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by

S. M. Gilinskiy, G. F. Telenin, and G. P. Tinyakov

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ABSTRACT

This paper deals with the development of a method for calculating supersonic flow past blunt bodies with a detached shock wave, which was proposed first by G. F. Telenin of the Scientific and Research Institute of Mechanics of Moscow State University in 1961. The first section contains a formulation of the problem of supersonic flow past a blunt body of an arbitrary shape and an investigation of basic gas-dynamics equations. A method is given for the selection of a differential system for solving a boundary problem. This is followed by sections on the construction of an algorithm for the numerical solution of a boundary value problem in a mixed region and on the method of solving a nonlinear problem. The final section gives examples illustrating the solution of different problems of flow past bodies of different shapes with a detached shock wave. Orig. art. has: 35 citations in the references at the end of the article, 16 figures and 72 equations.

U. S. BOARD ON GEOGRAPHIC NAMES TRANSLITERATION SYSTEM

Block	Italic	Transliteration	Block	Italic	Transliteration
А а	*А а*	A, a	Р р	*Р р*	R, r
Б б	*Б б*	B, b	С с	*С с*	S, s
В в	*В в*	V, v	Т т	*Т т*	T, t
Г г	*Г г*	G, g	У у	*У у*	U, u
Д д	*Д д*	D, d	Ф ф	*Ф ф*	F, f
Е е	*Е е*	Ye, ye; E, e*	Х х	*Х х*	Kh, kh
Ж ж	*Ж ж*	Zh, zh	Ц ц	*Ц ц*	Ts, ts
З з	*З з*	Z, z	Ч ч	*Ч ч*	Ch, ch
И и	*И и*	I, i	Ш ш	*Ш ш*	Sh, sh
Й й	*Й й*	Y, y	Щ щ	*Щ щ*	Shch, shch
К к	*К к*	K, k	Ъ ъ	*Ъ ъ*	"
Л л	*Л л*	L, l	Ы ы	*Ы ы*	Y, y
М м	*М м*	M, m	Ь ь	*Ь ь*	'
Н н	*Н н*	N, n	Э э	*Э э*	E, e
О о	*О о*	O, o	Ю ю	*Ю ю*	Yu, yu
П п	*П п*	P, p	Я я	*Я я*	Ya, ya

* ye initially, after vowels, and after ъ, ь; e elsewhere. When written as ё in Russian, transliterate as yё or ё. The use of diacritical marks is preferred, but such marks may be omitted when expediency dictates.

METHOD FOR CALCULATION OF SUPERSONIC FLOW PAST BLUNT BODIES WITH A DETACHED SHOCK WAVE

S. M. Silinskiy, G. F. Telenin
and G. P. Tinyakov

In 1961 at Moscow State University G. F. Telenin proposed a method for calculating supersonic flow past blunt bodies with a detached shock wave [23, 24]. During 1962 and 1963 systematic investigations were conducted on an ideal gas flowing past ellipsoids with a different ratio of semiaxes and bodies with an analytic contour, close in form to a face with a small degree of rounding of angles [27, 32], bodies with a concave contour in the region of critical point, bodies with discontinuity of contour curvature in a subsonic unit and contour fracture in the sonic point (type of face), and flat bodies [32, 33]; flow past bodies with a various form with M number striving for infinity and the adiabatic index striving for unity [27]; flow past a sphere by equilibrium carbon dioxide [26] and past bodies of different form by equilibrium air [25, 34]; flow past a sphere by nonequilibrium air [28], taking into account kinetics of excitation of fluctuations in oxygen and nitrogen, the flow of dissociation and exchange reactions (including $O_2 + N_2 = 2NO$) and ionization with the help of atomic collisions ($N + O = NO^+ + e$, $N + N = N_2^+ + e$, $O + O = O_2^+ + e$), satisfactorily describing the process up to M ∿ 30 [29, 30], and spatial flowing around with a detached shock wave [2, 35]. At the same time work on investigation of the method was conducted.

In the first, second, and third sections of this article the problem of supersonic flow past bodies with a detached shock wave is formulated, and on the basis of an analysis of properties of solutions of model linear equations of elliptic and mixed types the basic ideas of the method are expounded. In the fourth section a numerical method for solving nonlinear problems is described. In the fifth section examples are given which illustrate the application of the method for solving different problems of flowing around with a detached shock wave

§ 1. Formulation of Problem

We will examine the flow past an axisymmetrical body by a stream of an ideal gas with the parameters V_∞, p_∞, ρ_∞.

Using the stream function ψ, the system of equations for gas dynamics in a spherical system of coordinates r, θ, ϕ can be presented in the following form:

[illegible] (1.1)

Here [illegible] projection of velocity vector $\vec{W}$ on unit vectors $\vec{j}_r$ and $\vec{j}_\theta$, relative to V_{max}; p, ρ – pressure and density relative to $\rho_\infty V^2_{max}$ and ρ_∞ respectively; r is relative to the characteristic dimension L, and ψ to $\rho_\infty V^2_{max} L^2$.

The basic difficulty is in the transonic nature of the problem, since the solution has to be looked for simultaneously in the subsonic and in a certain part of supersonic regions. As is known, in the transonic region the Tricomi equation is found and is the most studied model of gas dynamics equations in the hodograph plane. It has been used to help in studying the formulation of ma..y transonic problems. It is natural to lean on these results also in examining the problem of flow past with a detached shock wave. According to the results obtained in the present work (Fig. 1a), a geometric picture is given for flow past a sphere by an ideal gas ($\gamma = 1.4$) at M = 6. The figure shows the shock wave ABC, sonic line DBQ, maximum characteristics DCK, limiting the region of influence of the supersonic fraction of flow behind the shock wave on subsonic. Also shown are a series of characteristics BK, MN, etc., of the first family, emanating from the sonic line, and the limiting line QP, starting at the point of intersection of sonic line with the axis and representing the envelope of characteristics of the first family. Solution for the region located before the shock wave is obtained numerically by continuation from data behind the shock wave with the help of the same program. With a decrease of M number and an increase of body flatness the form and arrangement of mutual disposition of the compression shock, sonic line, and maximum characteristics are changed. For clarification of the main positions we will limit ourselves to an examination of the simplest case which is depicted in Fig. 1a. In Figure 1b a representation of flow in the hodograph plane v_x, v_y (corresponding points are designated by the same letters with primes) is given for this case.

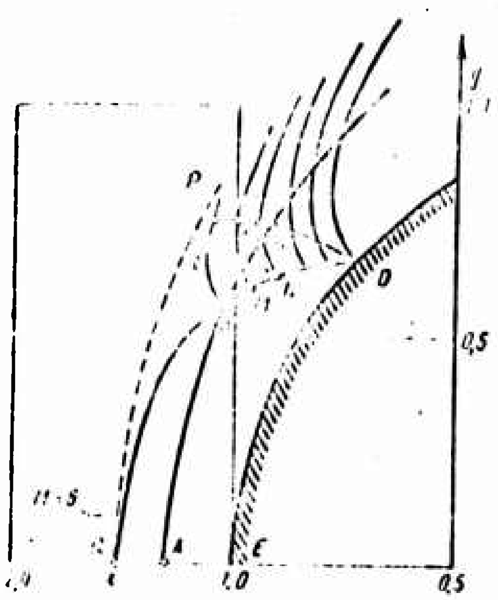

Fig. 1a.

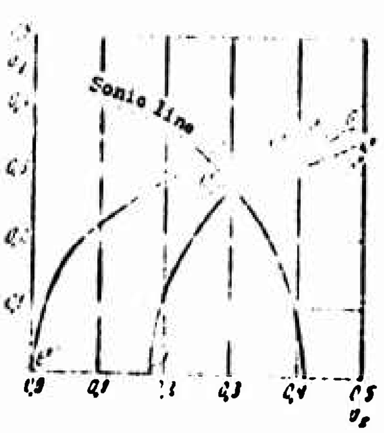

Fig. 1b.

Solution of system (1.1) should be subordinated to correlations for compression shock.

$$u = v_y \sin\theta - v_x \cos\theta, \quad v = v_y \cos\theta + v_x \sin\theta,$$

$$v_x = \frac{V_\infty}{V_{max}} \left[1 - \frac{2}{\gamma+1} \sin^2\sigma \left(1 - \frac{1}{M_\infty^2 \sin^2\sigma}\right)\right],$$

$$v_y = \left(\frac{V_\infty}{V_{max}} - v_x\right) \operatorname{ctg} \sigma,$$

$$p = \frac{2}{\gamma+1} \left[\left(\frac{V_\infty}{V_{max}}\right)^2 \sin^2\sigma - \left(1 - \frac{V_\infty^2}{V_{max}^2}\right) \frac{(\gamma-1)^2}{4\gamma}\right],$$

$$\rho = \frac{\gamma+1}{\gamma-1} \cdot \frac{V_\infty^2}{V_{max}^2} \cdot \frac{\sin^2\sigma}{1 - \frac{V_\infty^2}{V_{max}^2} \cos^2\sigma}, \qquad (1.2)$$

$$\psi = \frac{1}{2} \cdot \frac{V_\infty}{V_{max}} r_c^2 \sin^2\theta.$$

Here v_x and v_y are projections of velocity vector $\vec{W}$ on rectangular axes x and y (Fig. 1a), and $r = r_c(\theta)$ and σ the equation of shock wave and angle of inclination of it to the axis of symmetry x, to conditions of symmetry on segment of axis AE and the boundary conditon on body contour

$$u = v \frac{r_T'}{r_T}, \qquad (1.3)$$

where $r = r_T(\theta)$ – equation for body generator.

From the relationships (1.2) it is clear that if the form of shock wave is given, then $\sigma(\theta)$ is a known function.

In regions A'B'K'D'E'A' in the hodograph plane a boundary value problem of the Frankl-Tricomi [3] type appears: boundary conditions are assigned on the elliptic segment of the boundary of region B'A'E'D' and on the noncharacteristic curve B'C', located inside the characteristic triangle B'K'D' which is adjacent to the parabolic line D'B'. Line A'B'C', which is a form of compression shock, intersects each of the characteristics only once inside triangle B'K'D'. Line E'D', a form of body contour, is unknown beforehand and should be determined during the solving process. In an analogy with the Tricomi equation and flat potential problems of gas dynamics one may assume that in the hodograph plane the correlations for section A'C' of compression shock, conditions of symmetry on the segment of axis A'E', and the boundary conditions on part of body contour E'D' uniquely determine the solution in region A'B'K'D'E'A'. However, a physically possible flow does not satisfy every solution in a hodograph plane, since mapping from a hodograph plane into a physical plane can be single-valued, which leads to peculiarities in the physical plane. In the subsonic part of the hodograph plane mapping into a physical plane does not have any peculiarities besides the isolated branch points. In the supersonic region limiting lines can appear, along which the fastening of two sheets of the physical plane occurs. Along limiting lines the Jacobian

$$D = \frac{\partial(x, y)}{\partial(v_x, v_y)}$$

becomes zero, or which in our case is equivalent, derivatives of velocity components become infinite.

In order that the solution in the hodograph plane for region A'B'K'D'E'A' in the physical plane in the region ABKDEA is satisfied by the real flow, it is necessary to place an additional condition for the desired solution: nonreversal of the Jacobian D to zero, or what is the same, boundedness of derivatives of vectorial velocity components in the triangular region DKD. In accordance with this the condition of uniqueness in reference to the problem of supersonic flow past a blunt body with a detached shock wave can be formulated in the following way: in the class of functions with a continuous limited derivative of relationship for section AC of the compression shock, the conditions of symmetry for segment of axis AE and the boundary condition for a section of body contour uniquely determine the solution of a system of differential equations of gas dynamics in the region ABCDE of the physical plane (see Fig. 1).

It is necessary to note that since fulfillment of the characteristic equation along the characteristic curve is a corollary of differential equations, then assignment of this equation along limiting characteristic DK or some other characteristic can produce nothing for separation of the class of solutions with a limited derivative (such a condition is D ≠ 0 in the supersonic region), and consequently also for formulation of conditions of uniqueness. The appearance of a limiting line is conditioned by the boundary conditions of the problem. The characteristic equation contains derivatives only along the curve and is fulfilled for any characteristic independent of whether or not it has a common point with the limiting line and, consequently, whether or not the derivatives which are normal to characteristic remain limited (for example, formation of a limiting line during supersonic flow around a concave angle).

Making further use of well-known theorems for hyperbolic equations, our result is that in the class of functions with a limited derivative the solution is determined uniquely in a wider region ACGRFDE, if the conditions for compression shock and for the body are assigned accordingly up to points G and F (Fig. 2).

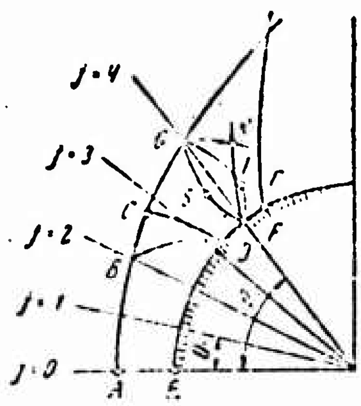

Fig. 2.

If the body contour up to point Γ is an analytic curve, then, as it follows in the region which is limited from above by the characteristic emanating from point Γ the field of gas-dynamic parameters behind the shock wave not only satisfies conditions of uniqueness but also is analytic. The angular point and discontinuity of curvature (or leading derivatives) on the contour in the subsonic sector of flow lead to appearance of isolated singular points on the surface of the body.

Let us trace in region ACGRFDE an analytic curve GSF which does touch upon a characteristics in one point. Then, if the unknown solution is analytical in a sufficiently large region, then in the region FSGTF which is bounded above by the arbitrary curve GTF, there exists an unique solution of the Cauchy problem with initial data for line GSF and, consequently, in the class of analytic functions the above-mentioned conditions of the boundary value problem determine the unique solution in region ACGTFDE.

§ 2. Selection of a Differential System For Solution of the Formulated Boundary Problem

For this purpose it is possible to use the method of establishment or some iteration method for solving a system of difference equations approximating a system of stationary equations (1.) simultaneously in all the subsonic and transonic regions (i.e., the solving of nonstationary equations). However both these methods require a great volume of computer storage and the carrying out of a large volume of calculations, since they essentially reduce the problem to a three-dimensional one. This limits the selection of high speed machines for the resolution of problems of supersonic flow past bodies with detached shock wave. Therefore it is natural to attempt [4, 6, 7] to construct a numerical algorithm, using the resolution of the Cauchy problem in some form or other. The difficulty is that in a general case the Cauchy problem is tactless for an elliptic region, i.e., solutions exist with Cauchy data for a length of boundary of the region differing as little as desired, but which differ as strongly as desired inside the region.

However, in a number of works (see, for example, [8, 9] it is shown that the problem becomes correct in a limited region if one were to narrow down the class of solutions examined. In the class of bounded functions the problem is correct in the sense of the root-mean-square norm, and if the function is moreover analytic, then in the usual sense.

Considering the analytic nature of the desired solution, it is natural to expect that by selecting an approximating system which takes this analyticity into consideration to a sufficient degree, it is possible to obtain a convergent and sufficiently stable method for the numerical resolution of the problem.

We will consider the selection of an approximating system in an example of the Laplace equation:

$$\frac{\partial^2\varphi}{\partial x^2}+\frac{\partial^2\varphi}{\partial y^2}=0. \tag{2.1}$$

Let it be necessary to find the solution of a Cauchy problem in the region $-1<x<1$, [illegible] with initial data (Fig. 3)

$$\varphi(x,0)=a(x),\quad \varphi_y(x,0)=b(x). \tag{2.2}$$

Let us assume that in a certain wider region ABCD the desired solution $\phi(x, y)$ is a harmonic function with a limited gradient $|\text{grad}\ \phi| \le M$, and the initial data (2.2) on segment KL is less than (1/2)m.

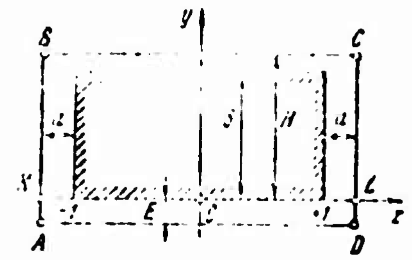

Fig. 3.

Then for the function of the complex variable $f(z) = \phi(x, y) + i\psi(x, y)$ ($\psi(0.0) = 0$) we have the estimate [illegible] in the region ABCD ($E < H$) and $|f(z)|\le 2m(1+a)=B$ for the initial line KL. Using the Corleman lemma [8, 9] we obtain the estimate for resolution of the Cauchy problem in the region [illegible]

$$|\varphi(x,y)|\le|f(z)|\le A^{1-\eta}B^{\eta}, \tag{2.3}$$

where

$$\eta=\left(\frac{H-y}{R}\right)^{\frac{1}{\alpha}},\quad \frac{\pi\alpha}{2}=\operatorname{arc\,tg}\frac{a}{H},\quad R=\frac{H}{\cos\frac{1}{2}\alpha\pi}.$$

Formula (2.3) shows that the greater a and H - h are, (the further peculiarities are located) then the more rapidly decreases the solution of $\phi(x, y)$ in the entire examined (crosshatched) region with a decrease of m. The value $E \neq 0$ is not essential.

Now let us consider on the same model the question of the selection of an algorithm for the numerical resolution of the Cauchy problem in an elliptic region.

If one were to continue the entire solution $\phi(x, y) = \phi_1(x, \eta, y) + i\phi_2(x, \eta, y)$ in the complex region $\zeta = x + iy$, then at a fixed x (x – parameter) determination of ϕ_1 and ϕ_2 is reduced to resolution of the Cauchy problem in the plane η, y for a hyperbolic equation (in the examined case wave [10, 11]) with initial data

$$\varphi_1(x, \eta) = \sigma_1(x, \eta),\ \varphi_{1y}(x, \eta) = \tau_1(x, \eta),\ \varphi_2(x, \eta) = \sigma_2(x, \eta),$$
$$\varphi_{2y}(x, \eta) = \tau_2(x, \eta).$$

Here $\sigma(\zeta) = \sigma_1(x, \eta) + i\sigma_2(x, \eta)$ and $\tau(\zeta) = \tau_1(x, \eta) + i\tau_2(x, \eta)$ are analytic continuations of initial data into the complex regions $\zeta = x + i\eta$.

Thus, in the three-dimensional region x, η, y it is possible to construct a stable algorithm for the numerical resolution of the Cauchy problem and to find the solution of the initial problem $\phi(x, \eta) = \phi_1(x, 0, y)$.

Such an approach has two deficiencies from the point of view of practical use in the numerical resolution of a Cauchy problem in an elliptic region. First, the process of continuation of initial data into a complex region is reduced to resolution of the Cauchy problem for the Laplace equation. For example, for determination of $\sigma_1(x, \eta)$ it is necessary to solve the problem with initial data

$$\sigma_1(x, 0) = \sigma(x),\quad \sigma_{1\eta}(x, 0) = 0.$$

Just as in a general case, the problem is reduced to only a simple Cauchy problem in an elliptic region, and in the examined case of a model equation this problem is equivalent to the initial one. Secondly, there is a strong increase in the volume of calculations, since actually again a problem is being solved in the three-dimensional region x, η, y. However, both these difficulties can be surmounted by the selection of a suitable difference system in the region of "physical" variables x, y, giving sufficient consideration to the analytic nature of the desired solution.

The general solution of the Cauchy problem (2.2) for a Laplace equation (2.1) can be presented in the form

$$\varphi(x,\eta,y) = \mathrm{Re}\{\sigma(z) - i\eta T(z)\}, \quad \text{[illegible]}$$
$$T(z) = \int_0^z \tau(\xi)\, d\xi. \qquad (2.4)$$

where σ(z), T(z) – analytic continuations in the complex region z = x + iy of functions σ(x) and T(x).

Let us assume that for the initial segment $-1 \leq x \leq 1$ the nodes 2n + 1 are selected. Based on the values of functions in the nodes for the initial data we construct Lagrangian interpolation polynomials in the complex region:

$$\sigma(\zeta) \approx \sigma_n^*(\zeta) = \sum_{j=0}^{2n} a_j \zeta^j, \quad \tau(\zeta) \approx \tau_n^*(\zeta) = \sum_{j=0}^{2n} b_j \zeta^j, \qquad (2.5)$$

where a_j and b_j are linear functions of $\sigma(x_j)$ and $\tau(x_j)$ values of functions in the nodes. If functions σ(x) and τ(x) are integer, then during arbitrary location of nodes the sequence of interpolation polynomials (2.5) converges evenly in any finite region of the complex variable.

If functions σ(ζ) and τ(ζ) have peculiarities in the complex plane and interpolation points coincide with zeroes of the Chebyshev polynomial, then the interpolation process (2.5) converges [illegible] in any closed region lying inside the ellipse $\rho = |\zeta + \sqrt{\zeta^2 - 1}| = \rho^*$. If all nodes coincide with the principle of coordinates, then polynomials (2.5) convert into segments of power series, and the region of uniform convergence is any closed region which is internal with respect to circle R = $|\zeta|$ = R*. Ellipse ρ = ρ* and circle R = R* pass through the nearest singular point (Fig. 4).[1]

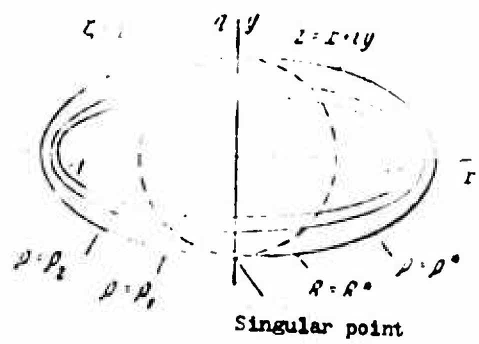

Fig. 4.

Let us note that the use of nodes which are disposed on the segment expands the convergence region of the interpolation process in direction of the initial line without changing its dimension in the direction η, when the peculiarity is disposed at x = 0 and a fixed η. Inside the region of uniform convergence the interpolation process gives an effective stable (in the examined class of functions) algorithm of continuation of initial data in the complex region.

Placing (2.5) in (2.4), we obtain an approximate solution of the Cauchy problem

$$\varphi(x, y) \approx \varphi_n^*(x, y) = \mathrm{Re}\left\{\sum_{j=0}^{n} a_j z^j + i \sum_{j=0}^{n} \frac{b_j}{j+1} z^{j+1}\right\}, \qquad (2.6)$$

satisfying the initial conditions (see 2.5) $\varphi_n^*(x, 0) = \varphi_n^*(x)$ and $\varphi_y(x, 0) = \psi_n^*(x)$.

In region G the sequence ϕ_n^* converges evenly to an exact resolution of the Cauchy problem ϕ(x, y) for the Laplace equation.

Now let us construct an algorithm for the numerical solution of the Cauchy problem which can be generalized for a case of more complex equations, including nonlinear ones. For this we approximate the desired solution by the Lagrangian polynomial for x:

$$\varphi(x, y) \approx \varphi_n^*(x, y) = \sum_{j=0}^{2n} \varphi_{jn}^0(y) x^j,$$

$$\frac{\partial^2 \varphi}{\partial x^2} \approx \frac{\partial^2 \varphi_n^*}{\partial x^2} = \sum_{j=2}^{2n} j(j-1) \varphi_{jn}^0(y) x^{j-2}, \tag{2.7}$$

where $\phi_{jn}^0(y)$ – linear function $\varphi_{jn}^*(y) = \varphi_n^*(x_j, y)$ of values of approximate solution for 2n + 1 rays of $x = x_j$. Placing (2.7) in (2.1) and requiring that the expression obtained is identically satisfied for all rays, we obtain the approximating system of ordinary differential equations

$$\frac{d^2 \varphi_{kn}^*}{dy^2} + \sum_{j=2}^{2n} j(j-1) \varphi_{jn}^0(y) x_k^{j-2} = 0 \quad (k = 0, 1, \ldots, 2n + 1) \tag{2.8}$$

and initial conditions

$$\varphi_{kn}^*(0) = \sigma(x_k), \quad \varphi_{kny}^*(0) = \tau(x_k). \tag{2.9}$$

It is obvious that any solution of initial equation (2.1), which at a fixed y is a polynomial of the power 2n based on x, for rays $x = x_k$ satisfies the approximating system (2.8). Since the solution $\phi_n^*(x, y)$ of the Laplace equation (2.6) constitutes a polynomial for x and y (for x to the 2n power), then

$$\varphi_{kn}^*(y) = \varphi_n^*(x_k, y) = \mathrm{Re}\left\{\sum_{j=0}^{2n} a_j z_k^j - i \sum_{j=0}^{2n} \frac{b_j}{j+1} z_k^{j+1}\right\} (z_k = x_k + iy) \tag{2.10}$$

constitutes a solution of the Cauchy problem for approximating system (2.8) with initial data (2.9). This can be easily checked also by direct substitution.

Thus, the solution of approximating system (2.8) with initial data (2.9) at $n \to \infty$ converges to an exact solution of the Cauchy problem for the initial partial differential equation. Dimensions of convergence range are determined by the distribution of singular points for initial data which are continued into the complex region.

In conventional difference systems the number of points by which derivatives are approximated is fixed. Thus, with an increase in the number of steps the accuracy of the approximating system increases, not due to an increase of established information about the analytic properties of the unknown functions, but due to an improvement of the quality (accuracy) of this information, inasmuch as it is gathered from all the smaller environs of the examined point. In contrast to this the difference system (2.8), a generalization of which is used in this work, may be called an "arrangement with increasing information," or an "analytic" system, since with an increase in the number of points on a layer there is an increase of analytic information about the function which is established in the system. This most fully considers the analytic nature of the solution. Let us note that it is not obligatory to perform an approximation of derivatives with respect to all points located on the layer. It is important that the number of these points increases according to a definite law together with a decrease of the step in the direction of the initial data.

In general the approximating system (2.8) can be integrated only numerically. During numerical integration the error of approximate solution $\delta\phi = \phi - \phi^*$ is composed of three parts: $\delta_a^x \varphi$ and $\delta_a^y \varphi$ – errors of approximation in direction x and y respectively, $\delta_b \phi$ – rounding error. Let us consider the properties of growth of these errors. For reduction of computations we will consider simplified initial data

$$\varphi(x, 0) = u(x), \quad \varphi_y(x, 0) = 0$$

and will dispose $N = 2n + 1$ nodes on initial line $y = 0$ in zeros of the Chebyshev polynomial $T_{2n+1}(x)$. The remaining member of the

Lagrangian interpolation polynomial in the complex region $\zeta = x + i\eta$ for function $\sigma(\zeta)$ can be represented in the form of the complex integral

$$R_n(\zeta) = \frac{1}{2\pi i} \int\limits_{\rho=\rho_2} \frac{T_{2n+1}(\zeta)}{T_{2n+1}(\xi)} \cdot \frac{\sigma(\xi)}{(\xi - \zeta)} d\xi, \qquad (2.11)$$

where T_{2n+1} – Chebyshev polynomial, $\rho = \rho_2 < \rho^*$ – ellipse, completely disposed (Fig. 4) in region of analyticity of interpolated function $\sigma(\zeta)$. Variable ζ is changed in the region $\rho \leqslant \rho_1 < \rho_2$. Parameters ρ_2 and ρ_1 can be taken as close as desired to ρ^*. Conducting an estimate of the integral with the use of the relationship

$$\lim_{n\to\infty} (|T_n(z)|)^{\frac{1}{n}} = \rho,$$

according to (2.6) and (2.4) we find that for as little a value of $\varepsilon > 0$ as desired there is such an n^*, that with $n > n^*$

$$|\delta_x^\xi \varphi| \leqslant |R_n(z)| \leqslant \frac{L_2 M_2}{2\pi\Delta} \cdot \frac{(\rho_1 + \varepsilon)^{2n+1}}{(\rho_2 - \varepsilon)^{2n+1}}. \qquad (2.12)$$

Here L_2 designates the length of curve $\rho = \rho_2$, M_2 is the maximum of modulus of function $\sigma(\zeta)$ on curve $\rho = \rho_2$, and Δ is the minimum distance between curves $\rho = \rho_2$ and $\rho = \rho_1$.

If it is assumed that $\varepsilon < \frac{1}{2}(\rho_2 - \rho_1)$, then according to (2.12) $\delta_x^\xi \varphi \to 0$ at $N = 2n + 1 \to \infty$. Thus, in the class of analytic functions the error connected with approximation in direction x decreases exponentially with an increase in the number of nodes. This is connected with the rapid decrease of error of approximation of initial data:

$$|\delta_0 \sigma(x)|_{-1 \leqslant x \leqslant 1} \leqslant \frac{L_2 M_2}{2\pi D_2} \cdot \frac{1}{(\rho_2 - \varepsilon)^{2n+1}}. \qquad (2.13)$$

Here D_2 is the minimum distance of segment $-1 \leqslant x \leqslant 1$ to curve $\rho = \rho_2$ and $\rho_2 - \varepsilon > 1$.

Now let us consider the increase of arbitrary errors in initial data. For this we will take the distribution function of errors in mesh nodes.

$$\delta\sigma(x_j) = \begin{cases} \varepsilon & \text{when } x = x_n = 0, \\ 0 & \text{when } x = x_j, \ j \neq n. \end{cases}$$

Initial data $\varphi(x_j, 0) = \delta\sigma(x_j),\ \varphi_y(x_j, 0) = 0$ is satisfied by the following solution of the approximating system:

$$\delta_\varepsilon\varphi = \varepsilon \operatorname{Re}\left\{\frac{(z - x_0)(z - x_1)\dots(z - x_{n-1})(z - x_{n+1})\dots(z - x_{2n})}{(x_n - x_0)(x_n - x_1)\dots(x_n - x_{n-1})(x_n - x_{n+1})\dots(x_n - x_{2n})}\right\}.$$

From here it is easy to obtain the asymptotic expression of behavior of error at large n and y > 0:

$$\delta_\varepsilon\varphi = \frac{(-1)^n \varepsilon}{2n+1} \operatorname{Re}\left\{\frac{T_{2n+1}(z)}{z}\right\} \sim \frac{\varepsilon}{2} \frac{(-1)^n}{2n+1} \times$$
$$\times \frac{\rho^{2n+1}}{x^2 + y^2} [x \cos(2n+1)\theta + y \sin(2n+1)\theta],$$
$$z + \sqrt{z^2 - 1} = \rho e^{i\theta}, \qquad (2.14)$$

which corresponds to Chebyshev nodes.

Thus, with an increase of the number of nodes N = 2n + 1 the arbitrary error in initial data grows exponentially with fixed x, y, just as when using the usual "nonanalytic" difference systems [14].

If during the integration of the approximating system of ordinary differential equations (2.8) stable difference systems are used, then increase of errors $\delta_3\varphi$ and $\delta_4\varphi$ is determined by the system of approximation in the direction x. According to (2.14) they increase by exponential law with an increase in the number of nodes N = 2n + 1. Error of approximation $\delta_1\varphi$ is found using a computer and at fixed N can be made sufficiently small so that its influence on the solution, taking growth into consideration, does not exceed ε.

Conversely, the rounding off error in each point cannot be decreased, since it is determined by the maximum numbers of digit positions utilized during calculation, i.e., the properties of the computer. During approximation of initial data the number of nodes N = 2n + 1 is determined with the required accuracy of approximation of the solution in this direction and depends on the nature of the problem. From this follows the important conclusion that in the elliptic region the Cauchy problem can be solved numerically only in that case, when the rounding off error is sufficiently small, so that taking into account growth at the selected N they do not exceed errors of approximation.[2] These considerations determine the selection of parameters of the difference grid and the practicable accuracy during solution of a specific problem.

§ 3. Construction of an Algorithm for the Numerical Solution of a Boundary Value Problem in a Mixed Region, when in One Part of the Region the Equation Has an Elliptic Nature, and the Other a Hyperbolic Nature

As a model we will consider the equation

$$\frac{\partial^2 \varphi}{\partial x^2} + (1 - y^2) \frac{\partial^2 \varphi}{\partial y^2} = 0. \qquad (3.1)$$

Basic questions of the existence and uniqueness have been studied for equations of such a type and more general ones in the papers [15, 16, 17]. If we disregard the vorticity of flow and in the first equation (1.1) introduce the velocity potential, then (3.1) can serve as a model of a problem dealing with supersonic flow past a blunt body with a detached shock wave in a physical plane. In variables ξ, θ, where

$$\xi = \frac{r - r_r(\theta)}{r_c(\theta) - r_r(\theta)},$$

the region of flow between surface of body and shock wave in Fig. 1 will be converted into a band (Fig. 5a).

Let us consider the band $0 \leq x \leq \frac{1}{2}$ (Fig. 5b). Lines $x = 0$ and $x = \frac{1}{2}$ correspond to shock and body surface. Regions $-1 < y < 1$ and $|y| > 1$ correspond to subsonic and supersonic units of flow. Coefficients of equation (3.1) just as the coefficients of the first equation (1.1) are even functions of θ. Equation of characteristics for (3.1) has the same form

$$y = \operatorname{ch}(x - c) \qquad (3.2)$$

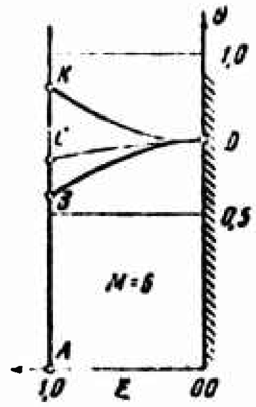

Fig. 5a.

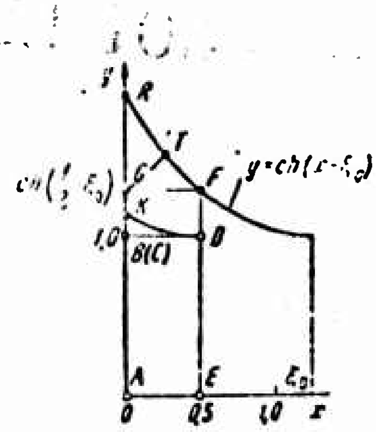

Fig. 5b.

Here c = const. With x < c we have the characteristic of the family, with x > c — of the secondary. All characteristics touch the parabolic line y = 1, therefore the triangular region BCD (Fig. 5a) degenerates into a line (Fig. 5b). In accordance with the formulation of the gas-dynamic problem we will consider as having "physical meaning" solutions with continuous and limited second derivatives. Making use of the separation of variables $\phi = F(x)f(y)$, we obtain two families of solutions

$$F_k''(x) - (\pm\lambda_k^2) F_k(x) = 0,$$

$$f_k''(y) + \frac{(\pm\lambda_k^2) f_k(y)}{1-y^4} = 0, \qquad (3.3)$$

corresponding to plus and minus signs λ_k^2. From (3.1) it is clear that a necessary condition of boundedness of second derivatives is the requirement [illegible] with y = 1. Considering the condition of symmetry for axis $\partial\phi/\partial y = 0$ at y = 0, we obtain for solutions of $f_k(y)$, having "physical meaning," in the region $0 \leq y \leq 1$ the boundary value problem

$$f_k(1) = 0, \quad f_k'(0) = 0, \qquad (3.4)$$

determining the fundamental values $\pm\lambda_k^2$. For solution of this boundary value problem we use the Cauchy problem with initial data for axis y = 0:

$$f_k(0) = \varepsilon_k, \quad f_k'(0) = 0, \qquad (3.5)$$

selecting by iteration ε_k such that it satisfies the first condition (3.4).

We will start from a case when in (3.3) the minus sign in front of λ_k^2 is removed. From the second equation we have

$$f_k''(y) = \frac{\lambda_k^2 f_k(y)}{1-y^2}, \quad \operatorname{sign} f_k'' = \operatorname{sign} \frac{f_k}{1-y^2} \tag{3.6}$$

and consequently $|f_k| \geq |\varepsilon_k|$ on segment $0 \leq y \leq 1$ for solutions satisfying initial data (3.5). Hence the immediate result is that the boundary value problem (3.4) has only a zero solution $f_k(y) \equiv 0$ at $0 \leq y \leq 1$. From the general theory of differential equations of the type (3.6) it follows further that this solution, being continued in region $y > 1$, is also identically equal to zero.

Let us now remover the plus sign in (3.3) before λ_k^2. In this case the fundamental values of boundary value problem (3.4) are easily determined $\lambda_k^2 = k(k + 1)$; as a result we obtain a family of particular solutions of equation (3.1) with a limited second derivative

$$\varphi_k(x, y) = (1 - y^2) P_k'(y) [A_k \operatorname{sh} \lambda_k x + B_k \operatorname{ch} \lambda_k x], \tag{3.7}$$

where $P_k'(y)$ – derivative of Lagranian polynomial.

Let us consider the question of the uniqueness of the solution of boundary value problem for equation (3.1). determined by boundary conditions

$$\varphi(0, y) = \delta_1(y), \quad \varphi\left(\frac{1}{2}, y\right) = \delta_2(y), \quad \varphi_y(x, 0) = 0. \tag{3.8}$$

This problem constitutes a model of supersonic flow past a blunt body with a detached shock wave. Let us assume that functions $\delta_1(y)$ and $\delta_2(y)$ are assigned only up to a parabolic line, i.e., on the segment $0 \leq y \leq 1$. We will assume that here there exist two solutions of equation (3.1) with a continuous limited second derivative $\phi_1(x, y)$ and $\phi_2(x, y)$, satisfying boundary conditions (3.8). Then their difference $\delta\phi$ is solution (3.1) with a limited second derivative, satisfying uniform conditions

$$\ldots \left(\frac{1}{2}, y\right) = \ldots \text{[illegible]} \tag{3.9}$$

We will look for $\delta\phi$ by solving the Cauchy problem from the axis of symmetry y = 0

$$\delta\varphi(x, 0) = \varepsilon(x), \quad \left.\frac{\partial\delta\varphi}{\partial y}\right|_{y=0} = 0.$$

Here function $\varepsilon(x)$ is determined from condition $\partial^2\delta\varphi/\partial x^2 = 0$ at y = 1, necessary for boundedness of the second derivative at y = 1. Considering $\lambda_k^2 = (2k\pi)^2$, from the first equation (3.3) we obtain

$$F_k(x) = C_k \cos(2k\pi x) + D_k \sin(2k\pi x).$$

Expanding function $\varepsilon(x)$ in accordance with boundary conditions (3.9) into a Fourier series based on sines

$$\varepsilon(x) = \sum_{k=1}^{\infty} \varepsilon_k \sin(2k\pi x),$$

we present the desired solution in the form

$$\delta\varphi = \sum_{k=1}^{\infty} f_k(y) \sin(2k\pi x).$$

Functions $f_k(y)$ should satisfy the second equation (3.3) and boundary conditions (3.4). According to the preceding it follows immediately from here that all $f_k(y) \equiv 0$ and in the elliptic region $0 \leq y \leq 1$ $\phi_1 \equiv \phi_2$. If boundary conditions (functions σ_1 and σ_2) are continued up to points G and F (Fig. 5), then we obtain $\phi_1 \equiv \phi_2$) in the whole region AGTFE. An analogous result can be obtained by assignment on lines x = 0 and x = $\frac{1}{2}$ of boundary conditions of a more general form.

Thus, in the class of functions with continuous and limited second derivatives the assignment of boundary conditions (Fig. 5a, 5b) on segments AB, ED and conditions of symmetry on segment AE uniquely determine the solution of equation (3.1) in elliptic region ABDE.

If boundary conditions are continued in the hyperbolic region up to points G and F, then a unique solution is determined in the entire region AGTFE, and consequently also in its section AGFE.

Thus, the formulation of a boundary value problem for model equation (3.1) possesses all the basic features of the gas-dynamic problem being examined (§ 1).

Let us consider the application of the Cauchy problem for construction of a boundary value problem (3.8). Two primarily different systems are possible.

1. If in addition to conditions (3.8) on the axis of symmetry $y = 0$ the distribution of unknown function $\phi(x, 0) = b(x)$, is assigned, then in band $0 \leqslant x \leqslant \frac{1}{2}$, $y \geqslant 0$ a mixed boundary value problem with initial data at $y = 0$ is obtained. Selecting with the help of iterative process the function $b(x) = b^0(x)$ in such a manner that the condition of boundedness of the second derivative is fulfilled, we obtain the desired solution of boundary value problem (3.8). This method is close to method of work [6].

We will examine a deviation from an exact solution $\delta\varphi = \varphi(x, y) - \varphi_\varepsilon(x, y)$, caused by a distinction in initial data of $b(x) - b^0(x)$ in the process of iteration. Function $\delta\phi(x, y)$ satisfies equation (3.1) and the uniform boundary conditions (3.9), so that in accordance with the preceding it has a limited second derivative only at $\delta\phi \equiv 0$, i.e., when the exact solution is found. In other words, since conditions (3.8) uniquely determine the solution, then any solution arbitrarily differing little from the desired one based on initial data has an infinite second derivative at $y = 1$. In this method the iterative process of approximation to a solution with a continuous and limited second derivative is constructed in the class of functions with an infinite derivative. Such incorrectness in the vicinity $y = 1$ during the numerical solution of a boundary value problem strongly aggravates the difficulties connected with the usual incorrectness of the Cauchy problem in an elliptic region. It can be surmounted in a class of sufficiently smooth functions. As can be seen from the second equation (3.3), difficulties increase progressively with an increase in the harmonic number k, i.e., with an increase of accuracy of the approximating system. Results of this analysis are well confirmed during the numerical solution of a nonlinear problem about supersonic flowing around with a detached shock wave [6].

2. The second system amounts to the problem being examined with initial data on line x = 0

$$\varphi(0, y) = \sigma_1(y), \quad \varphi_x(0, y) = \tau_1(y)$$

and even function $\tau_1(y) = \tau_1^0(y)$ will be selected by iteration in such a way that the second boundary condition (3.8) is satisfied on line x = 1/2. Since

$$y^2_* = 1 - P_1'(y)(1 - y^2),$$

$$y^4_* = 1 - \frac{6}{5} P_1'(y)(1 - y^2) - [illegible] (1 - y^2)$$

etc., then during approximation of function $\tau_1(y)$ by even polynomials the approximate solutions in the process of iterations will be linear combinations of particular solutions (3.7). During each iteration the condition of symmetry (third condition (3.8)) and condition of boundedness of second derivatives are satisfied automatically, since they are satisfied by the particular solutions (3.7). The possiblitity of constructing an iterative process in a class of functions with a continuous limited second derivative in this method is conditioned by the fact that approximate solutions are not subordinated to any condition at x = 1/2 (of three conditions (3.8) only two are subordinated), and consequently, according to conditions of uniqueness, can belong to this class of functions. From this analysis it is clear that during resolution of a problem on supersonic flow past a body with a detached shock wave one should use system 2, since it corresponds to the nature of the problem.

Let us now use the proposed difference system for solving a Cauchy problem and a boundary value problem based on system 2 for a model equation of mixed type (3.1). For this in accordance with our system we will trace the straight lines y = const = y_j based on m in the upper and lower half-planes. We will call these straight lines rays. All told, including the axis, we obtain 2m + 1 rays. Assuming the unknown solution even and approximating it with Lagrangian polynomials based on y, we obtain for the function

$$\varphi(x, y) \approx \varphi_m^*(x, y) = \sum_{l=0}^{m} \varphi_{lm}^0(x)\, y^{2l} \qquad (3.10)$$

and for derivatives with respect to y

$$\frac{\partial\varphi}{\partial y} \approx \frac{\partial \overset{\bullet}{\varphi}_m}{\partial y} = \sum_{j=1}^{m} 2j\,\varphi_m^0(x)\,y^{2j-1},$$

$$\frac{\partial^2\varphi}{\partial y^2} \approx \frac{\partial^2 \overset{\bullet}{\varphi}_m}{\partial y^2} = \sum_{j=1}^{m} 2j(2j-1)\,\varphi_{jm}^0(x)\,y^{2j-2} \qquad (3.11)$$

Here $\varphi_{jm}^0(x)$ – linear function $\overset{\bullet}{\varphi}_{jm}(x) = \overset{\bullet}{\varphi}_m(x, y_j)$ values of the unknown function for m + 1 rays in the upper half-plane. Placing (3.11) in (3.1) and requiring that the resulting expression is identically satisfied on all rays, we obtain an approximating system of ordinary differential equations

$$\frac{d^2 \overset{\bullet}{\varphi}_{km}}{dx^2} + (1 - y^2) \sum_{j=1}^{m} 2j(2j-1)\,\varphi_{jm}^0(x)\,y^{2j-2} = 0 \qquad (3.12)$$

$$(k = 0, 1, \ldots, m)$$

for determination of values of approximate solution $\overset{\bullet}{\varphi}_m(x, y)$ on the rays.

One of the evident properties of an approximating system is that if there exists a solution of the initial equation (3.1) $\phi_m(x, y)$ which at a fixed x is a polynomial of the power 2m with respect to y, then on the rays it satisfies the system (3.12), i.e., functions

$$\overset{\bullet}{\varphi}_{km}(x) = \varphi_m(x, y_k) \qquad (k = 0, 1, \ldots, m)$$

are the solution of a system (3.12) with any disposition of rays. This follows from the fact that for this solution $\overset{\bullet}{\varphi}_m = \varphi_m$ and correlations (3.11) are exact. Let us consider a Cauchy problem: in region AGFEA $(0 \leqslant x \leqslant \xi_0,\ 0 \leqslant y \leqslant \mathrm{ch}(\xi_1 - \xi_0))$ to find the solution of equation (3.1), satisfying the initial data (see Fig. 5b):

$$\varphi(0, y) = \Phi_{2m}(y) = a_0 + a_2 y^2 + \ldots + a_{2m} y^{2m}, \quad \varphi_x(0, y) = 0. \qquad (3.13)$$

Obviously,

$$\Phi_{2m} = \alpha_0 + \alpha_2 P_1'(y)(1 - y^2) + \ldots + \alpha_{2m} P_{2m-1}'(y)(1 - y^2).$$

where $\alpha_0, \alpha_1, \ldots, \alpha_{2m}$ are constants. The solution of the Cauchy problem for equation (3.1) with initial data (3.13) will be (see 3.7)

$$\varphi(x, y) = a_0 + \sum_{j=1}^{m} \{a_{2j} \dot{P}_{2j-1}(y)(1 - y^2) \operatorname{ch}[\sqrt{2j(2j-1)}\, x]\}, \tag{3.14}$$

and the solution for the approximating system (3.12) will be

$$\dot{\varphi}_{k,m}(x) = \varphi(x, y_k) \qquad (k = 0, 1, \ldots, m) \tag{3.15}$$

with any situation of rays.

Here the solution of approximating system (3.12) gives on the rays an exact solution of the initial equation (3.1). And if the Lagrangian interpolation polynomial (3.10) is used, then from the solution of the approximating system (3.15) we obtain an exact solution (3.14) in the entire region.

In the case examined the region of influence of differential equation (3.1) is ARFEA, while the region of influence of approximating system (3.12) formally is AGFEA, since only the values of unknown function in this region are bound in the difference approximation.

It is known that in the hyperbolic region in general the region of influence of the approximating system must coincide or envelop the region of influence of the initial system of differential equations. Otherewise there is no convergence of approximate solutions to exact, and the difference system itself turns out to be unstable during calculations. The meaning of this affirmation is that if the region of influence of the system of differential equations exceeds the bounds of the region of influence of the approximating system, then by changing initial data on part of the initial line (in our case on section GR), we do not affect the solution of the approximating system and at the same time we change the solution of the system of differential equations in the region of influence.

Although it is accurate in general, this affirmation becomes incorrect if it were to remain in the class of analytic solutions. This is already clear from the elementary consideration that in an analytic case it is impossible to change initial data on any section without having changed them in the appropriate manner on the entire initial curve, since in this case assignment of initial data on any section specifies them in the entire region of analyticity. This position can be given a more thorough foundation. Thus, in the paper by Dahlquist [18] for the wave equation during the usual three-point system of approximation of a derivative in direction of the initial straight line it is proven that in a class of analytic functions the solution of an approximating system converges to an exact one also when the region of influence of a differential equation exceeds the bounds of region of influence of the approximating system (i.e., at $\tau/h > 1$, where h is a step along the initial data, and τ in a perpendicular direction).

When in a class of analytic functions they are looking for a solution with the help of an "analytic" difference system, then, and in the hyperbolic region, the regions of influence of differential equations and the difference system are determined not by the position of extreme characteristic and points which are bound in the system, but by the region of analyticity of the solution and the convergence region of the difference system. Let us illustrate this in an example of model equation (3.1).

We will construct the solution of the Cauchy problem with the same initial data (3.13) in region of influence of equationn (3.1) ARFEA, bounded above by characteristic RF, the equation of which is $y = \mathrm{ch}(x - \xi_0)$. In order to use the proposed system we will pass from variables x, y to variables x, η, where

$$\eta = \frac{y}{\mathrm{ch}(x - \xi_0)}. \qquad (3.16)$$

Here region ARFEA will turn into rectangular $0 < x < \xi_0$, $0 \leq \eta \leq 1$, and differential equation (3.1) will take the form

$$\frac{\partial^2 \varphi}{\partial x^2} - 2\eta \frac{\mathrm{sh}(x - \xi_0)}{\mathrm{ch}(x - \xi_0)} \cdot \frac{\partial^2 \varphi}{\partial x \partial \eta} + \frac{1 - \eta^2}{\mathrm{ch}^2(x - \xi_0)} \cdot \frac{\partial^2 \varphi}{\partial \eta^2} - \eta \frac{1 - \mathrm{sh}^2(x - \xi_0)}{\mathrm{ch}^2(x - \xi_0)} \cdot \frac{\partial \varphi}{\partial \eta} = 0. \qquad (3.17)$$

Just as previously, by carrying out, in the upper half-plane m + 1 rays $\eta = \mathrm{const} = \eta_j$ $(0 \leq \eta_j \leq 1)$ and using approximation of the unknown solution with even Lagrangian polynomials (3.10), we obtain approximate expressions for derivatives with respect to η

in the form (3.11). Placing them in equation (3.17) and requiring that the resulting expression is fulfilled identically on the rays, we obtain an approximating system for ordinary differential equations

$$\frac{d^2\mathring{\varphi}_{km}}{dx^2} - 2\eta_k \frac{\operatorname{sh}(x-\xi_0)}{\operatorname{ch}(x-\xi_0)} \cdot \frac{d}{dx}\left[\sum_{j=1}^{m} 2j\varphi^0_{jm}(x)\,\eta_k^{2j-1}\right] +$$
$$+ \frac{1-\eta_k^2}{\operatorname{ch}^2(x-\xi_0)} \sum_{j=1}^{m} 2j(2j-1)\varphi^0_{jm}(x)\,\eta_k^{2j-2} -$$
$$- \eta_k \frac{1-\operatorname{sh}^2(x-\xi_0)}{\operatorname{ch}^2(x-\xi_0)} \sum_{j=1}^{m} 2j\varphi^0_{jm}(x)\,\eta_k^{2j-1} = 0 \quad (k=0,1,\dots,m) \tag{3.18}$$

for determination of values of approximate solution $\varphi = \mathring{\varphi}_m$ (x, y) on the rays. Since along the extreme ray η = 1, which is a characteristic of differential equation $\partial/\partial x = d/dx$, then the characteristic equation for it is written in the form

$$\frac{d^2\varphi}{dx^2} - 2\frac{\operatorname{sh}(x-\xi_0)}{\operatorname{ch}(x-\xi_0)} \cdot \frac{d}{dx}\frac{\partial\varphi}{\partial\eta} - \frac{1-\operatorname{sh}^2(x-\xi_0)}{\operatorname{ch}^2(x-\xi_0)} \cdot \frac{\partial\varphi}{\partial\eta} = 0. \tag{3.19}$$

It is easy to see that the equation of system (3.18) for extreme ray k = m, η = 1 constitutes a difference recording of characteristic equation (3.19). It is also obvious that fulfillment of (3.19) on line η = 1 directly follows from the differential equation (3.17). Placing $y = \eta\operatorname{ch}(x-\xi_0)$, in (3.7) we obtain a system of particular solutions of equation (3.17). The exact solution of our problem will obviously be (see 3.14)

$$\varphi(x, y) = a_0 + \sum_{j=1}^{m} \{a_{2j} P'_{2j-1}[\eta\operatorname{ch}(x-\xi_0)] \times$$
$$\times [1-\eta^2\operatorname{ch}(x-\xi_0)]\operatorname{ch}[\sqrt{2j(2j-1)}\,x]\}. \tag{3.20}$$

Since with a fixed x this solution will be a polynomial with respect to η to the 2m power, then the solution of the approximating system at any situation of rays can be presented in the form

$$\mathring{\varphi}_{km}(x) = \varphi(x, \eta_k) \quad (k=0,1,\dots,m). \tag{3.21}$$

If one were to approximate the solution with the help of Lagrangian interpolation polynomials, then from the solution of approximating system (3.21) we obtain the exact solution of the problem.

The solutions of approximating systems in the form (3.12) and (3.18) obviously agree, since they give values of one and the same exact solution on different lines ($y = const$ and $\eta = const$).

If one were to assign a perturbation on initial line $x = 0$ or any layer $x = const$, then the growth of errors during transition from layer to layer and increase in number of rays (growth m) will be the same for both solutions. In essence this fact is obvious in the class of analytic solutions (and consequently of initial data also). Remaining in the region of analytic solutions, the example cited, it is impossible to change the initial data (3.13) on section GR without changing them in a corresponding manner on section AG, so that the difference system (3.12) completely (exactly) considers the change of initial data on the entire section AR.

In the example considered due to the presence of polynomial solutions of equation (3.1) for initial data (3.13) it is easy to write out the exact solutions of approximating systems (3.12) and (3.18). However, the result obtained (independence of growth of errors on the selection of systems (3.12) or (3.18)) is not connected with the polynomial character of solutions. For equation (3.1) it is easily generalized for solutions presented in the form of converging series of polynomial solutions. By a somewhat more complex path an analogous result is obtained also for model equations which do not have polynomial solutions. This fact is also confirmed by carrying out numerical calculations of flow past blunt bodies by the method examined.

It is important to note that due to the analytic character of the difference system the approximating system (3.12) in elliptic and hyperbolic regions has the same properties. At a fixed value $x = H$ with an increase of accuracy of approximation with respect to y (increase of number of rays), errors in initial data grow according to the law expmH. During numerical determination of the approximate solution of a Cauchy problem errors of approximation and rounding off appear. In the examined case of analytic solution and difference system the errors of approximation on initial line $x = 0$ have an analytic character and decrease with a growth of m according to exponential law, so that at specific limitations on location of singular points of the unknown solution the errors evoked by them at $x > 0$ do not grow with an increase of $N = 2m + 1$ (see 2.12). Errors in rounding off have a nonanalytic character, therefore with an increase in number or rays they grow according to exponential law (see 2.13). Since in the class of arbitrary functions the Cauchy problem for elliptic equations is incorrect, then the exponential growth of rounding errors (and other arbitrary errors) with an increase in the number of steps on the initial line cannot be eliminated by a change in the difference system or the form of the region of influence of the system in the hyperbolic section of region (see note on page 41). The number of steps on the initial line (number of rays) is determined with the necessary accuracy of approximation of solution in this direction and depends on the nature of the problem. In connection with this, during the solving of mixed, just as of elliptic, problems (see § 2) for the construction of a numerical algorithm it is possible to use the Cauchy problem only in that case when the rounding error is sufficiently small, so that taking into account growth during the selected number of steps on the initial line they do not exceed the errors of approximation.

Considering everything said about properties of a difference system during a discussion of formulation of the problem. One should note that use of a characteristic, limiting the region of influence of the system of differential equations for gas dynamics as a boundary line when writing the approximating system of ordinary differential equations (here satisfaction of the characteristic condition is a result of differential equations, see, for example, (3.19), (3.17)), does nothing either for the formulation of conditions of uniqueness or from the point of view of improvement of convergence and stability of the difference system utilized here. On the other hand, the transformation of coordinates with the use of the characteristic leads to a noticeable complication of the approximating system (see (3.12) and (3.18)), to a dependence of grid on M number and other conditions of flowing around. This hampers analysis and the use of calculation data.

Now let us give an example of the solution of a boundary value problem (3.8) with boundary conditions

$$\varphi(0, y) = a_0 + a_2 y^2 [illegible] y^4 = \sigma_1(y),\ \varphi\left(\frac{1}{2}, y\right) = b_0 + [illegible] + b_4 y^4 = \sigma(y),$$
$$\varphi_y(x, 0) = 0 \qquad (3.22)$$

for model equation (3.1). Considering m = 2 (3 rays in upper half-plane y = 0, y_1 and y_2), we easily obtain the solution ϕ_k^* of the approximating system (3.12), satisfying the boundary conditions

$$\varphi_k^* = (0) = \sigma_1(y_k),\ \varphi_k^*\left(\frac{1}{2}\right) = \sigma_2(y_k) \quad (k = 0, 1, 2) \qquad (3.23)$$

in the form $\phi_k^* = \phi(x, y_k)$, where

$$\varphi(x, y) = M + Nx + P_1'(y)(y^2 - 1)(C_1 e^{\sqrt{2}x} + C_2 e^{-\sqrt{2}x}) +$$
$$+ \frac{2}{15} P_3'(y)(y^2 - 1)(A_1 e^{\sqrt{12}x} + A_2 e^{-\sqrt{12}x}),$$
$$C_1 = \frac{1}{2} \frac{\exp\left(-\frac{\sqrt{2}}{2}\right)\left(a_2 + \frac{6}{5} a_4\right) - \left(b_2 + \frac{6}{5} b_4\right)}{\mathrm{sh}\left(\frac{\sqrt{2}}{2}\right)}, \qquad (3.24)$$

$$C_2 = a_2 + \frac{6}{5} a_4 - C_1,$$

$$A_1 = \frac{1}{2} \frac{b_4 - a_4 [illegible]\left(-\frac{\sqrt{12}}{2}\right)}{\mathrm{sh}\left(\frac{\sqrt{12}}{2}\right)}, \qquad A_2 = a_4 - A_1,$$
$$M = a_0 + a_2 + a_4,$$
$$N = 2[(b_0 + b_2 + b_4) - M]$$

during any situation of rays. In the examined case the solution of approximating system (3.12) gives a precise solution of initial equation (3.1) on rays $\phi_k^*(x)$, and with the use of Lagrangian interpolation polynomials — in the entire region $\phi(x, y)$, since the number of rays corresponds to the power of polynomials in initial data (3.22). If initial data are set by polynomials of the 2m power, and number of rays N in the upper and lower half-plane (including axis y = 0) is less tnan 2m + 1, then the solution $\phi^*(x, y)$, obtained with the help of the approximating system, will be approximate. This makes it possible to study the properties of error $\delta\phi = \phi - \phi^*$.

We will limit ourselves to the simplest case, nonetheless making it possible to reveal the basic properties of $\delta\phi$. Let us take m = 2 and examine again the boundary value problem (3.22), with an exact solution $\phi(x, y)$ which is (3.24). During detecting of the approximate solution $\phi^*(x, y)$ we obtain the number of rays $N = 3 < 2m + 1$. Then, using (3.24) it is easy to obtain

$$\delta\varphi = \Delta M + x\Delta N + P_1'(y)(y^2 - 1)[\Delta C_1 e^{\sqrt{2}x} + \Delta C_2 e^{-\sqrt{2}x}] +$$
$$+ \frac{2}{15} P_3'(y)(y^2 - 1)(A_1 e^{\sqrt{12}x} + A_2 e^{-\sqrt{12}x}),$$
$$\Delta M = a_4(1 - h^2),$$
$$\Delta N = 2[b_4(1 - h^2) - \Delta M],$$
$$\Delta C_1 = -\frac{1}{2\,\mathrm{sh}\left(\frac{\sqrt{2}}{2}\right)}\left[\exp\left(-\frac{\sqrt{2}}{2}\right) a_4\left(\frac{6}{5} - h^2\right) - \right.$$
$$\left. - b_4\left(\frac{6}{5} - h^2\right)\right]. \qquad (3.25)$$
$$\Delta C_2 = a_4\left(\frac{6}{5} - h^2\right) - \Delta C_1,$$

where $y = h$ — position of extreme ray. Here the error in the approximation of boundary conditions is given by expressions

$$\delta\varphi(0, y) = a_4 y^2(y^2 - h^2), \quad \delta\varphi\left(\frac{1}{2}, y\right) = b_4 y^2(y^2 \ [illegible]) \qquad (3.26)$$

Thus, at a fixed number of rays N = 3 the parameter $q = \max(a_4, b_4)$ is a measure of accuracy of the approximating system. Expression (3.25) makes it possible to trace the nature of dependence of $\delta\phi$ on coordinates, the position of the extreme ray, and accuracy of the approximating system. Differentiating the expression for $\delta\phi$ based on h, will obtain

$$[illegible] - 2a_4 h + 4(a_4 \ [illegible]) \ [illegible] \ 1) \times$$
$$\times\left([illegible] - 2h \frac{b_4 - a_4 \exp\left([illegible]\right)}{\mathrm{sh}\left([illegible]\right)} \ [illegible] \sqrt{2}x\right). \qquad (3.27)$$

From (3.27) it is clear that the weaker the dependence of $\delta\phi$, and consequently also of the approximate solution $\phi^*(x, y)$ on the position of the extreme ray h, then the smaller is $q = \max(a_4, b_4)$, i.e., the higher the accuracy of the approximating system. At $q \to 0$ the solution of the approximating system aspires to be exact, and dependence on h vanishes. From (3.25) and (3.26) it is clear that $\delta\phi$ and $d\phi^*/dh$ decrease according to linear law together with q, so that

$$\delta\varphi = \varphi(x, y) - \varphi^*(x, y) = O\left(\frac{d\varphi^*}{dh}\right). \qquad (3.28)$$

Thus, $|d\phi^*/dh|$ (dependence of ϕ^* on situation of rays) can serve as a measure of accuracy of the solution of the approximating system.

§ 4. Method of Solving a Nonlinear Problem

For the solution of the nonlinear boundary value problem formulated in § 1 we will use the difference system which was studied in detail on model equation in § 2 and 3. From variables r, θ it is convenient to switch to variables ξ, θ, introducing the new variable

$$\xi = \frac{r - r_T(\theta)}{\varepsilon(\theta)}, \quad \varepsilon(\theta) = r_C(\theta) - r_T(\theta), \qquad (4.1)$$

where $r = r_T(\theta)$ and $r = r_C(\theta)$ are the equations for body contour and compression shock. In these variables the region of flow between shock wave and body will be converted into the band $0 \leq \xi \leq 1$ (see Fig. 5a).

Let us construct for m rays in the upper and lower half-planes, all told, including the axis of symmetry, 2m + 1 rays (see Fig. 2).

For obtaining explicit expressions we approximate the unknown functions with Lagrangian polynomials and then differentiate them. Considering the symmetry of flow with respect to the axis, we obtain for the functions

$$u \approx \sum_{j=0}^{m} u_j^0(\xi)\,\theta^{2j}, \quad v \approx \sum_{j=0}^{m} v_j^0(\xi)\,\theta^{2j+1}, \quad r_C \approx \sum_{j=0}^{m} r_j^0\,\theta^{2j} \qquad (4.2)$$

and for derivatives with respect to θ

$$u' \approx \sum_{j=1}^{m} 2j\,u_j^0(\xi)\,\theta^{2j-1}, \quad v' \approx \sum_{j=0}^{m} (2j+1)\,v_j^0(\xi)\,\theta^{2j}, \quad r_C' = \sum_{j=1}^{m} 2j\,r_j^0\,\theta^{2j-1}. \qquad (4.3)$$

Values u_j^0, v_j^0, and r_j^0 are linear functions of values u, v, and r_C on m + 1 rays in the upper half-plane.

Placing the expressions for derivatives with respect to θ in (1.1) and requiring that the resulting expressions are identically satisfied on each ray, we obtain the approximating system of ordinary differential equations with respect to u_k, v_k, p_k, ψ_k, of approximate values of these functions on the rays. Solving this system relative to derivatives with respect to ξ, we finally obtain:

$$\frac{du_k}{d\xi} = \frac{\varepsilon_k(r_{rk}+\xi\varepsilon_k)}{\Delta_k}\Big\{c_k[u_k v_k (r_{rk}+\xi\varepsilon_k)^2 + b_k(r'_{rk}+\xi\varepsilon'_k)\cdot(r_{rk}+\xi\varepsilon_k)] - d_k(2u_k + v_k \operatorname{ctg}\theta_k) - b_k v'_k + (u'_k - v_k)\frac{b_k(r'_{rk}+\xi\varepsilon'_k)}{r_{rk}+\xi\varepsilon_k} + (2u'_k - v_k)u_k v_k\Big\};$$

$$\frac{dv_k}{d\xi} = \varepsilon_k\left[\frac{1}{r_{rk}+\xi\varepsilon_k}\left(u'_k - \frac{r'_{rk}+\xi\varepsilon'_k}{\varepsilon_k}\frac{du_k}{d\xi}\right) - \frac{v_k}{r_{rk}+\xi\varepsilon_k} + c_k(r_{rk}+\xi\varepsilon_k)\right];$$

$$\frac{dp_k}{d\xi} = \frac{\varepsilon_k}{\vartheta(\psi_k)}\rho^{\frac{1}{\gamma}}\left[\frac{v_k^2}{r_{rk}+\xi\varepsilon_k} - \frac{v_k}{r_{rk}+\xi\varepsilon_k}\left(u_k\frac{r'_{rk}+\xi\varepsilon'_k}{\varepsilon_k} - \frac{du_k}{d\xi}\right) - \frac{u_k}{\varepsilon_k}\frac{du_k}{d\xi}\right];$$

$$\frac{d\psi_k}{d\xi} = \varepsilon_k(r_{rk}+\xi\varepsilon_k)\frac{v_k\sin\theta_k}{\vartheta(\psi_k)}\rho_k^{\frac{1}{\gamma}}; \qquad (4.4)$$

$$\rho_k = \left[\frac{p_k}{\vartheta(\psi_k)}\right]^{\frac{1}{\gamma}} \quad (k = 0, 1, \ldots, m),$$

where

$$a_k = \gamma p_k^{\frac{\gamma-1}{\gamma}}\vartheta(\psi_k) - u_k^2, \quad b_k = \gamma p_k^{\frac{\gamma-1}{\gamma}}\vartheta(\psi_k) - v_k^2,$$

$$c_k = -\sin\theta_k\frac{\gamma}{\gamma-1}\frac{p_k}{\vartheta(\psi_k)}\left.\frac{d\vartheta(\psi)}{d\psi}\right|_{\psi=\psi_k}, \quad d_k = \gamma p_k^{\frac{\gamma-1}{\gamma}}\vartheta(\psi_k), \qquad (4.5)$$

$$[illegible]\,(r_{rk}+\xi\varepsilon_k)^2 + 2(r'_{rk}+\xi\varepsilon'_k)(r_{rk}+\xi\varepsilon_k)u_k v_k + b_k(r'_{rk}+\xi\varepsilon'_k)^2;$$

subscript "k" denotes values of parameters on k ray, primes designate derivatives with respect to θ, enumerable by the formulas (4.3).

On zero ray (at k = 0) all the terms of the second and fourth equations (4.4) identically turn into zero. For increasing the accuracy of the system it is expedient here to introduce equation for [illegible] with θ = 0. Differentiating the second equation for θ and considering that with θ = 0

$$v_0 \quad \varepsilon_0' \cdot u_0' \cdot r_{T0}' \quad \theta_0 = 0,$$

we obtain

$$\frac{dv_0^0}{d\xi} = \varepsilon_0 \left[\frac{2(r_{T0}' + \xi\varepsilon_0') [illegible]}{[illegible](r_{T0} + \xi\varepsilon_0)^2} + \frac{2u_1^0 - v_0^0}{r_{T0} + \xi\varepsilon_0} \right] -$$

$$- \varepsilon_0 \frac{\gamma}{\gamma - 1} \frac{p_0}{\rho(0)} \left[\frac{d\varphi(\varphi)}{d\varphi} \right]_{\varphi = 0} (r_{T0} + \xi\varepsilon_0). \qquad (4.6)$$

To system (4.4)-(4.6) it is necessary to add a correlation connecting the departure of shock wave from the body ε with angle of inclination of the shock wave to axis of symmetry σ:

$$\frac{d\varepsilon_k}{d\theta} = -(r_{Tk} + \xi\varepsilon_k)\,\mathrm{ctg}(\sigma_k + \theta_k) - r_{Tk}'. \qquad (4.7)$$

Besides differential equations, in accordance with conditons of uniqueness the solution should be subordinated to boundary conditon (1.3) on the surface of the body at $\xi = 0$ and to correlations for compression shock (1.2) at $\xi = 1.0$ ($r_C = r_T + \varepsilon$) in mesh nodes, i.e., in intersection points of the ray with surface of the body and the shock wave. The algorithm for the numerical solution of the problem amounts to the following. Assignment (m + 1) of parameters r_j^0 ($j = 0, 1, \ldots, m$) according to (4.2) determines approximately the equation of the shock wave, and with the help of correlations (1.2) all the gas-dynamic parameters behind the compression shock. Then by solving the Cauchy problem for the system (4.4)-(4.7), we determine the values of parameters in nodes on the surface of the body, which generally speaking do not satisfy boundary condition (1.3). Selecting by iterations the values r_j^0 in such a way that in all the nodes on the surface of the body the boundary condition (1.3) is fulfilled with the assigned accuracy, we obtain the unknown solution of the approximating system in m approximation.

Thus the method of calculation amounts to the following. We construct a sequence of solutions satisfying all the boundary conditions in region AGFE: correlations for compression shock AG, conditions of symmetry on a segment of axis AE, and the boundary condition on contour of body AF.

An additional condition of boundedness of derivatives (nonconversion into zero of Jacobian $\frac{\partial(x, y)}{\partial(v_x, v_y)}$) is fulfilled automatically, since each term in the sequence of approximate solutions satisfies it. If at the beginning there is assigned or if in the process of iterations there develops a form of shock in which in the supersonic segment of region AGFE a limiting line emerges, then it will automatically be related by the program since the process of calculations is disrupted. The desired solution and the one obtained in the process of iterations belong to the class of analytic functions where the Cauchy problem is correct.

§ 5. Examples

Without dwelling here in detail on results of investigating the attributes of the method, growth of errors, and convergence and stability during the solving of a nonlinear problem on supersonic flow around with a detached shock wave, we will give examples illustrating the solution of different problems of flow around with a detached shock wave.

Figure 6-9 illustrate the influence of M number and body form on the geometric picture (relative position body surfaces, shock wave, and sonic line) of flow around by an ideal gas with $\gamma = 1.4$. Figure 6 shows the influence of a change in M number over a wide range during flow past on ellipsoid with a ration of semi-axis $\delta = 2.0$. Figure 7 gives a geometric picture of flow at $M = 3$ past a family of bodies with front section contours assigned by the equation $x^n + y^n = 1$. At $n > 2$ radius of curvature of contour in a critical point R_0 is equal to infinity, and with a change of n from 2 to 10 the ratio of the minimum angle in the vicinity, the radius of curvature, to body diameter $\frac{R_{min}}{D}$ is changed from 0.5 to 0.07. Figure 8, in a example of flow at $M = 3$ past Cassinian ovals with the contour equation $(x^2 + y^2)^2 + 2c^2(x^2 - y^2) = a^4 - c^4(a^2 + c^2 = 1)$, shows the influence of concavity of contour in region of critical point. Figure 9 shows flow at $M = 3$ past a, the contour of which has a discontinuity of curvature at point C and is formed by conjugate circumferences of radii $R_0/D = 1$ in region of critical point $R_1/D = 0.2$ in the vicinity of the midsection, and past body B, constituting a 60-degree segment with an angular point. Figure 10 gives the distribution of pressure on the surface of bodies of various form at flow past with M = 3. So that the curves do not merge, polar angle θ is accepted as an argument along the surface of the body (see Fig. 6, 7, 8). For body B $\theta = \theta_1$ (where θ_1 – angle with pole in center of circumference of radius R), for body a up to a point with $\theta = \theta_1$, and further $\theta = \theta_1 + \theta_2$ (see Fig. 9). Points plot the experimental data of Yu. Ya. Karpeyskiy.

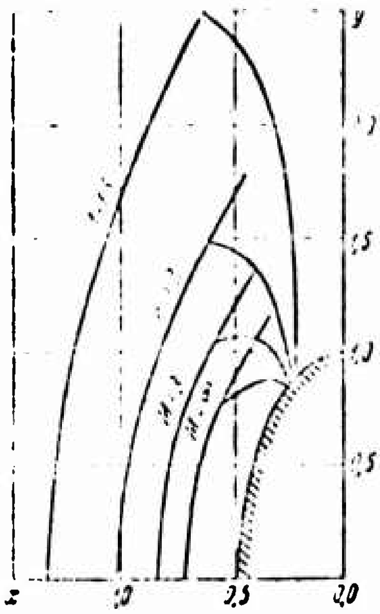

Fig. 6.

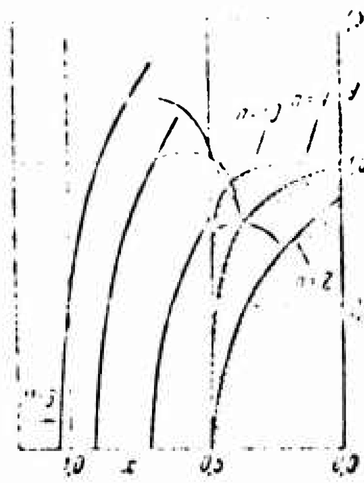

Fig. 7.

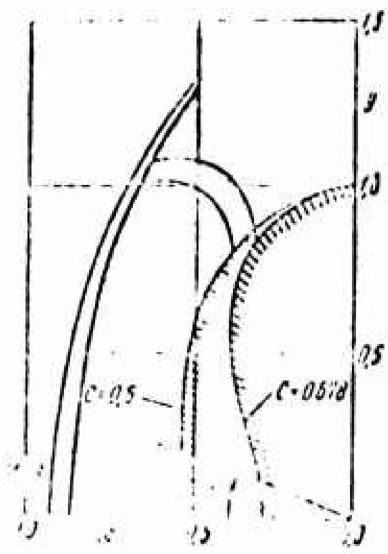

Fig. 8.

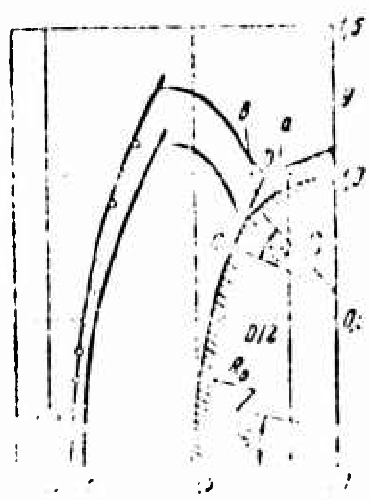

Fig. 9.

Investigation of supersonic flow past a sphere by nonequilibrium air was performed taking into account kinetics of excitation of fluctuations in N_2 and O_2, reactions of dissociation and exchange, including $O_2 + N_2 = 2NO$), and ionization with the help of atomic collisions $N + O = NO^+ + e$, $N + N = N_2^+ + e$, $O + O = O_2^+ + e$, satisfactorily describing the process up to M ~ 30. Influence of physical-chemical transformations occurring behind the shock waves on the departure of shock wave is shown in Fig. 11.

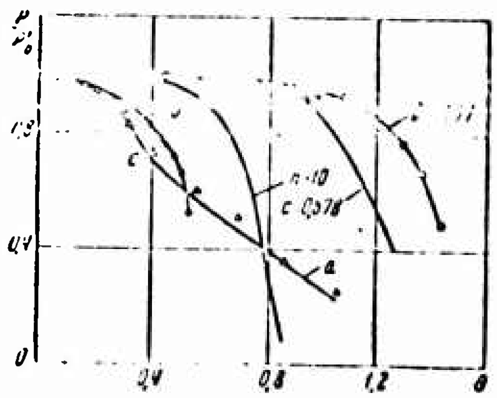

Fig. 10.

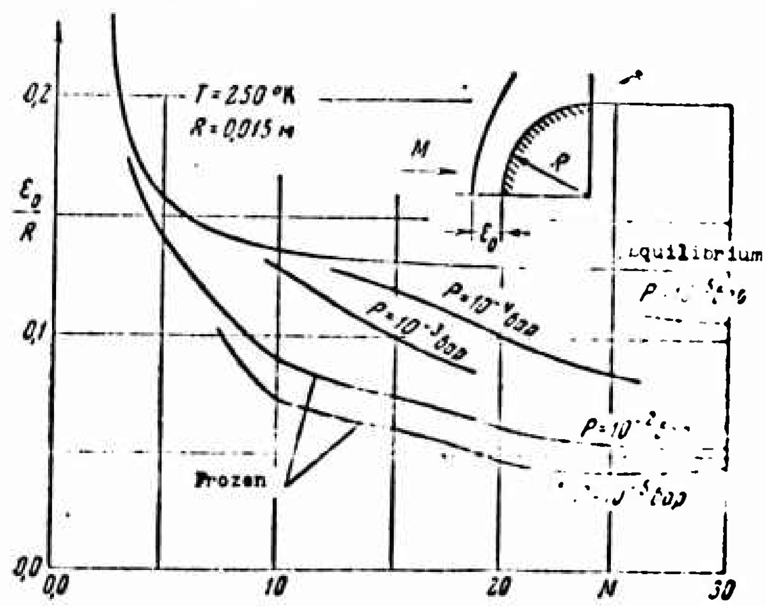

Fig. 11.

On Figs. 12 and 13 are shown the geometric picture and distribution of pressure on the surface during transverse flow past a cylinder (two-dimensional problem) with different M numbers.

In work [31] a method is generalized for the case of spatial flow past blunted bodies with detached shock wave. Figure 14 shows the influence of angle of attack on the geometric picture of flow past an ellipsoid with $\delta = 3.07$ at $M = 3$. For the same case Fig. 15 shows distribution of pressure on the surface of a body in a plane of symmetry. The points are plots of the experimental data of Yu. Ya. Karpeyskiy.

Fig. 12

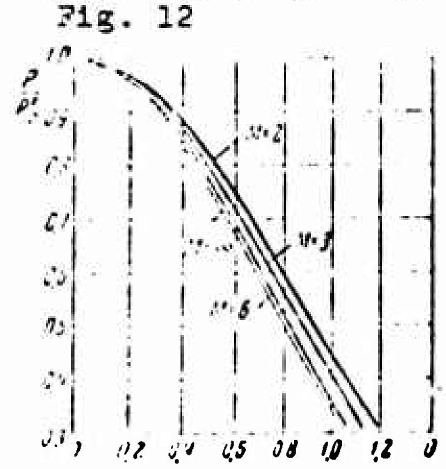

Fig. 13.

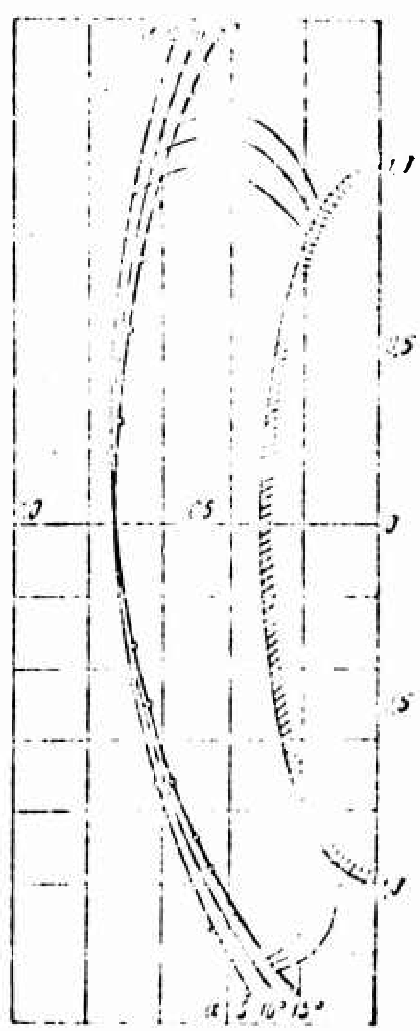

Fig. 14.

The calculations presented in Figs. 6-10 and 12 and 13 were conducted on a 9-point system (selection of 5 parameters), and in Fig. 11 on a 5-point system (selection of 3 parameters). During calculations of spatial flow past during iterations of the form of compression shock the selection 13 parameters was performed. Without dwelling here on the results of investigation of accuracy, [illegible]

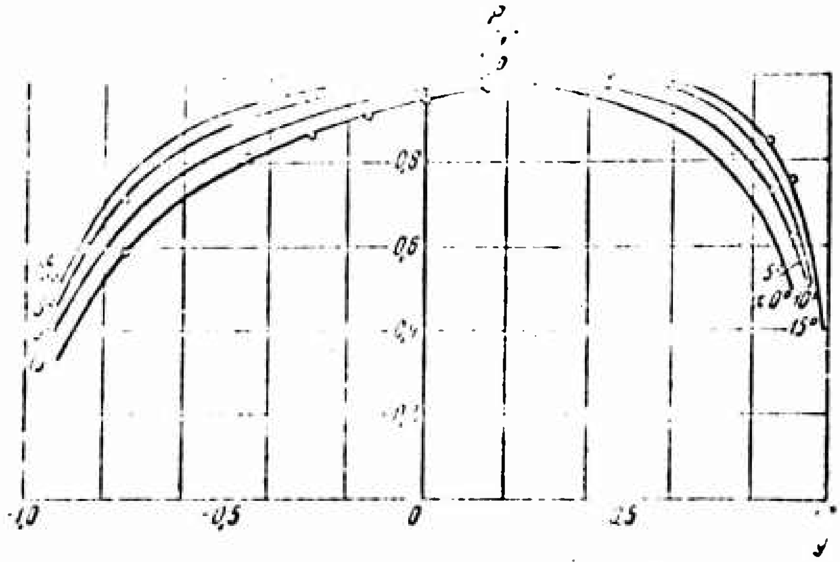

Fig. 15.

note that total error during calculation of the majority of variants does not exceed 1%. For example, in the case of flow past a sphere by an ideal gas at M = 3 the total error already during calculation based on a 5-point system does not exceed 0.2% in the entire field between wave and body. During calculation of flow past a body with a fracture of the generatrix in the sonic point (B in Fig. 9) in the vicinity of singular point an asymptotic solution was used. This was obtained by Guderley [19] for flat potential flows. Then the generalization in a number of works [21, 20] for vortex axisymmetrical flows was used.

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Footnotes

[1]Here two cases of disposition of nodes are examined. For an example, of theorems of convergence in a general case, see [12, 13].

[2]At present L. S. Frank and L. A. Chudov are developing methods based on the smoothing of the solution for each layer, thus making it possible to lessen the growth of rounding errors [22].