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# THE GIBBS-EINSTEIN TENSOR ANALYSIS WITH APPLICATION TO CONTINUUM MECHANICS AND CANONICAL FORMS OF GENERAL SECOND-ORDER TENSORS

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**Shunsuke Takagi**

November 1968



U.S. ARMY MATERIEL COMMAND
TERRESTRIAL SCIENCES CENTER
**COLD REGIONS RESEARCH & ENGINEERING LABORATORY**
HANOVER, NEW HAMPSHIRE

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## PREFACE

This paper was prepared by Dr. Shunsuke Takagi, Research Physical Scientist, of the Cold Regions Research and Engineering Laboratory, U.S. Army Terrestrial Sciences Center (USA TSC).

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## CONTENTS

## ABSTRACT

A new tensor analysis, called the Gibbs-Einstein tensor analysis, is developed based on the concept that directions are algebraic quantities subject to the rule of forming scalar products, tensor products, and linear combinations. The new tensor analysis is explained in this paper by way of reformulating continuum mechanics and the Hamilton-Cayley theorem in matrix theory. The latter reformulation yields an explanation of the deformation dyads introduced in the former reformulation. A scalar product of two deformation dyads yields the strain tensor, which is a thermodynamic state variable for thermodynamically reversible deformations. Mathematics dealing with directions in a flat space becomes much simpler and more understandable when the Gibbs-Einstein tensor expression is used.

# THE GIBBS-EINSTEIN TENSOR ANALYSIS WITH APPLICATION TO CONTINUUM MECHANICS AND CANONICAL FORMS OF GENERAL SECOND-ORDER TENSORS

by

Shunsuke Takagi

## INTRODUCTION

Three tensor expressions are used currently. The most prevalent is the expression by components $V^i$, $V_i$, $T^{ij}$, $T^i{}_j$, etc., which will be called the Einstein expression. The second expression, which will be called the Gibbs expression, consists of linear combinations of base vectors, of dyads, of triads, etc. (introduced by Gibbs and Wilson (1901)), whose coefficients, however, are not recognized as Einstein expressions. The third expression, which will be called the Gibbs-Einstein expression, is a combination of both the above expressions, expressing a vector **V** as $V^i\mathbf{e}_i = V_i\mathbf{e}^i$, a second-order tensor **T** as $T^{ij}\mathbf{e}_i\mathbf{e}_j = T^i{}_j\mathbf{e}_i\mathbf{e}^j = T_i{}^j\mathbf{e}^i\mathbf{e}_j = T_{ij}\mathbf{e}^i\mathbf{e}^j$, a third-order tensor **T** as $T^{ijk}\mathbf{e}_i\mathbf{e}_j\mathbf{e}_k = \ldots$, etc., in which $\mathbf{e}_i$ and $\mathbf{e}^i$ are covariant and contravariant base vectors, respectively, defined by

$$\mathbf{e}_i \cdot \mathbf{e}^j = \delta_i^j \tag{1}$$

where $\delta_i^j$ is a Kronecker delta. Coefficients $V^i$ and $V_i$ are contravariant and covariant components of the Einstein expression. Dyads $\mathbf{e}_i\mathbf{e}_j$, $\mathbf{e}_i\mathbf{e}^j$, $\mathbf{e}^i\mathbf{e}_j$ and $\mathbf{e}^i\mathbf{e}^j$ in the Gibbs expressions are bases for the second-order tensors whose Einstein expressions are $T^{ij}$, $T^i{}_j$, $T_i{}^j$, and $T_{ij}$, respectively. Similarly, triads $\mathbf{e}_i\mathbf{e}_j\mathbf{e}_k,\ldots$, in the Gibbs expression are bases for the third-order tensors whose Einstein expressions are $T^{ijk},\ldots$, etc.

The Gibbs-Einstein tensor expression was introduced first by Hessenberg (1917) and extended by Wills (1931). Recently this notation was used for the study of large deformation by Yoshimura (1957) and Sedov (1962) but it has not yet been widely accepted.

The Einstein expression can be used in a curved space without introducing normals to the curved space; therefore, it is convenient for the study of intrinsic properties of a manifold. A curved space, however, must be embedded in a flat space if the Gibbs-Einstein expression is to be applied. This is because the differentiation of a vector belonging to a curved space may yield a vector that has as a component a normal to the curved space.

Use of the Gibbs-Einstein expression is based on the recognition that identifying directions with sets of numbers is not a proper definition of directions. In terms of axiomatic geometry, a direction is an undefined quantity, like a point, a straight line, or a plane. In terms of abstract algebra, directions are algebraic quantities subject to the operations of scalar product, tensor product, and linear combinations of tensor bases. (A tensor product of vectors is a juxtaposition of vectors in a given order. Vectors in a tensor product are non-commutative. Juxtaposing a set of base vectors forming a dual basis forms a set of tensor bases. Coefficients of a linear combination of tensor bases, are, in general, functions of space and time.) Note that a different definition of scalar products defines a different geometry.

The Gibbs-Einstein notation yields simpler expressions and easier analysis of the quantities containing directions. Geometries, theory of functions of many variables, mechanics, and mathematical physics in flat spaces should be reformulated with this notation.

In the first part of this paper, the continuum mechanics reformulated with the Gibbs-Einstein tensor expression will be summarized. In the second part, the Hamilton-Cayley theorem will be reformulated with the Gibbs-Einstein tensor expression. Note that a matrix is the Einstein expression of a second-order tensor. The reformulated Hamilton-Cayley theorem is much simpler, directly yielding the minimal polynomial, and is more understandable. It also yields a new concept of deformation, defining the deformation dyad. A scalar product of two deformation dyads yields the strain tensor, which is a thermodynamic state variable for thermodynamically reversible deformations.

## PART I. APPLICATION TO CONTINUUM MECHANICS

Let $\xi^i$ $(i = 1, 2, 3)$ be the coordinates at time $t = 0$. A particle whose initial coordinates are $\xi^1, \xi^2, \xi^3$ will be called particle $\xi^i$. The position of particle $\xi^i$ at time $t$ is

$$\mathbf{x} = \mathbf{x}(\xi^1, \xi^2, \xi^3, t). \tag{2}$$

Covariant base vectors $\mathbf{e}_i (i = 1, 2, 3)$ are defined by

$$\mathbf{e}_i = \frac{\partial \mathbf{x}}{\partial \xi^i}. \tag{3}$$

Contravariant base vectors $\mathbf{e}^i (i = 1, 2, 3)$ are defined to satisfy eq 1. Vectors $\mathbf{e}_i$ represent deformation, because vector $\mathbf{e}_1$, for example, is a vector obtained by dividing the vector spanned by particles $(\xi^1 + d\xi^1, \xi^2, \xi^3)$ and $(\xi^1, \xi^2, \xi^3)$ by $d\xi^1$. Jacobian $\partial(X^1, X^2, X^3)/\partial(\xi^1, \xi^2, \xi^3)$, where $X^1, X^2, X^3$ are Cartesian components of $\mathbf{x}$, is equal to the volume of the parallelepiped $\mathbf{e}_1 \times \mathbf{e}_2 \cdot \mathbf{e}_3$. Assume that the initial coordinates are right-handed, then

$$\sqrt{g} = \mathbf{e}_1 \times \mathbf{e}_2 \cdot \mathbf{e}_3. \tag{4}$$

Unit tensor $\mathbf{1}$ is defined by

$$\mathbf{1} = \mathbf{e}_i \mathbf{e}^i = \mathbf{e}^i \mathbf{e}_i = g_{ij} \mathbf{e}^i \mathbf{e}^j = g^{ij} \mathbf{e}_i \mathbf{e}_j$$

in which

$$g_{ij} = \mathbf{e}_i \cdot \mathbf{e}_j \tag{6}$$

and

$$g^{ij} = \mathbf{e}^i \cdot \mathbf{e}^j. \tag{7}$$

Unit tensor $\mathbf{1}$ is the unit for dot multiplication

$$\mathbf{T}\cdot\mathbf{1} = \mathbf{1}\cdot\mathbf{T} = \mathbf{T} \tag{8}$$

where $\mathbf{T}$ is any tensor. Equation 8 will be proved by executing the dot multiplication in eq 8 when an appropriate expression of $\mathbf{1}$ in eq 5 is chosen, determined by the form of $\mathbf{T}$.

In a flat space, $\partial \mathbf{e}_i/\partial \xi^j$ is a vector belonging to the same space; therefore,

$$\frac{\partial \mathbf{e}_i}{\partial \xi^j} = \Gamma^h_{ij}\,\mathbf{e}_h \tag{9}$$

where components $\Gamma^h_{ij}$ are Christoffel symbols. Differentiating eq 1 and using the property of eq 8 of 1 yields

$$\frac{\partial \mathbf{e}^i}{\partial \xi^j} = -\Gamma^i_{jh}\,\mathbf{e}^h\,. \tag{10}$$

Covariant differentiation is derived by use of eq 9 and eq 10.

A differentiable tensor is a linear combination of Cartesian base tensors (base tensors formed of Cartesian base vectors) using differentiable functions of $\xi^i$ and $t$ as the coefficients. The order of partial differentiation of a differentiable tensor, therefore, is commutative,

$$\frac{\partial^2 \mathbf{T}}{\partial\xi^i\partial\xi^j} = \frac{\partial^2 \mathbf{T}}{\partial\xi^j\partial\xi^i}\,. \tag{11}$$

In terms of covariant base vectors, eq 11 is valid if and only if

$$\frac{\partial^2 \mathbf{e}_k}{\partial\xi^i\partial\xi^j} = \frac{\partial^2 \mathbf{e}_k}{\partial\xi^j\partial\xi^i}\,. \tag{12}$$

To show this, let $\mathbf{T}$ be of the $n$th order, and assume that the proposition is true for a tensor of $(n-1)$th order. Use of the unit tensor yields

$$\frac{\partial^2 \mathbf{T}}{\partial\xi^i\partial\xi^j} = \frac{\partial}{\partial\xi^i}\,\frac{\partial \mathbf{e}_k(\mathbf{e}^k\cdot\mathbf{T})}{\partial\xi^j}\,.$$

The function to be differentiated is a tensor product of $\mathbf{e}_k$ and $\mathbf{e}^k\cdot\mathbf{T}$; then we find

$$\frac{\partial^2 \mathbf{T}}{\partial\xi^i\partial\xi^j} = \frac{\partial^2 \mathbf{e}_k}{\partial\xi^i\partial\xi^j}\,\mathbf{e}^k\cdot\mathbf{T} + \frac{\partial \mathbf{e}_k}{\partial\xi^j}\,\frac{\partial \mathbf{e}^k\cdot\mathbf{T}}{\partial\xi^i} + \frac{\partial \mathbf{e}_k}{\partial\xi^i}\,\frac{\partial \mathbf{e}^k\cdot\mathbf{T}}{\partial\xi^j} + \mathbf{e}_k\,\frac{\partial^2 \mathbf{e}^k\cdot\mathbf{T}}{\partial\xi^i\partial\xi^j}$$

which shows that eq 11 is valid if and only if eq 12 is valid. Note that

$$\frac{\partial^2 \mathbf{e}_k}{\partial\xi^i \partial\xi^j} - \frac{\partial^2 \mathbf{e}_k}{\partial\xi^j \partial\xi^i} = \mathrm{B}^r_{kji}\mathbf{e}_r \tag{13}$$

where $\mathrm{B}^r_{kji}$ is a component of the Riemann-Christoffel tensor.

The nabla operator $\nabla$ is defined by

$$\nabla = \mathbf{e}^i \frac{\partial}{\partial\xi^i} . \tag{14}$$

Nabla is invariant under coordinate transformations; therefore, it is dependent on time $t$ only.

The gradient of a tensor $\mathbf{T}$ of any order is defined by

$$\mathrm{grad}\ \mathbf{T} = \mathbf{e}^i \frac{\partial \mathbf{T}}{\partial\xi^i} \tag{15}$$

where $\mathbf{e}^i$ is usually put at the extreme left of the base tensors of $\partial\mathbf{T}/\partial\xi^i$, but may be put anywhere in the base tensors of $\partial\mathbf{T}/\partial\xi^i$ to form a tensor of one order higher than $\mathbf{T}$.

Divergence of a tensor $\mathbf{T}$ of any order is defined by

$$\mathrm{div}\ \mathbf{T} = \mathbf{e}^i \cdot \frac{\partial \mathbf{T}}{\partial\xi^i} \tag{16}$$

where $\mathbf{e}^i$ is usually dotted with the base vectors at the left ends in the base tensors of $\partial\mathbf{T}/\partial\xi^i$, but may be dotted with any base vectors in the base tensors of $\partial\mathbf{T}/\partial\xi^i$ to form a tensor of one order lower than $\mathbf{T}$.

Curl of a tensor $\mathbf{T}$ of any order is defined by

$$\mathrm{curl}\ \mathbf{T} = \mathbf{e}^i \times \frac{\partial \mathbf{T}}{\partial\xi^i} \tag{17}$$

where $\mathbf{e}^i$ is usually crossed with the base vectors at the left ends in the base tensors of $\partial\mathbf{T}/\partial\xi^i$, but may be crossed with any base vectors in $\partial\mathbf{T}/\partial\xi^i$ to form a tensor of the same order as $\mathbf{T}$.

The use of nabla thus introduced allows us to extend use of almost all the integral and differential vector formulas to a tensor of any order in the Gibbs-Einstein expression (Takagi, 1968).

Time differentiation keeping $\xi^1$, $\xi^2$, $\xi^3$ constant is denoted by D/Dt. Thus,

$$\mathbf{v} = \frac{\mathrm{D}\mathbf{x}}{\mathrm{D}t} . \tag{18}$$

Differentiating eq 18 with respect to $\xi^i$ yields

$$\frac{\mathrm{D}\mathbf{e}_i}{\mathrm{D}t} = \frac{\partial \mathbf{v}}{\partial\xi^i} . \tag{19}$$

The symmetric part of grad **v** is denoted by

$$(\text{grad } \mathbf{v})^s = \dot{\varepsilon}_{ij}\mathbf{e}^i\mathbf{e}^j . \tag{20}$$

Components $\dot{\varepsilon}_{ij}$ satisfy

$$2\dot{\varepsilon}_{ij} = \frac{Dg_{ij}}{Dt} . \tag{21}$$

Strain tensor $\varepsilon$ is given by

$$\varepsilon = \int_0^t \dot{\varepsilon}_{ij}\mathring{\mathbf{e}}^i\mathring{\mathbf{e}}^j \, dt \tag{22}$$

where $\mathring{\mathbf{e}}^i$ is the initial value of $\mathbf{e}^i$ and is not dependent on $t$. When the deformation is the elongation of $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$, the integral

$$\int_0^t \dot{\varepsilon}_{ij}\,\mathbf{e}^i\mathbf{e}^j \, dt \tag{23}$$

whose integrand is a product of three time-dependent functions, yields a logarithmic strain. In general, however, the integral of eq 23 is dependent on the path of integration (Yoshimura, 1957), as may be shown by following elongations and rotations in different orders, and therefore is not a thermodynamic state variable. $\varepsilon$ in eq 22 is a thermodynamic state variable representing a thermodynamically reversible process (see the end of this part).

Note that

$$\varepsilon = \tfrac{1}{2}\Big[\underset{a\ 1}{\mathbf{e}^i\mathring{\mathbf{e}}_i} \cdot \underset{a\ 2}{\mathbf{e}^j\mathring{\mathbf{e}}_j}\Big]_0^t \tag{24}$$

where the numbers and letters under the tensor symbols indicate identical base vectors when they are on different sides and base vectors to be dotted when they are on the same side. The quantity in the brackets is a scalar product of *deformation dyads*, $\mathbf{e}^i\mathring{\mathbf{e}}_i$ and $\mathbf{e}^j\mathring{\mathbf{e}}_j$, which define the inverse deformation from time $t$ to time $t = 0$. To show this, let $\mathbf{e}_i d\xi^i$ be a material point in the neighborhood of a particle whose material bases are $\mathbf{e}_i$, $\mathbf{e}^i$ at time $t$ and $\mathring{\mathbf{e}}_i$, $\mathring{\mathbf{e}}^i$ at time $t = 0$. Dotting $\mathbf{e}_i d\xi^i$ from the left in $\mathbf{e}^j\mathring{\mathbf{e}}_j$ yields $\mathring{\mathbf{e}}_i d\xi^i$. Therefore, dotting from the left in $\mathbf{e}^i\mathring{\mathbf{e}}_i$ is equivalent to a deformation changing $\mathbf{e}_i d\xi^i$ to $\mathring{\mathbf{e}}_i d\xi^i$. In Part II, the more realistic interpretation of deformation dyads will be given.

Three vectors $\mathbf{e}_i$ ($i = 1, 2, 3$) of $\xi^1$, $\xi^2$, $\xi^3$ in a more-than-three-dimensional space span three-dimensional subspace, letting $\xi^1$, $\xi^2$, $\xi^3$ be a set of curvilinear coordinates of the subspace, if and only if $\mathbf{e}_i d\xi^i$ and $(\partial\mathbf{e}_i/\partial\xi^j)d\xi^j$ are total differentials. The latter condition, which yields the compatibility equations of components $\dot{\varepsilon}_{ij}$ requires that $d\mathbf{e}_i$ must be a linear combination of vectors $\mathbf{e}_i$, and therefore shows that the space spanned by vectors $\mathbf{e}_i$ is flat.

Let $\mathbf{a}_1,\ldots,\mathbf{a}_m$ be $n$-dimensional vectors, where $n \geq m$. An exterior product of vectors $\mathbf{a}_1,\ldots,\mathbf{a}_m$ (introduced by Grassmann, 1844), denoted by $\mathbf{a}_1 \wedge \cdots \wedge \mathbf{a}_m$, is defined, in the Gibbs-Einstein expression, by

$$\mathbf{a}_1 \cdots \mathbf{a}_m = \pi^{i_1 \ldots i_m} \mathbf{a}_{i_1} \cdots \mathbf{a}_{i_m} \tag{25}$$

where $\pi^{i_1 \ldots i_m}$ is a permutation symbol. Forming exterior products is called *wedge multiplication*, or *wedging* for short. Note that the right-hand side of eq 25 may be written

$$\begin{vmatrix} \mathbf{a}_1 \cdots \mathbf{a}_m \\ \cdots\cdots\cdots \\ \mathbf{a}_1 \cdots \mathbf{a}_m \end{vmatrix} \tag{26}$$

when the convention is applied that the determinant must be developed so that the elements of the first,...,$m$th row in the determinant become the first,...,$m$th base vectors, respectively, in the tensor products.

The geometric meaning of the exterior product is that

$$\mathbf{a}_1 \wedge \cdots \wedge \mathbf{a}_m = \epsilon V \tag{27}$$

where $\epsilon$ is the tensor expression of an $m$-dimensional cube (usually called orientation) used as the unit of measuring volume $V$ of the parallelepiped spanned by $\mathbf{a}_1,\ldots,\mathbf{a}_m$.

Expressed by a three-dimensional dual basis $\mathbf{e}_i$, $\mathbf{e}^i$ forming a right-handed skew coordinate system, $\epsilon$ becomes

$$\begin{aligned} \epsilon &= \epsilon^{ijk} \mathbf{e}_i \mathbf{e}_j \mathbf{e}_k \\ &= \epsilon_{ijk} \mathbf{e}^i \mathbf{e}^j \mathbf{e}^k \end{aligned} \tag{28}$$

where

$$\epsilon^{ijk} = \frac{1}{\sqrt{g}} \pi^{ijk}$$

and

$$\epsilon_{ijk} = \sqrt{g}\, \pi_{ijk} \; . \tag{29}$$

Note that $\sqrt{g}$ is the volume of the parallelepiped spanned by $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$.

$\epsilon$ is a constant tensor fulfilling

$$\frac{D\epsilon}{Dt} = 0 \tag{30}$$

and

$$\frac{\partial \boldsymbol{\epsilon}}{\partial \xi^i} = 0 . \tag{31}$$

Similarly to $\boldsymbol{\epsilon}$, $\mathbf{1}$ is also a constant tensor fulfilling

$$\frac{D\mathbf{1}}{Dt} = 0 \tag{32}$$

and

$$\frac{\partial \mathbf{1}}{\partial \xi^i} = 0. \tag{33}$$

An $n$-dimensional cross product may be defined by dotting with the $n$-dimensional $\boldsymbol{\epsilon}$

$$\underset{123\ldots n}{\boldsymbol{\epsilon}} \underset{12}{\cdot\, \mathbf{ab}} = \epsilon_{i_1 \ldots i_n} a^{i_1} b^{i_2} \underset{3}{\mathbf{e}^{i_3}} \ldots \underset{n}{\mathbf{e}^{i_n}} \tag{34}$$

where $\epsilon_{i_1 \ldots i_n}$ is a component of the $n$-dimensional $\boldsymbol{\epsilon}$. Because of the antisymmetric properties of $\boldsymbol{\epsilon}$, there are many other choices of dotting base vectors in $\boldsymbol{\epsilon}$ in the left-hand side that yield the same result as in the right-hand side, which, however, need not be shown here.

The exterior differentiation (introduced by Cartan (1922)) of a tensor $\mathbf{T}$ of any order is given, in the Gibbs-Einstein expression, by

$$\mathbf{e}^i \wedge \frac{\partial \mathbf{T}}{\partial \xi^i} \tag{35}$$

where $\mathbf{e}^i$ is usually wedged with the base vectors at the left ends in the base tensors of $\partial \mathbf{T}/\partial \xi^i$, but may be wedged with any base vectors in the base tensors of $\partial \mathbf{T}/\partial \xi^i$ to form a tensor of one order higher than $\mathbf{T}$.

The antisymmetric part of the three-dimensional gradient of $\mathbf{v}$ is equal to

$$\begin{aligned} [\text{grad } \mathbf{v}]^A &= \frac{1}{2} \mathbf{e}^i \wedge \frac{\partial \mathbf{v}}{\partial \xi^i} \\ &= \frac{1}{2}\left(\mathbf{e}^i \frac{\partial \mathbf{v}}{\partial \xi^i} - \frac{\partial \mathbf{v}}{\partial \xi^i} \mathbf{e}^i\right). \end{aligned} \tag{36}$$

The following remark shows that a material symmetry that existed at time $t = 0$ exists throughout the deformation.

**Remark**

Denote by $\mathring{\mathbf{e}}_i$ and $\mathring{\mathbf{a}}$ the base vectors and a vector at time $t = 0$, respectively,

$$\mathring{\mathbf{a}} = a^i \mathring{\mathbf{e}}_i \tag{37}$$

where scalars $a^i$ are components of $\mathring{\mathbf{a}}$ referred to $\mathring{\mathbf{e}}_i$. At time $t$, $\mathring{\mathbf{e}}_i$ and $\mathring{\mathbf{a}}$ become $\mathbf{e}_i$ and $\mathbf{a}$, respectively, but the components $a^i$ are the same,

$$\mathbf{a} = a^i \mathbf{e}_i. \tag{38}$$

*Proof.* Let initial coordinates $\eta^p$ different from $\xi^i$ define $\mathbf{f}_p$ and $\mathring{\mathbf{f}}_p$ different from $\mathbf{e}_i$ and $\mathring{\mathbf{e}}_i$, respectively. Then we have

$$\mathbf{e}_i d\xi^i = \mathbf{f}_p d\eta^p \tag{39}$$

at time $t$ and

$$\mathring{\mathbf{e}}_i d\xi^i = \mathring{\mathbf{f}}_p d\eta^p \tag{40}$$

at time $t = 0$. Therefore, the transformation from $\mathbf{e}_i$ to $\mathbf{f}_p$ is the same as the transformation from $\mathring{\mathbf{e}}_i$ to $\mathring{\mathbf{f}}_p$.

Letting $\mathring{\mathbf{a}}$ and $\mathbf{a}$ be one of $\mathring{\mathbf{f}}_p$ and $\mathbf{f}_p$, respectively, proves the theorem. The proof is thus completed.

The remark shows that constitutive equations must be written in terms of material coordinates.

Next, the axiom of objectivity will be given the Gibbs-Einstein expression. Let $\mathbf{c}_\alpha = \mathbf{c}^\alpha$ be a set of fixed orthogonal vectors and $\mathbf{a}_\alpha(t) = \mathbf{a}^\alpha(t)$ be a set of moving unit orthogonal vectors. Define

$$\underset{12}{\mathbf{Q}} = \underset{1\;\;2}{\mathbf{a}_\alpha \mathbf{c}^\alpha} = \underset{1\;\;2}{\mathbf{a}^\alpha \mathbf{c}_\alpha}. \tag{41}$$

The inverse of $\mathbf{Q}$ is

$$\underset{12}{\mathbf{Q}^{-1}} = \underset{12}{\mathbf{Q}^{\mathrm{T}}} = \underset{1\;\;2}{\mathbf{c}_\alpha \mathbf{a}^\alpha} = \underset{1\;\;2}{\mathbf{c}^\alpha \mathbf{a}_\alpha} \tag{42}$$

because they satisfy the relation

$$\underset{1a}{\mathbf{Q}} \cdot \underset{a2}{\mathbf{Q}^{-1}} = \underset{1a}{\mathbf{Q}^{-1}} \cdot \underset{a2}{\mathbf{Q}} = \underset{12}{\mathbf{1}}. \tag{43}$$

A rotation that changes $\mathbf{x} = x^\alpha \mathbf{c}_\alpha$ to $\mathbf{y} = x^\alpha \mathbf{a}_\alpha$ is given by

$$\mathbf{y} = \mathbf{Q} \cdot \mathbf{x} = \mathbf{x} \cdot \mathbf{Q}^{\mathrm{T}} \tag{44}$$

where nothing is shown under tensor symbols on the convention that two base vectors adjacent to the dot, one on the left and one on the right, shall be dotted when no indication for dotting is given.

Define

$$\mathbf{y} = \mathbf{Q} \cdot \mathbf{x} + \mathbf{b}(t) \tag{45}$$

where $\mathbf{b}(t)$ is a function of $t$. Vector $\mathbf{x}$ is referred to the fixed coordinates spanned by $\mathbf{c}_\alpha$, and vector $\mathbf{y}$ is referred to the moving coordinates rotating with $\mathbf{a}_\alpha(t)$ and translating with $\mathbf{b}(t)$.

Let $\mathbf{x}$ be a function of $\xi^i$ and $t$, and define

$$\mathbf{d}_i = \frac{\partial \mathbf{y}}{\partial \xi^i}. \tag{46}$$

Then we find

$$\left.\begin{aligned} \mathbf{d}_i &= \mathbf{Q} \cdot \mathbf{e}_i = \mathbf{e}_i \cdot \mathbf{Q}^T \\ \mathbf{e}_i &= \mathbf{Q}^T \cdot \mathbf{d}_i = \mathbf{d}_i \cdot \mathbf{Q} \\ \mathbf{d}^i &= \mathbf{e}^i \cdot \mathbf{Q}^T = \mathbf{Q} \cdot \mathbf{e}^i \\ \mathbf{e}^i &= \mathbf{d}^i \cdot \mathbf{Q} = \mathbf{Q}^T \cdot \mathbf{d}^i . \end{aligned}\right\} \tag{47}$$

Let

$$\frac{D\mathbf{y}}{Dt} = \mathbf{u}. \tag{48}$$

Then, operating D/D$t$ on $\mathbf{y}$ in eq 45 yields

$$\mathbf{u} = \mathbf{Q} \cdot \mathbf{v} + \frac{D\mathbf{Q}}{Dt} \cdot \mathbf{x} + \frac{D\mathbf{b}}{Dt}. \tag{49}$$

The nabla of $\mathbf{u}$ in the moving coordinates is given by

$$\mathbf{d}^i \frac{\partial \mathbf{u}}{\partial \xi^i} = \mathbf{e}^i \cdot \mathbf{Q}^T \left( \mathbf{Q} \cdot \frac{\partial \mathbf{v}}{\partial \xi^i} + \frac{D\mathbf{Q}}{Dt} \cdot \mathbf{e}_i \right)$$

a b 1 1a b2 2 b2 2

$$= \mathbf{Q} \cdot \mathbf{e}^i \frac{\partial \mathbf{v}}{\partial \xi^i} \cdot \mathbf{Q}^T + \mathbf{Q} \cdot \frac{D\mathbf{Q}^T}{Dt}. \tag{50}$$

a1 1 2 2b a1 1b

Similarly, we find

$$\frac{\partial \mathbf{u}}{\partial \xi^i} \mathbf{d}^i = \mathbf{Q} \cdot \frac{\partial \mathbf{v}}{\partial \xi^i} \mathbf{e}^i \cdot \mathbf{Q}^T + \frac{D\mathbf{Q}}{Dt} \cdot \mathbf{Q}^T \tag{51}$$

a b a1 1 2 2b a1 1b

Adding eq 50 and eq 51 yields

$$\mathbf{d}^i \frac{\partial \mathbf{u}}{\partial \xi^i} + \frac{\partial \mathbf{u}}{\partial \xi^i} \mathbf{d}^i = \mathbf{Q} \cdot \mathbf{e}^i \frac{\partial \mathbf{v}}{\partial \xi^i} + \frac{\partial \mathbf{v}}{\partial \xi^i} \mathbf{e}^i \cdot \mathbf{Q}^T \tag{52}$$

which shows that $(\text{grad } \mathbf{v})^S$ is objective. Subtracting eq 51 from eq 50 shows that $[\text{grad } \mathbf{v}]^A$ is not objective.

Base vectors $\mathbf{e}_i$, $\mathbf{e}^i$ are objective as shown in eq 47. An objective second-order tensor satisfies

$$\mathbf{T} = \mathbf{e}_i T^{ij} \mathbf{e}_j = \mathbf{d}_h S^{hk} \mathbf{d}_k \tag{53}$$

where $S^{hk}$ is the components in the moving coordinates. Operating D/D$t$ on $\mathbf{T}$ in eq 53 yields

$$\frac{\mathrm{D}\mathbf{T}}{\mathrm{D}t} = \left( \frac{\mathrm{D}T^{ij}}{\mathrm{D}t} + T^{pj} v^i{}_{,p} + T^{ip} v^j{}_{,p} \right) \mathbf{e}_i \mathbf{e}_j \tag{54}$$

which is again objective.

$\epsilon$ in eq 22 is a thermodynamic state variable representing a thermodynamically reversible process. To explain this, we first notice that d$U$, for example, in thermodynamics may be identified with (D$U$/D$t$)D$t$. This recognition leads us to a thermodynamic principle: A thermodynamic function $U$, for example, is a function of quantities $q^i$, if D$U$ is expressed as a linear combination of D$q^i$ when quantities $q^i$ are independent with each other.

From thermodynamics,

$$\mathrm{D}U = \mathrm{D}Q + \mathrm{D}W \tag{55}$$

where $U$ is the internal energy per unit mass, and D$Q$ and D$W$ are heat and work inputs, respectively, per unit mass per unit time. Divide D$W$ into two parts

$$\mathrm{D}W = (\mathrm{D}W)^{\mathrm{rev}} + (\mathrm{D}W)^{\mathrm{irrev}} \tag{56}$$

where $(\mathrm{D}W)^{\mathrm{rev}}$ and $(\mathrm{D}W)^{\mathrm{irrev}}$ represent reversible and irreversible work, respectively. Then we have

$$\mathrm{D}U = T\mathrm{D}S + (\mathrm{D}W)^{\mathrm{rev}} \tag{57}$$

$$T\mathrm{D}S = \mathrm{D}Q + (\mathrm{D}W)^{\mathrm{irrev}} \tag{58}$$

where $S$ is the entropy per unit mass.

When body couple and couple stress do not exist in the continuum under consideration, we have

$$(\mathrm{D}W)^{\mathrm{rev}} = \frac{1}{2} \sigma^{ij} \mathrm{D}g_{ij} . \tag{59}$$

Therefore, $U$ in eq 57 is a function of $g_{ij}$, $t$, and S. Because $g_{ij}$ is a tensor component, we may consider $g_{ij}\overset{\circ}{\mathbf{e}}^i\overset{\circ}{\mathbf{e}}^j$ as an independent variable of $U$, where the Gibbs-Einstein expression of eq 59

$$(\mathrm{D}W)^{rev} = \underset{1}{\overset{\circ}{\mathbf{e}}_p}\underset{a}{\mathbf{e}^p} \cdot \underset{ab}{\boldsymbol{\sigma}} \cdot \underset{b}{\mathbf{e}^q}\underset{2}{\overset{\circ}{\mathbf{e}}_q} \cdot \tfrac{1}{2}\mathrm{D}(g_{ij}\underset{1}{\overset{\circ}{\mathbf{e}}^i}\underset{2}{\overset{\circ}{\mathbf{e}}^j}) \tag{60}$$

is considered, in which $\boldsymbol{\sigma}$ is expressed with current base vectors. (The author was encouraged to use the expression on the right-hand side of eq 60 by Mindlin and Tiersten (1962).) $\boldsymbol{\epsilon}$ in eq 22 is integrated to

$$\begin{aligned} \boldsymbol{\epsilon} &= \tfrac{1}{2}(g_{ij} - \overset{\circ}{g}_{ij})\,\overset{\circ}{\mathbf{e}}^i\overset{\circ}{\mathbf{e}}^j \\ &= \tfrac{1}{2}(g_{ij}\overset{\circ}{\mathbf{e}}^i\overset{\circ}{\mathbf{e}}^j - \mathbf{1}) \end{aligned} \tag{61}$$

where $\overset{\circ}{g}_{ij} = \overset{\circ}{\mathbf{e}}_i \cdot \overset{\circ}{\mathbf{e}}_j$ and $\mathbf{1}$ is a constant tensor.

## PART II. CANONICAL FORMS OF GENERAL SECOND-ORDER TENSORS

The Hamilton-Cayley theorem in matrix theory is given the Gibbs-Einstein expression in the following. As shown below, the dual basis expression is more than suitable for discussing the canonical forms of general second-order tensors.

First, to give a summary of this part and to show how the results may be used, the results will be applied to three-dimensional tensors. Canonical forms of not necessarily symmetric real three-dimensional second-order tensors in the Gibbs-Einstein expression are classified into four categories:

**Category 1:** Eigenvalues $\lambda_i (i = 1, 2, 3)$ are all real, and determine three pairs of left and right eigenvectors which are never orthogonal with each other. Then eigenvectors span a dual basis $\mathbf{e}_i$, $\mathbf{e}^i$ satisfying

$$\mathbf{e}_i \cdot \mathbf{T} = \lambda_i \mathbf{e}_i \tag{62}$$

$$\mathbf{T} \cdot \mathbf{e}^i = \lambda_i \mathbf{e}^i \tag{63}$$

where $i = 1, 2, 3$. The summation convention is not applied on the right-hand sides. Juxtaposing $\mathbf{e}^i$ from the left in eq 62 and $\mathbf{e}_i$ from the right in eq 63 with the summation convention applied yields the same expression

$$\mathbf{T} = \sum_{i=0}^{3} \lambda_i \mathbf{e}^i\mathbf{e}_i \tag{64}$$

which is the canonical form of category 1. Note that eigenvectors are not necessarily unit nor orthogonal. When $\mathbf{e}_i = \mathbf{e}^i$ eigenvectors are unit orthogonal and $\mathbf{T}$ is symmetric. Eigenvalues in category 1 may be multiple roots.

**Category 2:** $\lambda_1$ is a double real root which determines an orthogonal pair. An *orthogonal pair* is a pair of left and right eigenvectors determined for an eigenroot and orthogonal with each other. Choose $\mathbf{e}_2$ and $\mathbf{e}^1$ as the left and right eigenvectors determined for $\lambda_1$; then

$$\mathbf{e}_2 \cdot \mathbf{T} = \lambda_1 \mathbf{e}_2 \tag{65}$$

$$\mathbf{T} \cdot \mathbf{e}^1 = \lambda_1 \mathbf{e}^1. \tag{66}$$

Let $\lambda_2$ be the remaining real eigenroot, and $\mathbf{e}_3$ and $\mathbf{e}^3$ be the left and right eigenvectors, respectively; then,

$$\mathbf{e}_3 \cdot \mathbf{T} = \lambda_2 \mathbf{e}_3 \tag{67}$$

$$\mathbf{T} \cdot \mathbf{e}^3 = \lambda_2 \mathbf{e}^3. \tag{68}$$

As proved later, $\mathbf{e}_1$ and $\mathbf{e}^2$ can be chosen to satisfy

$$\mathbf{e}_1 \cdot \mathbf{T} = \lambda(\mathbf{e}_1 + \mathbf{e}_2) \tag{69}$$

and

$$\mathbf{T} \cdot \mathbf{e}^2 = \lambda(\mathbf{e}^1 + \mathbf{e}^2). \tag{70}$$

Juxtaposing $\mathbf{e}^1$, $\mathbf{e}^2$, $\mathbf{e}^3$ from the left in eq 69, 65, 67, respectively, and summing the results, and juxtaposing $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ from the right into eq 66, 70, 68, respectively, and summing the results, yield the same expression,

$$\mathbf{T} = \lambda_1(\mathbf{e}^1\mathbf{e}_1 + \mathbf{e}^2\mathbf{e}_2 + \mathbf{e}^1\mathbf{e}_2) + \lambda_2 \mathbf{e}^3\mathbf{e}_3 \tag{71}$$

which is the canonical form of category 2. The canonical form has one off-diagonal term $\lambda_1\mathbf{e}^1\mathbf{e}_2$. As shown later, choice of base vectors for expressing the canonical form in category 2 is not unique.

**Category 3:** $\lambda$ is a triple real root which determines an orthogonal pair. Chose $\mathbf{e}_3$ and $\mathbf{e}^1$ as the left and right eigenvectors determined for $\lambda$; then

$$\mathbf{e}_3 \cdot \mathbf{T} = \lambda \mathbf{e}_3 \tag{72}$$

$$\mathbf{T} \cdot \mathbf{e}^1 = \lambda \mathbf{e}^1. \tag{73}$$

As proved later, $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}^2$ and $\mathbf{e}^3$ can be chosen to satisfy

$$\mathbf{e}_1 \cdot \mathbf{T} = \lambda(\mathbf{e}_1 + \mathbf{e}_2) \tag{74}$$

$$\mathbf{e}_2 \cdot \mathbf{T} = \lambda(\mathbf{e}_2 + \mathbf{e}_3) \tag{75}$$

$$\mathbf{T} \cdot \mathbf{e}^2 = \lambda(\mathbf{e}^1 + \mathbf{e}^2) \tag{76}$$

and

$$\mathbf{T} \cdot \mathbf{e}^3 = \lambda(\mathbf{e}^2 + \mathbf{e}^3). \tag{77}$$

Juxtaposing $\mathbf{e}^1$, $\mathbf{e}^2$, $\mathbf{e}^3$ from the left in eq 74, 75, 72, respectively, and summing the results, and juxtaposing $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$, from the right in eq 73, 76, and 77, respectively, and summing the results, yield the same expression,

$$\mathbf{T} = \lambda(\mathbf{e}^i\mathbf{e}_i + \mathbf{e}^1\mathbf{e}_2 + \mathbf{e}^2\mathbf{e}_3) \tag{78}$$

which is the canonical form of category 3. The canonical form has two off-diagonal terms, $\lambda\mathbf{e}^1\mathbf{e}_2$ and $\lambda\mathbf{e}^2\mathbf{e}_3$. As shown later, choice of base vectors for expressing canonical forms in category 3 is not unique.

**Category 4:** Two eigenroots are conjugate complex. As shown later, eigenvalues and eigenvectors in this case are expressed as $p(\cos\theta \mp i\sin\theta)$ and $\mathbf{e}_1 \pm i\mathbf{e}_2$, $\mathbf{e}^1 \mp i\mathbf{e}^2$; thus we find

$$(\mathbf{e}_1 \pm i\mathbf{e}_2) \cdot \mathbf{T} = p(\cos\theta \mp i\sin\theta)(\mathbf{e}_1 \pm i\mathbf{e}_2) \tag{79}$$

$$\mathbf{T} \cdot (\mathbf{e}^1 \mp i\mathbf{e}^2) = p(\cos\theta \mp i\sin\theta)(\mathbf{e}^1 \mp i\mathbf{e}^2) \tag{80}$$

where $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}^1$, $\mathbf{e}^2$ are real vectors, and $p$ and $\theta$ are real numbers. Decomposing eq 79 and 80 into the real and imaginary parts yields

$$\mathbf{e}_1 \cdot \mathbf{T} = p(\mathbf{e}_1\cos\theta + \mathbf{e}_2\sin\theta) \tag{81}$$

$$\mathbf{e}_2 \cdot \mathbf{T} = p(\mathbf{e}_2\cos\theta - \mathbf{e}_1\sin\theta) \tag{82}$$

$$\mathbf{T} \cdot \mathbf{e}^1 = p(\mathbf{e}^1\cos\theta - \mathbf{e}^2\sin\theta) \tag{83}$$

$$\mathbf{T} \cdot \mathbf{e}^2 = p(\mathbf{e}^2\cos\theta + \mathbf{e}^1\sin\theta). \tag{84}$$

Let $\mu$ be the remaining real eigenroot and $\mathbf{e}_3$, $\mathbf{e}^3$ be the left and right eigenvectors, respectively; then,

$$\mathbf{e}_3 \cdot \mathbf{T} = \mu\mathbf{e}_3 \tag{85}$$

and

$$\mathbf{T} \cdot \mathbf{e}^3 = \mu\mathbf{e}^3. \tag{86}$$

Juxtaposing $\mathbf{e}^1$, $\mathbf{e}^2$, $\mathbf{e}^3$ from the left in eq 81, 82, 85, respectively, and summing the results, and juxtaposing $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ from the right in eq 83, 84, and 86, respectively, and summing the results, yield the same expression

$$\mathbf{T} = p(\mathbf{e}^1\mathbf{e}_1 + \mathbf{e}^2\mathbf{e}_2)\cos\theta + p(\mathbf{e}^1\mathbf{e}_2 - \mathbf{e}^2\mathbf{e}_1)\sin\theta + \mu\mathbf{e}^3\mathbf{e}_3 \tag{87}$$

which is the real canonical form of category 4.

A general second-order tensor $\mathbf{T}$ can be written as a dyad $\mathbf{T} = \mathbf{a}^i\mathbf{b}_i$ and may be interpreted as a deformation dyad. Interpreted this way, the equations in eq 62 show that elongations in three directions $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$ have occurred. Equation 69 in category 2 shows that a slip has occurred in the $\mathbf{e}_1$, $\mathbf{e}_2$ plane along the $\mathbf{e}_2$ axis. Equations 74 and 75 in category 3 show that a double slip has occurred in the $\mathbf{e}_1$, $\mathbf{e}_2$ plane along the $\mathbf{e}_2$ axis and in the $\mathbf{e}_2$, $\mathbf{e}_3$ plane along the $\mathbf{e}_3$ axis. Equations 81 and 82 in category 4 show that a rotation by angle $\theta$ has occurred with $\mathbf{e}^3$ as the axis of rotation.

**Complex tensors**

In the following sections, general $n$-dimensional second-order tensors are given canonical forms. For that we must first extend theory of real tensors to theory of complex tensors.

A set of unit orthogonal vectors $\mathbf{c}_\alpha = \mathbf{c}^\alpha$ ($\alpha = 1,\ldots,n$) is fixed in the space and used as the standard of the coordinate systems. Vector

$$\mathbf{v} = v^\alpha\mathbf{c}_\alpha \tag{88}$$

is called a complex vector if components $v^\alpha = v_\alpha$ ($\alpha = 1,\ldots,n$) in the standard expression (eq 88) are complex numbers. Conjugate $\bar{\mathbf{v}}$ of $\mathbf{v}$ is defined by

$$\bar{\mathbf{v}} = \bar{v}^\alpha\mathbf{c}_\alpha . \tag{89}$$

Dotting ($\bar{\cdot}$) complex vectors $\mathbf{u}$ with $\mathbf{v}$, denoted by $\mathbf{u} \,\bar{\cdot}\, \mathbf{v}$, is defined by

$$\mathbf{u} \,\bar{\cdot}\, \mathbf{v} = \mathbf{u} \cdot \bar{\mathbf{v}} \tag{90}$$

where dotting ($\cdot$) on the right-hand side is the dotting in real Euclidean geometry applied to complex vectors. Vector $\mathbf{v}$ satisfying $\mathbf{v} \,\bar{\cdot}\, \mathbf{v} = 1$ is said to be of unit length. Vectors $\mathbf{u}$ and $\mathbf{v}$ satisfying $\mathbf{u} \,\bar{\cdot}\, \mathbf{v} = 0$ or $\mathbf{v} \,\bar{\cdot}\, \mathbf{u} = 0$ are said to be orthogonal. A dual basis $\mathbf{e}_i$, $\mathbf{e}^i$ for complex vectors satisfies

$$\mathbf{e}_i \cdot \bar{\mathbf{e}}^j = \delta_i^{\,j} \tag{91}$$

or

$$\mathbf{e}^i \cdot \bar{\mathbf{e}}_j = \delta^i_{\,j} \tag{92}$$

where $\delta_i^{\,j}$ and $\delta^i_{\,j}$ are Kronecker deltas.

The standard expression of an $n$-dimensional complex second-order tensor $\mathbf{T}$ is defined by

$$\mathbf{T} = T_i^{\,j}\bar{\mathbf{e}}^i\mathbf{e}_j \tag{93}$$

or

$$\mathbf{T} = T^i{}_j \tilde{\mathbf{e}}_i \mathbf{e}^j \tag{94}$$

in which the first and second members of the dyads are with and without tilde, respectively, and $i, j = 1,\ldots,n$. The standard form of a complex vector is given by using base vectors without tilde,

$$\mathbf{v} = v^k \mathbf{e}_k \tag{95}$$

or

$$\mathbf{v} = v_k \mathbf{e}^k . \tag{96}$$

Vector $\mathbf{v}$ in eq 95 and 96 can be readily dotted $(\cdot)$ from the left into $\mathbf{T}$ of eq 93 and 94, respectively.

In the following, we will derive equations by which a non-standard expression is transformed to a standard expression. Define $g_{ij}$ and $g^{ij}$ by

$$g_{ij} = \mathbf{e}_i \cdot \tilde{\mathbf{e}}_j \tag{97}$$

and

$$g^{ij} = \mathbf{e}^i \cdot \tilde{\mathbf{e}}^j \tag{98}$$

respectively.

**Lemma 1a.** The unit tensor $\mathbf{1}$ is given by

$$\mathbf{1} = \tilde{\mathbf{e}}^i \mathbf{e}_i = \tilde{\mathbf{e}}_i \mathbf{e}^i = g^{ij} \tilde{\mathbf{e}}_i \mathbf{e}_j = g_{ij} \tilde{\mathbf{e}}^i \mathbf{e}^j \tag{99}$$

and satisfies

$$\mathbf{v} \cdot \mathbf{1} = \mathbf{v} \tag{100}$$

and

$$\mathbf{1} \cdot \tilde{\mathbf{v}} = \tilde{\mathbf{v}}$$

where $\mathbf{v}$ is an arbitrary vector.

**Lemma 1b**

$$\mathbf{e}_i = g_{ij} \mathbf{e}^j \tag{102}$$

$$\mathbf{e}^i = g^{ij} \mathbf{e}_j \tag{103}$$

$$\tilde{\mathbf{e}}_j = \tilde{\mathbf{e}}^i g_{ij} \tag{104}$$

$$\tilde{\mathbf{e}}^j = \tilde{\mathbf{e}}_i g^{ij} \tag{105}$$

*Proof.* The foregoing two lemmas are proved in the following

Dotting $\tilde{\mathbf{e}}^i\mathbf{e}_i$ or $\tilde{\mathbf{e}}_i\mathbf{e}^i$ from the right in vector **v** expressed as $v^j\mathbf{e}_j$ or $v_j\mathbf{e}^j$ proves that **1** for the right multiplication is $\tilde{\mathbf{e}}^i\mathbf{e}_i$ or $\tilde{\mathbf{e}}_i\mathbf{e}^i$, respectively. Dotting $\tilde{\mathbf{e}}^i\mathbf{e}_i$ or $\tilde{\mathbf{e}}_i\mathbf{e}^i$ from the left in vector **v** expressed as $\tilde{v}_j\tilde{\mathbf{e}}^j$ or $\tilde{v}^j\tilde{\mathbf{e}}_j$ proves that **1** for the left multiplication is also $\tilde{\mathbf{e}}^i\mathbf{e}_i$ or $\tilde{\mathbf{e}}_i\mathbf{e}^i$. The proof that $\mathbf{e}^i\mathbf{e}_i = \mathbf{e}_i\mathbf{e}^i$ follows.

Juxtaposing $\mathbf{e}^j$ from the right in eq 97 yields eq 102. Equations 103, 104 and 105 are derived similarly. Substituting eq 105 into $\tilde{\mathbf{e}}^j\mathbf{e}_j$ yields

$$\tilde{\mathbf{e}}^j\mathbf{e}_j = g^{ij}\tilde{\mathbf{e}}_i\mathbf{e}_j. \tag{a}$$

Substituting eq 103 into $\tilde{\mathbf{e}}_i\mathbf{e}^i$ yields

$$\tilde{\mathbf{e}}_i\tilde{\mathbf{e}}^i = g^{ij}\tilde{\mathbf{e}}_i\mathbf{e}_j. \tag{b}$$

Comparing eq a and b proves a part of eq 99. The rest of eq 99 is proved similarly. The proof is thus completed.

Define the transformation of base vectors by

$$\mathbf{e}_i = a_i{}^\alpha\mathbf{c}_\alpha \tag{106}$$

and

$$\mathbf{e}^i = \mathbf{c}^\alpha b_\alpha{}^i \tag{107}$$

where $a_\alpha{}^i$ and $b_\alpha{}^i$ are complex numbers satisfying

$$a_i{}^\alpha\tilde{b}_\alpha{}^j = \delta_i{}^j. \tag{108}$$

Equations 106 and 107 are transformed in the following to their inverses.

**Lemma 1c**

$$\mathbf{c}^\alpha = \tilde{\mathbf{e}}^i a_i{}^\alpha \tag{109}$$

and

$$\mathbf{c}_\alpha = b_\alpha{}^j\tilde{\mathbf{e}}_j. \tag{110}$$

*Proof.* Dotting $\mathbf{c}^\beta$ into eq 106 yields

$$a_i^{\beta} = \mathbf{e}_i \cdot \mathbf{c}^\beta . \tag{a}$$

Juxtaposing $\bar{\mathbf{e}}^i$ from the left of eq a yields eq 109. Equation 107 transforms to eq 110 similarly.

**Theorem 1**

$$\mathbf{e}_i = a_i^{\alpha} b_\alpha^{\ j} \bar{\mathbf{c}}_j \tag{111}$$

$$\mathbf{a}^i = \bar{\mathbf{e}}^j a_j^{\alpha} b_\alpha^{\ i} . \tag{112}$$

*Proof.* Substituting eq 110 or 109 into eq 106 or 107 yields eq 111 or 112, respectively.

**Corollary 1**

$$\bar{\mathbf{e}}_i = \bar{a}_i^{\alpha} \bar{b}_\alpha^{\ j} \mathbf{e}_j \tag{113}$$

$$\bar{\mathbf{e}}_i = \mathbf{e}^j \bar{a}_j^{\alpha} b_\alpha^{\ i} . \tag{114}$$

*Proof.* Taking the conjugates of eq 111 or 112 proves eq 113 or 114, respectively. The proof is thus completed.

Equations 111 through 114 are the equations that must be used to transform non-standard expressions to standard expressions.

**Eigenvectors and eigenvalues**

Let $\mathbf{T}$ be an $n$-dimensional tensor in the standard form. A left or right eigenvector of $\mathbf{T}$ is an $n$-dimensional complex vector $\mathbf{x}$ or $\mathbf{y}$ such that dotting $(\cdot)$ $\mathbf{x}$ or $\bar{\mathbf{y}}$ from the left or right in $\mathbf{T}$ yields a vector in the direction of $\mathbf{x}$ or $\bar{\mathbf{y}}$, respectively; that is

$$\mathbf{x} \cdot \mathbf{T} = \lambda \mathbf{x} \tag{115}$$

or

$$\mathbf{T} \cdot \bar{\mathbf{y}} = \lambda \bar{\mathbf{y}} \tag{116}$$

where $\lambda$ is an eigenvalue.

Let $\mathbf{T}$ and a left eigenvector $\mathbf{x}$ be expressed as

$$\mathbf{T} = T_i^{\ j} \bar{\mathbf{e}}^i \bar{\mathbf{e}}_j \tag{117}$$

and

$$\mathbf{x} = x^h \mathbf{e}_h \tag{118}$$

respectively. Substituting eq 117 and 118 into eq 115 and equating the components on both sides of the transformed equation yields

$$x^i T_i^{\,j} = \lambda x^j . \tag{119}$$

Let a right eigenvector $\mathbf{y}$ be expressed as

$$\mathbf{y} = y_h \mathbf{e}^h. \tag{120}$$

Substituting eq 117 and 120 into eq 113 yields

$$T_i^{\,j} \tilde{y}_j = \lambda \tilde{y}_i . \tag{121}$$

Equations 119 and 121 determine the same characteristic equation

$$| T_i^{\,j} - \lambda \delta_i^{\,j} | = 0. \tag{122}$$

Therefore, one $\lambda$ determines at least one left and one right eigenvector. The rank of matrix $(T_i^{\,j} - \lambda\delta_i^{\,j})$ will be called the rank of root $\lambda$. Solutions of eq 115 and 116 do not depend on the choice of a type of $\mathbf{T}$ or of a dual basis.

**Canonical forms**

**Lemma 2a**

Let $\lambda$ and $\mu$ be two different eigenvectors of $\mathbf{T}$; let $\mathbf{x}$ and $\mathbf{y}$ be left and right eigenvectors determined for $\lambda$, respectively; and let $\mathbf{u}$ and $\mathbf{v}$ be left and right eigenvectors determined for $\mu$, respectively. Then $\mathbf{x}$ and $\mathbf{v}$ are orthogonal.

$$\mathbf{x} \cdot \tilde{\mathbf{v}} = 0 \tag{123}$$

and $\mathbf{y}$ and $\mathbf{u}$ are orthogonal,

$$\mathbf{y} \cdot \tilde{\mathbf{u}} = 0 . \tag{124}$$

*Proof.* Dotting $(\cdot)$ $\tilde{\mathbf{v}}$ from the right in eq 115 yields

$$\mathbf{x} \cdot \mathbf{T} \cdot \tilde{\mathbf{v}} = \lambda \mathbf{x} \cdot \tilde{\mathbf{v}}. \tag{a}$$

Dotting $(\cdot)$ from the left in the equation

$$\mathbf{T} \cdot \tilde{\mathbf{v}} = \mu \tilde{\mathbf{v}}$$

yields

$$\mathbf{x} \cdot \mathbf{T} \cdot \tilde{\mathbf{v}} = \mu \mathbf{x} \cdot \tilde{\mathbf{v}}. \tag{b}$$

Because $\lambda \neq \mu$ by assumption, eq a and b are compatible if and only if eq 123 is true. Equation 124 may be proved similarly.

**Lemma 2b**

If all the roots are single, no orthogonal pair exists.

The next lemma proves this lemma. Note that vectors $\mathbf{e}_i$ and $\mathbf{e}^i$ forming a dual basis are not orthogonal.

**Lemma 2c**

Assume that $n$ roots $\lambda_1,\ldots,\lambda_n$ of the characteristic eq 122 are distinct. Then, $\mathbf{T}$ can be reduced to the canonical form

$$\mathbf{T} = \sum_{i=1}^{n} \lambda_i \mathbf{e}^i \tilde{\mathbf{e}}_i . \tag{125}$$

The rank of $\lambda_i (i = 1,\ldots,n)$ is equal to $n - 1$.

*Proof.* Each $\lambda_i$ determines at least one left eigenvector. Let $\mathbf{e}_i$ be the left eigenvector determined for $\lambda_i (i = 1,\ldots,n)$

$$\mathbf{e}_1 \cdot \mathbf{T} = \lambda_1 \mathbf{e}_1 \tag{a}$$

..................

$$\mathbf{e}_n \cdot \mathbf{T} = \lambda_n \mathbf{e}_n . \tag{b}$$

Form a dual basis $\mathbf{e}_i$, $\mathbf{e}^i$ $(i = 1,\ldots,n)$. Juxtaposing $\tilde{\mathbf{e}}^1,\ldots,\tilde{\mathbf{e}}^n$ from the left in eq a,..., eq b, respectively, and summing the result yield eq 125. Dotting $(\cdot)$ $\tilde{\mathbf{e}}^i$ from the right in eq 125 shows that $\mathbf{e}^i$ is a right eigenvector determined for $\lambda_i (i = 1,\ldots,n)$.

$\mathbf{T}$ in eq 125 shows that each $\lambda_i$ determines one and only one pair of left and right eigenvectors. The rank of the roots is therefore all equal to $n - 1$. The proof is thus completed.

A multiple root which determines fewer eigenvectors than the number of multiplicity of the root is said to be *singular.*

**Lemma 2d**

At least one orthogonal pair belongs to a singular multiple root. If more than one orthogonal pair belongs to a singular multiple root, the number of orthogonal pairs can be reduced to one.

*Proof.* Assume that eigenvectors $\mathbf{x}$ and $\mathbf{y}$ determined for a multiple root are not orthogonal. Then, we can choose $\mathbf{e}_1 = \mathbf{x}$ and $\mathbf{e}^1 = \mathbf{y}$ after, if necessary, changing the lengths of $\mathbf{x}$ and $\mathbf{y}$ to make $\mathbf{x} \cdot \mathbf{y} = 1$. Then we have

$$\mathbf{e}_1 \cdot \mathbf{T} = \lambda \mathbf{e}_1 \tag{a}$$

and

$$\mathbf{T} \cdot \tilde{\mathbf{e}}^1 = \lambda \tilde{\mathbf{e}}^1 . \tag{b}$$

Form a dual basis $\mathbf{e}_i$, $\mathbf{e}^i$ $(i = 1,\ldots,n)$ by introducing a certain number of base vectors that are independent of $\mathbf{e}_1$, $\mathbf{e}^1$ and each other; express $\mathbf{T}$ with the dual basis $\mathbf{e}_i$, $\mathbf{e}^i$

$$\mathbf{T} = T_i{}^j \tilde{\mathbf{e}}^i \mathbf{e}_j \,. \tag{c}$$

Substituting eq c into eq a and b yields

$$T_1{}^1 = \lambda$$

$$T_1{}^2 = \dots = T_1{}^n = 0$$

$$T_2{}^1 = \dots = T_n{}^1 = 0.$$

Therefore, $\mathbf{T}$ in eq c becomes

$$\mathbf{T} = \lambda \tilde{\mathbf{e}}^1 \mathbf{e}_1 + \mathbf{S} \tag{d}$$

where

$$\mathbf{S} = T_p{}^q \tilde{\mathbf{e}}^p \mathbf{e}_q \tag{e}$$

in which $p$ and $q$ represent $2,\dots,n$. $\mathbf{S}$ is an $(n-1)$-dimensional tensor such that one of the eigenvalues is $\lambda$. Therefore, if the number of non-orthogonal pairs is equal to the number of multiplicity, eigenvectors determined for the multiple root can form a dual basis, which shows that at least one orthogonal pair must belong to a singular multiple root.

If more than one orthogonal pair belongs to $\lambda$, choose two pairs $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{u}$, $\mathbf{v}$. Form two non-orthogonal pairs by setting

$$\mathbf{e}_1 = \mathbf{x}, \qquad \mathbf{e}^1 = \mathbf{v}$$

$$\mathbf{e}_2 = \mathbf{u}, \qquad \mathbf{e}^2 = \mathbf{y}\,.$$

Then, $\mathbf{T}$ can be reduced to

$$\mathbf{T} = \lambda \tilde{\mathbf{e}}^1 \mathbf{e}_1 + \lambda \tilde{\mathbf{e}}^2 \mathbf{e}_2 + \mathbf{R} \tag{f}$$

where

$$\mathbf{R} = T_r{}^s \mathbf{e}^r \mathbf{e}_s \tag{g}$$

in which $r$ and $s$ represent $3,\dots,n$. Continuing this process the number of orthogonal pairs belonging to $\lambda$ can be reduced by an even number, but cannot be reduced to less than one. The proof is thus completed.

**Theorem 2**

Assume that $\lambda$ is a $(t+r)$-tuple root of rank $n-r-1$ of an $n$-dimensional tensor $\mathbf{T}$. Then, $\mathbf{T}$ can be reduced to

$$\mathbf{T} = \lambda(\tilde{\mathbf{e}}^1\mathbf{e}_1 + \ldots + \tilde{\mathbf{e}}^{t+r}\mathbf{e}_{t+r}) + \tilde{\mathbf{e}}^2\mathbf{e}_1 + \tilde{\mathbf{e}}^3\mathbf{e}_2 + \ldots + \tilde{\mathbf{e}}^t\mathbf{e}_{t-1} + \mathbf{S} \tag{126}$$

where $\mathbf{S}$ does not have $\lambda$ as an eigenvalue and is $(n - t - r)$-dimensional. Then, the left eigenvectors determined for $\lambda$ are $\mathbf{e}_1, \mathbf{e}_{t+1},\ldots,\mathbf{e}_{t+r}$. The right eigenvectors determined for $\lambda$ are $\mathbf{e}^t, \mathbf{e}^{t+1},\ldots,\mathbf{e}^{t+r}$. Vectors $\mathbf{e}_2,\ldots,\mathbf{e}_t$ satisfy

$$\mathbf{e}_\phi \cdot \mathbf{T} = \lambda\mathbf{e}_\phi + \mathbf{e}_{\phi-1} \tag{127}$$

where $\phi$ represents $t - 1$ integers

$$2 \leqq \phi \leqq t\,.$$

Vectors $\mathbf{e}^1,\ldots,\mathbf{e}^{t-1}$ satisfy

$$\mathbf{T} \cdot \mathbf{e}^\psi = \lambda\mathbf{e}^\psi + \mathbf{e}^{\psi+1} \tag{128}$$

where $\psi$ represents $t - 1$ integers

$$1 \leqq \psi \leqq t - 1\,.$$

The number of off-diagonal terms is $t - 1$.

*Proof.* Because the rank of $\lambda$ is $n - r - 1$, $(r + 1)$ pairs of independent left and right eigenvectors exist, of which, if $t \leqq 2$, one is an orthogonal pair. Denote by $\mathbf{e}_1$, $\mathbf{e}^t$ the orthogonal pair, by $\mathbf{e}_1, \mathbf{e}_{t+1},\ldots,\mathbf{e}_{t+r}$ the left eigenvectors, and by $\mathbf{e}^t, \mathbf{e}^{t+1},\ldots,\mathbf{e}^{t+r}$ the right eigenvectors. Then

$$\mathbf{e}_1 \cdot \mathbf{T} = \lambda\mathbf{e}_1 \tag{a}$$

$$\mathbf{e}_{t+1} \cdot \mathbf{T} = \lambda\mathbf{e}_{t+1} \tag{b}$$

$$\vdots$$

$$\mathbf{e}_{t+r} \cdot \mathbf{T} = \lambda\mathbf{e}_{t+r} \tag{c}$$

$$\mathbf{T} \cdot \tilde{\mathbf{e}}^t = \lambda\tilde{\mathbf{e}}^t \tag{d}$$

$$\mathbf{T} \cdot \tilde{\mathbf{e}}^{t+1} = \lambda\tilde{\mathbf{e}}^{t+1} \tag{e}$$

$$\vdots$$

$$\mathbf{T} \cdot \tilde{\mathbf{e}}^{t+r} = \lambda\tilde{\mathbf{e}}^{t+r}. \tag{f}$$

Form a dual basis $\mathbf{e}_i, \mathbf{e}^i$ $(i = 1,\ldots,n)$ by arbitrarily introducing independent vectors $\mathbf{e}_2,\ldots,\mathbf{e}_t$, $\mathbf{e}_{t+r+1},\ldots,\mathbf{e}_n$, $\mathbf{e}^1,\ldots,\mathbf{e}^{t-1}$, $\mathbf{e}^{t+r\ 1},\ldots,\mathbf{e}^n$. Express $\mathbf{T}$ as

$$\mathbf{T} = T_i^{\,j}\,\tilde{\mathbf{e}}^i\,\mathbf{e}_j \tag{g}$$

Substituting eq g into eq a through f yields

$$T_1^{\;j} = \lambda\delta_1^{\;j} \tag{a'}$$

$$T_{t+1}^{\;j} = \lambda\delta_{t+1}^{\;j} \tag{b'}$$

$$\vdots$$

$$T_{t+r}^{\;j} = \lambda\delta_{t+r}^{\;r} \tag{c'}$$

$$T_i^{\;t} = \lambda\delta_i^{\;t} \tag{d'}$$

$$T_i^{\;t+1} = \lambda\delta_i^{\;t+1} \tag{e'}$$

$$\vdots$$

$$T_i^{\;t+r} = \lambda\delta_i^{\;t+r} . \tag{f'}$$

The matrix of $\mathbf{T} - \lambda\mathbf{1}$ is shown in Figure 1, in which all the elements on the straight lines are zero, submatrix $A$ is composed of 2nd,..., $t$th row and 1st,..., $(t-1)$th column, and the main diagonal of $A$ is not on the main diagonal of the matrix of $\mathbf{T} - \lambda\mathbf{1}$.

Define $\mathbf{x}$ for a given vector $\mathbf{a}$ by

$$\mathbf{x} \cdot \mathbf{T} = \lambda\mathbf{x} + \mathbf{a}. \tag{h}$$

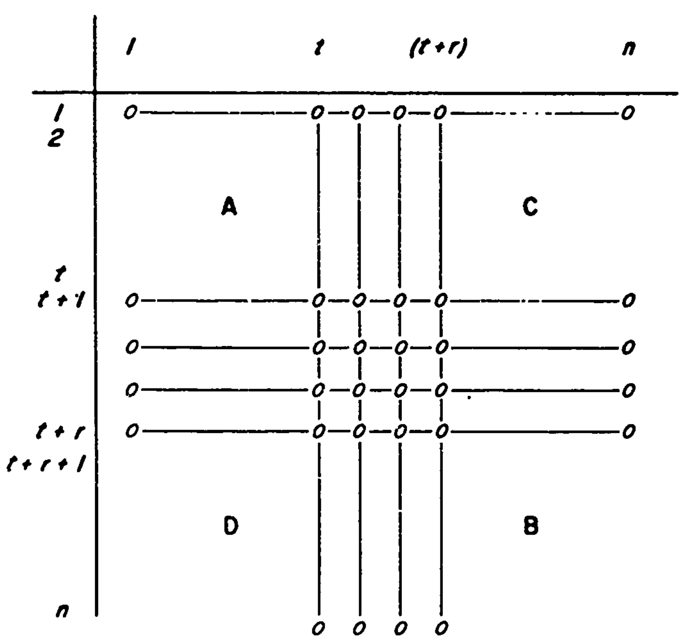


*Figure 1. Matrix of* $\mathbf{T} - \lambda\mathbf{1}$.

Equation h becomes

$$x^i(T_i^{\,j} - \lambda\delta_i^{\,j}) = a^j \tag{i}$$

by substituting eq g for $\mathbf{T}$, eq 118 for $\mathbf{x}$, and $\mathbf{a} = a^j\mathbf{e}_j$ .

Because of conditions in eq d' through f', equations in eq i are simultaneous $n - r - 1$ equations given for $j = 1,\ldots, t-1,\ t{+}r{+}1,\ldots, n$. Vector $\mathbf{a}$ must satisfy a condition that components $a^t,\ldots, a^{t+r}$ are zero. Because of conditions in eq a' through c', equations in eq i have $n - r - 1$ unknowns $x^2,\ldots, x^t,\ x^{t+r+1},\ldots, x^n$ . The determinant of the simultaneous equations (eq i) is of rank $n - r - 1$, and is the only non-zero $(n - r - 1)$-dimensional submatrix containing $A$. Then $\mathbf{x}$ is determined with $(r + 1)$ arbitrary components $x^1,\ x^{t+1},\ldots, x^{t+r}$.

Put $\mathbf{a} = \mathbf{e}_1$ and define $\mathbf{x}_2$ by

$$\mathbf{x}_2 \quad \mathbf{T} = \lambda\mathbf{x}_2 + \mathbf{e}_1 . \tag{j}$$

$\mathbf{x}_2$ is linearly independent of $\mathbf{e}_1$. To show this, define

$$\mathbf{y} = y^1\mathbf{e}_1 + y^2\mathbf{x}_2 . \tag{k}$$

We find

$$\mathbf{y} \cdot \mathbf{T} = \lambda\mathbf{y} + y^2\mathbf{e}_1 \tag{l}$$

by use of eq a and j. Therefore, if $\mathbf{y} = 0$, we necessarily have $y^1 = y^2 = 0$. Let $\mathbf{x}_2$ be chosen as new $\mathbf{e}_2$. Then eq j becomes one of eq 127.

Put $\mathbf{a} = \mathbf{e}_2$ and define $\mathbf{x}_3$ by

$$\mathbf{x}_3 \cdot \mathbf{T} = \lambda\mathbf{x}_3 + \mathbf{e}_2 . \tag{m}$$

$\mathbf{x}_3$ is linearly independent of $\mathbf{e}_2$ and $\mathbf{e}_1$, as may be shown similarly to the above. Let $\mathbf{x}_3$ be chosen as new $\mathbf{e}_3$. Then eq m becomes one of eq 127.

Continuing this process, vectors $\mathbf{e}_4,\ldots, \mathbf{e}_t$ are defined and eq 127 is proved. All the vectors represented by $\mathbf{e}_{\phi-1}$ in eq 127 satisfy the condition which must be satisfied by vector $\mathbf{a}$ in eq h.

Suppose that eq g is expressed with the new base vectors thus introduced. Substituting eq b thus determined into eq 127 shows that matrix $A$ in Figure 1 is a unit matrix and that submatrix C in Figure 1 is a zero matrix, yielding eq 126.

Dotting $(\cdot)\ \tilde{\mathbf{e}}^{\psi}$ from the right into $\mathbf{T}$ in eq 126 yields eq 128.

Substituting eq g into eq 128 shows that submatrix $D$ in Figure 1 is a zero matrix. The proof is thus completed.

Integer $t$ is called the grade of $\lambda$ (Turnbull and Aitken, 1932). Vectors $\mathbf{e}_2,\ldots,\mathbf{e}_t$ are called left *pseudo-eigenvectors* determined for $\lambda$. Vectors $\mathbf{e}^1,\ldots,\mathbf{e}^{t-1}$ are called right *pseudo-eigenvectors* determined for $\lambda$. $\mathbf{T}-\mathbf{S}$ in eq 126 is called the canonical part belonging to $\lambda$.

**Corollary 2a**

The canonical part belonging to $\lambda$ does not change form when dual basis $\mathbf{e}_i$, $\mathbf{e}^i$ is transformed to another dual basis defined by

$$\mathbf{f}_1 = p^1\mathbf{e}_1 \tag{129}$$

$$\mathbf{f}_2 = p^2\mathbf{e}_1 + p^1\mathbf{e}_2 \tag{130}$$

$$\vdots$$

$$\mathbf{f}_{t-1} = p^{t-1}\mathbf{e}_1 + p^{t-2}\mathbf{e}_2 + \ldots + p^1\mathbf{e}_{t-1} \tag{131}$$

$$\mathbf{f}_t = \sum_{\xi=1}^{t} p^{t+1-\xi}\mathbf{e}_\xi + \sum_{\zeta=t+1}^{t+r} f_t^{\zeta}\mathbf{e}_\zeta \tag{132}$$

$$\mathbf{f}_{t+1} = \mathbf{f}_{t+1}^1\,\mathbf{e}_1 + \sum_{\xi=t+1}^{t+r} \mathbf{f}_{t+1}{}^{\zeta}\mathbf{e}_\zeta \tag{133}$$

$$\mathbf{f}_{t+r} = \mathbf{f}_{t+r}^1\,\mathbf{e}_1 + \sum_{\xi=t+1}^{t+r} \mathbf{f}_{t+r}{}^{\zeta}\mathbf{e}_\zeta \tag{134}$$

where $p^1,\ldots,p^t$, $f_t{}^{\zeta}$, $f_\omega{}^1$, $f_\omega{}^{\zeta}$ $(\omega, \zeta = t+1,\ldots,t+r)$ are arbitrary if they make $\mathbf{f}_1,\ldots,\mathbf{f}_{t+r}$ linearly independent.

*Proof.* If vectors $\mathbf{f}_1,\ldots,\mathbf{f}_{t+r}$ satisfy

$$\mathbf{f}_1 \cdot \mathbf{T} = \lambda\mathbf{f}_1 \tag{a}$$

$$\mathbf{f}_\phi \cdot \mathbf{T} = \lambda\mathbf{f}_\phi + \mathbf{f}_{\phi-1}, \quad (2 \leqq \phi \leqq t) \tag{b}$$

$$\mathbf{f}_\zeta \cdot \mathbf{T} = \lambda\mathbf{f}_\zeta, \quad (t+1 \leqq \zeta \leqq t+r) \tag{c}$$

the canonical part determined by $\mathbf{f}_1,\ldots,\mathbf{f}_{t+r}$ is the same as that determined by $\mathbf{e}_1,\ldots,\mathbf{e}_{t+r}$.

Let index $a$ represent integers $1,\ldots,t+r$. Substituting

$$\mathbf{f}_1 = f_1{}^a\,\mathbf{e}_a$$

and $\mathbf{T}$ in eq 126 into eq a shows that vector $\mathbf{f}_1$ must be such as in eq 129.

Substituting

$$\mathbf{f}_\phi = f_\phi{}^a \mathbf{e}_a$$

and $\mathbf{T}$ in eq 126 into eq b yields the conditions,

$$f_\phi{}^2 = f_{\phi-1}{}^1$$
$$\vdots$$
$$f_\phi{}^t = f_{\phi-1}{}^{t-1}$$
$$f_{\phi-1}{}^t = \dots = f_{\phi-1}{}^{t+r} = 0$$

which show that vectors $\mathbf{f}_2,\dots,\mathbf{f}_t$ must be such as in eq 130,...,eq 132, respectively. Substituting

$$\mathbf{f}_\zeta = f_\zeta{}^a \mathbf{e}_a$$

and $\mathbf{T}$ in eq 126 into eq c shows that vectors $\mathbf{f}_{t+1},\dots,\mathbf{f}_{t+1}$ must be such as in eq 133,...,eq 134. The proof is thus completed.

**Corollary 2b**

Left and right eigenvectors determined for a single root are not orthogonal.

*Proof.* Denote by $\mathbf{S}$ the sum of all the canonical parts belonging to the multiple roots. Then, all the eigenvalues of $\mathbf{T} - \mathbf{S}$ are single, and, as proved by Lemma 2b, no orthogonal pair exists for $\mathbf{T} - \mathbf{S}$.

**Hamilton-Cayley Theorem**

**Lemma 3**

Let $\mathbf{e}_1,\dots,\mathbf{e}_n$ be $n$ independent vectors and $\mathbf{S}$ be an $n$-dimensional complex tensor of any order. If the relation

$$\mathbf{e}_i \cdot \mathbf{S} = 0 \tag{135}$$

is true for all $\mathbf{e}_i (i = 1,\dots,n)$, then $\mathbf{S}$ is identically equal to zero.

*Proof.* Form a dual basis $\mathbf{e}_j$, $\mathbf{e}^i$. Juxtaposing $\mathbf{e}^i$ from the left into eq 135 and summing over $i$ yields the required property of $\mathbf{S}$. The proof is thus completed.

Let $\mathbf{T}$ be a complex second-order tensor in a standard form

$$\mathbf{T} = \mathbf{e}^i T_i{}^j \mathbf{e}_j \, . \tag{136}$$

Then, $m$ time scalar product of $\mathbf{T}$

$$\mathbf{T} \cdot \dots \cdot \mathbf{T} = \mathbf{T}^m \tag{137}$$

abbreviated to $\mathbf{T}^m$, is a second-order tensor

$$\mathbf{T}^m = \bar{\mathbf{e}}^i T_i^{j_1} T_{j_1}^{j_2} \ldots T_{j_{m-1}}^{k} \mathbf{e}_k \tag{138}$$

where $m$ is any positive integer.

**Theorem 3**

Let $\lambda_\nu$ $(\nu = 1,\ldots,p)$ be the roots of the characteristic equation of $\mathbf{T}$. Assume that $\lambda_\nu$ is $(t_\nu + r_\nu)$-tuple root of rank $n - r_\nu - 1$, where

$$\sum_{\nu=1}^{p} (t_\nu + r_\nu) = n . \tag{139}$$

Then a second-order tensor

$$f(\mathbf{T}) = \prod_{\nu=1}^{p} (\mathbf{T} - \lambda_\nu \mathbf{1})^{t_\nu} \tag{140}$$

is identically equal to zero.

*Proof.* Let the collection of eigenvectors and pseudo-eigenvectors belonging to all the eigenvalues $\lambda_1,\ldots,\lambda_p$ form a dual basis $\mathbf{e}_i$, $\mathbf{e}^i$

Equations for $\lambda_1$ are

$$\mathbf{e}_1 \cdot (\mathbf{T} - \lambda_1 \mathbf{1}) = 0 \tag{a}$$

$$\mathbf{e}_2 \cdot (\mathbf{T} - \lambda_1 \mathbf{1}) = \mathbf{e}_1 \tag{b}$$

$$\vdots$$

$$\mathbf{e}_{t_1} \cdot (\mathbf{T} - \lambda_1 \mathbf{1}) = \mathbf{e}_{t_1 - 1} \tag{c}$$

$$\mathbf{e}_{t_1+1} \cdot (\mathbf{T} - \lambda_1 \mathbf{1}) = 0 \tag{d}$$

$$\vdots$$

$$\mathbf{e}_{t_1+r_1} \cdot (\mathbf{T} - \lambda_1 \mathbf{1}) = 0. \tag{e}$$

Equation b through c can be changed to

$$\mathbf{e}_2 \cdot (\mathbf{T} - \lambda_1 \mathbf{1})^2 = 0 \tag{f}$$

$$\vdots$$

$$\mathbf{e}_{t_1} \cdot (\mathbf{T} - \lambda \mathbf{1})^{t_1} = 0 . \tag{g}$$

Therefore, we find

$$\mathbf{e}_{\xi_1} \cdot (\mathbf{T} - \lambda\ \mathbf{1})^{t_1} = 0 \qquad \text{(h)}$$

where

$$\xi_1 = 1,\ldots,t_1 + r_1 .$$

Similarly, we find

$$\mathbf{e}_{\xi_\nu} \cdot (\mathbf{T} - \lambda\ \mathbf{1})^{t_\nu} = 0 \qquad \text{(i)}$$

where

$$\xi_\nu = \sum_{\mu=1}^{\nu-1} (t_\mu + r_\mu) + 1,\ldots, \sum_{\mu=1}^{\nu} (t_\mu + r_\mu) .$$

Form $f(\mathbf{T})$ in eq 132. The order of dotting in eq 132 may be exchanged, because the results of dotting in different orders are the same. We therefore find

$$\mathbf{e}_i \cdot f(\mathbf{T}) = 0 \qquad \text{(j)}$$

where $i = 1,\ldots,n$. Then, $f(\mathbf{T})$ must be identically equal to zero. The proof is thus completed.

Polynomial $f(x)$ obtained by substituting scalar $x$ for $\mathbf{T}$ is the minimal polynomial defined for $\mathbf{T}$.

**Real canonical forms of real tensors**

A tensor is said to be real if there exists a transformation that changes all the components to real numbers and all the base vectors to real vectors at the same time. In the following, $\mathbf{T}$ is a second-order tensor that has real components and real base vectors.

It is obvious that a real eigenvalue of $\mathbf{T}$ determines real eigenvectors, and that conjugate complex eigenvalues of $\mathbf{T}$ determine conjugate complex eigenvectors. Only the latter case need be discussed.

**Theorem 4**

Assume that $\alpha \pm i\beta$ are $(t + r)$-tuple roots of rank $n - r - 1$ of a real second-order tensor $\mathbf{T}$, where

$$2(t + r) \leq n .$$

Then, $\mathbf{T}$ can be reduced to

$$\mathbf{T} = \alpha \sum_{p=1}^{2(t+r)} \mathfrak{f}^p \mathfrak{f}_p + \beta \sum_{q=1}^{t+r} (\mathfrak{f}^{2q} \mathfrak{f}_{2q-1} - \mathfrak{f}^{2q-1} \mathfrak{f}_{2q}) +$$

$$+ \sum_{\phi=1}^{t} (\mathfrak{f}^{2\phi} \mathfrak{f}_{2\phi-2} + \mathfrak{f}^{2\phi-1} \mathfrak{f}_{2\phi-3}) + \mathbf{S} \tag{141}$$

where $\mathfrak{f}_p$, $\mathfrak{f}^p\,[1 \leqq p \leqq 2(t+r)]$ are real vectors, forming a dual basis, and $\mathbf{S}$ does not have $\alpha \pm i\beta$ as eigenvalues and is $2(n - t - r)$-dimensional.

Vectors $\mathfrak{f}_p$, $\mathfrak{f}^p$ $(p = 1, \ldots, 2(t+r))$ satisfy

$$\mathfrak{f}_1 \cdot \mathbf{T} = \alpha \mathfrak{f}_1 - \beta \mathfrak{f}_2 \tag{142}$$

$$\mathfrak{f}_2 \cdot \mathbf{T} = \alpha \mathfrak{f}_2 + \beta \mathfrak{f}_1 \tag{143}$$

$$\vdots$$

$$\mathfrak{f}_{2\phi-1} \cdot \mathbf{T} = \alpha \mathfrak{f}_{2\phi-1} - \beta \mathfrak{f}_{2\phi} + \mathfrak{f}_{2\phi-3} \tag{144}$$

$$\mathfrak{f}_{2\phi} \cdot \mathbf{T} = \alpha \mathfrak{f}_{2\phi} + \beta \mathfrak{f}_{2\phi-1} + \mathfrak{f}_{2\phi-2} \qquad (2 \leqq \phi \leqq t) \tag{145}$$

$$\vdots$$

$$\mathfrak{f}_{2\xi-1} \cdot \mathbf{T} = \alpha \mathfrak{f}_{2\xi-1} - \beta \mathfrak{f}_{2\xi} \tag{146}$$

$$\mathfrak{f}_{2\xi} \cdot \mathbf{T} = \alpha \mathfrak{f}_{2\xi} + \beta \mathfrak{f}_{2\xi-1} \qquad (t+1 \leqq \xi \leqq t+r) \tag{147}$$

$$\mathbf{T} \cdot \mathfrak{f}^{2\gamma-1} = \alpha \mathfrak{f}^{2\gamma-1} + \beta \mathfrak{f}^{2\gamma} + \mathfrak{f}^{2\gamma+1} \tag{148}$$

$$\mathbf{T} \cdot \mathfrak{f}^{2\gamma} = \alpha \mathfrak{f}^{2\gamma} - \beta \mathfrak{f}^{2\gamma-1} + \mathfrak{f}^{2\gamma+2} \qquad (1 \leqq \gamma \leqq t-1) \tag{149}$$

$$\vdots$$

$$\mathbf{T} \cdot \mathfrak{f}^{2\eta-1} = \alpha \mathfrak{f}^{2\eta-1} + \beta \mathfrak{f}^{2\eta} \tag{150}$$

$$\mathbf{T} \cdot \mathfrak{f}^{2\eta} = \alpha \mathfrak{f}^{2\eta} - \beta \mathfrak{f}^{2\eta-1} \qquad (t \leqq \eta \leqq t+r)\,. \tag{151}$$

Complex vectors $\frac{1}{\sqrt{2}}(\mathfrak{f}_{2q-1} \pm i\mathfrak{f}_{2q})$, $\frac{1}{\sqrt{2}}(\mathfrak{f}^{2q-1} \mp i\mathfrak{f}^{2q})$ $(q = 1, \ldots, t+r)$ form a dual basis.

*Proof.* Let $\mathbf{e}_1, \ldots, \mathbf{e}_{t+r}$ be complex eigenvectors and pseudo-eigenvectors belonging to $\alpha + i\beta$

$$\mathbf{e}_1 \cdot \mathbf{T} = (\alpha + i\beta)\mathbf{e}_1 \tag{a}$$

$$\mathbf{e}_2 \cdot \mathbf{T} = (\alpha + i\beta)\mathbf{e}_1 \qquad \text{(b)}$$

$$\vdots$$

$$\mathbf{e}_t \cdot \mathbf{T} = (\alpha + i\beta)\mathbf{e}_t + \mathbf{e}_{t-1} \qquad \text{(c)}$$

$$\mathbf{e}_{t+1} \cdot \mathbf{T} = (\alpha + i\beta)\mathbf{e}_{t+1} \qquad \text{(d)}$$

$$\vdots$$

$$\mathbf{e}_{t+r} \cdot \mathbf{T} = (\alpha + i\beta)\mathbf{e}_{t+r} \, . \qquad \text{(e)}$$

Write $\mathbf{e}_q (1 \leqq q \leqq t+r)$ with real vectors $\mathbf{f}_{2q-1}, \mathbf{f}_{2q}$ as

$$\mathbf{e}_q = \frac{1}{\sqrt{2}} (\mathbf{f}_{2q-1} + i\mathbf{f}_{2q}). \qquad \text{(f)}$$

Substituting eq f into eq a through e yields eq 142 through 147.

Determine $\mathbf{f}^p [1 \leqq p \leqq 2(t+r)]$ to form a dual basis $\mathbf{f}_p, \mathbf{f}^p$. Juxtaposing $\mathbf{f}^1, \ldots, \mathbf{f}^{2(t+r)}$ from the left into eq 142 through 147, respectively, and adding the equations thus formed yield

$$\mathbf{T} = \mathbf{f}^1(\alpha\mathbf{f}_1 - \beta\mathbf{f}_2) + \mathbf{f}^2(\alpha\mathbf{f}_2 + \beta\mathbf{f}_1) +$$
$$+ \sum_{\phi=2}^{t} \left[\mathbf{f}^{2\phi-1}(\alpha\mathbf{f}_{2\phi-1} - \beta\mathbf{f}_{2\phi} + \mathbf{f}_{2\phi-3}) + \mathbf{f}^{2\phi}(\alpha\mathbf{f}_{2\phi} + \beta\mathbf{f}_{2\phi-1} + \mathbf{f}_{2\phi-2})\right] +$$
$$+ \sum_{\xi=t+1}^{t+r} \left[\mathbf{f}^{2\xi-1}(\alpha\mathbf{f}_{2\xi-1} - \beta\mathbf{f}_{2\xi}) + \mathbf{f}^{2\xi}(\alpha\mathbf{f}_{2\xi} + \beta\mathbf{f}_{2\xi-1})\right] + \mathbf{S} \qquad \text{(g)}$$

where $\mathbf{S}$ does not have $\alpha \pm i\beta$ as eigenvalues. $\mathbf{T}$ in eq g can be transformed to $\mathbf{T}$ in eq 141.

Dotting $\mathbf{f}^1, \ldots, \mathbf{f}^{2(t+r)}$ from the right in $\mathbf{T}$ in eq g yields eq 148 through 151, respectively. The proof is thus completed.

$\mathbf{T}$ in eq g may be expressed as

$$\mathbf{T} = \mathbf{S} + [\mathbf{f}^1 \ldots \mathbf{f}^{2(t+r)}]T \begin{bmatrix} \mathbf{f}_1 \\ \vdots \\ \mathbf{f}_{2(t+r)} \end{bmatrix} \qquad (152)$$

where $T$ is a $2(t+r)$ by $2(t+r)$ matrix

$$T = \begin{bmatrix} A & C \\ D & B \end{bmatrix}. \qquad (153)$$

in which $A$ is a $2t$ by $2t$ matrix

$$A = \begin{bmatrix} \alpha & -\beta & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ \beta & \alpha & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 1 & 0 & \alpha & -\alpha & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 1 & \beta & \alpha & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & \alpha & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 1 & 0 & \alpha & -\beta \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 & \beta & \alpha \end{bmatrix}$$

$B$ is a $2r$ by $2r$ matrix

$$B = \begin{bmatrix} \alpha & -\beta & 0 & \cdots & 0 & 0 & 0 \\ \beta & \alpha & 0 & & 0 & 0 & 0 \\ 0 & 0 & \alpha & & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & & 0 & \alpha & -\beta \\ 0 & 0 & 0 & & 0 & \beta & \alpha \end{bmatrix}$$

$C$ and $D$ are composed of zero only. In matrix $A$, all the elements on the line parallel to the main diagonal and containing 1 are 1.

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13. ABSTRACT

A new tensor analysis, called the Gibbs-Einstein tensor analysis, is developed based on the concept that directions are algebraic quantities subject to the rule of forming scalar products, tensor products, and linear combinations. The new tensor analysis is explained in this paper by way of reformulating continuum mechanics and the Hamilton-Cayley theorem in matrix theory. The latter reformulation yields an explanation of the deformation dyads introduced in the former reformulation. A scalar product of two deformation dyads yields the strain tensor, which is a thermodynamic state variable for thermodynamically reversible deformations. Mathematics dealing with directions in a flat space becomes much simpler and more understandable when the Gibbs-Einstein tensor expression is used.

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