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HYPERSONIC STRONG INTERACTION SIMILARITY SOLUTIONS FOR FLOW PAST A VERY SLENDER AXISYMMETRIC BODY

W. B. Bush

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The RAND Corporation
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PREFACE

Hypersonic strong-interaction theory provides a framework for the analysis of the flow past a slender body during portions of its hypersonic flight through the atmosphere. Performance predictions and scaling laws following from such an analysis can be used in trajectory and discrimination studies.

In this Memorandum, a theoretical study is made of the laminar flow past a very slender symmetric body when the Mach and Reynolds numbers are large, in which case Navier-Stokes equations are reasonable models for the equations of motion for a viscous heat-conducting gas. It is well known that the flow-field structure for two-dimensional bodies exhibits important qualitative differences from that for axisymmetric bodies, and one objective of this study is to investigate these differences and to develop a consistent theoretical solution of the equations of motion. Matched asymptotic expansion techniques are employed to develop the significant features of the flow field, and simple expressions are presented for drag and for heat transfer in the extreme case where the body diameter is much less than the thickness of the viscous layer supported by the body.

In addition to its relevance for discrimination and performance studies, the present Memorandum should be of interest to those involved in the numerical computation of viscous flows, since it provides a difficult test case that can be used to make needed checks on numerical work.

This study, sponsored by the Advanced Research Projects Agency, is part of continuing RAND Corporation research on hypersonic gasdynamics for application to early reentry phenomenology.

SUMMARY

The Navier-Stokes hypersonic strong-interaction theory is presented for the flow of a viscous, heat-conducting, compressible fluid past a very slender axisymmetric body, where D_b, the measure of the ratio of the radius of the body to the thickness of the viscous layer(s) (which the body produces and supports), goes to zero. It is assumed that the fluid is a perfect gas having constant specific heats, a constant Prandtl number, σ, whose numerical value is of order unity, and viscosity coefficients varying as a power, ω, of the absolute temperature. Limiting forms of the solutions are studied as the free-stream Mach number, M, the free-stream Reynolds number based on the body's axial length, R_L, and the modified interaction parameter, $\chi' = \left\{M^{2(2+\omega)}/R_L \log(1/D_b)\right\}^{1/2}$, go to infinity.

By means of matched asymptotic expansions, it is shown that, for $(1 - \omega) > 0$, four distinct layers make up the region between the shock wave and the body. The self-similar leading approximations for the behavior of the flow in these four regions are analyzed, and it is found that an arbitrary body supports 3/4-power viscous and shock layers.

ACKNOWLEDGMENTS

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CONTENTS

SYMBOLS*

$C_{D,f}$ = friction drag coefficient in Eq. (47a)

$C_{D,p}$ = pressure drag coefficient in Eq. (47b)

C_f, C_q = friction, heat-transfer coefficients in Eqs. (46b) and (46c)

C_p = pressure coefficient in Eq. (46a)

c_{v_1}, c_{p_1} = dimensional specific heats

D_b = δ_b/δ, thickness ratio parameter

k = k_1/k_∞, nondimensional heat-conduction coefficient

L = characteristic longitudinal body length

M = $u_\infty/(\gamma p_\infty/\rho_\infty)^{1/2}$, free-stream Mach number

p = p_1/p_∞, nondimensional pressure

R_L = $\rho_\infty u_\infty L/\mu_\infty$, free-stream Reynolds number

r = r_1/L, nondimensional radial distance, measured from axis of symmetry of body

s_b, t_b = nondimensional independent variables, BVL in Eq. (26a)

s_k, t_k = nondimensional independent variables, PVL in Eq. (15)

T = T_1/T_∞, nondimensional temperature

u, v = u_1/u_∞, v_1/u_∞, nondimensional axial and radial velocity components

$\bar{V}_L$ = $\left(M^{2\omega}/R_L\right)^{1/2}$, rarefaction parameter

$\bar{V}_x$ = $\left(M^{2\omega}/R_L x\right)^{1/2}$, rarefaction parameter

x = x_1/L, nondimensional axial distance, measured from vertex of body

α_k = expansion parameter, PVL

δ = thickness ratio of "effective body"

δ_b = thickness ratio of actual body

*The basic nomenclature is given here. Additional quantities are defined as they are introduced.

ϵ = $(\gamma-1)/(\gamma+1)$, Newtonian parameter

θ_S = $\epsilon\{(1+\epsilon)/(1-\epsilon)\}\ M^2$, stagnation temperature parameter

Λ_k = $[\epsilon\{(1+\epsilon)/(1-\epsilon)\}]^{1+\omega}\ \bar{V}_L^2\ \tau_b/\delta^4$, parameter, PVL

λ = λ_1/μ_∞, nondimensional second viscosity coefficient

μ = μ_1/μ_∞, nondimensional first viscosity coefficient

ρ = ρ_1/ρ_∞, nondimensional density

σ = $\mu_1 c_{p_1}/k_1$, Prandtl number

τ_b = $[\log(1/D_b)]^{-1}$, modified thickness ratio parameter

φ_w = T_w/θ_S, nondimensional wall temperature

χ' = $\left\{M^{2(2+\omega)}/R_L\ \log\ (1/D_b)\right\}^{1/2}$, modified interaction parameter

ω = viscosity exponent

Subscripts

b body viscous layer (BVL)

h inviscid shock layer (ISL)

ht ISL-TL matching

k principal viscous layer (PVL)

kb PVL-BVL matching

kt PVL-TL matching

t viscous transition layer (TL)

w body surface

1 dimensional quantity

∞ free-stream dimensional quantity

I. INTRODUCTION

The purpose of this Memorandum is to enlarge upon the existing hypersonic strong-interaction theory (HSIT) for viscous, compressible flow past a slender axisymmetric body, whose cross-sectional radius is much less than the thickness of the viscous layer(s), which it produces and supports.

The existing HSIT for such a flow regime was first formulated by Stewartson (1964) for a slender cone using modified von Mises coordinates. [Though Stewartson's formulation is for a cone, his approach is a modification of the formulation for a "3/4-power" body -- the body for which similarity solutions have been shown to exist when the body radius and the thickness of the viscous layer are of the same magnitude (Yasuhara 1956, 1962; Luniev 1960).] Succeeding analyses by Solomon (1967) and Ellinwood and Mirels (1967) have followed the path laid down by Stewartson, employing the Lees-Dorodnitsyn "boundary layer coordinates," which are not especially appropriate for the flow problem under consideration (as the latter authors point out).

In the Stewartson-Solomon-Ellinwood and Mirels (SSEM) formulation a shock wave is postulated, and between the shock wave and the body there are three principal zones of interest: (1) the thin inviscid zone, or inviscid shock layer (ISL), described by hypersonic small-disturbance theory (HSDT), bounded on one side by the shock wave and on the other by the viscous layers on the actual body; (2) the thin outer viscous zone, or principal viscous layer (PVL), comparable in thickness to the ISL, in which the fluid velocity and temperature are slightly changed from their ISL values; and (3) the inner viscous zone, or body viscous layer (BVL) -- which is implicitly of the thickness of the body, and, hence, thinner than either the ISL or the PVL -- in which the shear and heat-conduction terms dominate the tangential momentum and energy equations, respectively.

In the SSEM formulation, the matching of the solutions for the various zones, or layers, is not treated formally in either the ISL-PVL or the PBL-BVL case.

In the following sections, a rigorous and more complete treatment of the very slender axisymmetric body HSIT is presented for the case where self-similarity exists throughout the flow field. In Section II, the Navier-Stokes equations are given in cylindrical polar coordinates. In Section III, the similarity HSDT for the ISL is given under the presumption that the "effective body," made up of the viscous layers and the actual body, is a power-law body. The treatment is essentially that given by Stewartson (1964). In Section IV, the similarity formulation of the "Oseen-like" PVL is presented. This formulation follows quite closely that for Stewartson's outer viscous zone, except that it introduces a modified radial variable, which simplifies the analysis of the axisymmetric Navier-Stokes equations. In Section V, the similarity formulation of the "Stokes-like" BVL is given. As in the case of the PVL, the proper choice of the modified radial variable greatly simplifies the analysis of this layer, which follows, in part, that of Stewartson's inner viscous zone. The details of the PVL-BVL matching are also presented.

It is determined that the theory presented is valid for

$$\delta_b \ll D_b \left[\log\left(\frac{1}{D_b}\right)\right]^{-1/2} = \tau_b^{1/2} \exp\left(-\frac{1}{\tau_b}\right) \ll 1$$

$$\bar{V}_L \ll \left[\log\left(\frac{1}{D_b}\right)\right]^{-1/2} = \tau_b^{1/2} \ll 1$$

Further, it is found that, to the order of approximation considered for the flow quantities, the PVL-BVL matching requires the "effective body" shape, i.e., the outer edge of the PVL, to be

$$r = \delta N_h x^{3/4}, \qquad N_h = N_h(\epsilon) = \text{const.}$$

but puts no requirement on the actual body shape, which may be represented as

$$r = \delta_b F_b(x), \qquad F_b(x) = \text{arbitrary fnc}(x)$$

The pressure, friction, heat transfer, and drag coefficients are

$$C_p = C_{p,o}\bar{V}_x \left[\log\left(\frac{\bar{V}_x}{\delta_b^2}\right)\right]^{-1/2} + \ldots$$

$$C_f = C_{f,o}\left(\frac{\bar{V}_x^2}{\delta_b}\right)\left[\log\left(\frac{\bar{V}_x}{\delta_b^2}\right)\right]^{-1}\left[\frac{F_b(x)}{x}\right]^{-1} + \ldots$$

$$C_q = C_{q,o}\left(\frac{\bar{V}_x^2}{\delta_b}\right)\left[\log\left(\frac{\bar{V}_x}{\delta_b^2}\right)\right]^{-1}\left[\frac{F_b(x)}{x}\right]^{-1} + \ldots$$

$$C_D = 2C_{f,o}\left(\frac{\bar{V}_x^2}{\delta_b^2}\right)\left[\log\left(\frac{\bar{V}_x}{\delta_b^2}\right)\right]^{-1}\left[\frac{F_b(x)}{x}\right]^{-2} + \ldots$$

where $C_{p,o}$, $C_{f,o}$, $C_{q,o}$ = fncs(ϵ, σ, ω, T_w/θ_S) = const. These expressions for tha coefficients are equivalent to those found through the SSEM formulation.

In tha Appandix, the formulation of the required viscous transition layar (TL), intarmadiate to the ISL and PVL, is given (cf. Bush 1966). Tha ISL-TL-PVL matching for the temperature is demonstrated.

A schematic diagram of the hyperaonic strong-interaction layers for flow paat a very slender axiaymmetric body is given in Fig. 1.

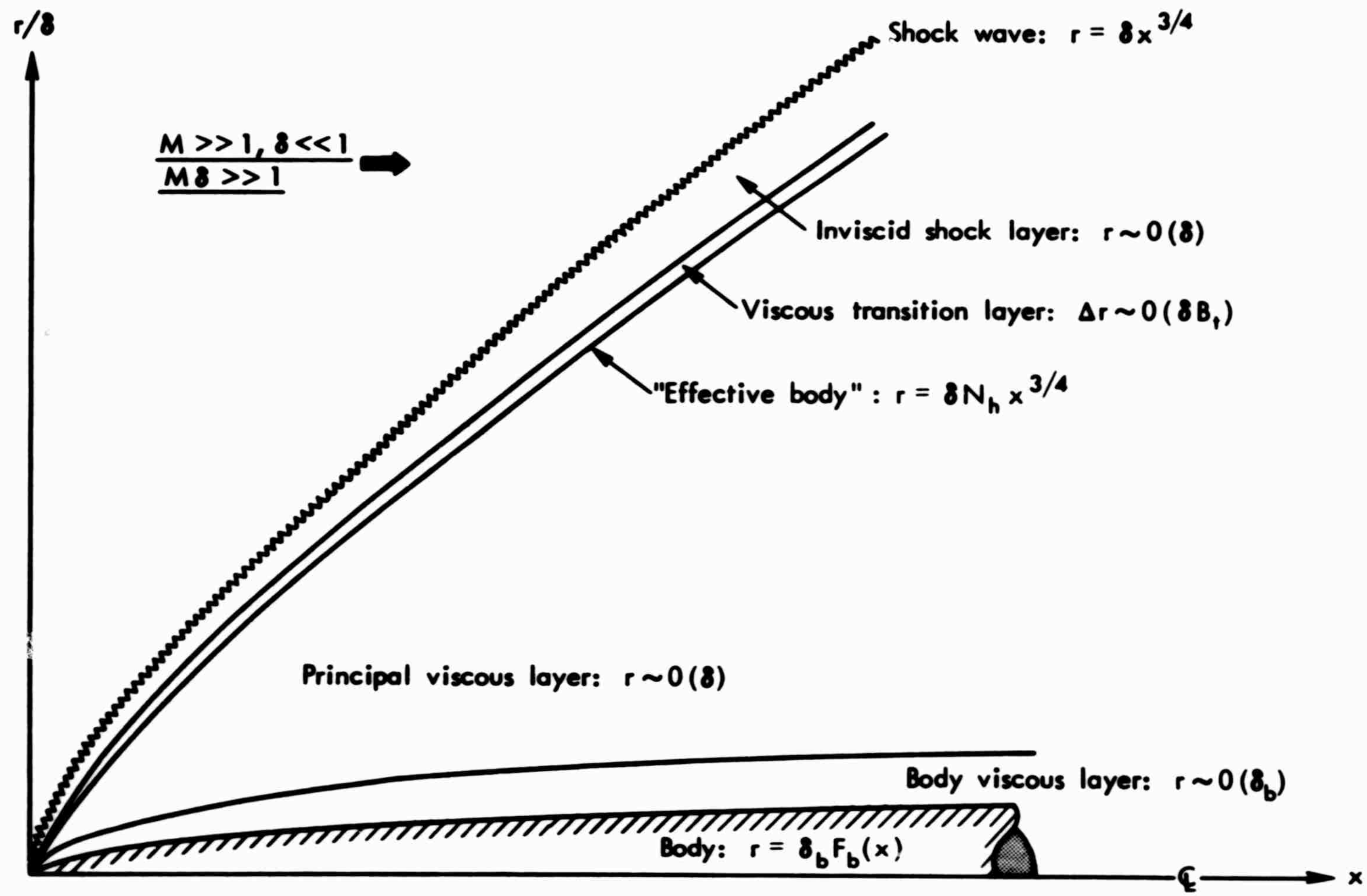

Fig. 1 -- Schematic diagram of hypersonic strong-interaction layers for flow past a very slender axisymmetric body

II. THE EQUATIONS OF MOTION

Consider the flow of a viscous, compressible gas past a very slender axisymmetric body. Let $x_1 = Lx$ and $r_1 = Lr$ represent the cylindrical polar coordinates along the axis of symmetry from the vertex of the body and normal to this axis, respectively. The length L is chosen so that x is of order unity in the region where the strong-interaction theory is valid.

Under this formulation, the equation of the surface of this slender body is

$$r = \delta_b F_b(x), \quad \text{with } \delta_b = \delta D_b \ll \delta \ll 1, \quad F_b(x) = O(1)$$

where δ represents the scaling of the effective thickness of the viscous layer(s) supported by the body. The velocity components in the x_1- and r_1-directions are $u_1 = u_\infty u$ and $v_1 = u_\infty v$, and the pressure, temperature, and density, respectively, are $p_1 = p_\infty p$, $T_1 = T_\infty T$, and $\rho_1 = \rho_\infty \rho$, where u_∞, p_∞, T_∞, and ρ_∞ are, respectively, the velocity in the x_1-direction, pressure, temperature, and density in the undisturbed region upstream of the body.

A perfect gas ($p = \rho T$) is assumed, having (1) constant specific heats, c_{v_1} and c_{p_1}, with $\gamma = (c_{p_1}/c_{v_1})$ = const.; (2) a constant Prandtl number of order unity (σ = const. = $O(1)$); and (3) its first and second viscosity coefficients proportional to a power, ω, of the absolute temperature [$\mu_1 = \mu_\infty \mu = \mu_\infty T^\omega$, with $1/2 \le \omega < 1$; $\lambda_1 = \mu_\infty \lambda = K\mu_\infty \mu = K\mu_\infty T^\omega$, K = const. = $O(1)$].

The Navier-Stokes equations of motion in cylindrical polar co-ordinates for the flow of such a gas are

$$\frac{\partial}{\partial x}(\rho u r) + \frac{\partial}{\partial r}(\rho v r) = 0 \tag{1}$$

$$\rho\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial r}\right) + \frac{1-\epsilon}{1+\epsilon}\frac{1}{M^2}\frac{\partial p}{\partial x}$$

$$= \frac{1}{R_L}\left[\frac{1}{r}\frac{\partial}{\partial r}\left(rT^{\omega}\left\{\frac{\partial u}{\partial r} + \frac{\partial v}{\partial x}\right\}\right) + \frac{\partial}{\partial x}\left(T^{\omega}\left\{(2+K)\frac{\partial u}{\partial x} + \frac{K}{r}\frac{\partial}{\partial r}(rv)\right\}\right)\right] \tag{2}$$

$$\rho\left(u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial r}\right) + \frac{1-\epsilon}{1+\epsilon}\frac{1}{M^2}\frac{\partial p}{\partial r}$$

$$= \frac{1}{R_L}\left[\frac{\partial}{\partial r}\left(T^{\omega}\left\{(2+K)\frac{\partial v}{\partial r} + K\frac{v}{r} + K\frac{\partial u}{\partial x}\right\}\right) + 2T^{\omega}\frac{\partial}{\partial r}\left(\frac{v}{r}\right) + \frac{\partial}{\partial x}\left(T^{\omega}\left\{\frac{\partial u}{\partial r} + \frac{\partial v}{\partial x}\right\}\right)\right] \tag{3}$$

$$\rho\left(u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial r}\right) - \frac{2\epsilon}{1+\epsilon}\left(u\frac{\partial p}{\partial x} + v\frac{\partial p}{\partial r}\right)$$

$$= \frac{1}{\sigma}\frac{1}{R_L}\left[\frac{1}{r}\frac{\partial}{\partial r}\left(rT^{\omega}\frac{\partial T}{\partial r}\right) + \frac{\partial}{\partial x}\left(T^{\omega}\frac{\partial T}{\partial x}\right)\right]$$

$$+ \frac{2}{1+\epsilon}\frac{\theta_S}{R_L}\left[T^{\omega}\left\{\left(\frac{\partial u}{\partial r} + \frac{\partial v}{\partial x}\right)^2 + 2\left[\left(\frac{\partial v}{\partial r}\right)^2 + \left(\frac{v}{r}\right)^2 + \left(\frac{\partial u}{\partial x}\right)^2\right]\right\}\right]$$

$$+ \frac{2}{1+\epsilon}\frac{\theta_S}{R_L}K\left[T^{\omega}\left\{\frac{1}{r}\frac{\partial}{\partial r}(rv) + \frac{\partial u}{\partial x}\right\}^2\right] \tag{4}$$

where $\epsilon = (\gamma-1)/(\gamma+1) = O(1)$, the Newtonian approximation (of $\epsilon \ll 1$) not being invoked; $M^2 = (\rho_\infty u_\infty^2/\gamma p_\infty) \gg 1$; $\theta_S = \epsilon\{(1+\epsilon)/(1-\epsilon)\}M^2 \gg 1$; and $R_L = (\rho_\infty u_\infty L/\mu_\infty) \gg 1$.

III. THE INVISCID SHOCK LAYER

According to present hypersonic strong-interaction theory (HSIT), the slender body, whose surface is given by $r = \delta_b F_b(x)$, combines with the thin, viscous, heat-conducting layer(s) at the body surface to disturb the uniform external flow. This combination of the body and the viscous layer(s), whose outer edge is given by $r = \delta F_k(x)$, with δ, the thickness parameter of this combination (for the flow regime under consideration, $\delta_b = \delta D_b \ll \delta \ll 1$) acts as a slender "effective body," producing an oblique shock wave, $r = \delta F_{sh}(x) > \delta F_k(x)$, and an inviscid shock layer (ISL) between the shock wave and the "effective body." For this ISL, whose lateral extent is

$$\delta F_k(x) \le r \le \delta F_{sh}(x)$$

the approximations of hypersonic small-disturbance theory (HSDT) are assumed to apply.

Introducing the coordinates x_h and r_h, defined by

$$x_h = x, \qquad r_h = \frac{r}{\delta} \tag{5}$$

the HSDT expansions of the dependent variables have the form

$$\left.\begin{aligned} u &= 1 + \delta^2 u_h + \ldots \\ v &= \delta v_h + \ldots \\ p &= M^2 \delta^2 p_h + \ldots \\ T &= \theta_S \delta^2 T_h + \ldots \end{aligned}\right\} \tag{6}$$

with $f_h = f_h(x_h, r_h) = O(1)$. Thus, the first-approximation equations of HSDT, taking

$$\left[\left(\epsilon\left\{\frac{(1+\epsilon)}{(1-\epsilon)}\right\}\right)^{1+\omega}\frac{M^{2\omega}}{R_L\delta^{2(1-\omega)}}\right] \ll 1^*$$

are

$$\left.\begin{aligned}
\frac{\partial}{\partial x_h}\left(\frac{p_h r_h}{T_h}\right) + \frac{\partial}{\partial r_h}\left(\frac{p_h v_h r_h}{T_h}\right) &= 0 \\
\frac{p_h}{T_h}\left(\frac{\partial u_h}{\partial x_h} + v_h\frac{\partial u_h}{\partial r_h}\right) + \epsilon\frac{\partial p_h}{\partial x_h} &= 0 \\
\frac{p_h}{T_h}\left(\frac{\partial v_h}{\partial x_h} + v_h\frac{\partial v_h}{\partial r_h}\right) + \epsilon\frac{\partial p_h}{\partial r_h} &= 0 \\
\frac{p_h}{T_h}\left(\frac{\partial T_h}{\partial x_h} + v_h\frac{\partial T_h}{\partial r_h}\right) - \frac{2\epsilon}{1+\epsilon}\left(\frac{\partial p_h}{\partial x_h} + v_h\frac{\partial p_h}{\partial r_h}\right) &= 0
\end{aligned}\right\} \qquad (7)$$

The Rankine-Hugoniot shock relations, which determine the boundary conditions for the flow quantities of Eq. (6) at $r_h = F_{sh}(x_h)$, are

$$\left.\begin{aligned}
(u_h)_{sh} &= -(1-\epsilon)[F'_{sh}(x_h)]^2, & (v_h)_{sh} &= (1-\epsilon)F'_{sh}(x_h) \\
(p_h)_{sh} &= (1+\epsilon)[F'_{sh}(x_h)]^2, & (T_h)_{sh} &= (1-\epsilon)[F'_{sh}(x_h)]^2
\end{aligned}\right\} \qquad (8)$$

To obtain similar solutions to the general HSIT problem for a very slender axisymmetric body, it is necessary to obtain similar solutions for the ISL. For $M^2\delta^2 \gg 1$, the flow in the ISL is self-similar if the shock associated with this flow is described by a power law; i.e., if

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*In the notation of Section IV, $[(\epsilon\{(1+\epsilon)/(1-\epsilon)\})^{1+\omega}M^{2\omega}/R_L\delta^{2(1-\omega)}] = \Lambda_k(\delta^2/\alpha_k)^{1+\omega}$. In the formulation of Section IV, it is necessary that $\Lambda_k = O(1)$, $(\delta^2/\alpha_k) \ll 1$; hence, the requirement expressed above for the ISL is consistent with the requirements of the PVL.

$$F_{sh}(x_h) = x_h^m \tag{9}$$

For such similar solutions, the appropriate independent and dependent variables are

$$\xi_h = x_h, \quad \eta_h = \frac{r_h}{x_h^m} \tag{10}$$

$$\left.\begin{aligned} u_h &= -(1-\epsilon)m^2\xi_h^{-2(1-m)}U_h(\eta_h), \quad & v_h &= (1-\epsilon)m\xi_h^{-(1-m)}V_h(\eta_h) \\ p_h &= (1+\epsilon)m^2\xi_h^{-2(1-m)}P_h(\eta_h), \quad & T_h &= (1-\epsilon)m^2\xi_h^{-2(1-m)}\Theta_h(\eta_h) \end{aligned}\right\} \tag{11}$$

When Eqs. (10) and (11) are introduced, the equations of motion, Eq. (7), reduce to

$$\left.\begin{aligned} &\frac{d}{d\eta_h}\left(\frac{\eta_h P_h W_h}{\Theta_h}\right) + 2\frac{\eta_h P_h}{\Theta_h} = 0 \\ &\frac{P_h}{\Theta_h}\left[W_h\frac{dU_h}{d\eta_h} - 2\left(\frac{1-m}{m}\right)U_h\right] + \epsilon\left[\eta_h\frac{dP_h}{d\eta_h} + 2\left(\frac{1-m}{m}\right)P_h\right] = 0 \\ &\frac{P_h}{\Theta_h}\left[W_h\left\{\frac{dW_h}{d\eta_h} + \left(\frac{2m-1}{m}\right)\right\} - \left(\frac{1-m}{m}\right)\eta_h\right] + \epsilon(1-\epsilon)\frac{dP_h}{d\eta_h} = 0 \\ &\frac{P_h}{\Theta_h}\left[W_h\frac{d\Theta_h}{d\eta_h} - 2\left(\frac{1-m}{m}\right)\Theta_h\right] - \frac{2\epsilon}{1+\epsilon}\left[W_h\frac{dP_h}{d\eta_h} - 2\left(\frac{1-m}{m}\right)P_h\right] = 0 \end{aligned}\right\} \tag{12}$$

where $W_h = [(1-\epsilon)V_h - \eta_h]$. The boundary conditions for these equations, obtained from the shock relations, are

$$U_h, V_h, P_h, \Theta_h \to 1, \quad W_h \to -\epsilon, \quad \text{as} \quad \eta_h \to 1 \tag{13}$$

The solutions of Eq. (12), subject to Eq. (13), determine $N_h = N_h(\epsilon, m) < 1$, the value of η_h at the outer edge of the "effective body," where $W_h = 0$ (cf., e.g., Lees and Kubota 1957). For $1/2 < m < 1$, as $\eta_h \to N_h$, the similarity solutions yield

$$\left.\begin{aligned}
P_h &= P_{h,o}(\epsilon,m) + \dots \\
V_h &= \{N_h(\epsilon,m)/(1-\epsilon)\} + \dots \\
\Theta_h &= \Theta_{h,o}(\epsilon,m)(\eta_h - N_h)^{-E} + \dots, \\
E &= E(\epsilon,m) = \left[\left(\frac{m}{1-m}\right)\left(\frac{1+\epsilon}{1-\epsilon}\right) - 1\right]^{-1}
\end{aligned}\right\} \quad (14a)$$

Hence, the expressions for p_h, v_h, and T_h, as $r_h \to F_k = N_h x_h^m$, are

$$\left.\begin{aligned}
p_h &= (1+\epsilon)m^2 P_{h,o}\xi_h^{-2(1-m)} + \dots \\
&= \Pi_h(\epsilon,m)\xi_h^{-2(1-m)} + \dots \\
v_h &= mN_h\xi_h^{-(1-m)} + \dots \\
T_h &= (1-\epsilon)m^2\Theta_{h,o}\xi_h^{-2(1-m)}(\eta_h - N_h)^{-E} + \dots \\
&= \Phi_h(\epsilon,m)\xi_h^{-2(1-m)}(\eta_h - N_h)^{-E} + \dots
\end{aligned}\right\} \quad (14b)$$

For $m = 3/4$, $\epsilon = 1/6$, $N_h(1/6, 3/4) = 0.875$, and $\Pi_h(1/6, 3/4) = 0.549$; $E(1/6, 3/4) = 0.313$.

IV. THE PRINCIPAL VISCOUS LAYER

Just as there is a principal viscous layer, usually designated as the viscous boundary layer, which acts as an "effective body" in the HSIT for flow past a flat plate, it is postulated that there is a principal viscous layer (PVL), having a thickness ratio of $O(\delta)$ (with $\delta_b = \delta D_b \ll \delta \ll 1$), which acts as an "effective body" in the HSIT for flow past a very slender axisymmetric body with a thickness ratio of $O(\delta_b)$. As in the case of the HSIT viscous boundary layer for a flat plate, the HSIT PVL for a very slender axisymmetric body, which supports the ISL, is considered to be a high temperature, low density region, across which the pressure is constant, at whose outer edge, $r = \delta F_k(x) = \delta N_h x^m$, the flow quantities u, T, v, and p have the behavior: $u \to 1$, $T/\theta_S \to 0$, $v \to \delta m N_h x^{-(1-m)}$, and $p \to M^2 \delta^2 \Pi_h x^{-2(1-m)}$, where N_h and Π_h are constants defined in Section III.

The appropriate coordinates of the PVL are taken to be s_k and t_k, given by

$$s_k = x, \qquad t_k = \log\left\{\frac{\delta F_k(x)}{r}\right\} = \log\left\{\frac{\delta N_h x^m}{r}\right\} \tag{15}$$

From the definition of t_k, it follows that

$$t_k \to 0 \quad \text{as} \quad r \to \delta F_k; \qquad t_k \to \log\left\{\frac{F_k/F_b}{D_b}\right\} \to \infty \quad \text{as } r \to \delta D_b F_b$$

The expansions for the flow quantities in this layer, following, <u>in part</u>, Stewartson (1964), are

$$u = 1 + \alpha_k u_k + \ldots$$

$$v = \delta F'_k \exp(-t_k) v_k + \ldots = \delta(mN_h s_k^{-(1-m)}) \exp(-t_k) v_k + \ldots$$

$$T = \theta_S \alpha_k T_k + \ldots$$

$$p = M^2 \delta^2 p_k + \ldots \qquad (16)$$

with α_k = parameter much less than 0(1) to be determined, and f_k = $f_k(s_k, t_k)$ = 0(1).

For these representations, with

$$\Lambda_k = \left(\epsilon\left\{\frac{1+\epsilon}{1-\epsilon}\right\}\right)^{1+\omega} \frac{M^{2\omega}}{R_L \delta^4} \alpha_k^{1+\omega} = 0(1), \quad \delta^2 \ll \alpha_k \ll 1^*$$

the leading terms in the equations of motion for the PVL are

$$\frac{\partial p_k}{\partial t_k} = 0, \quad \text{i.e.,}\ p_k = P_k(s_k) = \Pi_k s_k^{-2(1-m)} \qquad (17a)$$

$$\frac{\partial}{\partial t_k}\left(\frac{1-v_k}{T_k}\right) - 2\left(\frac{1-v_k}{T_k}\right) = -\frac{1}{m}\left[2(2m-1)\frac{1}{T_k} + s_k \frac{\partial}{\partial s_k}\left(\frac{1}{T_k}\right)\right] \qquad (17b)$$

---

*The requirements of (1) $M^{2\omega}\alpha_k^{1+\omega}/R_L\delta^4$ = 0(1), (2) $\delta^2 \ll \alpha_k \ll 1$, and (3) $M^2\delta^2 \gg 1$ yield the following individual inequalities:

$$M^{-2/(1-\omega)} \ll \delta \ll 1, \qquad M^{-4/(1-\omega)} \ll \alpha_k \ll 1$$

$$1 \ll M^{2\omega} \ll R_L \ll M^{2(2+\omega)}, \quad \text{or} \quad M^{-2} \ll \bar{V}_L = (M^{2\omega}/R_L)^{1/2} \ll 1$$

Further, the modified interaction parameter, χ', obtained from (1) and (3) is

$$\chi' = \left\{M^{2(2+\omega)}\alpha_k^{1+\omega}/R_L\right\}^{1/2} \gg 1$$

$$\Pi_k N_h^2 s_k^{4m-3} \exp(-2t_k)\left[\frac{s_k}{T_k}\frac{\partial u_k}{\partial s_k} + m\left(\frac{1-v_k}{T_k}\right)\frac{\partial u_k}{\partial t_k} - 2\epsilon(1-m)\right]$$

$$= \Lambda_k \frac{\partial}{\partial t_k}\left(T_k^\omega \frac{\partial u_k}{\partial t_k}\right) \qquad (17c)$$

$$\Pi_h N_h^2 s_k^{4m-3} \exp(-2t_k)\left[\frac{s_k}{T_k}\frac{\partial T_k}{\partial s_k} + m\left(\frac{1-v_k}{T_k}\right)\frac{\partial T_k}{\partial t_k} + \frac{4\epsilon}{1+\epsilon}(1-m)\right]$$

$$= \frac{\Lambda_k}{\sigma}\frac{\partial}{\partial t_k}\left(T_k^\omega \frac{\partial T_k}{\partial t_k}\right) \qquad (17d)$$

Equations (17b-d) must satisfy the boundary conditions

$$u_k(s_k,\ 0) = T_k(s_k,\ 0) = 0, \qquad v_k(s_k,\ 0) = 1 \qquad (18)$$

at the outer edge of the PVL in order that the PVL solutions may match to those of the ISL.

For

$$\left.\begin{aligned} u_k(s_k,t_k) &= A_k(s_k)U_k(t_k) \\ T_k(s_k,t_k) &= A_k(s_k)\Theta_k(t_k) \\ v_k(s_k,t_k) &= V_k(t_k) \end{aligned}\right\} \qquad (19a)$$

where

$$A_k(s_k) = \left(\frac{m\Pi_h N_h^2}{\Lambda_k}\right)^{1/(1+\omega)} s_k^{(4m-3)/(1+\omega)}$$

$$\Rightarrow A_k = \left(\frac{3\Pi_h N_h^2}{4\Lambda_k}\right)^{1/(1+\omega)} \quad \text{for } m = 3/4 \qquad (19b)$$

the self-similar ordinary differential equations and boundary conditions governing the flow in the PVL are

$$\frac{d}{dt_k}\left(\frac{1 - V_k}{\Theta_k}\right) - 2\left(\frac{1 - V_k}{\Theta_k}\right) = -\frac{1}{m}\left[2(2m - 1) - \left(\frac{4m - 3}{1 + \omega}\right)\right]\frac{1}{\Theta_k} \tag{20a}$$

$$\frac{d}{dt_k}\left(\Theta_k^{\omega}\frac{dU_k}{dt_k}\right) = \frac{\exp(-2t_k)}{m}\left[\left(\frac{4m - 3}{1 + \omega}\right)\frac{U_k}{\Theta_k} + m\left(\frac{1 - V_k}{\Theta_k}\right)\frac{dU_k}{dt_k} - 2\epsilon(1 - m)\right] \tag{20b}$$

$$\frac{d}{dt_k}\left(\Theta_k^{\omega}\frac{d\Theta_k}{dt_k}\right) = \frac{\sigma\exp(-2t_k)}{m}\left[\left(\frac{4m - 3}{1 + \omega}\right) + m\left(\frac{1 - V_k}{\Theta_k}\right)\frac{d\Theta_k}{dt_k} + \frac{4\epsilon}{1 + \epsilon}(1 - m)\right] \tag{20c}$$

$$U_k(0) = \Theta_k(0) = 0, \qquad V_k(0) = 1 \tag{21}$$

Indeed, it is found that, in conjunction with Eqs. (20a) and (21), Eqs. (20b) and (20c) yield, upon integration,

$$\Theta_k^{\omega}\frac{dU_k}{dt_k} = 2\left(\frac{2m - 1}{m}\right)\int_0^{t_k}\exp(-2\tilde{t}_k)\frac{U_k}{\Theta_k}d\tilde{t}_k - \epsilon\left(\frac{1 - m}{m}\right)\left[1 - \exp(-2t_k)\right] + \frac{U_k}{\Theta_k}(1 - V_k)\exp(-2t_k) \tag{22a}$$

$$\frac{1}{\sigma}\Theta_k^{\omega}\frac{d\Theta_k}{dt_k} = \left[\left(\frac{2m - 1}{m}\right) + \frac{2\epsilon}{1 + \epsilon}\left(\frac{1 - m}{m}\right)\right]\left[1 - \exp(-2t_k)\right] + (1 - V_k)\exp(-2t_k) \tag{22b}$$

Near the outer edge of the PVL, where $t_k \to 0$, it is found that, for $(1 - \omega) = O(1) > 0$, the asymptotic solutions for the flow quantities in Eqs. (20) are

$$\Theta_k = \Theta_{k,o} t_k^{2/(1+\omega)} + \ldots$$

$$U_k = U_{k,o} t_k^{2/(1+\omega)} + \ldots$$

$$V_k = 1 - V_{k,o} t_k + \ldots \tag{23a}$$

where

$$\Theta_{k,o} = \left\{\frac{\sigma}{m}\left(\frac{1+\omega}{1-\omega}\right)\left[2(2m-1) - \frac{(4m-3)}{2} + \frac{2\epsilon}{1+\epsilon}(1-m)(1-\omega)\right]\right\}^{1/(1+\omega)}$$

$$U_{k,o} = (1-m)(1-\omega)\left[\frac{1-\sigma}{\epsilon}\left\{2(2m-1) - \frac{(4m-3)}{2}\right\} - \frac{2\sigma}{1+\epsilon}(1-m)(1-\omega)\right]^{-1} \Theta_{k,o}$$

$$V_{k,o} = \frac{1}{m}\left(\frac{1+\omega}{1-\omega}\right)\left[2(2m-1) - \frac{(4m-3)}{1+\omega}\right] \tag{23b}$$

When $m = 3/4$,

$$\Theta_{k,o} = \left\{\frac{4\sigma}{3}\left(\frac{1+\omega}{1-\omega}\right)\left[1 + \frac{\epsilon}{1+\epsilon}\left(\frac{1-\omega}{2}\right)\right]\right\}^{1/(1+\omega)}, \text{ etc.}$$

A comparison of Eqs. (14) and (23) shows clearly that the functional behaviors of the temperature in the ISL and the PVL, as $\eta_h \to N_h$ and $t_k \to 0$, respectively, do not permit direct matching between these two layers, and the introduction of an intermediate transition layer* is necessary to complete the matching.

Near the inner edge of the PVL, where $t_k \to \infty$, it is seen from Eqs. (22a-b), that

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*The details of the formulation of this transition layer, which is consistent with that introduced in the HSIT for flow past a flat plate (Bush 1966), are presented in the Appendix.

$$\Theta_k^{\omega} \frac{dU_k}{dt_k} \rightarrow \frac{1}{m}\left[2(2m - 1) \int_0^{\infty} \exp(-2t_k) \frac{U_k}{\Theta_k} dt_k - \varepsilon(1 - m)\right]$$

$$\frac{1}{\sigma} \Theta_k^{\omega} \frac{d\Theta_k}{dt_k} \rightarrow \frac{1}{m}\left[(2m - 1) + \frac{2\varepsilon}{1 + \varepsilon}(1 - m)\right]$$

Hence, the shear and heat-conduction terms are found to dominate the momentum and energy equations, respectively, yielding, in this limit, the following asymptotic solutions of the similarity equations:

$$\Theta_k = \Theta_{k,\infty} t_k^{1/(1+\omega)} + \ldots \Rightarrow T = \theta_S \alpha_k A_k \Theta_{k,\infty} t_k^{1/(1+\omega)} + \ldots$$

$$U_k = U_{k,\infty} t_k^{1/(1+\omega)} + \ldots \Rightarrow u = 1 + \alpha_k A_k U_{k,\infty} t_k^{1/(1+\omega)} + \ldots$$

$$V_k = V_{k,\infty} + \ldots \Rightarrow v = \delta\left(mN_h s_k^{-(1-m)}\right) V_{k,\infty} \exp(-t_k) + \ldots \quad (24a)$$

where

$$\Theta_{k,\infty} = \left\{\frac{\sigma}{m}(1 + \omega)\left[(2m - 1) + \frac{2\varepsilon}{1 + \varepsilon}(1 - m)\right]\right\}^{1/(1+\omega)}$$

$$U_{k,\infty} = \frac{1 + \omega}{m}\left[2(2m - 1) \int_0^{\infty} \exp(-2t_k) \frac{U_k}{\Theta_k} dt_k - \varepsilon(1 - m)\right] \Theta_{k,\infty}^{-\omega}$$

$$V_{k,\infty} = \left(\frac{1 - m}{m}\right) + \frac{1}{1 + \omega}\left(\frac{4m - 3}{2m}\right) \quad (24b)$$

With the possible exception of the asymptotic solution for v, the solutions of Eqs. (24a-b) do not satisfy the usual nonslip, temperature-specified boundary conditions at the body surface,

$$u,v \to 0, \quad \frac{T}{\theta_S} \to \varphi_w = \text{specified fnc}(s_k), \qquad \text{as } t_k \to \log\left(\frac{F_k/F_b}{D_b}\right) \to \infty \tag{25}$$

Therefore, the analysis of an additional layer, interior to the PVL, in which $t_k = O\left([\log(1/D_b)]\right) \to \infty$, is introduced in the next section in order to describe the adjustment of the flow in the vicinity of the body surface to satisfy the surface boundary conditions of Eq. (25).

V. THE BODY VISCOUS LAYER

For the viscous layer adjacent to the body surface, referred to in this Memorandum as the body viscous layer (BVL), the appropriate coordinates are s_b and t_b, given by

$$s_b = x, \qquad t_b = \tau_b \log\left[\frac{\delta F_b(x)}{r}\right] = \tau_b\left[t_k + \log\left(\frac{F_b}{F_k}\right)\right] \tag{26a}$$

where

$$\tau_b = \left[\log\left(\frac{1}{D_b}\right)\right]^{-1} \ll 1 \tag{26b}$$

From the definition of t_b, it follows that

$$t_b \to 1 \quad \text{as} \quad r \to \delta D_b F_b; \qquad t_b \to \tau_b \log \frac{F_b}{F_k} \to 0 \quad \text{as} \quad r \to \delta F_k$$

The expansions for the flow quantities in this layer [cf. Stewartson, et al. (1964)] are

$$\left.\begin{aligned}
u &= u_b + \ldots \\
v &= \delta F_b' \exp\left(-\frac{t_b}{\tau_b}\right) v_b + \ldots \\
&= \delta\left[mN_h s_b^{-(1-m)}\right] \exp(-t_k)\left[\frac{1}{m}\left(\frac{s_b}{F_b}\frac{dF_b}{ds_b}\right)\right] v_b + \ldots \\
T &= \theta_S T_b + \ldots \\
p &= M^2 \delta^2 p_b + \ldots
\end{aligned}\right\} \tag{27}$$

with $f_b = f_b(s_b, t_b) = O(1)$.

Subject to later verification as to consistency, for τ_b, $\delta^2/\alpha_k^{1+\omega}$, and $(\alpha_k^{1+\omega}/\tau_b^2)\exp(-2/\tau_b) \ll 1$, the leading terms in the equations of motion for the BVL are

$$\frac{\partial p_b}{\partial t_b} = 0, \qquad \text{i.e., } p_b = P_b(s_b) = \Pi_h s_b^{-2(1-m)} \tag{28a}$$

$$v_b = \left(\frac{s_b}{F_b}\frac{dF_b}{ds_b}\right)^{-1}\left[(1-m) + \frac{1}{2}\left(\frac{s_b}{T_b}\frac{\partial T_b}{\partial s_b} - \frac{s_b}{u_b}\frac{\partial u_b}{\partial s_b}\right)\right]u_b \tag{28b}$$

$$\frac{\partial}{\partial t_b}\left(T_b^{\omega}\frac{\partial u_b}{\partial t_b}\right) = 0 \tag{28c}$$

$$\frac{\partial}{\partial t_b}\left(T_b^{\omega}\frac{\partial T_b}{\partial t_b}\right) + \frac{2\sigma}{1+\epsilon}T_b^{\omega}\left(\frac{\partial u_b}{\partial t_b}\right)^2 = 0 \tag{28d}$$

The boundary conditions to be satisfied at the body surface are

$$u_b(s_b,1) = 0, \qquad T_b(s_b,1) = \varphi_w \tag{29}$$

Taking the wall temperature to be constant so that

$$\varphi_w = \Theta_{b,w} = \text{const.} \tag{30a}$$

and taking the flow variables to be of the form

$$u_b(s_b, t_b) = U_b(t_b)$$

$$T_b(s_b, t_b) = \Theta_b(t_b)$$

$$v_b(s_b, t_b) = \left[(1-m)\left(\frac{s_b}{F_b}\frac{dF_b}{ds_b}\right)^{-1}\right]V_b(t_b) \tag{30b}$$

the self-similar equations and boundary conditions governing the flow in the BVL are

$$V_b = U_b \tag{31a}$$

$$\frac{d}{dt_b}\left(\Theta_b^{\omega}\frac{dU_b}{dt_b}\right) = 0 \tag{31b}$$

$$\frac{d}{dt_b}\left(\Theta_b^{\omega}\frac{d\Theta_b}{dt_b}\right) + \frac{2\sigma}{1+\epsilon}\Theta_b^{\omega}\left(\frac{dU_b}{dt_b}\right)^2 = 0 \tag{31c}$$

$$U_b(1) = 0, \qquad \Theta_b(1) = \Theta_{b,w} \tag{32}$$

The first integral of Eq. (31b) is

$$\Theta_b^{\omega}\frac{dU_b}{dt_b} = -S_b \tag{33a}$$

where S_b, the shear function, is a positive constant. The first integral of Eq. (31c) is

$$\Theta_b^{\omega}\frac{d\Theta_b}{dt_b} = \pm 2\left(\frac{\sigma}{1+\epsilon}\right)^{1/2} S_b\,(\Theta_{b,m} - \Theta_b)^{1/2} \tag{33b}$$

where $\Theta_{b,m}$, the maximum value of Θ_b in the BVL, is a positive constant. The positive branch of Eq. (33b) represents the temperature field between the point of maximum temperature, $t_b = t_{b,m}$, and the outer edge of the BVL, $t_b \to 0$; the negative branch of Eq. (33b) represents the temperature field between the point of maximum temperature, $t_b = t_{b,m}$, and the body surface, $t_b \to 1$.

For $U_b \to 1$ and $\Theta_b \to 0$ as $t_b \to 0$, Eqs. (33) combine to yield the Crocco-like second integral,

$$U_b = 1 - \left(\frac{1 + \epsilon}{\sigma}\right)^{1/2} \left|\Theta_{b,m}^{1/2} \mp (\Theta_{b,m} - \Theta_b)^{1/2}\right| \tag{34}$$

whose negative branch is valid for $0 \le t_b < t_{b,m}$, and whose positive branch is valid for $t_{b,m} < t_b \le 1$. From an evaluation of the positive branch at $t_b = 1$, i.e., for $U_b = 0$, $\Theta_b = \Theta_{b,w}$, it is determined that

$$\Theta_{b,m} = \frac{1}{4}\frac{\sigma}{1 + \epsilon}\left(1 + \frac{1 + \epsilon}{\sigma}\Theta_{b,w}\right)^2 > 0 \tag{35}$$

When Eq. (33b) is solved to determine $\Theta_b(t_b)$, the solutions $U_b(t_b)$ and $V_b(t_b)$ follow directly from Eqs. (34) and (31a).

From direct integration of Eq. (33b), taking $\Theta_b(0) = 0$ and $\Theta_b(1) = \Theta_{b,w}$, it is found that the shear function, S_b, is

$$S_b = \frac{1}{4^{1+\omega}}\left(\frac{\sigma}{1 + \epsilon}\right)^{\omega}\left(1 + \frac{1 + \epsilon}{\sigma}\Theta_{b,w}\right)^{2\omega+1}\left[\int_0^1 Z^{\omega}(1 - Z)^{-1/2}\,dZ + \int_{Z_0}^1 Z^{\omega}(1 - Z)^{-1/2}\,dZ\right]$$

$$Z_0 = 4\left(\frac{1 + \epsilon}{\sigma}\right)\Theta_{b,w}\left(1 + \frac{1 + \epsilon}{\sigma}\Theta_{b,w}\right)^{-2} \tag{36*}$$

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*To give an idea of the magnitude of S_b, it is noted that

$$S_b\left(\omega = \frac{1}{2},\ \Theta_{b,w} = 0\right) = \left(\frac{\pi}{8}\right)\left[\frac{\sigma}{(1 + \epsilon)}\right]^{1/2} = 0.31 \quad \text{for } \sigma = \frac{3}{4},\ \epsilon = \frac{1}{6}$$

Thus, near the outer edge of the BVL, where $t_b \to 0$, it is seen from Eq. (33b) that

$$\Theta_b^{\omega} \frac{d\Theta_b}{dt_b} \to \frac{\sigma}{1+\epsilon} S_b \left(1 + \frac{1+\epsilon}{\sigma} \Theta_{b,w}\right)$$

and, hence, it follows that, at the outer edge of the BVL,

$$\Theta_b = \Theta_{b,o} t_b^{1/(1+\omega)} + \ldots \Rightarrow \quad T = \theta_S \Theta_{b,o} t_b^{1/(1+\omega)} + \ldots$$

$$U_b = 1 - U_{b,o} t_b^{1/(1+\omega)} + \ldots \Rightarrow \quad u = 1 - U_{b,o} t_b^{1/(1+\omega)} + \ldots$$

$$V_b = V_{b,o} + \ldots \Rightarrow$$

$$v = \delta(1-m) \frac{F_b}{s_b} V_{b,o} \exp\left(-\frac{t_b}{\tau_b}\right) + \ldots$$

$$= \delta \left[m N_h s_b^{-(1-m)}\right] \left[\left(\frac{1-m}{m}\right) V_{b,o}\right] \exp(-t_k) + \ldots \qquad (37a)$$

where

$$\Theta_{b,o} = \left[(1+\omega) S_b \left(\frac{\sigma}{1+\epsilon}\right)\left(1 + \frac{1+\epsilon}{\sigma} \Theta_{b,w}\right)\right]^{1/(1+\omega)}$$

$$U_{b,o} = (1+\omega) S_b \Theta_{b,o}^{-\omega}$$

$$V_{b,o} = 1 \qquad (37b)$$

To verify the similarity formulations of the PVL and BVL, at least insofar as these two layers are interrelated, it must be shown that the solutions of Eq. (24a-b) match to those of Eq. (37a-b). The solutions of the PVL and the BVL are matched through the introduction of the intermediate limit, $\lim_{kb}$, defined by

$$s_{kb} = x \text{ fixed}, \qquad t_{kb} = \tau_{kb} \log\left(\frac{\delta F_{kb}(x)}{r}\right) \text{ fixed;}$$

$$\tau_b \ll \tau_{kb} \ll 1, \quad F_{kb}(x) = \text{arbitrary fnc}(x) = 0(1)$$

In this limit,

$$t_k = \frac{1}{\tau_{kb}}\left[t_{kb} + \tau_{kb} \log\left(\frac{F_k}{F_{kb}}\right)\right] \to \frac{1}{\tau_{kb}} t_{kb} \to \infty$$

$$t_b = \frac{\tau_b}{\tau_{kb}}\left[t_{kb} + \tau_{kb} \log\left(\frac{F_b}{F_{kb}}\right)\right] \to \frac{\tau_b}{\tau_{kb}} t_{kb} \to 0 \tag{38}$$

The PVL-BVL temperature matching requires that

$$\lim_{kb}\left[\!\left[\left\{\theta_S \alpha_k T_k\left(s_{kb}, \frac{t_{kb}}{\tau_{kb}} + \ldots\right) + \ldots\right\} - \left\{\theta_S T_b\left(s_{kb}, \frac{\tau_b t_{kb}}{\tau_{kb}} + \ldots\right) + \ldots\right\}\right]\!\right] = 0 \tag{39a}$$

From Eqs. (24) and (37), it is seen that Eq. (39a) reduces to

$$\lim_{kb}\left[\!\left[\left\{\theta_S \alpha_k\left[A_k \Theta_{k,\infty}\left(\frac{t_{kb}}{\tau_{kb}}\right)^{1/(1+\omega)} + \ldots\right] + \ldots\right\} - \left\{\theta_S\left[\Theta_{b,o}\left(\frac{\tau_b t_{kb}}{\tau_{kb}}\right)^{1/(1+\omega)} + \ldots\right] + \ldots\right\}\right]\!\right] = 0 \tag{39b}$$

and, hence, the temperature matches if

$$\alpha_k = \tau_b^{1/(1+\omega)} = \left[\log\left(\frac{1}{D_b}\right)\right]^{-1/(1+\omega)} \ll 1 \tag{40a}$$

$$A_k\Theta_{k,\infty} = \Theta_{b,o} \Rightarrow m = \frac{3}{4}, \quad \Lambda_k = \left(\frac{1+2\epsilon}{2}\right)\frac{\Pi_h N_h^2}{S_b\left(1+\frac{1+\epsilon}{\sigma}\Theta_{b,w}\right)}$$

(40b)

The PVL-BVL longitudinal velocity matching requires that

$$\lim_{kb}\left[\!\left[\left\{1+\alpha_k u_k\left(s_{kb}, \frac{t_{kb}}{\tau_{kb}} + \ldots\right) + \ldots\right\} - \left\{u_b\left(s_{bk}, \frac{\tau_b t_{kb}}{\tau_{kb}} + \ldots\right) + \ldots\right\}\right]\!\right] = 0$$

(41a)

i.e., that

$$\lim_{kb}\left[\!\left[\left\{1+\alpha_k\left[A_k U_{k,\infty}\left(\frac{t_{kb}}{\tau_{kb}}\right)^{1/(1+\omega)} + \ldots\right] + \ldots\right\}\right.\right.$$

$$\left.\left. - \left\{\left[1 - U_{b,o}\left(\frac{\tau_b t_{kb}}{\tau_{kb}}\right)^{1/(1+\omega)} + \ldots\right] + \ldots\right\}\right]\!\right] = 0 \qquad (41b)$$

Thus, the longitudinal velocity matching reconfirms the requirement that $\alpha_k = \tau_b^{1/(1+\omega)}$, and yields the additional requirement that

$$A_k U_{k,\infty} = -U_{b,o} \Rightarrow$$

$$m = \frac{3}{4}, \int_o^\infty \exp(-2t_k)\frac{U_k}{\Theta_k}dt_k = -\left[\left(\frac{1+2\epsilon}{2}\right)\left(1+\frac{1+\epsilon}{\sigma}\Theta_{b,w}\right)^{-1} - \frac{\epsilon}{4}\right]$$

(42)

The PVL-BVL radial velocity matching reconfirms the requirement that $m = 3/4$.

An important fact, <u>implicit</u> in the results of Eqs. (40a-b) and (42), is that, although, to the order of approximation considereu for the flow quantities, the PVL-BVL matching requires that m = 3/4, and, hence, the "effective body" shape is given by

$$r = \delta N_h\left(\epsilon, m = \frac{3}{4}\right) x^{3/4}$$

This matching does not require the specification of the (actual) body shape, and so the equation for this body remains

$$r = \delta_b F_b(x), \qquad F_b(x) = \text{arbitrary fnc}(x)$$

Therefore, in the present first-order analysis, the flow in the PVL is independent of the exact body shape. This point is not made explicitly in the first-order analyses of Stewartson, Solomon, and Ellinwood and Mirels.

Further, since, from Section IV,

$$\Lambda_k = \left(\epsilon\left\{\frac{1+\epsilon}{1-\epsilon}\right\}\right)^{1+\omega} \bar{V}_L^2 \frac{\alpha_k^{1+\omega}}{\delta^4}$$

where

$$\alpha_k = \tau_b^{1/(1+\omega)}, \qquad \delta = \left(\frac{1}{D_b}\right)\delta_b = \left[\exp\left(\frac{1}{\tau_b}\right)\right]\delta_b$$

it follows that the parameters D_b and τ_b are given (in terms of the measure of the actual body thickness ratio, δ_b, etc.) by

$$D_b\left[\log\left(\frac{1}{D_b}\right)\right]^{-1/4} = \tau_b^{1/4}\left[\exp\left(-\frac{1}{\tau_b}\right)\right]$$

$$= \left[\left(\epsilon\left\{\frac{1+\epsilon}{1-\epsilon}\right\}\right)^{-(1+\omega)} \Lambda_k\right]^{1/4} \frac{\delta_b}{\bar{V}_L^{1/2}} \ll 1 \tag{43a}$$

i. e.,

$$\tau_b = \left|\log\left(\frac{1}{D_b}\right)\right|^{-1} \sim 2\left[\log\left(\frac{\bar{V}_L}{\delta_b^2}\right)\right]^{-1} \tag{43b}$$

In addition, the inequalities required in the formulation of Eqs. (28a-d),

$$\tau_b, \quad \frac{\delta^2}{\alpha_k^{1+\omega}}, \quad \left(\frac{\alpha_k^{1+\omega}}{\tau_b^2}\right) \exp\left(-\frac{2}{\tau_b}\right) \ll 1$$

make it necessary that $\bar{V}_L$ satisfy

$$\bar{V}_L \ll \left|\log\left(\frac{1}{D_b}\right)\right|^{-1/2} = \tau_b^{1/2} \ll 1 \tag{44}$$

From Eqs. (43) and (44), δ_b, itself, must satisfy

$$\delta_b \ll D_b \left|\log\left(\frac{1}{D_b}\right)\right|^{-1/2} = \tau_b^{1/2} \exp\left(-\frac{1}{\tau_b}\right) \ll 1 \tag{45}$$

which requires that the actual body must be very slender for the theory presented to be valid. For example, if $\tau_b = 10^{-1}$, then $\delta_b \ll 10^{-5}$.

It is found that the pressure, friction, and heat transfer coefficients are, respectively,

$$C_p = \frac{2[p_1]_w}{\rho_\infty u_\infty^2}$$

$$= \left\{4\left\{\frac{1-\epsilon}{1+\epsilon}\right\}\left[\frac{1}{1+2\epsilon}\left(\epsilon\left\{\frac{1+\epsilon}{1-\epsilon}\right\}\right)^{1+\omega}\left(\frac{\Pi_h}{N_h^2}\right)S_b\left(1+\frac{1+\epsilon}{\sigma}\Theta_{b,w}\right)\right]^{1/2}\right\} \times$$

$$\bar{V}_x\left[\log\left(\frac{\bar{V}_x}{\delta_b^2}\right)\right]^{-1/2} + \ldots$$

$$= C_{p,o}\,\bar{V}_x\left[\log\left(\frac{\bar{V}_x}{\delta_b^2}\right)\right]^{-1/2} + \ldots \tag{46a}$$

$$C_f = \frac{2\left[\mu_1\left(\frac{\partial u_1}{\partial r_1}\right)\right]_w}{\rho_\infty u_\infty^2}$$

$$= \left[4\left(\epsilon\left\{\frac{1+\epsilon}{1-\epsilon}\right\}\right)^\omega S_b\right] \frac{\bar{V}_x^2}{\delta_b}\left[\log\left(\frac{\bar{V}_x}{\delta_b^2}\right)\right]^{-1}\left[\frac{F_b(x)}{x}\right]^{-1} + \ldots$$

$$= C_{f,o}\,\frac{\bar{V}_x^2}{\delta_b}\left[\log\left(\frac{\bar{V}_x}{\delta_b^2}\right)\right]^{-1}\left[\frac{F_b(x)}{x}\right]^{-1} + \ldots \qquad (46b)$$

$$C_q = \frac{2\left[k_1\left(\frac{\partial T_1}{\partial r_1}\right)\right]_w}{\rho_\infty u_\infty^3}$$

$$= \frac{1}{2}\left(1 - \frac{1+\epsilon}{\sigma}\,\Theta_{b,w}\right) C_f + \ldots$$

$$= C_{q,o}\,\frac{\bar{V}_x^2}{\delta_b}\left[\log\left(\frac{\bar{V}_x}{\delta_b^2}\right)\right]^{-1}\left[\frac{F_b(x)}{x}\right]^{-1} + \ldots \qquad (46c)$$

The friction drag coefficient is

$$C_{D,f} = \frac{4\int_0^{x_1}\left[\mu_1\left(\frac{\partial u_1}{\partial r_1}\right)\right]_w [r_1]_w\, dx_1}{\rho_\infty u_\infty^2\, [r_1]_w^2}$$

$$= 2C_{f,o}\,\frac{\bar{V}_x^2}{\delta_b^2}\left[\log\left(\frac{\bar{V}_x}{\delta_b^2}\right)\right]^{-1}\left[\frac{F_b(x)}{x}\right]^{-2} + \ldots \qquad (47a)$$

The pressure drag coefficient is

$$C_{D,p} = \frac{4\int_o^{x_1} [p_1]_w [r_1]_w \left(\frac{d[r_1]_w}{dx_1}\right) dx_1}{\rho_\infty u_\infty^2 [r_1]_w^2}$$

$$= C_{p,o}\bar{V}_x \left[\log\left(\frac{\bar{V}_x}{\delta_b^2}\right)\right]^{-1/2}$$

$$\times \left\{1 + \frac{1}{2} x^{-3/2} \left[\frac{F_b(x)}{x}\right]^{-2} \int_o^x \tilde{x}^{1/2} \left[\frac{F_b(\tilde{x})}{\tilde{x}}\right]^2 d\tilde{x}\right\} + \dots \qquad (47b)$$

Thus, the principal contribution to the drag coefficient is from friction, since

$$\frac{C_{D,p}}{C_{D,f}} = \left(\frac{C_{p,o}}{2C_{f,o}}\right) \frac{\delta_b^2}{\bar{V}_x} \left[\log\left(\frac{\bar{V}_x}{\delta_b^2}\right)\right]^{1/2}$$

$$\times \left\{\left[\frac{F_b(x)}{x}\right]^2 + \frac{1}{2} x^{-3/2} \int_o^x \tilde{x}^{1/2} \left[\frac{F_b(\tilde{x})}{\tilde{x}}\right]^2 d\tilde{x}\right\} + \dots$$

where

$$\frac{\delta_b^2}{\bar{V}_x} \ll 1$$

In terms of the parameters introduced in this Memorandum, the interaction parameter of Ellinwood and Mirels, $\Lambda_{(EM)}$, is given by

$$\Lambda_{(EM)} = \frac{\bar{V}_x}{\delta_b^2} \left[\frac{F_b(x)}{x}\right]^{-2}, \qquad \log \Lambda_{(EM)} = \log\left(\frac{\bar{V}_x}{\delta_b^2}\right) + \dots \qquad (48)$$

From Eq. (48), then, it follows that the expressions for C_p, C_f, C_q, $C_{D,f}$ given in this Memorandum are equivalent to those found by Ellinwood and Mirels for these coefficients. (Further, since, in turn,

$$\tau_b = 2\,[\log \Lambda_{(EM)}]^{-1} + \ldots$$

it also follows that the expansion parameter of the PVL, α_k, is linearly proportional to the expansion parameter of Ellinwood and Mirels, $\epsilon_{(EM)}$.)

When the (actual) body is a cone, the body studied by Stewartson,

$$\left|\frac{F_b(x)}{x}\right|_{cone} = 1$$

and the pressure, friction, heat transfer, and drag coefficients are

$$C_p = C_{p,o}\bar{V}_x \left[\log\left(\frac{\bar{V}_x}{\delta_b^2}\right)\right]^{-1/2} + \ldots$$

$$C_f = C_{f,o}\left(\frac{\bar{V}_x}{\delta_b}\right)^2 \left[\log\left(\frac{\bar{V}_x}{\delta_b^2}\right)\right]^{-1} + \ldots$$

$$C_q = C_{q,o}\left(\frac{\bar{V}_x}{\delta_b}\right)^2 \left[\log\left(\frac{\bar{V}_x}{\delta_b^2}\right)\right]^{-1} + \ldots$$

$$C_D = 2C_{f,o}\left(\frac{\bar{V}_x}{\delta_b^2}\right)^2 \left[\log\left(\frac{\bar{V}_x}{\delta_b^2}\right)\right]^{-1} + \ldots \tag{49}$$

The foregoing expressions for C_f and C_q are equivalent to those found by Stewartson.

Appendix

THE VISCOUS TRANSITION LAYER

To span the distance between the ISL and PVL, a viscous transition layer (TL) is introduced, whose coordinates are taken to be x_t and r_t, defined by

$$x_t = x, \qquad r_t = \frac{\left\{\left(\frac{r}{\delta}\right) - F_k(x)\right\}}{B_t} = \frac{\left\{\left(\frac{r}{\delta}\right) - N_h\, x^{3/4}\right\}}{B_t} \qquad \text{(A-1)}$$

where B_t is a parameter much less than unity to be determined. The expansions for the flow quantities in this layer are taken to be

$$u = 1 + \alpha_k B_t^{2/(1+\omega)}\, u_t + \ldots$$

$$v = \delta\left\{\frac{3}{4} N_h\, x_t^{-1/4} + B_t v_t + \ldots\right\}$$

$$T = \theta_S \alpha_k B_t^{2/(1+\omega)}\, T_t + \ldots$$

$$p = M^2 \delta^2 p_t + \ldots \qquad \text{(A-2a)}$$

The behavior of u and T in the ISL ?VL requires

$$\delta^2 \ll B_t^{2/(1+\omega)}\, \alpha_k \ll \alpha_k = \tau_b^{1/(1+\omega)} \ll 1 \qquad \text{(A-2b)}$$

Then, the equations of motion for this TL reduce to

$$\frac{\partial p_t}{\partial r_t} = 0, \quad \text{i.e.,}\ p_t = P_t(x_t) = \Pi_h x_t^{-1/2}$$

$$\frac{\partial}{\partial r_t}\left(\frac{v_t}{T_t}\right) + \frac{\partial}{\partial x_t}\left(\frac{1}{T_t}\right) + \frac{1}{4x_t}\frac{1}{T_t} = 0$$

$$\frac{x_t^{-1/2}}{T_t}\left(\frac{\partial u_t}{\partial x_t} + v_t\frac{\partial u_t}{\partial r_t}\right) - \frac{\epsilon}{2}x_t^{-3/2} = \frac{\Lambda_k}{\Pi_h}\frac{\partial}{\partial r_t}\left(T_t^{\omega}\frac{\partial u_t}{\partial r_t}\right)$$

$$\frac{x_t^{-1/2}}{T_t}\left(\frac{\partial T_t}{\partial x_t} + v_t\frac{\partial T_t}{\partial r_t}\right) + \frac{\epsilon}{1+\epsilon}x_t^{-3/2} = \frac{1}{\sigma}\frac{\Lambda_k}{\Pi_h}\frac{\partial}{\partial r_t}\left(T_t^{\omega}\frac{\partial T_t}{\partial r_t}\right) \tag{A-3}$$

To find a self-similar solution for T_b,* take

$$T_t = \xi_t^a\,\Theta_t(\zeta_t) \tag{A-4a}$$

where

$$\xi_t = x_t, \quad \zeta_t = \frac{r_t}{N_h x_t^b} \tag{A-4b}$$

where a is a constant to be determined, and $b = b(a) = 3/4 + [(1+\omega)/2]a$. Then, the continuity and energy equations combine to yield

$$\frac{\Lambda_k}{\sigma\Pi_h N_h^2}\frac{d}{d\zeta_t}\left[\frac{\frac{d}{d\zeta_t}\left(\Theta_t^{\omega}\frac{d\Theta_t}{d\zeta_t}\right)}{\frac{d\Theta_t}{d\zeta_t}}\right] - \left\{a + \frac{\epsilon}{1+\epsilon}\right\}\frac{d}{d\zeta_t}\left(\frac{1}{\frac{d\Theta_t}{d\zeta_t}}\right) + \left\{1 - \left(\frac{1-\omega}{2}\right)a\right\}\frac{1}{\Theta_t} = 0 \tag{A-5}$$

---

*For the sake of brevity, self-similar solutions for u_t and v_t are not considered in this presentation. It can be seen, however, from Eq. (A-3), that, once the self-similar solution for T_t has been determined, such solutions follow directly.

(This equation is invariant under the transformation $\zeta_t \rightarrow \zeta_t + \zeta_t^*$, where $\zeta_t^* =$ const.)

Considering that the TL temperature solution should approach the PVL temperature solution for $\zeta_t \rightarrow -\infty$, it is found that one asymptotic solution for the TL temperature, based upon a balance of the inviscid and viscous terms of Eq. (A-5), is

$$\Theta_t = \Theta_{t,k}(-\zeta_t)^{2/(1+\omega)} + \ldots, \qquad \text{with } \Theta_{t,k} = (A_k\Theta_{k,o})_{m=3/4}$$

$$T_t = \Theta_{t,k}(-\eta_t)^{2/(1+\omega)} + \ldots, \qquad \text{for } \eta_t = \frac{r_t}{N_h x_t^{3/4}} \tag{A-6}$$

Considering that the TL temperature solution should approach the ISL temperature solution for $\zeta_t \rightarrow +\infty$, it is found that one asymptotic solution for the TL temperature, based upon the domination of the inviscid terms in Eq. (A-5), is

$$\Theta_t = \Theta_{t,h}\zeta_t^{-c} + \ldots,$$

$$\text{where } c = c(a) = -\frac{2[a(1+\epsilon)+\epsilon]}{(3+\omega)(1+\epsilon)+2\epsilon}, \quad \Theta_{t,h} = \text{const. to be determined}$$

$$T_t = \Theta_{t,h}\xi_t^{-g}\eta_t^{-c} + \ldots, \qquad \text{where } g = g(c) = \left[\frac{(1+2\epsilon)c+\epsilon}{1+\epsilon}\right] \tag{A-7}$$

The temperature solutions of the PVL and the TL are matched through the introduction of the intermediate limit, $\lim_{kt}$, defined by

$$\eta_{kt} = - \frac{\left[\left(\frac{r}{\delta N_h x^{3/4}}\right) - 1\right]}{B_{kt}} \text{ fixed,} \qquad B_t \ll B_{kt} \ll 1 \tag{A-8}$$

In this limit,

$$t_k = \log\left\{(1 - B_{kt}\eta_{kt})^{-1}\right\} = B_{kt}\eta_{kt} + \ldots \to 0$$

$$\eta_t = -\left(\frac{B_{kt}}{B_t}\right)\eta_{kt} \to -\infty$$

The PVL-TL temperature matching requires that

$$\lim_{kt}\left\| \theta_S\alpha_k T_k(s_k, \log\{(1 - B_{kt}\eta_{kt})^{-1}\}) + \ldots - \theta_S\alpha_k B_t^{2/(1+\omega)} T_t\left(\xi_t, -\frac{B_{kt}\eta_{kt}}{B_t}\right) + \ldots \right\| = 0 \tag{A-9}$$

From inspection of Eqs. (19), (23), and (A-6), it can be seen that the matching follows directly.

The temperature solutions of the ISL and the TL are matched through the introduction of the intermediate limit, $\lim_{ht}$, defined by

$$\eta_{ht} = \frac{\left[\left(\frac{r}{\delta N_h x^{3/4}}\right) - 1\right]}{B_{ht}} \text{ fixed,} \qquad B_t \ll B_{ht} \ll 1 \tag{A-10}$$

In this limit,

$$\eta_h = N_h(1 + B_{ht}\eta_{ht}) \rightarrow N_h$$

$$\eta_t = \left(\frac{B_{ht}}{B_t}\right)\eta_{ht} \rightarrow \infty$$

The ISL-TL temperature matching requires that

$$\lim_{ht} \Big\| \theta_S\delta^2 T_h(\xi_h, N_h\{1 + B_{ht}\eta_{ht}\}) + \ldots$$

$$- \theta_S\alpha_k B_t^{2/(1+\omega)} T_t\left(\xi_t, \frac{B_{ht}\eta_{ht}}{B_t}\right) + \ldots \Big\| = 0 \qquad \text{(A-11)}$$

From Eqs. (14) and (A-7), it can be shown that the matching requires that

$$B_t = \left[\frac{\tau_b}{\delta^{2(1+\omega)}}\right]^a, \quad \text{with} \quad a = -\frac{1}{2}\left|1 + \left(\frac{1+\omega}{4}\right)\left(\frac{1-\epsilon}{1+2\epsilon}\right)\right|^{-1}$$

$$\Theta_{t,h} = \Phi_h N_h^{-E} \qquad \text{(A-12)}$$

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10. ABSTRACT

The Navier-Stokes hypersonic strong-interaction theory is presented for the flow of a viscous, heat-conducting, compressible fluid past a very slender axisymmetric body, when the ratio of the radius of the body to the radial thickness of the viscous layer(s) (which the body produces and supports), goes to zero. It is assumed that the fluid is a perfect gas having constant specific heats, a constant Prandtl number of order unity, and viscosity coefficients varying as a power of the absolute temperature. Solutions are studied for the free-stream Mach number, the free-stream Reynolds number based on the body's axial length, and a modified strong-interaction parameter going to infinity. It is shown that when the viscosity-temperature exponent is less than one, four distinct layers span the region between the shock wave and the body. The self-similar leading approximations for the behavior of the flow in these four regions are analyzed, and it is found that an arbitrary body supports "3/4-power" viscous and shock layers.

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