

II. INITIAL VALUE FORMULATION

As mentioned in the Introduction, this section is intended primarily as a brief outline of the treatment by the authors of the initial value problem, the details of which appear in Cole and Greifinger (1968). The problem considered is the motion in an isothermal atmosphere, above a ground plane, produced by the instantaneous release of energy at a point on the ground (the origin of Fig. 1). For an isothermal atmosphere in a uniform gravitational field, the equilibrium conditions are characterized by the usual exponential distributions

$$\frac{P_0(z)}{P^*} = \frac{\rho_0(z)}{\rho^*} = e^{-z/h} \quad (1)$$

where P^* and ρ^* are the pressure and density in the plane $z = 0$, with the positive z direction taken upward. The scale height h is given in terms of the temperature T^* , the specific heat ratio γ , and the gravitational constant g by the relation $h = RT^*/g$. The isentropic sound speed is $c^* = (\gamma RT^*)^{1/2} = \gamma^{1/2} c_g$, where $c_g = (gh)^{1/2}$ is the gravity wave speed.

The basic parameter of the problem is

$$\epsilon = \frac{(\gamma-1)Q_0}{P^* h^3} \quad (2)$$

where Q_0 is the energy released at $t = 0$. This parameter roughly measures the energy release compared to the internal energy in a scale height volume. For sea level explosions, where the scale height is of

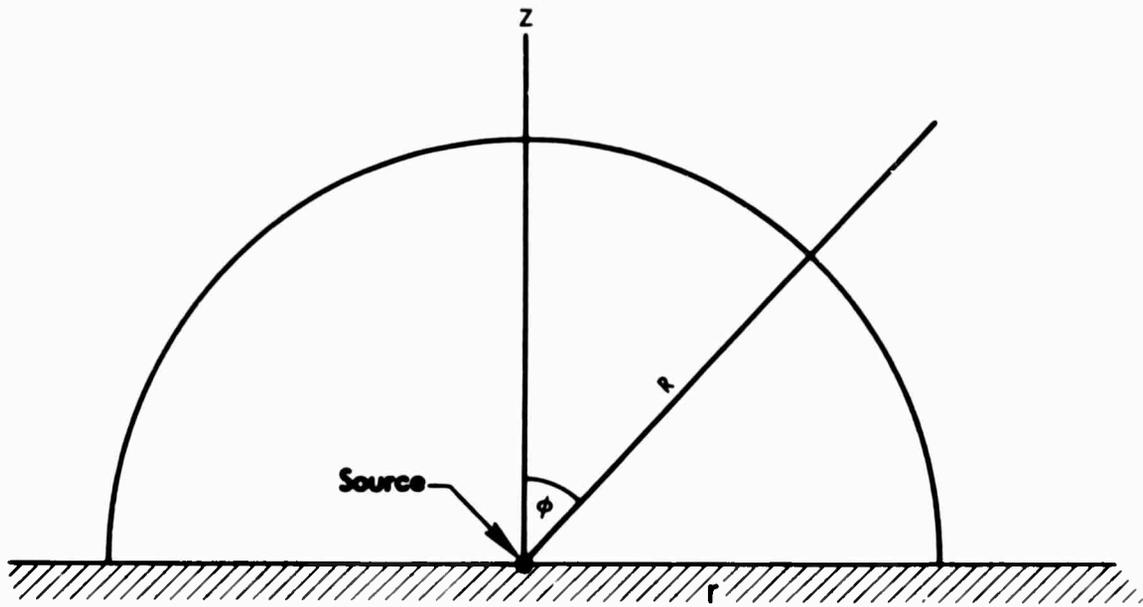


Fig. 1 - Co-ordinate system, with the origin at the point of energy release.

the order of 10 km, $\epsilon \approx 1.5 \times 10^{-5} Y_H$, where Y_H is the hydrodynamic yield in KT. Thus, $\epsilon \ll 1$ for explosions of up to several MT or so.

With h , h/c^* , P^* and ρ^* as the units of length, time, pressure, and density, respectively, an expansion in the small parameter ϵ is carried out. This represents the flow as small changes superimposed on the ambient state, viz.,

$$\begin{aligned} P &= e^{-z} (1 + \epsilon p + \dots) \\ \rho &= e^{-z} (1 + \epsilon \sigma + \dots) \\ \vec{q} &= \epsilon \vec{v} + \dots \\ \vec{v} &= (u, w) \end{aligned} \tag{3}$$

where u is the radial component of the velocity and w is the vertical component. A linearized system is obtained for (p, σ, \vec{v}) , with a δ -function energy source.

A formal integral representation of the solution of the equations of motion is constructed by a combination of Laplace and Hankel transformation. For the pressure field, for example, this has the form

$$p(r, z, t) = e^{z/2} \frac{1}{2\pi} \int_0^\infty J_0(kr) F(k; z, t) k dk \tag{4}$$

$$F(k; z, t) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} e^{-st - \mu(s, k)z} \left[\frac{2-\gamma + \mu}{s^2 + k^2} \right] s ds \tag{5}$$

$$\mu(s, k) = \frac{1}{s} [(s^2 + w_1^2(k))(s^2 + w_2^2(k))]^{1/2}$$

$$\text{Re}(\mu) \geq 0$$

$$\omega_{1,2}(k) = \frac{1}{2} \left[\left(\frac{1}{4} + k^2 + 2\beta k \right)^{1/2} \pm \left(\frac{1}{4} + k^2 - 2\beta k \right)^{1/2} \right]$$

$$\beta = \frac{(\gamma-1)^{1/2}}{\gamma} = \text{Brunt-Väisälä frequency (dimensionless)}$$

where r is the cylindrical radius (Fig. 1).

The pressure field is the main quantity of interest, but similar expressions are readily obtained for the other variables. The usual amplification factor $e^{z/2}$ appears in these expressions. The properties of the medium are contained in the exponential factor $e^{-\mu z}$, while the boundary conditions and the properties of the source are contained in the form of the square brackets. The initial conditions are satisfied by taking the path of integration in (5) to the right of all singularities in the s -plane.

A closed form approximation to the pressure field, valid in the main wave zone for large r and t , can be obtained by asymptotic integration of (4) and (5). The procedure involves the transformation of (5) to a Fourier integral over those frequencies which can propagate for a given (real) k , followed by two successive applications of the method of stationary phase. The method requires, in addition, that the asymptotic approximation for the Bessel function of large argument be used.

The transformation is accomplished by a suitable deformation of the path of integration in the s -plane. The integrand of (5) has branch points at $s = \pm i\omega_1(k)$ and $s = \pm i\omega_2(k)$, poles at $s = \pm ik$, and an essential singularity (due to the μ in the exponent) at $s = 0$. If the s -plane is cut in the manner shown in Fig. 2 and the phase of

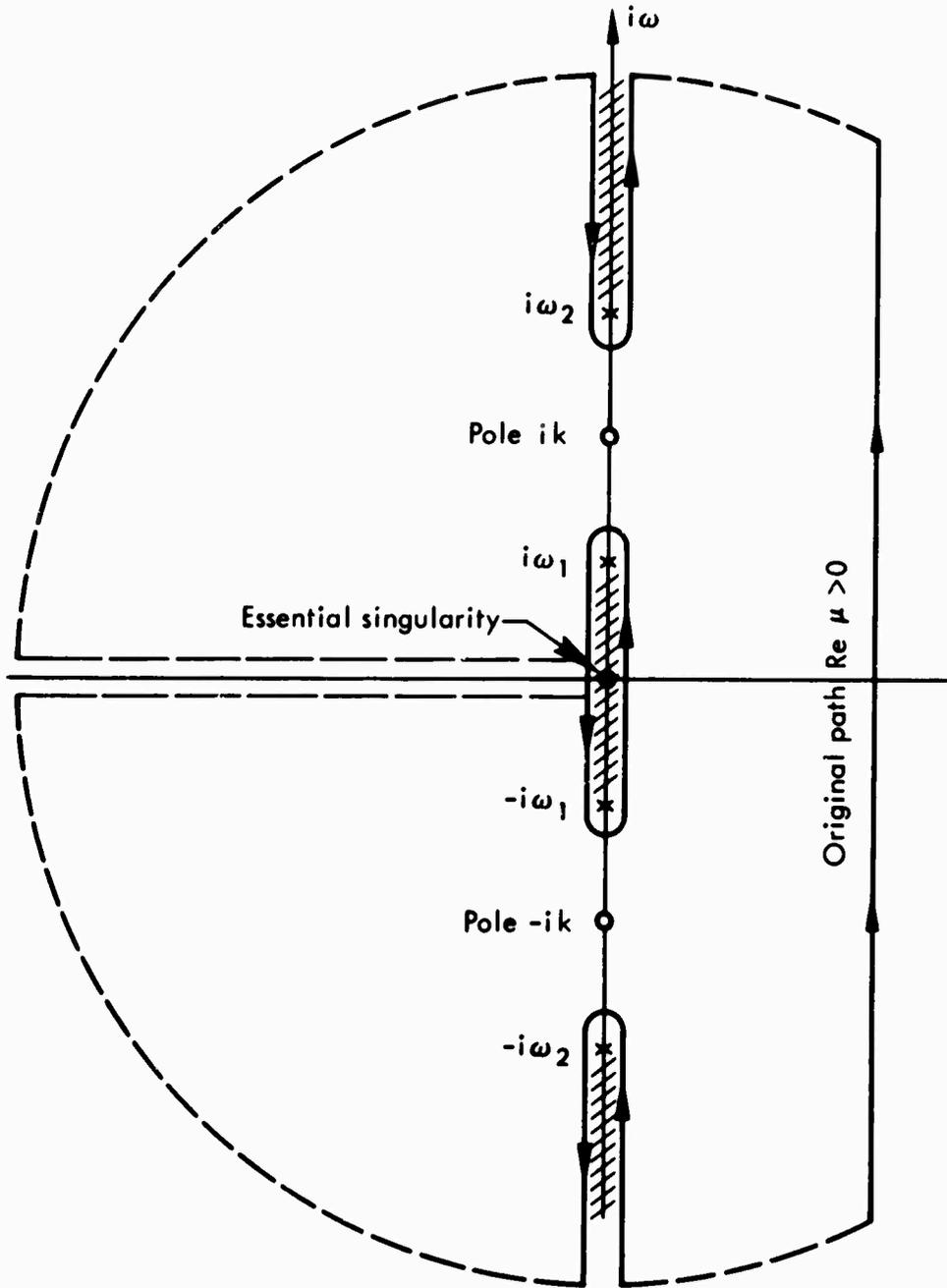


Fig. 2 - Contour for carrying out the integration in the s-plane, with the location of poles, branch points, and essential singularity as shown.

μ suitably defined, the integrand becomes a single valued function of s on and inside the contour. By Cauchy's theorem, then, the integral in (5) is equal to the sum of the integrals around the branch lines plus the contribution from the residues at the poles. The essential singularity is outside the contour, and thus does not contribute a residue.

The contribution from the poles, which arises from the presence of the ground, is easily calculated, and requires no approximation of the Bessel function. The resulting pressure field has the form

$$P_{GR} = -\frac{1}{2\pi} \frac{(2-\gamma)}{\gamma} e^{\frac{(\gamma-1)}{\gamma} z} \frac{t}{(t^2-r^2)^{3/2}} \quad (6)$$

This "ground wave" is a non-oscillatory cylindrical wave, which exists only behind the spherical acoustic front.

The dispersive waves arise from the integrals around the branch lines. A change of the order of integration makes it possible to express these as integrals over positive frequencies in two frequency bands. The pressure field in the low-frequency, or "acoustic-gravity," band is given by

$$P_{AG} = \frac{e^{z/2}}{2\pi^2} \operatorname{Im} \left\{ \int_0^B \omega e^{i\omega t} \int_{k_1(\omega)}^{\infty} G(\omega, k) dk d\omega \right\} \quad (7)$$

while the pressure field in the high-frequency, or "acoustic," band is given by

$$P_A = -\frac{e^{z/2}}{2\pi^2} \operatorname{Im} \left\{ \int_{1/2}^{\infty} \omega e^{i\omega t} \int_0^{k_1(\omega)} G(\omega, k) dk d\omega \right\} \quad (8)$$

where

$$G(\omega, k) = \frac{k J_0(kr)}{(k^2 - \omega^2)} \left[\left(\frac{2-\gamma}{2\gamma} + i\mu \right) e^{-i\mu z} - \left(\frac{2-\gamma}{2\gamma} - i\mu \right) e^{i\mu z} \right]$$

$$k_1(\omega) = \omega \left[\frac{\omega^2 - \frac{1}{4}}{\omega^2 - \beta^2} \right]^{1/2}$$

For r sufficiently large that the asymptotic approximation for the Bessel function is valid, the integral over k may be carried out by the method of stationary phase. The resulting expressions for the pressure field are

$$P_{AG} = e^{z/2} \frac{\cos \phi}{2\pi^2 R} \operatorname{Im} \left\{ \int_{\beta \cos \phi}^{\beta} \frac{\omega (\beta^2 - \omega^2) (1/4 - \omega^2)^{1/2}}{D(\omega, \phi)} d\omega \right\} \quad (9)$$

$$\left\{ \left(\frac{2-\gamma}{2\gamma} + i\mu_s \right) e^{i(\omega t + \Omega R)} - \left(\frac{2-\gamma}{2\gamma} - i\mu_s \right) e^{i(\omega t - \Omega R)} \right\} d\omega$$

$$P_A = e^{z/2} \frac{\cos \phi}{2\pi^2 R} \operatorname{Re} \left\{ \int_{1/2}^{\infty} \frac{\omega (\omega^2 - \beta^2) (\omega^2 - 1/4)^{1/2}}{D(\omega, \phi)} d\omega \right\}$$

$$\left\{ \left(\frac{2-\gamma}{2\gamma} + i\mu_s \right) e^{i(\omega t - \Omega R)} + \left(\frac{2-\gamma}{2\gamma} - i\mu_s \right) e^{i(\omega t + \Omega R)} \right\} d\omega$$

(10)

where

$$D(\omega, \phi) = (\omega^2 - \beta^2)^2 \cos^2 \phi + \left(\frac{1}{4} - \beta^2\right) \omega^2 \sin^2 \phi$$

$$\mu_B = \cos \phi \left[\frac{(\frac{1}{4} - \omega^2)(\beta^2 - \omega^2)}{(\omega^2 - \beta^2 \cos^2 \phi)} \right]^{1/2}$$

$$\Omega(\omega, \phi) = \left[\frac{(\frac{1}{4} - \omega^2)(\omega^2 - \beta^2 \cos^2 \phi)}{(\beta^2 - \omega^2)} \right]^{1/2}$$

This representation is an expansion in spherical waves, with a spherical phase and group velocity depending on the pole angle ϕ . It is interesting to note that the exponents in the spherical wave representations (9) and (10) are identical with those in the exact free-space Green's function obtained by Pierce (1963) and Row (1967).

One final point worthy of note is that (9) and (10) both contain the factor $\cos \phi$. Thus, in this approximation, the dispersive wave train vanishes at the ground, where the signal consists entirely of the non-oscillatory cylindrical wave given by (6).

III. PROPERTIES OF ASYMPTOTIC PRESSURE FIELD
AND NUMERICAL RESULTS

The pressure field associated with the dispersive waves has been expressed in terms of integrals of the form

$$\int f(\omega, \phi) e^{i(\omega t \pm \Omega R)} d\omega \quad (11)$$

over the two bands of propagating frequencies. For large R (spherical radius) and t , the exponentials are rapidly oscillating functions of ω , and the integrals therefore permit a second application of the method of stationary phase. It is easily shown that stationary points can exist only for the exponentials with the minus sign. Such points must be solutions of

$$\frac{t}{R} - \frac{d}{d\omega} \Omega(\omega, \phi) = 0 \quad (12)$$

and are therefore functions of R/t and ϕ , only. Some important properties of the flow field can be inferred from the distribution of stationary points in $(R/t, \phi)$ -space. This has been discussed in some detail by Cole and Greifinger (1968), and will be briefly summarized at this point to provide a better understanding of the numerical results to follow.

Obviously, all points of stationary phase must lie behind the acoustic front, i.e., inside a circle of radius $R/t = 1$ in the $(R/t, \phi)$ -plane. In the acoustic band, $1/2 \leq \omega < \infty$, there is a single stationary point associated with each point in the $(R/t, \phi)$ -plane

behind the acoustic front. Along any ray in this plane, i.e., for any fixed value of ϕ , the frequency of stationary phase decreases monotonically from $\omega \rightarrow \infty$ at the acoustic front to $\omega = 1/2$ at the origin.

In the acoustic-gravity band, $\beta \cos \phi \leq \omega \leq \beta$, the situation is quite different. For any given angle ϕ , there are two stationary points for all R/t smaller than some maximum value. At this maximum value of R/t , the two stationary points coincide, while for larger values of R/t (at the given ϕ), there are no stationary points. Along a ray in the $(R/t, \phi)$ -plane, the two frequencies of stationary phase diverge from their common value at coincidence, one increasing toward β and the other decreasing toward $\beta \cos \phi$ as R/t tends toward the origin.

The situation for the acoustic-gravity band is identical to that which arises in the case of incompressible flow in a density stratified liquid considered by Mowbray and Rarity (1967). As in that case, the locus of double (coincident) points of stationary phase defines a front, or "caustic," representing the onset of the disturbance. Between the caustic and the acoustic front, the acoustic-gravity band is exponentially small, and the main contribution comes from the acoustic band.

The location of the caustic can be calculated, parametrically, from its definition as the locus of double stationary points. The result is shown in Fig. 3, where the caustic has been plotted in cylindrical $(r/t, z/t)$ -space. Also plotted (dashed lines) are lines of constant $\phi = \frac{1}{t} (\omega t - R\Omega)$. These are lines along which the phase

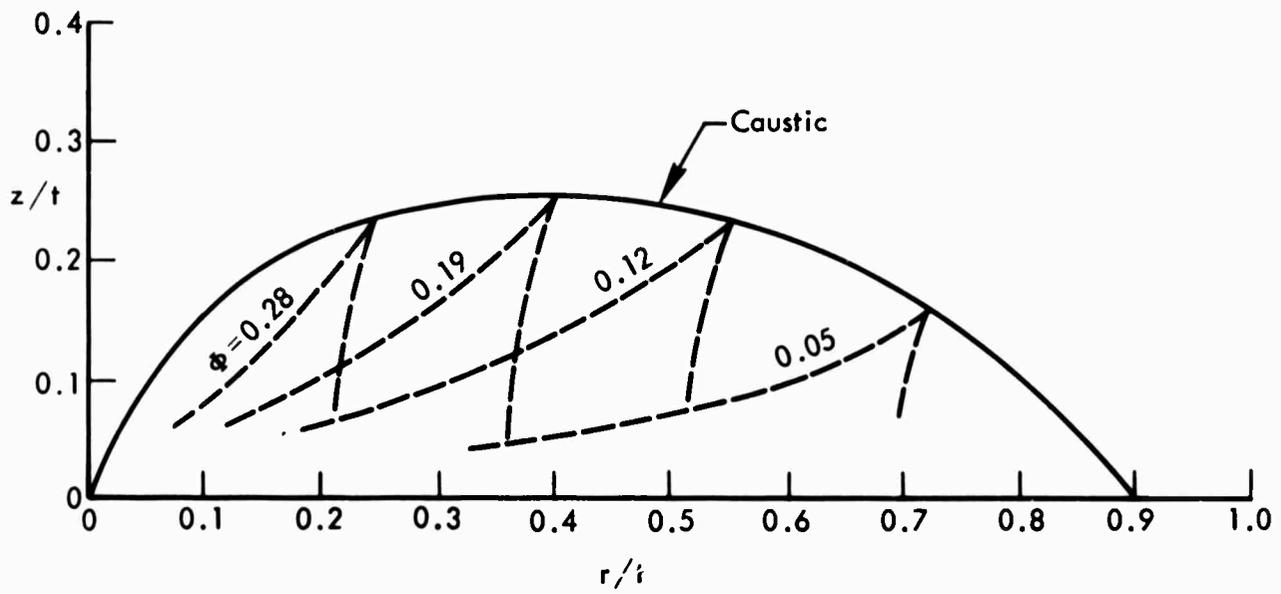


Fig. 3 - Location of the caustic in $(r/t, z/t)$ -space. The dashed lines are lines along which the phase is constant at any instant.

is constant at a given t , and intersect the caustic in a cusp. Crests (or troughs), as particular lines of constant phase, are therefore representable by such curves. However, since lines of constant phase are lines along which ϕ decreases with time, crests are not stationary in this diagram, but move across it obliquely from left to right. The number of crests in the diagram increases linearly with time, new crests being continuously created at the origin.

From the above considerations, the frequency-time history of the signal at any location can be described qualitatively. It is to be remembered that the frequency of stationary phase is the principal frequency associated with the group of waves at the location in question. Furthermore, a given point (R, ϕ) in physical space corresponds, as time increases, to points approaching the origin along a ray in the $(R/t, \phi)$ -plane. The variation of the frequency of stationary phase along such a ray thus provides the frequency-time history of the signal at the point (R, ϕ) . Clearly, this history is qualitatively the same for all locations at the same angle ϕ , which all map into the same ray in $(R/t, \phi)$ -space. Thus, the first disturbance to arrive at any location (R, ϕ) is the acoustic spherical front, containing the very high frequencies. The principal frequency in this part of the disturbance decreases with time, tending asymptotically towards $\omega = 1/2$. Some time after the passage of the spherical front, the caustic arrives, carrying the frequency of the double stationary point associated with the angle ϕ . This part of the signal then splits into two frequencies, one increasing toward β and the other decreasing toward $\beta \cos \phi$ as time progresses. Thus, the signal at any (R, ϕ) consists,

after long times, of the three angular frequencies $1/2$, β , and $\beta \cos \phi$.

Such considerations, of course, cannot provide the shape of the signal, which depends on the relative amplitudes and phases of the contributing frequencies. This can be determined only by actual evaluation of (9) and (10). These integrals have been calculated by the method of stationary phase, and the results are presented in Figs. 4, 5, and 6. The numerical results are based on a specific heat ratio of $\gamma = 1.4$ and an atmospheric scale height of $h = 8$ km, values which are more appropriate to sea level than to ionospheric altitudes. However, no attempt is being made here to correlate these results with observation, the purpose being primarily to illustrate the dispersion of the pulse with distance from the source.

In Fig. 4, the three separate frequency components are shown for the signal at a typical location. It is apparent that, after the arrival of the acoustic-gravity components, the dominant frequency is the lowest frequency component, i.e., the component that tends toward $\omega = \beta \cos \phi$ as time increases. It is this part of the signal which has been calculated approximately by Row (1967) for an unbounded isothermal atmosphere. Figure 5 shows how the pulse spreads out with lateral distance from the source at a fixed altitude, while Fig. 6 illustrates the change in shape of the pulse with altitude at a fixed lateral distance from the source. Although the principal contribution to the signal is from the lowest frequency component, the shape of the signal is noticeably affected by the presence of the higher frequency components. The break in each curve, which occurs at the

arrival of the caustic, is a consequence of the inapplicability of the stationary phase approximation in the vicinity of the caustic. A more accurate approximation in the neighborhood of the caustic would show a large, but continuous, change in amplitude with passage of the caustic.

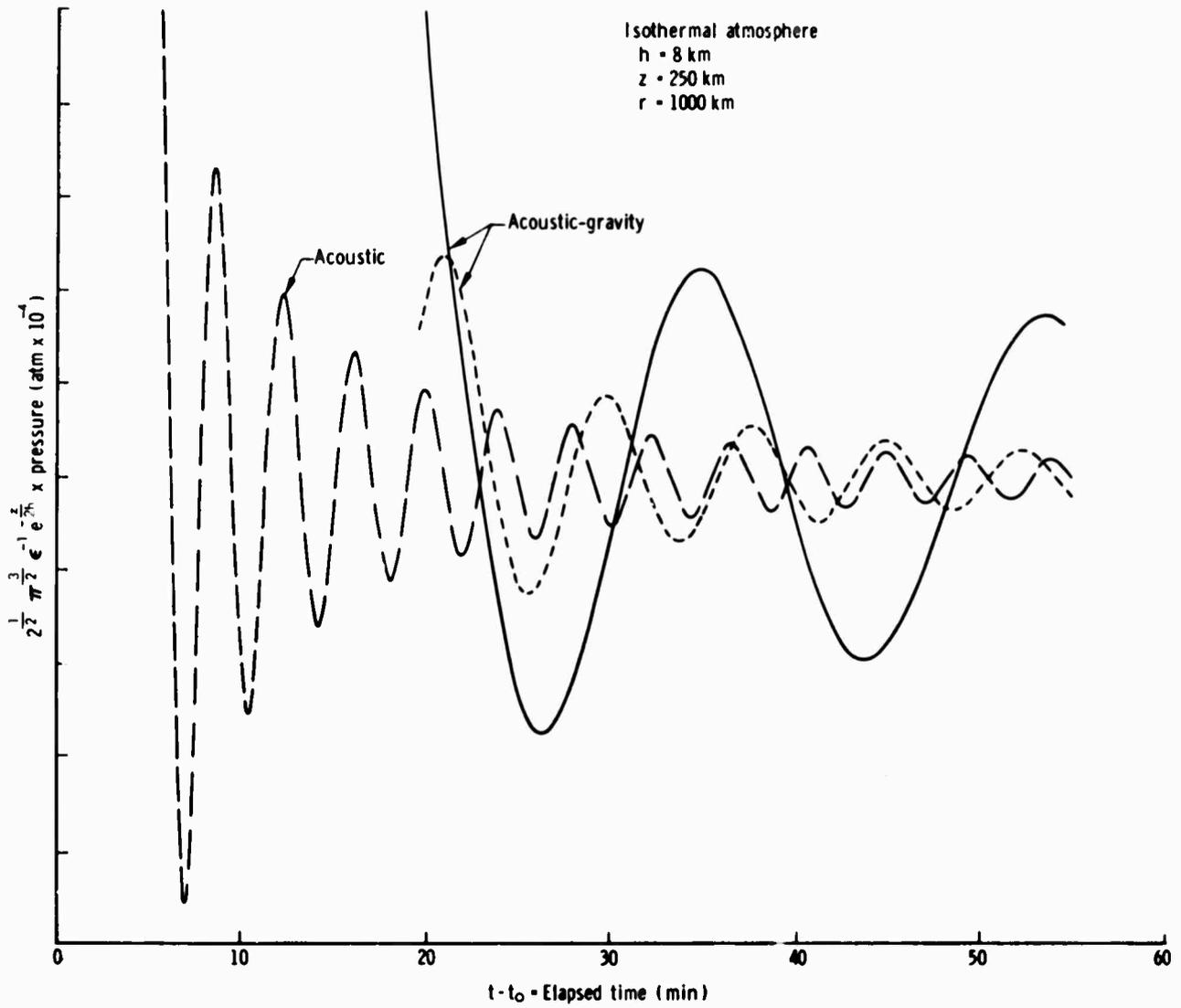


Fig. 4 - Resolution of the pressure amplitude at a typical location into its three components. The elapsed time measures the time after the arrival of the acoustic front.

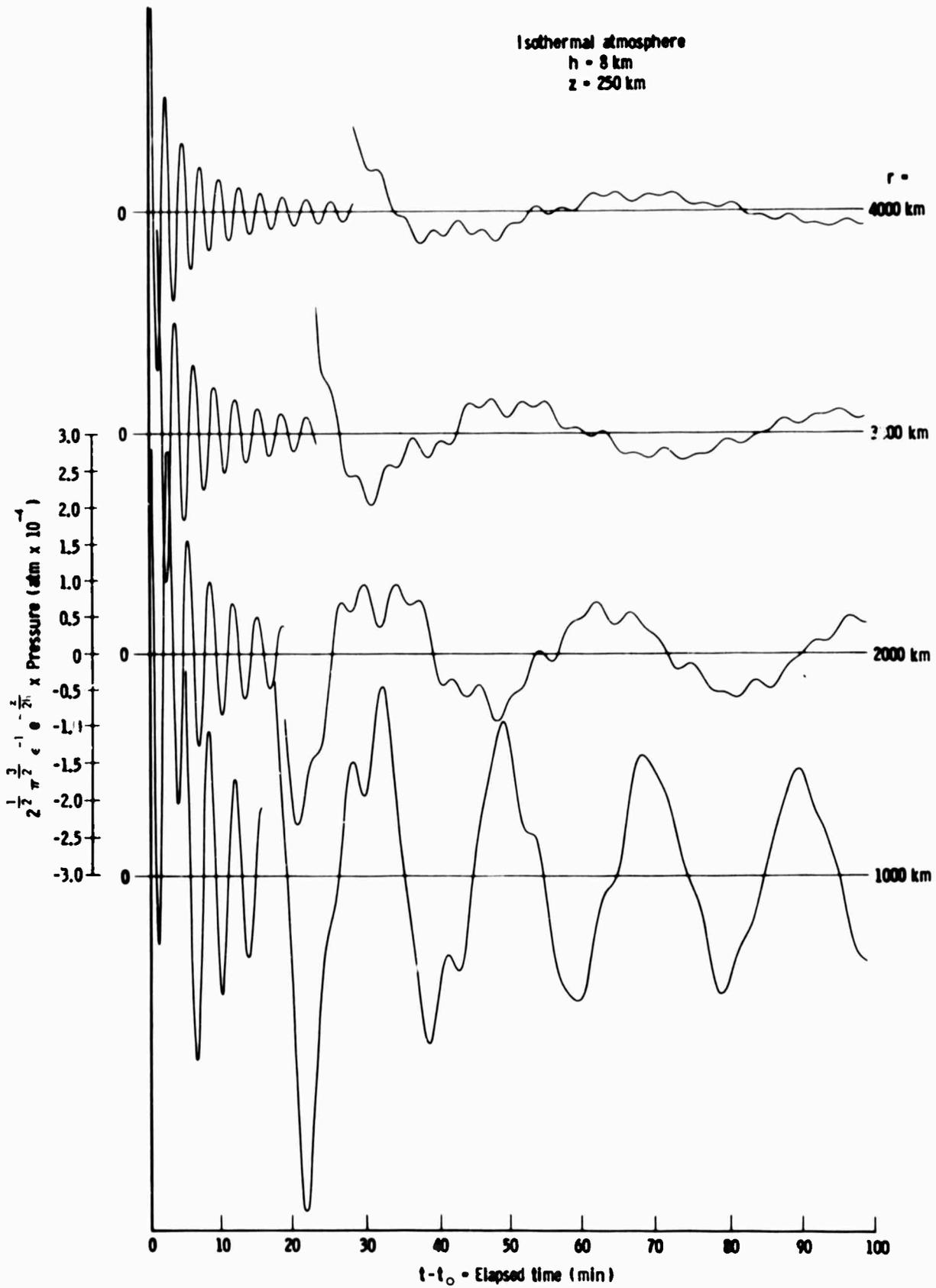


Fig. 5 - Response of an isothermal atmosphere to an energy source at the ground, for fixed altitude and varying range. The elapsed time measures the time after the arrival of the acoustic front at the given location.

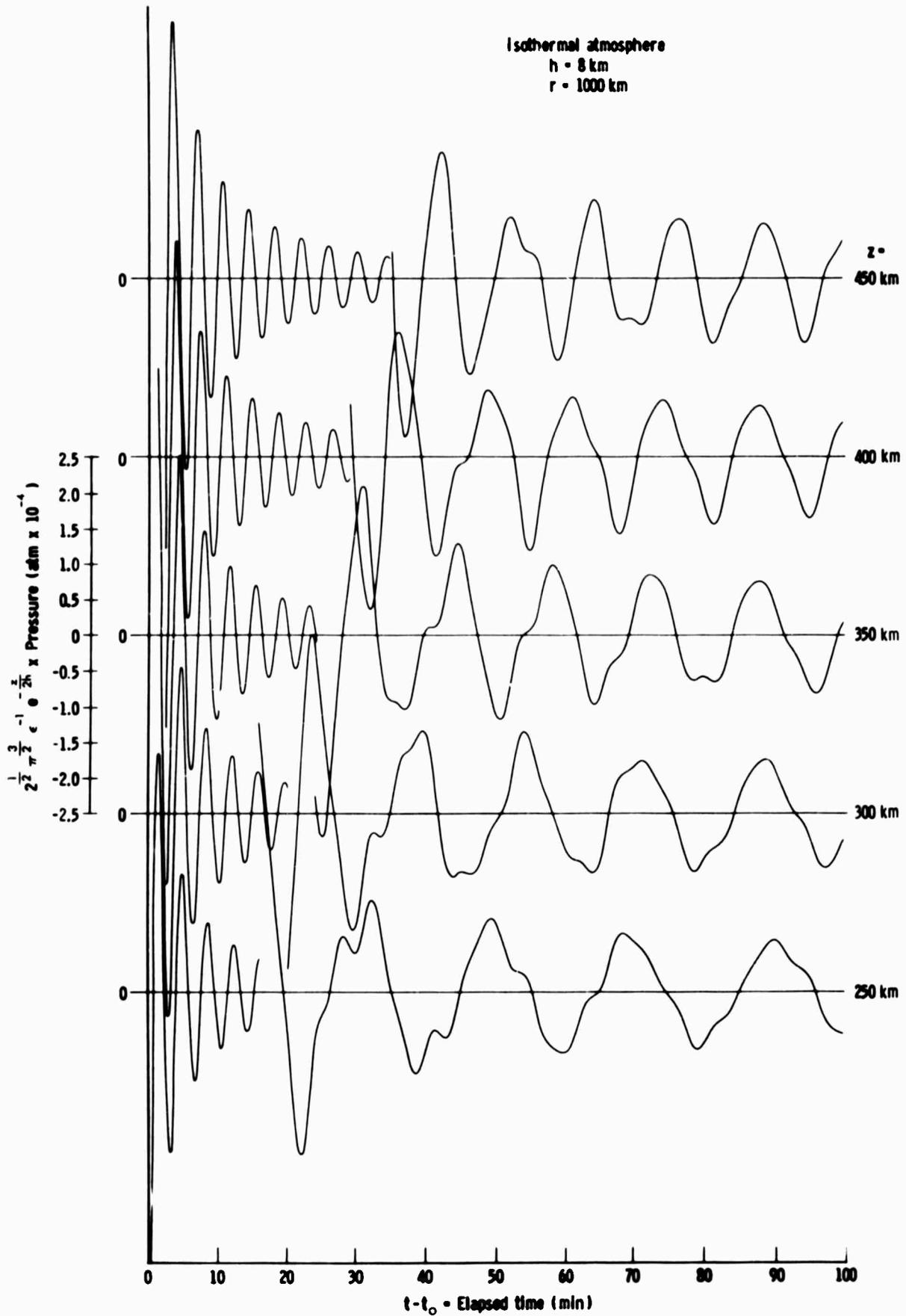


Fig. 6 - Response of an isothermal atmosphere to an energy source at the ground, for fixed range and varying altitude. The elapsed time measures the time after the arrival of the acoustic front at the given location.

IV. KINEMATIC DESCRIPTION OF DISPERSIVE WAVE TRAIN AND CAUSTIC

In this section, it will be shown how the overall geometric features of the dispersive wave pattern in the asymptotic field can be obtained from the dispersion relation and the kinematic theory of Whitham (1961). In fact, although it will not be demonstrated explicitly here, inclusion in the theory of energy considerations (Whitham, 1961) makes it possible even to determine the relative amplitudes of the various frequency components. Thus, only the precise phase relationships are not directly obtainable from the theory. In every other respect, the results obtained by kinematic theory are identical with those obtained by the method of stationary phase. This is not surprising in view of the close connection between the assumptions underlying the two methods (Lighthill, 1965). However, the equivalence is of more than academic interest, since kinematic theory seems more readily adaptable to the case of a slightly non-uniform* medium than is the method of stationary phase.

As far as kinematic theory is concerned, all of the dynamics is contained in the dispersion relation

$$\omega = \omega(\vec{k}) \quad (13)$$

which is a functional relationship between frequency and wave number \vec{k} . The principal result of the kinematic theory is that, for a homogeneous medium with a dispersion relation of the form (13), groups of waves of given wave number (and therefore also of given frequency)

*The isothermal atmosphere is, of course, not a uniform medium. However, it can be made to "look" uniform by a suitable choice of dependent variables.

propagate with constant group velocity \vec{c}_g , given by the gradient of the dispersion relation

$$\vec{c}_g = \vec{\nabla}_k \omega(\vec{k}) . \quad (14)$$

Thus, if all the waves originate at the origin, a given group travels radially outward from the origin with constant speed (provided no reflections take place at a boundary, as would be the case if the source were above the ground). Thus, the group velocity is $\vec{c}_g = \vec{R}/t$, where \vec{R} is the position of the group at time t . These relations make it possible to determine the location in (\vec{R}, t) of groups of given wave number and frequency. This procedure will now be carried out in some detail for the case in question.

It can be shown from Cole and Greifinger (1968) (cf Pierce, 1963) that the homogeneous wave equation for the quantity $\bar{p} = e^{-z/2} p$, where p is the pressure perturbation defined by (3), is

$$\left[\frac{\partial^4}{\partial t^4} - \frac{\partial^2}{\partial t^2} \left(\frac{\partial^2}{\partial z^2} - \frac{1}{4} \right) - \left(\frac{\partial^2}{\partial t^2} + \beta^2 \right) \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \right] \bar{p} = 0. \quad (15)$$

For plane wave solutions of the form $\exp[i(k_x x + k_y y + k_z z - \omega t)]$, the resulting dispersion relation can be written

$$\omega^2 = \frac{1}{2} (k_r^2 + k_z^2 + \frac{1}{4}) \pm \left[\frac{1}{4} (k_r^2 + k_z^2 + \frac{1}{4})^2 - \beta^2 k_r^2 \right]^{1/2} \quad (16)$$

$$k_r^2 = k_x^2 + k_y^2 .$$

The plus sign identifies the high frequency acoustic branch, while the minus sign is associated with the low frequency acoustic-gravity branch. The kinematics apply separately to each branch, leading to the possibility of coexisting families of waves.

For the purposes of calculation, it is more convenient to write the dispersion relation in the form

$$k_r^2 = \omega^2 \frac{(k_z^2 + \frac{1}{4} - \omega^2)}{(\beta^2 - \omega^2)} \quad (17)$$

treating k_r as the dependent variable and ω and k_z as the independent variables. In this form, the acoustic branch corresponds to frequencies $\omega^2 \geq k_z^2 + \frac{1}{4}$ and the acoustic-gravity branch to frequencies $\omega^2 \leq \beta^2$. In terms of these variables, the components of the group velocity take the form

$$c_{g_r} = \frac{r}{t} = \frac{k_r}{\omega \left[\frac{\partial(k_r^2)}{\partial(\omega^2)} \right]} \quad (18)$$

$$c_{g_z} = \frac{z}{t} = \frac{k_z \left[\frac{\partial(k_r^2)}{\partial(k_z^2)} \right]}{\omega \left[\frac{\partial(k_r^2)}{\partial(\omega^2)} \right]}$$

Furthermore, since the groups travel radially outward, the direction (polar angle) ϕ in which the group propagates is given by $\tan \phi = r/z$. From these relationships, it is possible to express the components

of the wave number of a given group in terms of the frequency and the angle at which the group propagates; thus,

$$k_r = \left[\frac{\omega^2 \left(\frac{1}{4} - \omega^2 \right) \sin^2 \phi}{(\beta^2 - \omega^2)(\omega^2 - \beta^2 \cos^2 \phi)} \right]^{1/2} \quad (19)$$

$$k_z = \left[\frac{\left(\frac{1}{4} - \omega^2 \right) (\beta^2 - \omega^2) \cos^2 \phi}{(\omega^2 - \beta^2 \cos^2 \phi)} \right]^{1/2}$$

The requirement that k_r and k_z be real serves to define the allowed frequency bands for groups of waves propagating in a given direction. These bands are shown in Fig. 7, where k_r and k_z are plotted against ω for a given ϕ . The frequency range of the acoustic band is $1/2 \leq \omega < \infty$, while for the acoustic-gravity band the range is $\beta \cos \phi \leq \omega \leq \beta$. The cutoff frequencies for the propagation of groups in a given direction are thus identical with those obtained by the method of stationary phase described earlier.

With k_r and k_z given by (19), it is now possible to express the components of the group velocity (18) in terms of ω^2 and ϕ . The resultant group velocity is

$$c_g = \frac{R}{t} = \frac{[(\frac{1}{4} - \omega^2)(\beta^2 - \omega^2)(\omega^2 - \beta^2 \cos^2 \phi)]^{1/2}}{\omega [(\beta^2 - \omega^2)^2 + \beta^2 (\frac{1}{4} - \omega^2) \sin^2 \phi]} \quad (20)$$

This result agrees exactly with that derived by Pierce (1963) from the free-space Green's function.

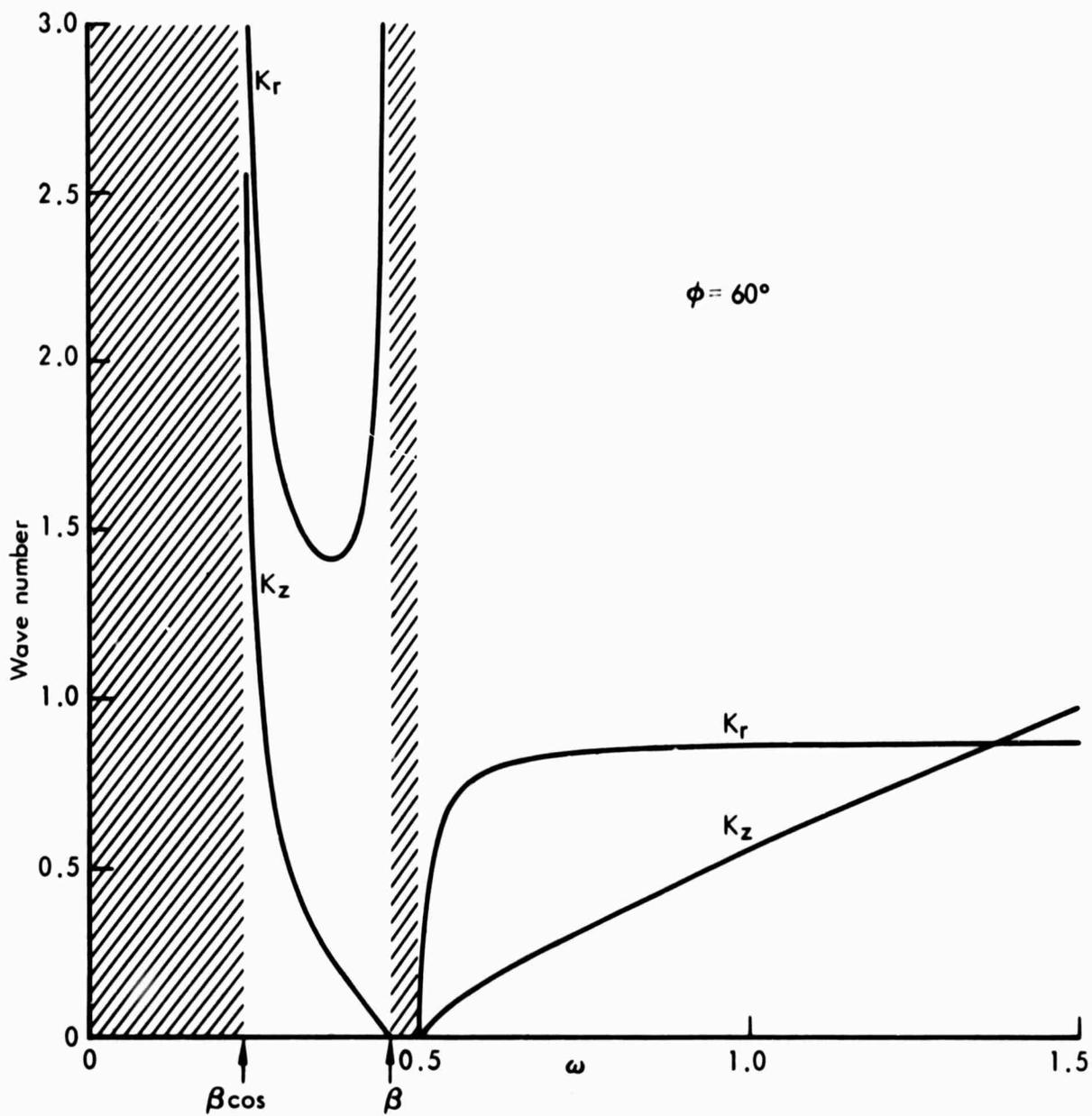


Fig. 7 - Vertical and radial wave number (units of 1/h) as a function of angular frequency (units of c^*/h).

In Fig. 8, c_g is plotted against ω for a given value of ϕ . Equivalently, this is a plot of R/t against ω , where (R,t) is the location in physical space (at the angle ϕ) of the group of frequency ω . In the acoustic band, the group velocity increases monotonically from 0 to 1 as the frequency increases from $1/2$ to ∞ . Thus, there is one, and only one, frequency in this band at each location between the origin and the acoustic front. In the acoustic-gravity band, on the other hand, the group velocity is zero at both ends and has a single maximum in between. Thus, there is a maximum $R/t < 1$ to which waves in this band can propagate, or, equivalently, a fastest group of waves. For all smaller values of R/t , there are two groups of waves from the acoustic-gravity band (in addition to the single group from the acoustic band). These results are identical with those obtained by the stationary phase approximation.

The occurrence of a front, or caustic, appears in the kinematic theory as a maximum in the group velocity in any direction. The condition for the caustic is therefore

$$\frac{\partial c_g(\omega, \phi)}{\partial \omega} = 0 \quad (21)$$

which serves to define the frequency $\omega_c(\phi)$ at the caustic in a given direction. If this is combined with (18), an equation for the caustic can be obtained. The result again agrees exactly with the stationary phase calculation of Cole and Greifinger (1968).

It can, in fact, be demonstrated more generally that the equation for a caustic obtained by kinematic theory is identical with that

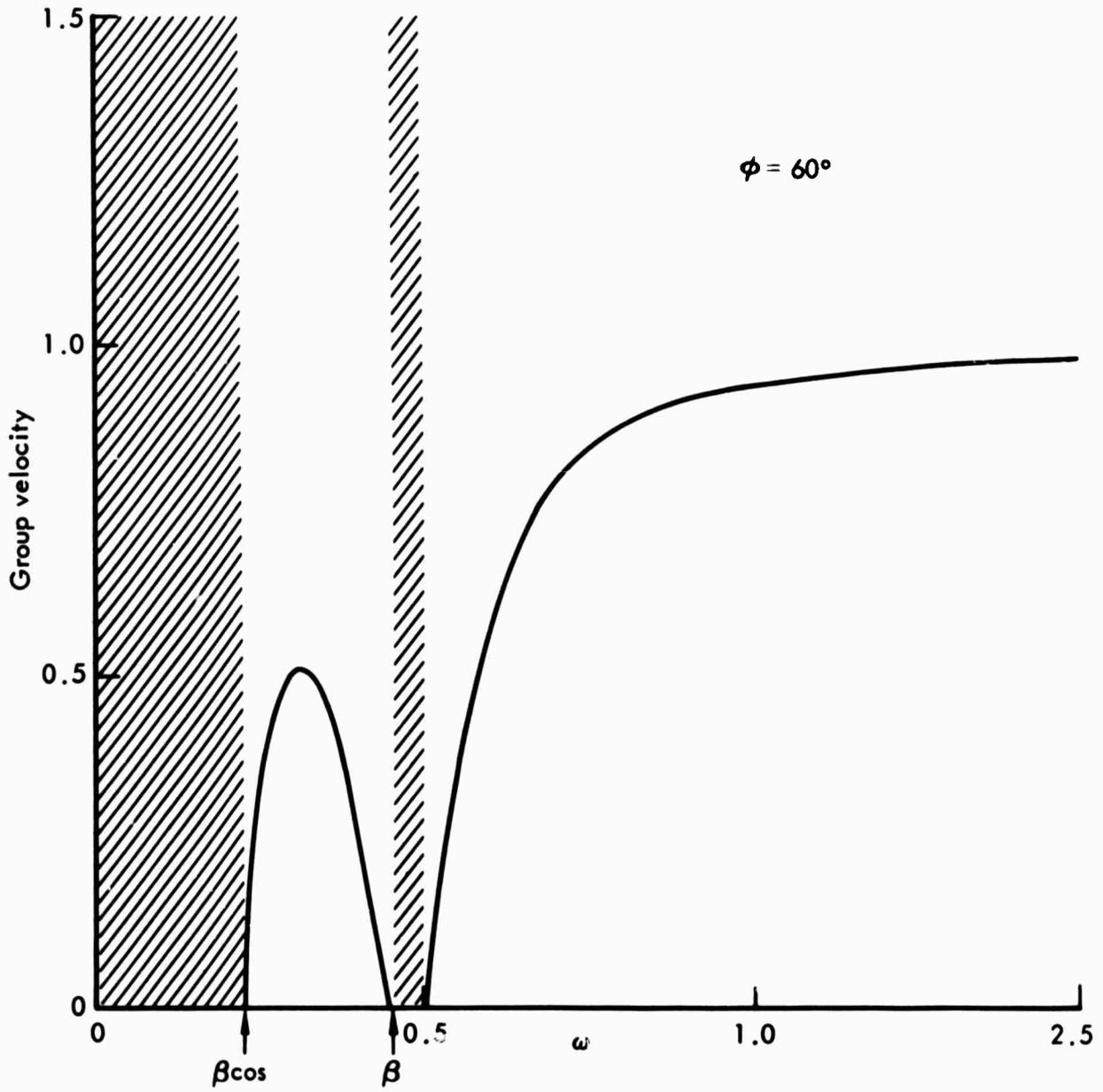


Fig. 8 - Group velocity (units of c^*) as a function of angular frequency (units of c^*/h).

obtained by the method of stationary phase. Since the group velocity is in the radial direction, the condition that the group velocity be a maximum in a given direction is equivalent to the condition that the derivative of the group velocity be zero in the direction of the group velocity. The condition for a caustic can thus be written (in k-space)

$$\vec{c}_g \cdot \vec{\nabla}_k \vec{c}_g = \vec{\nabla}_k \left(\frac{c^2}{2} \right) = 0 \quad (22)$$

where the first equality follows from the usual vector relationships and $\vec{\nabla}_k \times \vec{c}_g = 0$. In terms of Cartesian components, this becomes

$$\sum_{j=1}^3 \frac{\partial^2 \omega}{\partial k_i \partial k_j} \frac{\partial \omega}{\partial k_j} = 0 \quad (i = 1, 2, 3). \quad (23)$$

The system (23) for the three components $\frac{\partial \omega}{\partial k_j}$ of the group velocity has non-trivial solutions only if the determinant of the coefficients vanishes, i.e.,

$$\left| \frac{\partial^2 \omega}{\partial k_i \partial k_j} \right| = 0. \quad (24)$$

This condition for a caustic is identical with that deduced from very general considerations of stationary phase (Lighthill, 1965).

In the kinematic theory, there is no acoustic-gravity signal ahead of the caustic. As mentioned in the preceding section, in the stationary phase approximation of the exact integral representation, the signal ahead of the fastest group is exponentially small. Moreover, a more sophisticated evaluation of such integrals (Lighthill,

1965) provides an accurate description in the immediate neighborhood of the front. The kinematic theory, however, does give the correct qualitative picture of the dispersive wave trains behind the front as well as the exact location of the front itself. In that it can be easily generalized to a medium with slowly varying properties, it should prove a useful method in treating such problems.

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10. ABSTRACT An analysis by two different methods --stationary phase and kinematic theory --of the pressure pulse generated by a nuclear explosion near the ground in an isothermal atmosphere. The present study builds upon work reported in RM-5738-ARPA/AFT . The shape of the signal is shown, as a function of time, at several distances from the source for a fixed altitude, and at several altitudes for a fixed lateral distance. The first signal to arrive at any location is a high-frequency acoustic wave, followed by a low-frequency acoustic-gravity wave. The onset of the latter is marked by a sharp front, or caustic. At any instant, after the arrival of the caustic, there are three principal frequency components at any location, the lowest of which becomes dominant as time progresses. It is shown how the qualitative features of the flow, as well as the exact location of the caustic, can be obtained from kinematic theory.	11. KEY WORDS Physics Nuclear explosions Nuclear effects Geophysics Sound Wave propagation	