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AN EXPOSITION OF HILBERT SPACE AND LINEAR OPERATORS FOR ENGINEERS AND SCIENTISTS

Dr. F. M. Reza

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93

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Syracuse, New York

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FOREWORD

This report was written by Dr. F.M. Reza, 117 Borden Avenue, Syracuse, New York, for Rome Air Development Center, Griffiss Air Force Base, New York, and the Air Force Office of Aerospace Research, under Contract F30602-68-C-0062, Project 8505. Haywood E. Webb, Jr., EMIIS, was the RADC Project Engineer.

Project 8505, Research in Systems Theory, is addressed to both fundamental research in System Theory relevant to Air Force problems, and to "coupling" of abstract results to Engineering Areas where it can more profitably be used in the form of expository technical reports.

This report represents the combining of two expository reports into a single report. The first chapter treats Elementary Hilbert Space. The second chapter treats Linear Operators in Space. Both are treated from an engineering point of view. They are motivated by the power of some of the concepts, the realization that the digital computer is here, and the hope that many of the concepts which are abstract in the general engineering community can be applied to many more real Signal Processing problems.

The report(s) here is the last of a sequence of reports in the development of engineering concepts from mathematical concepts of Linear Spaces. The reader may also be interested in RADC-TR-65-399, "Engineering Applications of Function Space Concepts to Signals and Systems," AD 638 633; RADC-TR-66-595, "Some Applications of Linear Differential Vector Equations," AD 648 247; RADC-TR-67-376, "Functions of a Matrix," AD 663 739; and RADC-TR-67-648, "Elements of Approximation Theory," AD 666 938.

This technical report has been reviewed by the Foreign Disclosure Policy Office (EMLI) and the Office of Information (EMLS) and is releasable to the Clearinghouse for Federal Scientific and Technical Information.

This Technical Report has been reviewed and is approved.

Approved: HAYWOOD E. WEBB, JR
Project Engineer

Approved: EDWARD N. MUNZER
Chief, Intelligence Applications Branch

ABSTRACT

The vast and rapid advancement in telecommunications, computers, controls and aerospace science has necessitated major changes in our basic understanding of the theory of electrical signals and processing systems. There is strong evidence that today's engineer needs to extend and to modernize his analytical techniques. The latest fundamental analytical approach for the study of signals and systems seems to have its roots in the mathematics of Functional Analysis.

This report contains a bird's-eye view of the elements of Hilbert spaces and their associated linear operators. The first chapter of the report gives an exposition of the most essential properties of Hilbert spaces. The second chapter presents the elements of linear operators acting on such spaces.

The report is addressed to engineers and scientists interested in the theory of signals and systems. The applications of the theory will be undertaken in a separate report.

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TABLE OF CONTENTS

CHAPTER 1 ELEMENTARY HILBERT SPACE

CHAPTER 2 LINEAR OPERATORS IN HILBERT SPACE

TABLE OF CONTENTS (continued)

Chapter I

ELEMENTARY HILBERT SPACE THEORY

1-1. Introduction

In the report TR-65-399 we have discussed the finite-dimensional linear space and metric space in some generality. The reader recalls that the concept of distance between pairs of points of the space, such as the familiar Euclidean distance, played an important role in our studies. In this chapter, we wish to investigate a more general type of space, that is, metric spaces of infinite dimensions.

A vector space of finite dimensions is said to be an n-space if it contains a maximum of n linearly independent elements. A vector space of infinite dimensions in its simple form is a generalization of the n-space when the number of linearly independent elements becomes arbitrarily large. An understanding of the concept of a space with infinite dimensions in itself requires some preliminary preparations. The introduction of "infinity" is accompanied by certain problems of convergence and continuity. Therefore, our first job is to make some inquiry about the continuity of the metric associated with pairs of elements in this space. Moreover, when the dimension of the space is an uncountable number (nondenumerably infinite), the structure of the space becomes considerably more complex. There lies a professional area of mathematics beyond our aim. A comprehensive mathematical study of abstract Hilbert spaces requires certain specialized preparations beyond the scope of this undertaking. In view of the fact that Hilbert space, Banach space, and function space often appear in engineering literature, a rudimentary knowledge of the subject seems to be indispensible to the engineer. To comply with this need, we will give an elementary account of the generalization of the concept of a finite-dimensional Euclidean space to the case where the dimension of the space may become countably infinite. We will limit ourselves to what may be hopefully termed, in the words of mathematician P. R. Halmos*, "a glimpse into Hilbert space".

1-2. Continuity in Metric Spaces

A set X of elements of any kind is called a metric space if to any ordered pair of elements $\underline{x}$, $\underline{y} \in X$ there corresponds a real number $d(\underline{x},\underline{y})$ with the properties:

$$
\begin{aligned}
&(1)\quad d(\underline{x},\underline{y}) \geq 0 \\
&(2)\quad d(\underline{x},\underline{y}) = 0 \qquad \text{if and only if } \underline{x} = \underline{y} \\
&(3)\quad d(\underline{x},\underline{y}) = d(\underline{y},\underline{x}) \\
&(4)\quad d(\underline{x},\underline{y}) \leq d(\underline{x},\underline{z}) + d(\underline{z},\underline{y}) \quad \text{(triangle inequality)}
\end{aligned}
\tag{1-1}
$$

The presence of a metric on the elements of X allows us to talk naturally about convergence, and continuity in the sense of the metric.

This section deals with the general concept of continuity in a metric space. For the sake of brevity, we merely restrict ourselves to the most pertinent definitions and basic theorems. For our limited purposes the definitions relevant to metric spaces may be considered as generalizations of the alike familiar concepts of Euclidean spaces.

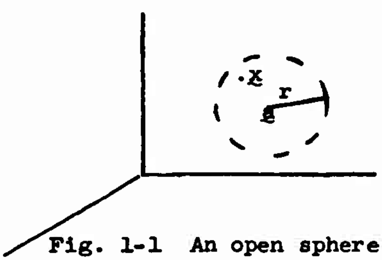

Fig. 1-1 An open sphere
$d(\underline{a},\underline{x}) < r$

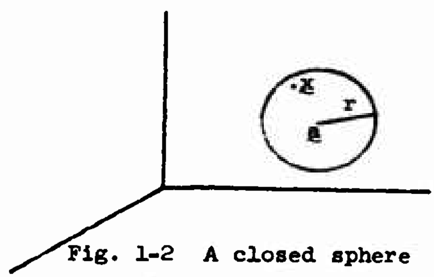

Fig. 1-2 A closed sphere
$d(\underline{a},x) \leq r$

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*Halmos studies in Lectures on Modern Mathematics, T.L. Saaty, Editor, Vol. 1, John Wiley and Son, New York, pages 1-22, 1963.

(The words "element", "point" and "vector" are used synonomously; vectors are underlined with a wavy line.)

Open Sphere - An <u>open sphere</u> in a metric space X, with center at point $\underset{\sim}{a}$ and radius r, is defined by

$$S(\underset{\sim}{a},r) = \{\underset{\sim}{x} \in X \mid d(\underset{\sim}{a},\underset{\sim}{x}) < r\} \qquad (1\text{-}2)$$

Closed Sphere - A <u>closed sphere</u> with center at point $\underset{\sim}{a}$ and radius r is defined by

$$\bar{S}(\underset{\sim}{a},r) = \{\underset{\sim}{x} \in X \mid d(\underset{\sim}{a},\underset{\sim}{x}) \leq r\} \qquad (1\text{-}3)$$

The closed sphere includes the points on its surface, i.e., points $\underset{\sim}{x} \in X$ such that

$$d(\underset{\sim}{a},\underset{\sim}{x}) = r \qquad (1\text{-}4)$$

The pictorial representations are used to facilitate an understanding of the concepts involved by analogy of the Euclidean space. They should not be employed, however, as substitutes for the formal definitions. A limitation of pictorial, representation is, for instance, that pictures may assume a variety of forms, depending on the definition of the distance function. For example, in the space of continuous functions C[0,1] with Chebychev norm (see Sec. 4-2), Fig. 1-3 may be envisaged for the open sphere $d(\underset{\sim}{x}-\underset{\sim}{a}) < \underset{\sim}{r}$. (Compare with Fig. 1-1.)

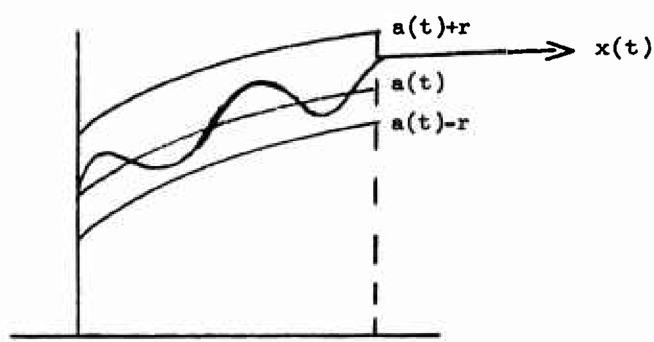

Fig. 1-3 Open sphere in C[0,1]

Neighborhood

Let $\underline{a}$ be a point of the metric space X; a set of points in X is said to be a neighborhood of $\underline{a}$, denoted by $N(\underline{a})$, if there exist a $\delta > 0$ such that $N(\underline{a})$ contains the sphere $S(\underline{a},\delta)$, i.e.:

$$N(\underline{a}) \supset \left\{ \underline{x} \in X \mid d(\underline{a},\underline{x}) < \delta \right\} \qquad (1\text{-}5)$$

The collection of all neighborhoods of a point is referred to as the complete or fundamental system of neighborhoods of that point. The characterization of the continuity of a function at a point is based on this concept. From this definition it follows that the open sphere $S(\underline{a},r)$ is a neighborhood for each of its points. Geometrically, the latter statement implies that for any point $\underline{b}$ in $S(\underline{a},r)$ a number r' can be found such that $S(\underline{b},r')$ is contained in the sphere $S(\underline{a},r)$.

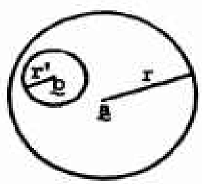

Fig. 1-4 An open sphere is a neighborhood for each of its points.

Boundedness

A non empty set in a metric space X is called bounded if it is contained in some open sphere in X.

Limit of a sequence

Let $\left\{ \underline{a}_1, \underline{a}_2, \ldots \right\}$ be a sequence of points in a metric space X. A point $\underline{a}$ in X is said to be the limit of this sequence, if for any specified $\epsilon > 0$ there exists an integer N, generally depending on ϵ, such that whenever $n > N$, then $d(\underline{a},\underline{a}_n) < \epsilon$

or equivalently $\underline{a}_n$ lies in the open sphere $S(\underline{a},\epsilon)$. This statement, sometimes, is abbreviated by writing

$$\lim_{n \to \infty} d(\underline{a},\underline{a}_n) = 0 \tag{1-6}$$

Alternatively, it is said that the sequence converges to $\underline{a}$

$$\lim_{n \to \infty} \underline{a}_n = \underline{a} \tag{1-7}$$

Function-continuous mapping

Let X and X' be metric spaces with distance functions d and d', and let f be a mapping of X into X'. A function of a vector argument $f(\underline{x})$ defined over subset D of a metric space X is said to be a continuous mapping function at a point $\underline{x}_0 \in D$, if for any specified $\epsilon > 0$ there exist a $\delta > 0$ such that whenever

$$d(\underline{x},\underline{x}_0) < \delta$$

then in the metric space of the images X',

$$d'(f(\underline{x}),\ f(\underline{x}_0)) < \epsilon$$

Alternatively, the function is said to be continuous at a point $\underline{a}$, if whenever $\underline{a}$ is the limit for a sequence of points in X, then $f(\underline{a})$ is the limit for the corresponding sequence of image points in the image space X'. A mapping is said to be a continuous mapping in D if it is continuous at each point of D. In particular, if $f(\underline{x})$ is a numerical function, i.e., D is mapped into R' or C', then we have the more familiar definition of a numerical function continuous at a point $\underline{x}_0 \in X$; i.e.,

$$d(\underline{x},\underline{x}_0) < \delta \Longrightarrow |f(\underline{x}) - f(\underline{x}_0)| < \epsilon$$

Fundamental sequence

A sequence of points $\{\underline{a}_1,\underline{a}_2,\ldots\}$ of a metric space X is called a fundamental, or a Cauchy sequence, if for each $\epsilon > 0$ there is a positive integer N_ϵ such that $d(\underline{a}_n,\underline{a}_m) < \epsilon$ whenever n and $m > N_\epsilon$. It is not difficult to see that every convergent sequence is a fundamental sequence. In fact, if the sequence $\{\underline{f}_n\}$ converges to $\underline{f}$, then for any specified $\epsilon > 0$, one can find N_ϵ such that

$$d(\underline{f}_n,\underline{f}) < \epsilon/2 \qquad n > N_\epsilon \tag{1-8}$$

Now consider a point $\underline{f}_m$ for $m > N_\epsilon$

$$d(\underline{f}_m,\underline{f}) < \epsilon/2 \tag{1-9}$$

But due to the triangle inequality

$$d(\underline{f}_m,\underline{f}_n) \leq d(\underline{f}_m,\underline{f}_n) + d(\underline{f}_n,\underline{f}) < \epsilon \tag{1-10}$$

Thus a converging sequence is a Cauchy sequence.

Complete Space

A metric space X is said to be a complete space if every fundamental sequence in X converges to some element in that space.

Separable Space

A thorough discussion of the separability property of a metric space requires some mathematical preparations beyond the scope of the present undertaking. Since the terms separability and separable metric spaces appear frequently in the statements of theorems on Hilbert space, an introductory notion of this concept is included. In the space of real numbers R, let us consider the set of rational numbers; this set has a very important property. Every real number $x \in R$ can be expressed in the form of a limit of a sequence of rational numbers. Take, for instance, the number π; the following sequence of rational numbers converges to π:

$$3,\ 3.1,\ 3.14,\ 3.141,\ 3.1415,\ 3.14159,\ \ldots$$

The set of rational numbers is denumerable, i.e., the rational numbers can be put into a one-to-one correspondence with the set of positive integer. Furthermore, the set of rational numbers is <u>dense</u> in the space of real numbers. In other words, between any two distinct real numbers there is a rational number.

A metric space, consisting of an infinite set of elements is said to be a separable space if there is a denumerable subset of elements dense* in X:

$$\left\{ \underset{\sim}{x}_1,\ \underset{\sim}{x}_2,\ \ldots,\ \underset{\sim}{x}_n,\ \ldots \right\} \tag{1-11}$$

That is, for any specified $\underset{\sim}{x} \in X$ and $\epsilon > 0$ there exists an element $\underset{\sim}{x}_{n_o}$ in the above sequence such that

$$d(\underset{\sim}{x}, \underset{\sim}{x}_{n_o}) < \epsilon \tag{1-12}$$

In the finite-dimensional real Euclidean space R^n, the set of all points with rational coordinates is a countable set dense in R^n. Therefore, R^n is a separable space. Same is true for a finite-dimensional complex space.

<u>Compact Set</u>

A set E of a metric space X is said to be compact if every infinite sequence of elements in E contains a subsequence which converges to some $\underset{\sim}{x} \in X$. The require-

---

*A more formal definition of the term dense results from the following. Let X be a metric space, and E is a subset of X, then:

(1) $\underset{\sim}{x} \in X$ is a <u>limit point</u> of E if every open sphere about $\underset{\sim}{x}$ contains a point $\underset{\sim}{y} \neq \underset{\sim}{x}$ such that $\underset{\sim}{y} \in E$.

(2) E is <u>dense</u> in X if every point of X is either a limit point of E, or a point of E (or both).

Rudin, W., <u>Principles of Mathematical Analysis</u>, McGraw-Hill Book Co., New York, 1964, p. 28.

ment that a space be compact is very rigid and more restrictive than the requirement of separability and completeness. The set of points $I = \{0,1,2,3,\ldots\}$ of R^1 is not a compact set, because the sequence $\{0,1,2,3,\ldots\} \in I$ does not contain any convergent subsequence. Therefore, the Euclidean space R^1 (and likewise R^n) is not a compact metric space. One can show that every subset of points on the real line R^1, or of the Euclidean space R^n is a compact set if and only if it is closed and bounded. (For instance, the set $a \leq x \leq b$ is a compact set.)

Example 1

Show that the set of all continuous real-valued functions defined on the real interval $[a,b]$ with distance function

$$d(\underline{f},\underline{g}) = \max \left\{ |f(t) - g(t)| \right\} , \quad a \leq t \leq b$$

forms a separable metric space.

Solution: It is readily seen that the suggested distance function satisfies the requirements of axioms 1 to 4. To see whether a metric space is a separable space, consider all polynomials with rational coefficients. There are countably many such polynomials. By virtue of the Weierstrass approximation theorem, any element of the space can be uniformly approximated by a polynomial with rational coefficients. Therefore, this is a separable metric space.

1-3. Normed Linear and Banach Spaces

In Chapter 4 of TR-65-399 we have defined metric spaces of finite dimensions. In this chapter the same basic concepts are presented in a more general fashion, and without regard to finiteness or infiniteness of dimensions. Consider a linear vector space V with finite or infinite dimensions. A norm in a linear vector space is defined as a real number associated with every element $\underline{x}$ of the space (denoted by $\|\underline{x}\|$) having the following properties.

1. $\|\underline{x}\| \geq 0$ for every $\underline{x} \in V$

2. $\|\underline{x}\| = 0$ if and only if $\underline{x} = \underline{0}$

3. $\|\lambda\underline{x}\| = |\lambda| \cdot \|\underline{x}\|$ for any $\underline{x} \in V$ and any scalar λ

4. $\|\underline{x}+\underline{y}\| \leq \|\underline{x}\| + \|\underline{y}\|$ for any pair of elements in V.

The relation between normed linear spaces and metric spaces is a simple one. Let $\|\underline{x}\|$ be the norm of an element $\underline{x}$ of a linear vector space V; define the following distance function over the elements of that space.

$$d(\underline{x},\underline{y}) = \|\underline{x}-\underline{y}\| = \|\underline{y}-\underline{x}\| \tag{1-13}$$

The fact that this is a permissible distance function is an immediate consequence of the aforementioned properties. In fact, (1), (2), and (3) can be easily verified. To show the validity of (4), note that

$$\begin{aligned} d(\underline{x},\underline{y}) &= \|\underline{x}-\underline{y}\| = \|(\underline{x}-\underline{z}) + (\underline{z}-\underline{y})\| \\ &\leq \|\underline{x}-\underline{z}\| + \|\underline{z}-\underline{y}\| = d(\underline{x},\underline{z}) + d(\underline{z},\underline{y}) \end{aligned} \tag{1-14}$$

Thus the normed linear spaces are metric spaces.

In view of the material of Section 1-2, we are now in a position to study metric spaces where a concept of convergence of vectors has been introduced. We say that $\underline{x}$ is the limit of a sequence $\underline{x}_n$, i.e.,

$$\underline{x} = \lim_{n \to \infty} \underline{x}_n \ (\text{or } \underline{x}_n \to \underline{x}) \text{ if } \|\underline{x}_n - \underline{x}\| \underset{n \to \infty}{\to} 0 \tag{1-15}$$

This type of convergence is convergence in the norm. If every fundamental sequence of a linear normed space has a limit in that space, then the space is said to be a <u>complete linear normed space</u>. A linear normed space which is complete with respect to its norm is also referred to as a <u>Banach</u> space (named after the Polish mathematician S. Banach). Our plans do not require a detailed mathematical study of normed spaces and Banach spaces. Nonetheless, we note in passing that many familiar properties of ordinary Euclidean spaces are valid for this broader class of Banach spaces. The key to the generalization lies in the fact that Banach spaces, like ordinary real Euclidean lines, are linear, normed and complete. Some of the familiar concepts which may be directly extended from the ordinary R^n to Banach spaces are: the concepts of linear dependence of elements, linear manifold, subspace, plane and sphere, and convergence **in norm.** The concept of orthogonality, the elegant properties of projection, and the least square distance criterion are not inadvertently maintainable in a Banach space. The applicability of these latter concepts is restricted to the Hilbert space which is a subclass of the Banach spaces. This matter will be discussed shortly.

<u>Example 2</u>

Show that the space R^n is a Banach space.

Solution: R^n is, of course, a linear normed space. It remains to show that it is complete with respect to the Euclidean norm. Let

$$\{\underline{x}_i\} = \{\underline{x}_1, \underline{x}_2, \cdots\}, \quad \underline{x}_k = \{x_{1k}, x_{2k}, \ldots, x_{nk}\}$$

be a fundamental sequence of elements in this space; that is, for every $\epsilon > 0$ there exist an N_ϵ such that for p, g $> N_\epsilon$ we have

$$d(\underline{x}_p, \underline{x}_g) = \left[\sum_{k=1}^{n} (x_{kp} - x_{kg})^2\right]^{1/2} < \epsilon$$

If the sum of a finite number of non negative terms is smaller than ϵ^2 then everyone of these terms must be smaller than ϵ^2; hence,

$$|x_{kp} - x_{kg}| < \epsilon \qquad k = 1,2,\ldots,n$$

That is, $\left\{ x_{k1}, x_{k2}, \dots \right\}$ is a fundamental sequence of real numbers.

Let

$$\xi_k = \lim_{p \to \infty} x_{kp} \qquad k = 1,2,\dots,n$$

and

$$\underset{\sim}{x} = \left\{ \xi_1, \xi_2, \dots, \xi_n \right\}$$

Then it becomes clear that

$$\lim_{n \to \infty} \underset{\sim}{x}_n = \underset{\sim}{x}$$

Example 3

Consider the half-open set of real numbers between zero and one, including one but excluding zero. Show that in this space the sequence $x_n = \frac{1}{n}$ is a Cauchy sequence which is not converging.

Solution: The above Cauchy sequence has no limit point in the specified space. However, if the space was to include zero, then the Cauchy sequence would converge to a point in the space. The latter space, $\left\{ x \colon 0 \leq x \leq 1 \right\}$, is a complete metric space. This is an example of a metric space which can be made complete by adjoining additional elements.

Example 4

a) Show that the set of rational numbers form a metric space with respect to the norm

$$\|\underset{\sim}{x}\| = |x|$$

where x is a rational number.

b) Is this a Banach space?

Solution: Part (a) is straight forward. In order to answer (b), take, for instance, the sequence:

$$\{r_n\} = \left\{\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots\right\}$$

This is a fundamental sequence since it converges to zero which is an element of the space. On the other hand, the set

$$\{r_m\} = \left\{(1 + \frac{1}{1})^1, (1 + \frac{1}{2})^2, (1 + \frac{1}{3})^3, \ldots\right\}$$

has no limit in the space of rational numbers, since

$$\lim_{n \to \infty} (1 + \frac{1}{m})^m = e$$

This is a simple example of a metric space in which there are fundamental sequences which do not converge to a limit in that space. This is not a B-space.

Example 5

The space of continuous function $C[a,b]$ is a Banach space. The distance between vectors $x(t)$ and $y(t)$ is assumed to be the absolute value of their largest deviation in the interval $[a,b]$.

$$d(\underline{x},\underline{y}) = \text{Max } |x(t) - y(t)|, \quad a \leq t \leq b$$

Solution: Consider a fundamental sequence

$$\{x_1(t), x_2(t), \ldots\}$$

Thus for a given $\epsilon > 0$ there exists an N_ϵ such that for $n,m > N_\epsilon$

$$\|\underline{x}_n - x_n\| < \epsilon$$

or

$$|x_n(t) - x_m(t)| < \epsilon \qquad a \leq t \leq b$$

This implies that the sequence converges uniformly and its limit is a continuous function $x(t)$ such that

$$|x_n(t) - x(t)| < \epsilon \qquad \begin{array}{l} n > N \\ a \leq t \leq b \end{array}$$

This implies the convergence in the sense of the applied metric.

Example 6

Show that the space of square integrable continuous function $C^2[a,b]$ is not a Banach space. The distance being defined by:

$$d(\underline{x},\underline{y}) = \left\{ \int_a^b [x(t) - y(t)]^2 \, dt \right\}^{1/2}$$

Solution: It is not difficult to show that the space of all continuous functions $C^2[a,b]$ is a metric space.

In order to show that this metric space is not a B-space, let us simply construct a Cauchy sequence in the space $C^2[-1,1]$ which does not converge to a vector in the same space.

Let

$$\begin{array}{ll} f_n(t) = 0 & \text{for } -1 \leq t \leq 0 \\ f_n(t) = nt & \text{for } 0 \leq t \leq 1/n \\ f_n(t) = 1 & \text{for } 1/n < t \leq 1 \end{array}$$

The sequence $\{f_n(t)\}$ is indeed a sequence of continuous functions. But as n approaches infinity, we find:

$$\lim_{n \to \infty} f_n(t) = 0 \qquad \text{for } -1 \leq t \leq 0$$

$$\lim_{n \to \infty} f_n(t) = 1 \qquad \text{for } 0 < t \leq 1$$

The limit vector is a discontinuous function. You may wish to show as an exercise that the sequence

$$\{g_n(t)\} = \arctan nt \qquad \text{for } -1 \leq t \leq 1$$

is also a fundamental sequence which converges to the discontinuous function:

$$g_\infty(t) = -\pi/2 \qquad \text{for } t < 0$$

$$g_\infty(t) = \pi/2 \qquad \text{for } t > 0$$

1-4. Abstract Hilbert Spaces

As far as the application of the concepts of abstract space to physical sciences is concerned, the so-called Hilbert space, commonly denoted by H, occupies the dominant place. Hilbert space, or Hilbert function-space, is a space of finite or infinite dimensions defined over the field of complex numbers having the following main characters. There should be a suitable distance function defined, that is, the metric should arise from an inner product form. The space must be complete, i.e., it should possess the convergence property for all its fundamental sequences with respect to its metric. Furthermore, the elements of the space must have a certain property of "closeness" which is referred to as the separability requirement.

Hilbert space as defined here is an inner product space which is complete with respect to its norm, and may be separable or non separable. More specifically, the following axioms, 1, 2, 3, and 4 are required:

1. H is a linear vector space over the field of complex numbers.
2. With every pair of elements, $\underline{x},\underline{y}$ of H, there is associated a complex number $(\underline{x},\underline{y})$ called the scalar or inner product of $\underline{x}$ and $\underline{y}$, with the properties:

a) $(\underset{\sim}{x},\underset{\sim}{y}) = \overline{(\underset{\sim}{y},\underset{\sim}{x})}$

(Bar denotes the complex conjugates; note that $(\underset{\sim}{x},\underset{\sim}{y})$ is real, and the number $(\underset{\sim}{x},\underset{\sim}{x})^{1/2} = \|\underset{\sim}{x}\|$ is called the norm of $(\underset{\sim}{x})$.

b) $(\underset{\sim}{x}+\underset{\sim}{z},\underset{\sim}{y}) = (\underset{\sim}{x},\underset{\sim}{y}) + (\underset{\sim}{z},\underset{\sim}{y})$

c) $(\lambda\underset{\sim}{x},\underset{\sim}{y}) = \lambda(\underset{\sim}{x},\underset{\sim}{y})$ for arbitrary complex number λ

d) $(\underset{\sim}{x},\underset{\sim}{x}) \geq 0$

e) $(x,\underset{\sim}{x}) = 0$ if and only if $\underset{\sim}{x} = \underset{\sim}{0}$

3. H is complete with respect to the metric

$$d(\underset{\sim}{x},\underset{\sim}{y}) = \|\underset{\sim}{x}-\underset{\sim}{y}\|$$

4. For every positive integer n, the space contains n linearly independent elements.

5. H is a separable space. (This property is optional as a great part of the theory Hilbert function-space is applicable without regard to separability.)

We note in passing that if properties (1) and (2) are satisfied, the space is referred to as a Pre-Hilbert space, also a Unitary space. An n-dimensional unitary space is a complex Euclidean space. In the definition of Hilbert spaces, it is not unusual to forego the separability requirement, or even the requirement of infinite dimensions, at times. In the present work, however, we shall always adhere to the first three requirements, but will not insist on the separability condition for every problem discussed. A Hilbert space which satisfies conditions (1), (2), (3), and (4), but not (5), is referred to as a non separable Hilbert space. Hilbert space is a subclass of B-space which arises as a direct generalization of Euclidean space. In view of the fact that Hilbert space is an inner product space, its geometry is closer to the geometry of the ordinary Euclidean space than to any other B-space.

The following two important inequalities for elements of Hilbert spaces are direct consequences of the defined axioms.

a) <u>Schwarz's inequality</u>

$$|(\underset{\sim}{x},\underset{\sim}{y})| \leq \|\underset{\sim}{x}\| \ \|\underset{\sim}{y}\| \tag{1-16}$$

To prove this inequality, note that for any arbitrary λ the norm

$\|\underset{\sim}{x}+\lambda\underset{\sim}{y}\|$ is a non negative number, that is

$$(\underset{\sim}{x} + \lambda\underset{\sim}{y},\ \underset{\sim}{x} + \lambda\underset{\sim}{y}) \geq 0 \tag{1-17}$$

Note that this inequality holds trivially if $\underset{\sim}{y} = \underset{\sim}{0}$. So we assume $\underset{\sim}{y} \neq \underset{\sim}{0}$.

$$(\underset{\sim}{x},\underset{\sim}{x}) + \bar{\lambda}(\underset{\sim}{x},\underset{\sim}{y}) + \lambda(\underset{\sim}{y},\underset{\sim}{x}) + |\lambda|^2\ (\underset{\sim}{y},\underset{\sim}{y}) \geq 0 \tag{1-18}$$

By letting

$$\lambda = -\frac{(\underset{\sim}{x},\underset{\sim}{y})}{(\underset{\sim}{y},\underset{\sim}{y})} \tag{1-19}$$

One obtains

$$(\underset{\sim}{x},\underset{\sim}{x}) - \frac{|(\underset{\sim}{x},\underset{\sim}{y})|^2}{(\underset{\sim}{y},\underset{\sim}{y})} \geq 0 \tag{1-20}$$

or

$$|(\underset{\sim}{x},\underset{\sim}{y})| \leq \|\underset{\sim}{x}\| \ \|\underset{\sim}{y}\| \tag{1-21}$$

b) Triangle inequality

$$\|\underset{\sim}{x} + \underset{\sim}{y}\| \leq \|\underset{\sim}{x}\| + \|\underset{\sim}{y}\| \tag{1-22}$$

This inequality can be derived in a similar vein.

$$\|(\underset{\sim}{x} + \underset{\sim}{y})\|^2 = (\underset{\sim}{x} + \underset{\sim}{y}, \underset{\sim}{x} + \underset{\sim}{y}) \tag{1-23}$$

$$= (\underset{\sim}{x}, \underset{\sim}{x}) + (\underset{\sim}{y}, \underset{\sim}{x}) + (\underset{\sim}{x}, \underset{\sim}{y}) + (\underset{\sim}{y}, \underset{\sim}{y})$$

$$\leq \|\underset{\sim}{x}\|^2 + 2\|\underset{\sim}{x}\| \|\underset{\sim}{y}\| + \|\underset{\sim}{y}\|^2$$

$$\|\underset{\sim}{x} + \underset{\sim}{y}\| \leq \|\underset{\sim}{x}\| + \|\underset{\sim}{y}\| \tag{1-24}$$

1-5. Infinite-Dimensional Euclidean Space - Space ℓ^2

In the previous section, we have defined the basic requirements for abstract Hilbert spaces. In this section, we propose to investigate in some detail the infinite-dimensional Euclidean space as an example of Hilbert space. This space is often referred to as the space ℓ^2. The coordinates of every vector may be real or complex numbers; the usual vector operations and the norm of an element are defined as follows:

$$\underset{\sim}{x} = (\xi_1, \xi_2, \ldots, \xi_n, \ldots)$$

$$\underset{\sim}{y} = (\eta_1, \eta_2, \ldots, \eta_n, \ldots)$$

$$\underset{\sim}{x} + \underset{\sim}{y} = (\xi_1 + \eta_1, \xi_2 + \eta_2, \ldots, \xi_n + \eta_n, \ldots)$$

$$\lambda\underset{\sim}{x} = (\lambda\xi_1, \lambda\xi_2, \ldots, \lambda\xi_n, \ldots) \quad , \lambda \text{ a complex number}$$

The inner product and the norm are respectively:

$$(\underset{\sim}{x},\underset{\sim}{y}) = \sum_{i=1}^{\infty} \xi_i \bar{\eta}_i \tag{1-26}$$

$$\|\underline{x}\| = \left[\sum_{i=1}^{\infty} |\xi_i|^2 \right]^{1/2} < +\infty \quad \text{for any } \underline{x} \in \ell^2 \tag{1-27}$$

The condition of finiteness of length is essential for convergence requirement, and must be included in the definition of the space ℓ^2. The first thing is to show that this space is actually a linear normed (inner product) space. The first three conditions of normed spaces are obviously met. The completeness of this space can be inferred from Example 2, for a real Euclidean space of infinite dimensions. The same line of reasoning is essentially valid when the components are complex numbers. It can be also shown that the space ℓ^2 is a separable space, but this fact is not of an immediate concern to our studies.

The following two basic inequalities, of course, are valid for ℓ^2.

$$\left| \sum_{i=1}^{\infty} \xi_i \bar{\eta}_i \right|^2 \leq \left(\sum_{i=1}^{\infty} \xi_i \bar{\xi}_i \right) \left(\sum_{i=1}^{\infty} \eta_i \bar{\eta}_i \right) \tag{1-28}$$

$$\left[\sum_{i=1}^{\infty} |\xi_i + \eta_i|^2 \right]^{1/2} \leq \left[\sum_{i=1}^{\infty} |\xi_i|^2 \right]^{1/2} + \left[\sum_{i=1}^{\infty} |\eta_i|^2 \right]^{1/2}$$

These inequalities are rather interesting and often lend themselves to useful physical interpretation in applications. For this reason, a slightly different derivation is presented below. In order to show directly the validity of the inequality

$$\|\underline{x}+\underline{y}\| \leq \|\underline{x}\| + \|\underline{y}\|$$

one may use a simple and well-known variational procedure. Let us first prove the validity of these inequalities for E^n -- an extension to the case of ℓ^2 will present no difficulty. Consider the finite sets of real numbers

$$\{a_1, a_2, \ldots, a_n\} \quad \text{and} \quad \{b_1, b_2, \ldots, b_n\} \quad .$$

A familiar variational method suggests the calculation of the quadratic function

$$f(x) = \sum_{i=1}^{n} (a_i x + b_i)^2 = Ax^2 + 2Bx + C \tag{1-30}$$

where x is an arbitrary real quantity, and

$$A = \sum_{i=1}^{n} (a_i^2) \quad , \quad B = \sum_{i=1}^{n} a_i b_i \quad , \quad C = \sum_{i=1}^{n} (b_i^2) \tag{1-31}$$

Since f(x) must remain non negative for all real values of x, the condition

$$B^2 - AC \leq 0 \tag{1-32}$$

promptly gives

$$\left(\sum_{i=1}^{n} a_i b_i\right)^2 \leq \left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right) \tag{1-33}$$

This is a useful inequality which was originally obtained by Cauchy. Now the triangle inequality can be derived without difficulty. In fact, by taking the square root of both sides of Cauchy's inequality, multiplying by two, and adding A + C we find

$$2\sqrt{B} + (A+C) \leq 2\sqrt{A}\sqrt{C} + (A+C) \tag{1-34}$$

$$\sum_{i=1}^{n} (a_i^2 + 2a_i b_i + b_i^2) \leq (\sqrt{A} + \sqrt{C})^2 \tag{1-35}$$

or, the triangle inequality:

$$\left[\sum_{i=1}^{n} (a_i + b_i)^2\right]^{1/2} \leq \left[\sum_{i=1}^{n} a_i{}^2\right]^{1/2} + \left[\sum_{i=1}^{n} b_i^2\right]^{1/2} \tag{1-36}$$

The extension of Cauchy's inequality to the case where $n = \infty$, (also for complex ℓ^2 spaces) is not difficult. The only restriction is that A and C should remain finite; this is guaranteed by virtue of the assumption of finite norm for all elements of ℓ^2.

Separability of Space ℓ^2

Consider the set D of all elements of ℓ^2 having only a finite number n of non-zero coordinates; all such coordinates being rational numbers. We will show that D is dense in ℓ^2. To this end, for any $\underset{\sim}{x}$ and $\epsilon > 0$ we set forth to find a suitable converging sequence, by means of a point $\underset{\sim}{z} \in$ D.

$$\underset{\sim}{x} = \{x_1, x_2, \dots\} \qquad \underset{\sim}{x} \in \ell^2$$

$$\underset{\sim}{z} = \{z_1, z_2, \dots, 0, 0, \dots\} \quad \underset{\sim}{z} \in D$$

$$\|\underset{\sim}{x} - \underset{\sim}{z}\|^2 = \sum_{k=1}^{n} |x_k - z_k|^2 + \sum_{k=n+1}^{\infty} |x_k|^2$$

Note that the series $\displaystyle\sum_{k=n+1}^{\infty} |x_k|^2$ is convergent by hypothesis. Therefore, we can find an $n = n_o$ such that

$$\sum_{k=n_o+1}^{\infty} |x_k|^2 \leq \frac{1}{2}\epsilon^2$$

Consequently, in the finite sum $\displaystyle\sum_{k=1}^{n_o} |x_k - z_k|^2$ we may choose the rational numbers z_k close enough to x_k such that

$$\sum_{k=1}^{n_o} |x_k - z_k|^2 \leq \frac{1}{2}\epsilon^2$$

$$\|\underset{\sim}{x}-\underset{\sim}{z}\|^2 < \epsilon^2, \quad \|\underset{\sim}{x}-\underset{\sim}{z}\| < \epsilon$$

This inequality proves that D is dense in ℓ^2.

Example 7

The unit sphere of the infinite-dimensional Euclidean space (ℓ^2) is not a compact set.

Solution: To show the validity of this statement, consider for instance the infinite set of orthogonal vectors $\{\underset{\sim}{e}_k\}$: $\underset{\sim}{e}_1 = (1,0,0,\ldots)$, $\underset{\sim}{e}_2 = (0,1,0,0,\ldots)$

$$\|\underset{\sim}{e}_k\| = 1 \qquad\qquad k = 1,2,3,\ldots$$

We cannot select a convergent subsequence from $\{\underset{\sim}{e}_k\}$ since the distance between any pair of elements is $\sqrt{2}$.

$$\|\underset{\sim}{e}_n - \underset{\sim}{e}_m\|^2 = (\underset{\sim}{e}_n - \underset{\sim}{e}_m, \underset{\sim}{e}_n - \underset{\sim}{e}_m) = 2$$

1-6. The Space L^2 - Function Space

An important realization of Hilbert space is provided by the so-called L^2 space. Let [a,b] denote a finite or infinite interval on the real axis. Consider the set of all real valued square integrable functions f(t),* i.e.,

$$\int_a^b [f(t)]^2 \, dt < +\infty \qquad (1\text{-}37)$$

Under the usual definition of sum, and product by a scalar, and defining the zero element as a function which is "equal" to zero (almost everywhere) in [a,b], we have a linear vector space. In this vector space induce the inner product

*f(t) is denoted by lower-case letters and is represented in the vector space by a vector $\underset{\sim}{f}$.

$$(f,g) = \int_a^b f(t)\, g(t)\, dt \qquad (1\text{-}38)$$

The integral is to be taken in the sense of Riemann. The functions space thus defined is referred to as L^2 space. It will be shown now that the space L^2 is a linear vector space.

1. L^2 is a linear vector space.

   i) The product of any two elements of L^2 is integrable, since

   $$|f(t)\, g(t)| \leq \frac{1}{2} [f(t)]^2 + \frac{1}{2} [g(t)]^2 \qquad (1\text{-}39)$$

   observation is needed for the proof of (ii) below.

   ii) The sum of any two elements of L^2 is an element of L^2.

   Proof:

   $$[f(t) + g(t)]^2 \leq [f(t)]^2 + 2\,|f(t)\, g(t)| + [g(t)]^2 \qquad (1\text{-}40)$$

   iii) If $f \in L^2$ and λ a constant, then evidently $\lambda f \in L^2$.

   $$\int [\lambda\, f(t)]^2\, dt = \lambda^2 \int [f(t)]^2\, dt < \infty \qquad (1\text{-}41)$$

2. Properties a,b, and c for the suggested inner product function are easy to verify. For d one notes that the integral of a non **negative** quantity. Part e is more difficult to verify. The difficulty arises on account of some **mathematical** subtleties. Take, for instance, square integrable functions $\mu(t)$ whose integrals are zero but the functions can take on positive values. The difficulty can be circumvented by re-sorting to the fact that the set of t for which $\mu(t) > 0$ has the measure zero; that is, a set of points on [a,b] which can be covered by a finite, or countably infinite, set of intervals with an arbitrarily small total length. This amounts to saying that any non negative func-

tion whose integral is zero differs from zero element on a set of measure zero. One may say that such an element is equivalent to the zero element. In other words, the null element of L^2 is the collection of all functions defined on the real axis which are equivalent to zero. Two elements differing on at most a set of zero measure are referred to as "equivalent". The integrals of the square of equivalent elements are equal. Therefore, from the strict mathematical point of view the space of functions considered here is rather a space of "equivalence classes". The distance between two functions f(t) and g(t) equals zero precisely when f(t) = g(t) for almost all t. To depict this situation, let us write $\underline{f} \sim \underline{g}$ if and only if $d(\underline{f},\underline{g}) = 0$. We do not pursue this matter further; a presentation encompassing measure theoretic consideration (Theory of Lebesque integral) is not within the scope of our present undertaking.

3. The L^2 space is a complete space.
For proof, see standard texts, such as W. Rudin, Real and Complex Analysis: McGraw-Hill Book Co.; Akhiezer and Glazman, Theory of Linear Operators in Hilbert Space, Vol. 1, pages 21-23 of the English translation.

4 & 5. For a discussion of the validity of these properties for L^2, see Chapter 1 of the last-cited reference, or Chapter VIII of Functional Analysis, by Kolmogorov and Fomin. Detailed proof of separability with its implications can be found in Chapter 2 of L.V. Kantorovich and A.P. Akilov, Functional Analysis in Normed Spaces, Macmillan Co., New York, 1964.

For the sake of exercise, Schwarz' inequality and the triangle inequalities for vectors of L^2 will be derived directly. Let $\underline{f}$ and g be elements of L^2 not equivalent to the zero element, and λ an aribtrary parameter.

$$\int_a^b (\lambda f+g)^2 \, dt = \lambda^2 \int_a^b f^2 \, dt + 2\lambda \int_a^b fg \, dt + \int_a^b g^2 \, dt \tag{1-42}$$

This quadratic function must remain non negative for all real values of λ, hence

$$\left[\int_a^b fg \, dt\right]^2 \leq \left[\int_a^b f^2 \, dt\right] \left[\int_a^b g^2 \, dt\right] \tag{1-43}$$

This inequality is known as Cauchy-Bunyakovski's inequality. By taking square roots, multiplying by two and adding $\int_a^b f^2 \, dt + \int_a^b g^2 \, dt$ to both sides, one obtains an inequality bearing the name of Minkowski.

$$\int_a^b (f+g)^2 \, dt \leq \left\{\left[\int_a^b f^2 \, dt\right]^{1/2} + \left[\int_a^b g^2 \, dt\right]^{1/2}\right\}^2 \tag{1-44}$$

These inequalities are readily recognized as the Schwarz and the triangle inequalities discussed earlier for Hilbert spaces, i.e.,

$$|(\underline{f},g)| \leq \|\underline{f}\| \, \|g\| \tag{1-45}$$

$$\|\underline{f}+g\| \leq \|\underline{f}\| + \|g\| \tag{1-46}$$

In the definition of the space $L^2[a,b]$, for simplicity, the assertion was made that these functions are real valued. This restriction can be easily removed. Let $\phi(t) > 0$ be a non negative square integrable function on $[a,b]$ and denote by $L^2_\phi[a,b]$ all complex-valued functions square integrable with respect to the weighting function $\phi(t)$, that is:

$$\int_a^b \phi(t)\ |x(t)|^2\ dt < \infty \tag{1-47}$$

A simple extension of the foregoing material will enable one to show that the space $L^2_\phi[a,b]$ is also a separable Hilbert space. The chosen inner product is

$$(\underline{x},\underline{y}) = \int_a^b \phi(t)\ [x(t)\ \overline{y(t)}]\ dt \tag{1-48}$$

Very often, in application, one deals with the case $\phi(t) = 1$; that is $L^2[a,b]$

$$(\underline{x},\underline{y}) = \int_a^b x(t)\ \ y(t)\ dt \tag{1-49}$$

1-7. Continuity of Scalar Product

We wish to show that the inner product is a continuous function with respect to the norm of the Hilbert space. Let $\underline{x}$ and $\underline{y}$ be the limits of the sequence $\{\underline{x}_n\}$ and $\{\underline{y}_n\}$ respectively. By virtue of the preceding inequalities one can write:

$$|(\underline{x}_n,\underline{y}_n) - (\underline{x},\underline{y})| \le |(\underline{x}_n,\underline{y}_n) - (\underline{x}_n,\underline{y})| + |(\underline{x}_n,\underline{y}) - (\underline{x},\underline{y})| \tag{1-50}$$

$$= |(\underline{x}_n,\underline{y}_n-\underline{y})| + |(\underline{x}_n-\underline{x},\underline{y})|$$

$$\le \|\underline{x}_n\|\ \|\underline{y}_n-\underline{y}\| + \|\underline{y}\|\ \|\underline{x}_n-\underline{x}\|$$

$$\le M\|\underline{y}_n-\underline{y}\| + \|\underline{y}\|\ \|\underline{x}_n-\underline{x}\|$$

where M denotes the upper bound of $\|\underset{\sim}{x}_n\|$ and $\|\underset{\sim}{y}_n\|$. As n is infinitely increased, we find

$$\lim_{n\to\infty}\|\underset{\sim}{x}_n-\underset{\sim}{x}\| \to 0 \quad , \quad \lim_{n\to\infty}\|\underset{\sim}{y}_n-\underset{\sim}{y}\| \to 0 \tag{1-51}$$

whence

$$|(\underset{\sim}{x}_n,\underset{\sim}{y}_n) - (\underset{\sim}{x},\underset{\sim}{y})| \to 0 \tag{1-52}$$

This proves that the inner product in a Hilbert space is a continuous function with respect to the norm.

1-8. Linearly Independent Vectors

The reader is already familiar with the definition of a finite set of linearly independent elements of a finite dimensional metric space.

A finite set of elements $\{\Phi_1,\Phi_2,\ldots,\Phi_n\}$ of a Hilbert space is said to be linearly dependent if there exists scalars $\{\alpha_1,\alpha_2,\ldots,\alpha_n\}$, not all zero, such that

$$\sum_{i=1}^{n} \alpha_i\Phi_i = \underset{\sim}{0} \tag{1-53}$$

when the finite set is not linearly dependent, then is said to be linearly independent. The following statement is self-explanatory.

Theorem: The necessary and sufficient condition for a finite set of n vectors of an inner product space to be linearly independent is that its Gram Determinant be different from zero, i.e.:

$$G(\Phi_1,\Phi_2,\ldots,\Phi_n) = \begin{vmatrix} (\Phi_1,\Phi_1) & (\Phi_1,\Phi_2) & \cdots & (\Phi_1,\Phi_n) \\ (\Phi_2,\Phi_1) & (\Phi_2,\Phi_2) & \cdots & (\Phi_2,\Phi_n) \\ (\Phi_n,\Phi\ ) & (\Phi_n,\Phi_2) & \cdots & (\Phi_n,\Phi_n) \end{vmatrix} \neq 0 \tag{1-54}$$

An infinite set of vectors $\{\phi_1, \phi_2, \ldots\}$ of a Hilbert space is said to be linearly independent if every finite subset of it is linearly independent. It is natural to extend the concept of <u>basis</u> from a **finite-dimensional** metric space to a Banach space of infinite dimension. When the number of elements in a set of linearly independent elements of the space becomes infinite, the space is said to be of infinite dimension. This number may be countable or uncountable. In this elementary exposition we only consider Hilbert spaces of countable dimension, or simply Euclidean spaces of infinite dimension.

Let $\{\underline{e}_1, \underline{e}_2, \ldots, \underline{e}_n, \ldots\}$ be a countable set of independent elements of a Banach space X such that every element $\underline{x} \in X$ admits a unique representation

$$\underline{x} = \sum_{k=1}^{\infty} x_k \underline{e}_k \tag{1-55}$$

That is, to each $\underline{x}$ there corresponds a unique sequence $x_1, x_2, \ldots, x_n, \ldots$ of scalars such that

$$\lim_{m \to \infty} \left\| \sum_{k=1}^{\infty} x_k e_k - \underline{x} \right\| = 0 \tag{1-56}$$

Under these conditions we say that X has a <u>countable basis</u>. In our study we simply adhere to the case where the space has a countable basis. A discussion of B-spaces with uncountable basis is indeed beyond our plans. It can be proved that any Banach space with a countable basis is also a separable space. Also, we are told that the following question is an unsolved problem of modern mathematics.

> "Determine whether for every separable Banach space there exists a countable basis."

See, for instance, A.E. Taylor, <u>General Theory of Functions and Integration,</u>

Blaisdale Publishing Co., New York, 1965, page 152. The set of all linear combinations of subsets of an infinite set of independent elements $\{\phi\}$ is referred to as the linear subspace generated by $\{\phi\}$ (also linear hull).

1-9. Linear Manifold and Subspace

A non empty subset of elements of H is said to form a linear manifold L if for any $\underline{f} \in L$, $\underline{g} \in L$ and arbitrary numbers α and β we have

$$\alpha \underline{f} + \beta\, \underline{g} \in L \tag{1-57}$$

A subset of elements in a linear space is said to be closed if the set contains all its limit points. In view of this definition, we note that in contrast with the finite-dimensional inner product space, a linear manifold in a Hilbert space may not be necessarily closed. A closed subspace is itself a Hilbert space. Trivial examples of closed subspaces are the whole space H, and the null space of H containing the zero element only.

As an example of a linear manifold which is also a closed subspace, consider the following type of vectors of ℓ^2:

$$\underline{f} = \{f_1, f_2, f_3, f_4, f_5, \ldots\} \quad , \quad f_1 = f_2 = f_3 \tag{1-58}$$

$$\underline{g} = \{g_1, g_2, g_3, g_4, g_5, \ldots\} \quad , \quad g_1 = g_2 = g_3 \tag{1-59}$$

Evidently all points $\underline{x} = \alpha\underline{f} + \beta\underline{g}$ will have coordinates of the form

$$\underline{x} = \{x_1, x_2, x_3, x_4, x_5, \ldots\} \quad , \quad x_1 = x_2 = x_3 \tag{1-60}$$

The points of H satisfying $x_1 = x_2 = x_3$ form a linear manifold L which is also a closed subspace.

As an example of a linear manifold which is not a closed subspace, consider the set L of all vectors in ℓ^2 which have a finite number of non zero coordinates relative to some basis. Then the sequence of vectors $\underline{x}_1, \underline{x}_2, \underline{x}_3, \ldots$ where

$$x_1 = \{1,0,0,0,\ldots\}$$
$$x_2 = \{1,\tfrac{1}{2},0,0,\ldots\}$$
$$x_3 = \{1,\tfrac{1}{2},\tfrac{1}{4},0,\ldots\}$$
$$\vdots$$

is a converging sequence of vectors in L. However, the limit vector of this sequence does not belong to L.

$$x = \lim_{n \to \infty} x_n = \{1,\tfrac{1}{2},\tfrac{1}{4},\tfrac{1}{8},\ldots\}$$

1-10. <u>Orthogonality</u>

The elements x and y of H are said to be orthogonal if

$$(x,y) = 0 \tag{1-61}$$

Orthogonal elements are denoted as $x \perp y$. An element $x \in H$ is said to be orthogonal to a subspace S of H if x is orthogonal to every element of S, and we write $x \perp S$. If elements of two sets S_1 and S_2 are pairwise orthogonal, the sets will be referred to as orthogonal: $S_1 \perp S_2$.

The sets of all elements orthogonal to a given set S is a subspace of H. This subspace is called the <u>orthogonal complement</u> of S. It is not difficult to see that if $x \perp y$ and $x \perp z$, then $x \perp \alpha_1 y + \alpha_{||} z$. Also in view of the continuity of the scalar product, we find that if $x \perp y_n$ $(n=1,2,\ldots)$ and $y_n \to y$, then $x \perp y$.

The definition of length of a vector, and the angle between vectors in Hilbert space are identical with these definitions in finite dimensional Euclidean space, i.e.,

$$\|x\| = +\sqrt{(x,x)}$$

$$\text{cosine of the angle between } x \text{ and } y = \frac{(x,y)}{\|x\|\ \|y\|}$$

A set of elements $e_1, e_2, \ldots, e_n$ of a Hilbert space is said to be orthonormal if

$$(e_j, e_i) = \begin{Bmatrix} 1 & \text{for } i = j \\ 0 & \text{for } i \neq j \end{Bmatrix} \tag{1-62}$$

It is easy to show that every orthonormal set of vectors is an independent set. A set of orthogonal or orthonormal elements $\{\phi_1, \phi_2, \ldots, \phi_n, \ldots\}$ of H is said to be complete if there does not exist a non zero vector in H orthogonal to each vector of the set. Problems concerning the completeness of a set of orthogonal vectors in a metric space are rather complex. Here we quote some of the results without proof.

1. There is a complete orthonormal set in any non empty inner-product space.
2. If H is a separable space then there exists in H a complete orthonormal set with at most denumerable elements.

Every finite or infinite set $\{h_1, h_2, \ldots, h_n, \ldots\}$ of linearly independent elements of H can be used to obtain a set of orthonormal elements by an orthogonalization process. This process, in essence, was described in an earlier chapter for inner produce spaces with finite dimensions.

In Section 1-13 we shall present a procedure for construction of a set of orthonormal vectors in a Hilbert space. A Hilbert space may contain a countable, or an uncountable number of such orthonormal vectors. In the first case, the space is a Hilbert space of countable dimension. Finite-dimensional spaces and spaces of countable dimensions can be also defined as separable spaces.

Example 1-8

Let h be an arbitrary point of H except 0; show that the set of all points of H orthogonal to h forms a subspace of H.

Solution: In fact, if $f \perp h$ and $g \perp h$, then due to the bilinear property of the inner product function, we find that $(\alpha f + \beta g) \perp h$.

1-11 Isomorphism of Separable Hilbert Spaces

The definition of isomorphism between two **separable** Hilbert spaces can be formulated as a generalization of the definition of isomorphism between finite dimensional Euclidean spaces. Two separable Hilbert spaces H and H' **are said** to be isomorphic if there exists a one-to-one correspondence between their elements such that:

1. If $\underset{\sim}{x}' \in H'$ corresponds to $\underset{\sim}{x} \in H$, and $\underset{\sim}{y}' \in H'$ corresponds to $\underset{\sim}{y} \in H$, then

   $\underset{\sim}{x}' + \underset{\sim}{y}' \in H'$ should correspond to $\underset{\sim}{x} + \underset{\sim}{y} \in H$.

2. For any arbitrary number λ of the field,

   $\lambda \underset{\sim}{x}' \in H'$ should correspond to $\lambda \underset{\sim}{x} \in H$.

Moreover, the two spaces are said to be isometric if

3. $d(\underset{\sim}{x},\underset{\sim}{y}) = d(\underset{\sim}{x}',\underset{\sim}{y}')$

It is not difficult to show that all Hilbert spaces of countable dimensions are isomorphic to each other. In particular, the spaces $L^2[a,b]$ and ℓ^2 are isomorphic and isometric.

Let $\{\underset{\sim}{e}_1,\underset{\sim}{e}_2,\ldots\}$ be a complete orthonormal system in a separable Hilbert space H, and $\underset{\sim}{x}$ an arbitrary element of H. Then $\underset{\sim}{x}$ admits the (Fourier) representation:

$$\underset{\sim}{x} = \sum_{k=1}^{\infty} x_k \, \underset{\sim}{e}_k$$

where

$$x_k = (\underset{\sim}{x}, e_k) \text{ and } \sum_{k=1}^{\infty} |x_k|^2 < \infty$$

We can establish a one-to-one correspondence between points of H and ℓ^2, for instance by selecting $\underset{\sim}{x}' \in \ell^2$ to correspond to $\underset{\sim}{x} \in H$ such that $\underset{\sim}{x}'$ has the same coordinates as $\underset{\sim}{x}$, i.e.,

$$\underset{\sim}{x} = \{x_1, x_2, \ldots\} \qquad \text{relative to } \{\underset{\sim}{e}_1, \underset{\sim}{e}_2, \ldots\}$$

$$\underset{\sim}{x}' = \{x_1, x_2, \ldots\} \qquad \text{relative to } \{\underset{\sim}{e}_1', \underset{\sim}{e}_2', \ldots\}$$

This correspondence obviously establishes an isomorphism between H and ℓ^2. In order to see whether the two spaces are isometric, let us calculate the inner product function for two arbitrary elements $\underset{\sim}{x}$ and $\underset{\sim}{y}$ of H, and their corresponding images $\underset{\sim}{x}'$ and $\underset{\sim}{y}'$ of ℓ^2. With some mathematical care about the convergence of the incurring sequences, we find:

$$(\underset{\sim}{x}', \underset{\sim}{y}') = \left(\sum_{k=1}^{\infty} x_k \underset{\sim}{e}_k' \; , \; \sum_{k=1}^{\infty} y_k \underset{\sim}{e}_k' \right) = \sum_{k=1}^{\infty} x_k \bar{y}_k$$

$$(\underset{\sim}{x}, \underset{\sim}{y}) = \int_a^b x(t)\, \bar{y}(t)\, dt = \int_a^b \left(\sum_{k=1}^{\infty} x_k \underset{\sim}{e}_k \right) \left(\sum_{k=1}^{\infty} \bar{y}_k \bar{\underset{\sim}{e}}_k \right) dt = \sum_{k=1}^{\infty} x_k \bar{y}_k$$

Thus, any separable Hilbert space is isometric with ℓ^2.

The separability argument enters the discussion in view of the necessity of representing every vector of the space by its coordinates (the so-called Fourier coefficients). (See, for example, Liusternik-Sobolev, page 79 of English translation, Ungar Publishing Co.)

1-12 Projection of a Point on a Subspace

In many problems concerned with application, one wishes to find the "shortest" distance between a given point $\underset{\sim}{x}$ of a Hilbert space and a subspace S of that space. An intuitive, but rather illuminating way of looking into this problem is to search for the point $\underset{\sim}{y}$ which may be intuitively referred to as the "projection" of $\underset{\sim}{x}$ on S. This is schematically illustrated in the figure below. In a **rigorous** mathematical approach, one has to show that there exists **a unique** point $\underset{\sim}{y} \in S$ which could justify the definition of projection. To fulfill this, we require (a) that $\underset{\sim}{x} - \underset{\sim}{y}$ be orthogonal to the subspace S; and (b) that $\underset{\sim}{y}$ be the "nearest" point of S to $\underset{\sim}{x}$, in symbols:

$$\text{(a)} \quad x - y \perp y' \qquad\qquad y' \in S \tag{1-63}$$

$$(\underset{\sim}{x} - \underset{\sim}{y}, \underset{\sim}{y}') = 0$$

$$\text{(b)} \quad \|\underset{\sim}{x}-\underset{\sim}{y}\| = \inf_{y' \in S} \|\underset{\sim}{x}-\underset{\sim}{y}'\| \leq \|\underset{\sim}{x}-\underset{\sim}{y}'\| \tag{1-64}$$

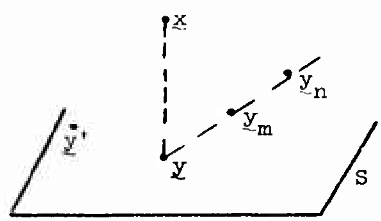

Fig. 1-5. Projection on a Subspace

We prove formally:

<u>Projection Theorem</u>. Given $\underset{\sim}{x} \in H$ and S a complete subspace of H, there exists a unique point $\underset{\sim}{y} \in S$ satisfying requirements a and b above.

Since $\|x-y'\|$, (for all $y' \in S$) is bounded below by 0, it has a greatest lower bound, $\inf \|\underset{\sim}{x}-\underset{\sim}{y}'\| = d \geq 0$. Hence there exist a sequence $d_n = \|\underset{\sim}{x}-\underset{\sim}{y}_n\|^2$ which converges to d. Let $\{y_n\}$ be a sequence in S such that $d_n = \|\underset{\sim}{x}-\underset{\sim}{y}_n\|^2$ approaches d as $n \to \infty$. Furthermore, let $\underset{\sim}{h}$ be an arbitrary element of S, and λ an arbitrary complex number. Of course, $y_n + \lambda h$ is an element of S and by hypothesis

$$\|x - (y_n + \lambda h)\|^2 \geq d \tag{1-66}$$

Since the space possesses an inner product, we may write:

$$\|x-y_n\|^2 - \lambda(h,x-y_n) - \bar{\lambda}(x-y_n,h) + |\lambda|^2 \; \|h\|^2 \geq d \tag{1-67}$$

This inequality must hold for any desired value of λ. In particular, letting

$$\lambda = \frac{(x-y_n,h)}{\|h\|^2} \tag{1-68}$$

yields

$$\|x-y_n\|^2 - \frac{|(x-y_n,h)|^2}{\|h\|^2} \geq d \tag{1-69}$$

$$(d_n-d) \; \|h\|^2 \geq |(x-y_n,h)|^2 \tag{1-70}$$

$$\|h\| \sqrt{d_n-d} \geq |(x-y_n,h)| \tag{1-71}$$

Thus, for any y_n and y_m of the S we have

$$\begin{aligned} |(y_n-y_m,h)| &\leq |(y_n-x,h)| + |(x-y_m,h)| \leq \\ &\left(\sqrt{d_n-d} + \sqrt{d_m-d}\right) \quad \|h\| \end{aligned} \tag{1-72}$$

For $h = y_n - y_m$, we find

$$\|y_n - y_m\| \leq \sqrt{d_n-d} + \sqrt{d_m-d} \tag{1-73}$$

This inequality leads to the conclusion that the sequence $y_1, y_2, \ldots, y_n, \ldots$ is a fundamental sequence, and converges to a point y. We will now show* that $\underline{y} \in S$ is in fact the desired projection point defined by $d = \|\underline{x}-\underline{y}\|$ $= \inf_{\underline{y}' \in S} \|\underline{x}-\underline{y}'\|$ as follows:

$$\|\underline{x}-\underline{y}\| \leq \|\underline{x}-\underline{y}_n\| + \|y-y_n\| \qquad (1\text{-}74)$$

Taking the limit of both sides yields:

$$\|\underline{x}-\underline{y}\| \leq d + 0 \qquad (1\text{-}75)$$

but by hypothesis $\|\underline{x}-\underline{y}\| \geq d$ for all $\underline{y} \in S$, hence

$$\|x-y\| = d \qquad (1\text{-}76)$$

Taking the limit in inequality 1-71 yields:

$$(\underline{x}-\underline{y},\underline{h}) = 0$$

$$\underline{x}-\underline{y} \perp \underline{h}$$

Since $\underline{h}$ is an arbitrary element of S, it follows that $\underline{x}-\underline{y} \perp S$.

In order to show the uniqueness of this element $\underline{y}$, let $\underline{x}-\underline{y} = \underline{z}$ and assume that $\underline{y}' \neq \underline{y}$ is another point of S with the desired property. We have

$$\underline{x} = \underline{y} + \underline{z} \qquad \underline{y} \in S \quad , \qquad \underline{z} \perp S \qquad (1\text{-}77)$$

$$\underline{x} = \underline{y}' + \underline{z}' \qquad \underline{y}' \in S \quad , \qquad \underline{z}' \perp S \qquad (1\text{-}78)$$

These relations suggest that

$$\underline{y} - \underline{y}' \in S \quad , \qquad\qquad \underline{z} - \underline{z}' \perp S \qquad (1\text{-}79)$$

whence

$$\|\underline{y}-\underline{y}'\|^2 = (\underline{y}-\underline{y}',\underline{y}-\underline{y}') = (\underline{z}'-\underline{z},\underline{y}-\underline{y}') = 0 \qquad (1\text{-}80)$$

The latter equality implies that $\underline{y}$ and $\underline{y}\_$ are coincident. This unique point $\underline{y}$ with the described property is the projection of $\underline{x}$ on the subspace S.

---

*An alternative proof may be based on the continuity of the inner product, that is:

$$d = \lim_{n\to\infty} d_n = \lim_{n\to\infty} \|\underline{x}-\underline{y}_n\| = \|\underline{x} - \lim_{n\to\infty} \underline{y}_n\| = \|\underline{x}-\underline{y}\|$$

In particular, if the subspace S is the linear manifold generated by a sole vector g, then the projection of $\underset{\sim}{x}$ on that manifold is the vector $\underset{\sim}{y}$ such that

$$(\underset{\sim}{x}-\underset{\sim}{y},\underset{\sim}{g}) = 0 \tag{1-81}$$

Now $\underset{\sim}{y}$ can be expressed as $\underset{\sim}{y} = \lambda g$, whence

$$\lambda = \frac{(\underset{\sim}{x},\ \underset{\sim}{g})}{(\underset{\sim}{g},\ \underset{\sim}{g})} \tag{1-82}$$

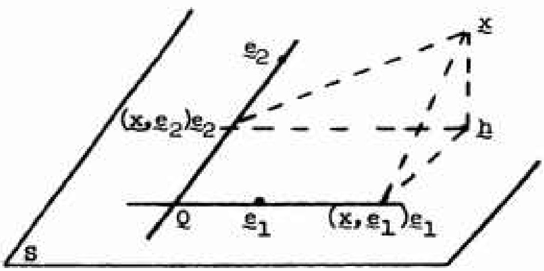

Fig. 1-7
Projection on a multi-dimensional subspace

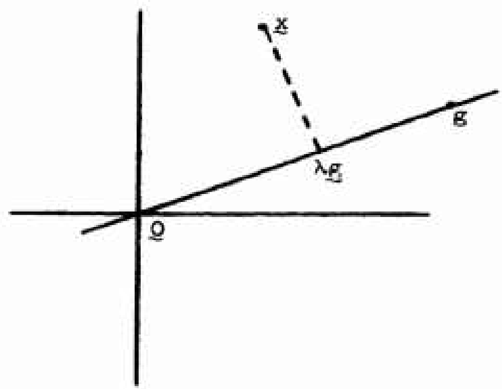

Fig. 1-6
Projection on a one-dimensional subspace

The projection of $\underset{\sim}{x}$ on $\underset{\sim}{g}$ is:

$$\text{Proj. } \underset{\sim}{x} \text{ on } \underset{\sim}{g} = \frac{(\underset{\sim}{x},\ \underset{\sim}{g})}{(\underset{\sim}{g},\ \underset{\sim}{g})}\, \underset{\sim}{g} \tag{1-83}$$

Moreover, when $\underset{\sim}{g}$ is a vector of unit length, one arrives at the simplified familiar relation

$$\text{Proj. } \underset{\sim}{x} \text{ in } \underset{\sim}{g} = (\underset{\sim}{x},\underset{\sim}{g})\ \underset{\sim}{g} \tag{1-84}$$

If a subspace S is generated by the linear manifold $\sum_{k=1}^{n} a_k \underset{\sim}{e}_k$, where $\underset{\sim}{e}_k$ is an orthonormal set, it is not hard to arrive at the simpler result

$$\text{Proj. } \underset{\sim}{x} \text{ on } S = \sum_{k=1}^{n} (x, \underset{\sim}{e}_k)\, \underset{\sim}{e}_k \tag{1-85}$$

1-13 A Procedure of Orthonormalization

Two sets of vectors in a Hilbert space are said to be equivalent if and only if each element of one set is a finite linear combination of the elements of the other set. If every pair of vectors in a set of elements of a Hilbert space are mutually orthogonal vectors, then the set is said to be an orthogonal set. Moreover, if the elements of an orthogonal set have unit length, the set will be referred to as an orthonormal set.

In the sequel a method is presented for constructing an equivalent orthonormal set, $\{\underset{\sim}{e}_1, \underset{\sim}{e}_2, \ldots, \underset{\sim}{e}_n, \ldots\}$ for a given finite or infinite set of independent vectors $\{\underset{\sim}{g}_1, \underset{\sim}{g}_2, \ldots, \underset{\sim}{g}_n, \ldots\}$. Let

$$e_1 = \frac{\underset{\sim}{g}_1}{\|\underset{\sim}{g}_1\|} \tag{1-86}$$

Denote by E_1 the one-dimensional space generated by either vector $\underset{\sim}{e}_1$ or $\underset{\sim}{g}_1$. Next, find the projection of $\underset{\sim}{g}_2$ on space E_1, and calculate the non zero vector $\underset{\sim}{h}_2$ orthogonal to E_1.

$$\underset{\sim}{h}_2 = \underset{\sim}{g}_2 - (\underset{\sim}{g}_2, \underset{\sim}{e}_1)\, \underset{\sim}{e}_1 \neq 0 \tag{1-87}$$

Thus $\underset{\sim}{h}_2$ and $\underset{\sim}{e}_1$ are orthogonal pairs, and $h_2 \neq 0$ due to linear independence of $\underset{\sim}{g}_1$ and $\underset{\sim}{g}_2$. In order to construct an orthonormal pair, $\{\underset{\sim}{e}_2, \underset{\sim}{e}_1\}$ let simply

$$\underline{e}_2 = \frac{\underline{h}_2}{\|\underline{h}_2\|} \tag{1-88}$$

Now $\underline{e}_1$ and $\underline{e}_2$ are orthonormal basis of a two-space E_2. We proceed likewise by finding the projection of g_3 on E_2, and by continuing the process described earlier we find a vector $\underline{h}_3 \neq 0$ orthogonal to E_2.

$$\underline{h}_3 = \underline{g}_3 - (\underline{g}_3, \underline{e}_1)\underline{e}_1 - (\underline{g}_3, \underline{e}_2)\underline{e}_2 \neq 0 \tag{1-89}$$

Normalization yields

$$\underline{e}_3 = \frac{\underline{h}_3}{\|\underline{h}_3\|} \tag{1-90}$$

The vector $\underline{e}_n$, for any positive integer, can be constructed in a similar fashion.

$$\underline{h}_n = \underline{g}_n - \sum_{k=1}^{n-1} (\underline{g}_n, \underline{e}_k)\underline{e}_k \neq 0 \tag{1-91}$$

$$\underline{e}_n = \frac{\underline{h}_n}{\|\underline{h}_n\|} \tag{1-92}$$

A set of orthonormal vectors of H is said to be a <u>complete</u> orthonormal system, if the system is maximal, that is, there exists no other element different from 0 and orthogonal to every Hilbert space. For example, for $L^2[-\pi, \pi]$ by using vectors of the form sin nt, cos nt, we can derive the following complete orthonormal system:

$$\frac{1}{\sqrt{2\pi}}, \frac{\cos t}{\sqrt{\pi}}, \frac{\sin t}{\sqrt{\pi}}, \frac{\cos 2t}{\sqrt{\pi}}, \frac{\sin 2t}{\sqrt{\pi}}, \ldots, \frac{\cos nt}{\sqrt{\pi}}, \frac{\sin nt}{\sqrt{\pi}}, \ldots \tag{1-93}$$

It is impossible to add any other orthonormal non-identically zero element to this system.

1-14 Fourier Representation in a Hilbert Space

As discussed before, the problem of representation is to ascertain the possibility of characterizing every element of the space as a linear combination of the elements of a given basis of that space. In other words. to represent a vector by its so-called "coordinates" with respect to a certain basis. Accordingly, in this section we study the representation of any vector $\underline{h} \in H$ with respect to a complete orthonormal set of vectors

$$\{\underline{e}\} = \{\underline{e}_1, \underline{e}_2, \ldots, \underline{e}_n, \ldots\} .$$

It will be shown that the problem of determination of the coordinates of $\underline{h}$, and the problem of least square approximation of $\underline{h}$ by a linear combination of elements of $\underline{e}$ are essentially the same. Both these problems in turn coincide with the finding of the projection of $\underline{h}$ on the subspace specified by the orthonormal set.

The problem is to search for the "best" linear expansion of the vector $\underline{h}$ in terms of the elements of the specified orthonormal set

$$\underline{h} \sim \sum_{i=1}^{\infty} a_i \, \underline{e}_i \tag{1-94}$$

What are the "best" coordinates $\{a_1, a_2, \ldots, a_n, \ldots\}$ for representation of $\underline{h}$? By the "best" it is meant that no other set of $\{a_1, a_2, \ldots, a_n\}$ should lead to a lower error in norm, i.e., an error smaller than:

$$\|\underline{h} - \sum_{i=1}^{\infty} a_i \, \underline{e}_i\| \tag{1-95}$$

This is the problem of least square approximation, or Fourier series expansion in Hilbert space.

We begin by studying the problem of the least square approximation in spaces of finite dimensions.

Theorem

Let $\underset{\sim}{h}$ be a vector of a finite-dimensional inner product space with an orthonormal set of vectors $\{\underset{\sim}{e}_1, \underset{\sim}{e}_2, \ldots, \underset{\sim}{e}_n\}$, the minimum of the expression

$$\|\underset{\sim}{h} - \sum_{i=1}^{n} a_i \underset{\sim}{e}_i\| \tag{1-96}$$

over all possible values of a_i's corresponds to the selection of

$$a_i = (\underset{\sim}{h}, \underset{\sim}{e}_i) \qquad i=1,2,\ldots,n \tag{1-97}$$

For any other values of a_i's we will have

$$\|\underset{\sim}{h} - \sum_{i=1}^{n} (\underset{\sim}{h}, \underset{\sim}{e}_i)\underset{\sim}{e}_i\| < \|\underset{\sim}{h} - \sum_{i=1}^{n} a_i \underset{\sim}{e}_i\| \tag{1-98}$$

Proof

$$\|\underset{\sim}{h} - \sum_{i=1}^{n} a_i \underset{\sim}{e}_i\|^2 = (\underset{\sim}{h} - \sum_{i=1}^{n} a_i \underset{\sim}{e}_i \,,\, \underset{\sim}{h} - \sum_{i=1}^{n} a_i \underset{\sim}{e}_i) \tag{1-99}$$

$$= (\underset{\sim}{h}, \underset{\sim}{h}) - \sum_{i=1}^{n} a_i (\underset{\sim}{e}_i, \underset{\sim}{h}) - \sum_{i=1}^{n} \bar{a}_i (\underset{\sim}{h}, \underset{\sim}{e}_i)$$

$$+ \sum_{i=1}^{n} \sum_{j=1}^{n} a_i \bar{a}_j \, (\underset{\sim}{e}_i, \underset{\sim}{e}_j)$$

$$= (\underset{\sim}{h}, \underset{\sim}{h}) - \sum_{i=1}^{n} a_i (\underset{\sim}{e}_i, \underset{\sim}{h}) - \sum_{i=1}^{n} \bar{a}_i (\underset{\sim}{h}\underset{\sim}{e}_i)$$

$$+ \sum_{i=1}^{n} |a_i|^2$$

By adding and subtracting the expression $\sum_{i=1}^{n} (e_i, h)(h, e_i)$ to the right side, one finds

$$\|h - \sum_{i=1}^{n} a_i e_i\|^2 = (h,h) - \sum_{i=1}^{n} |(h,e_i)|^2 + \sum_{i=1}^{n} |a_i - (h,e_i)|^2$$

The first two terms of the right side do not depend on the coefficients a_i; therefore, the minimization of the left side expression requires that

$$a_i = (h, e_i) \qquad i=1,2,\ldots,n \tag{1-100}$$

Furthermore, the solution to this problem of least square approximation is unique, and

$$\underset{a_i}{\text{minimum}} \|h - \sum_{i=1}^{n} a_i e_i\|^2 = \|h\|^2 - \sum_{i=1}^{n} |(h,e_i)|^2 = \delta^2 \tag{1-101}$$

This is the square of the distance δ of the point h to the linear manifold spanned by an orthonormal set $\{e_1, e_2, \ldots, e_n\}$. Since the minimum of a distance cannot be a negative quantity, we have

$$\sum_{i=1}^{n} |(h,e_i)|^2 \leq \|h\|^2 \tag{1-102}$$

This inequality is referred to as Bessel's inequality. The preceding theorem can be generalized to the case of infinite-dimensional Hilbert space.

Let $\{e\} = \{e_1, e_2, \ldots, e_n, \ldots\}$ be a complete set of orthonormal elements of a Hilbert space H, and L the linear manifold generated by the first n vectors of $\{e\}$. For any given element h and specified $\epsilon > 0$, we can approximate h by a linear combination of elements of L

$$\sum_{i=1}^{n} a_i \underset{\sim}{e}_i ,$$

such that

$$\|\underset{\sim}{h} - \sum_{i=1}^{n} a_i \underset{\sim}{e}_i\| < \epsilon \qquad \text{for } n > N, \text{ some } N$$

That is, this representation converges in the metric. The least value of the expression to the left is obtained by the so-called Fourier coefficients, i.e., when

$$a_i = (\underset{\sim}{h}, \underset{\sim}{e}_i) \qquad i=1,2,\ldots,n \tag{1-103}$$

We now show that the series $\sum_{k=1}^{\infty} a_k \underset{\sim}{e}_k$ converges. Let $\underset{\sim}{s}_p$ and $\underset{\sim}{s}_q$ be respectively the sum of the first p and the first q terms of the Fourier series. For $q > p$ we will have

$$\|\underset{\sim}{s}_q - \underset{\sim}{s}_p\|^2 = \|\sum_{p+1}^{q} a_k \underset{\sim}{e}_k\|^2 = \sum_{p+1}^{q} |a_k|^2 \tag{1-104}$$

As p and q are increased infinitely the sum $\sum_{p+1}^{q} |a_k|^2$ converges to zero.

Therefore, the sums $\underset{\sim}{s}_p$ form a fundamental sequence in the H-space. Let $\underset{\sim}{s}$ be the limit for the sequence. Since the space H is required to be complete, the point $\underset{\sim}{s}$ must belong to H. The fact that h = s can be seen from the following. In view of the continuity of the inner product, the Fourier coefficient $(\underset{\sim}{h}, \underset{\sim}{e}_m)$ can be obtained as follows:

$$(\underset{\sim}{s}, \underset{\sim}{e}_m) = \lim_{p \to \infty} (\underset{\sim}{s}_p, \underset{\sim}{e}_m) = \lim_{p \to \infty} \left(\sum_{i=1}^{p} a_i \underset{\sim}{e}_i, \underset{\sim}{e}_m \right) \tag{1-105}$$

$$= a_m = (\underset{\sim}{h}, \underset{\sim}{e}_m)$$

For any fixed m and $p > m$, we find

$$(\underline{h}-\underline{s},\underline{e}_m) = (\underline{h},\underline{e}_m) - (\underline{s},\underline{e}_m) = 0 \tag{1-106}$$

The identity of $\underline{h}$ and $\underline{s}$ follows from this relation and the fact that the set $\{e\}$ is complete. Whence,

$$\underline{h} = \lim_{p \to \infty} \underline{s}_p = \sum_{n=1}^{\infty} a_n \underline{e}_n \tag{1-107}$$

<u>Theorem</u>

Let H be a separable Hilbert space, with a complete set of orthonormal vectors $\{\underline{e}_1,\underline{e}_2,\underline{e}_3,\dots\}$. We can associate with any element $\underline{x} \in H$ a Fourier series $\sum_{k=1}^{\infty} (\underline{x},\underline{e}_k)\,\underline{e}_k$ converging in the norm to $\underline{x}$, and such that the series $\sum_{k=1}^{\infty} |(\underline{x},\underline{e}_k)|^2$ converges to

$$\sum_{k=1}^{\infty} |(\underline{x},\underline{e}_k)|^2 = (\underline{x},\underline{x}) \tag{1-108}$$

Alternatively, we state that every vector $\underline{x} \in H$ can be approximated as closely as desired, in the mean square sense, by the above Fourier series. An element can be exactly expressed by a Fourier sum $\sum_{k=1}^{\infty} a_k \underline{e}_k$ if Bessel's inequality is changed into the so-called Parseval's equality; that is,

$$\sum_{k=1}^{\infty} |(\underline{x},\underline{e}_k)|^2 = (\underline{x},\underline{x}) \tag{1-109}$$

Conversely, one may state a criterion for completeness of a set of orthonormal vectors in a Hilbert space as follows:

Theorem

In order that the set of orthonormal vectors $\underset{\sim}{e}$ be a complete set, it is sufficient that Parseval's equation holds for every $\underset{\sim}{x} \in H$.

$$(\underset{\sim}{x},\underset{\sim}{x}) = \|\underset{\sim}{x}\|^2 = \sum_{k=1}^{\infty} |(\underset{\sim}{x},\underset{\sim}{e}_k)|^2$$

If H is also a complete space (as in our present case), then this condition is necessary and sufficient for completeness of $\{\underset{\sim}{e}\}$.*

In case of the space $L^2[a,b]$, any arbitrary nonzero vector f(t) may be represented in terms of its Fourier expansion as:

$$f(t) \sim \sum_{k=1}^{\infty} a_k e_k(t)$$

But this representation does not necessarily converge. However, when $\{\underset{\sim}{e}_k(t)\}$ is a complete orthonormal set, then it is well known that the series will converge.

In the spirit of the elementary scope of this chapter we have primarily considered separable Hilbert spaces, which are obtainable as a simple extension of the Euclidean space. In reality, Hilbert space may have countable dimensions. Then the left side of the Bessel inequality, Eq. (1-102), will represent an uncountable set. The impact of such a generalization, on the selection of a complete orthonormal set, and related convergence consideration for Fourier expansion, will require a full understanding of Lebesgue integral; this is beyond our plans.

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*The completeness of a metric space X and the completeness of a set of orthonormal vectors in X are to be distinguished.

In brief, as long as we remain content with a direct countable infinite-dimensional generalization of the Euclidean space, things are under control. A step beyond that into the most general type of Hilbert and Banach space requires more specialized mathematical preparations.

1-15 Representation by Fourier Trigonometric Polynomials in a Function Space

In this section we will apply the content of the previous section to a problem of common interest. Let h(t) be an arbitrary vector of the real function space $L^2[0,2\pi]$. It is desired to obtain the best Fourier representation of h(t), in the sense of minimizing the norm of error, in terms of real trigonometric polynomials of degree n or less:

$$f_n(t) = \frac{a_0}{2} + a_1 \cos t + b_1 \sin t + \ldots + a_n \cos nt + b_n \sin nt \tag{1-110}$$

The totality of these polynomials forms a subspace S of dimension 2n+1 over the field of real numbers.

We set forth to derive the trigonometric polynomials $f_n(t)$ which is the best linear approximation to h(t) in the least square sense, that is minimizing:

$$\|\underline{h}-\underline{f}_n\|^2 = \sum_0^{2\pi} [h(t)-f_n(t)]^2 \, dt \tag{1-111}$$

In other words, the problem is that of projecting $\underline{h}$ on S. Let us choose a normalized version of the common Fourier trigonometric series as the maximal set $\{\underline{e}\}$ of an orthonormal basis for S,

$$\underline{e}_0 = \frac{1}{\sqrt{2\pi}}, \quad \underline{e}_1 = \frac{\cos t}{\sqrt{\pi}}, \quad \underline{e}_2 = \frac{\sin t}{\sqrt{\pi}}, \quad \ldots,$$
$$e_{2n-1} = \frac{\cos nt}{\sqrt{\pi}}, \quad e_{2n} = \frac{\sin nt}{\sqrt{\pi}} \tag{1-112}$$

It is to be noted that the orthonormal set $\{\underline{e}\}$ is "complete", that is, no additonal non zero vector of S can be found which is linearly independent of the elements of $\{\underline{e}\}$. Whence,

$$\text{Prog. } \underline{h} \text{ on } S = \sum_{k=0}^{2n} c_k \underline{e}_k \tag{1-113}$$

where

$$c_k = (\underline{e}_k, \underline{h}) \tag{1-114}$$

Recalling the usual definition of the inner product, we find

$$c_0 = \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} h(t)\, dt, \quad c_{2k-1} = \frac{1}{\sqrt{\pi}} \int_0^{2\pi} h(t) \cos kt\, dt$$
$$c_{2k} = \frac{1}{\sqrt{\pi}} \int_0^{2\pi} h(t) \sin kt\, dt \tag{1-115}$$

These constants specify the best trigonometric polynomial approximation for elements of $L^2[0,2\pi]$ relative to the $\{\underline{e}\}$ basis. Since the question was formulated relative to $\underline{f}_n$, from these relations we derive the familiar Fourier series coefficients:

$$a_0 = \sqrt{\frac{2}{\pi}}\, c_0 = \frac{1}{\pi} \int_0^{2\pi} h(t)\, dt \tag{1-116}$$

$$a_k = \sqrt{\frac{1}{\pi}}\, c_{2k-1} = \frac{1}{\pi} \int_0^{2\pi} h(t) \cos kt\, dt$$
$$b_k = \sqrt{\frac{1}{\pi}}\, c_{2k} = \frac{1}{\pi} \int_0^{2\pi} h(t) \sin kt\, dt \tag{1-117}$$

The square of the error in approximating $\underline{h}$ by $\underline{f}$ is:

$$\|\underline{h}-\underline{f}_n\|^2 = \|\underline{h}\|^2 - \sum_{i=0}^{2n} |(\underline{h}\underline{e}_i)|^2 \tag{1-118}$$

No other trigonometric polynomial of the above type $f_m(t)$ with $m \leq n$ may offer a lower error. When n is increased, the error is reduced and when $n \to \infty$ the converging Fourier series expansion approaches the exact representation of $\underline{h}$.

Example 1-9

Calculate the Fourier coefficients for the function

$$f(x) = x \qquad -\pi < x < \pi$$

in $[-\pi, \pi]$ using the orthonormal set

$$\underline{e}_1 = \frac{\sin x}{\sqrt{\pi}}, \ \underline{e}_2 = \frac{\sin 2x}{\sqrt{\pi}}, \ \underline{e}_3 = \frac{\sin 3x}{\sqrt{\pi}}, \ \ldots$$

Solution:

$$c_k = (\underline{f}, e_k) = \frac{1}{\pi} \int_{-\pi}^{\pi} x \sin kx \, dx \qquad k=1,2,\ldots$$

$$= \frac{1}{\sqrt{r}} \left[- \frac{x \cos kx}{k} \Bigg| \int_{-\pi}^{\pi} + \frac{1}{k} \int_{-\pi}^{\pi} \cos kx \, dx \right]$$

$$= \frac{2\sqrt{\pi}}{\lambda} \cos kx$$

The projection of $\underline{x}$ on the linear manifold of the above orthonormal set is:

$$x \sim 2 \left(\sin x - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \ldots\right)$$

This orthonormal set is not a basis for the infinite dimensional space. A complete set of orthonormal basis containing the above set can be obtained by the addition of elements

$$\left\{ \frac{1}{\sqrt{2\pi}}, \frac{\cos x}{\sqrt{\pi}}, \frac{\cos 2x}{\sqrt{2\pi}}, \ldots \right\}$$

But the projection of $\underset{\sim}{x}$ on the latter set is the vector $\underset{\sim}{0}$. Therefore, the exact representation of $\underset{\sim}{x}$ in the infinite-dimensional space is

$$\underset{\sim}{x} = 2 \sum_{k=1}^{\infty} (-1)^{k-1} \frac{\sin kx}{k}$$

1-16 Optimality and Closed Convex Sets

A set of points X in a vector space is said to be convex if for any arbitrary pairs $\underset{\sim}{x}$ and $\underset{\sim}{y}$ of X and $0 < t < 1$ we have $\underset{\sim}{z} = [(1-t)\underset{\sim}{x} + t\underset{\sim}{y}] \in X$. A simple geometric interpretation of this property may be visualized by thinking that all points $\underset{\sim}{z}$ on the line segment connecting $\underset{\sim}{x}$ and $\underset{\sim}{y}$ and between $\underset{\sim}{x}$ and $\underset{\sim}{y}$ must belong to X. The figure below sketches a non-convex set of points:

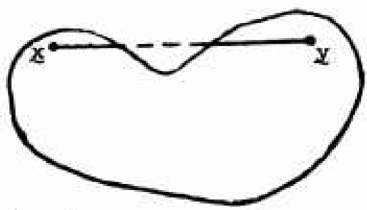

Fig. 1-8 A non-convex set of points

In view of this definition the following important theorem may be proved.

Theorem

Every non-empty closed convex set X in a Hilbert space H contains a unique elements $\underset{\sim}{h}_0$ of smallest norm.

Proof

Let $\delta = \inf \|\underset{\sim}{x}\|$ for $\underset{\sim}{x} \in X$ denote the least value of $\|\underset{\sim}{x}\|$ for all points in the convex set X. Consider a sequence $\{\underset{\sim}{x}_n\}$ of points in X such that $\|\underset{\sim}{x}_n\| \to \delta$ and $n \to \infty$. A proof of the theorem requires the following two steps:

1. $\{\underset{\sim}{x}_n\}$ is a Cauchy sequence.
2. $\underset{\sim}{x}_n \to \underset{\sim}{h}_0$ and $n \to \infty$.

For any arbitrary points $\underline{x}$ and $\underline{y}$ of H in view of the existence of an inner product, the so-called parallelogram law gives:

$$\|\underline{x}+\underline{y}\|^2 + \|\underline{x}-\underline{y}\|^2 = 2\|\underline{x}\|^2 + 2\|\underline{y}\|^2 \qquad \underline{x},\underline{y} \in H \tag{1-119}$$

$$\frac{1}{4}\|\underline{x}-\underline{y}\|^2 = \frac{1}{2}\|\underline{x}\|^2 + \frac{1}{2}\|\underline{y}\|^2 - \left\|\frac{\underline{x}+\underline{y}}{2}\right\|^2 \tag{1-120}$$

Now by considering points $\underline{x}$, $\underline{y}$ in X, in view of the convexity of X, the vector $\frac{\underline{x}+\underline{y}}{2}$ will be also in X, whence:

$$\|\underline{x}-\underline{y}\|^2 \leq 2\|\underline{x}\|^2 + 2\|\underline{y}\|^2 - 4\delta^2 \qquad \underline{x},\underline{y} \in X \tag{1-121}$$

This inequality may be applied for points $\underline{x}_n$ and $\underline{x}_m$ for showing that $\{\underline{x}_n\}$ is a Cauchy sequence:

$$\|\underline{x}_n-\underline{x}_m\|^2 \leq 2\|\underline{x}_n\|^2 + 2\|\underline{x}_m\|^2 - 4\delta^2 \tag{1-122}$$

As $n \to \infty$ and $m \to \infty$ the right side of this inequality by definition tends to zero. Therefore $\{\underline{x}_n\}$ is a Cauchy sequence. The limit point of this sequence must be in H since H is a complete space. Let $\underline{h}_0$ be the limit point of the sequence $\{\underline{x}_n\}$, then

$$\|\underline{x}_n-\underline{h}_0\| \to 0 \qquad \text{as } n \to \infty \tag{1-123}$$

The point $\underline{h}_0$ belongs to X since X is assumed to be a closed convex set. It remains to show that $\|\underline{h}_0\| = \lim_{n \to \infty} \|\underline{x}_n\| = \delta$, i.e., the norm is a continuous function. To prove this consider once again the fundamental triangle inequalities:

$$\|\underline{x}\| \leq \|\underline{x}-\underline{y}\| + \|\underline{y}\| \tag{1-124}$$

$$\|\underline{x}\| - \|\underline{y}\| \leq \|\underline{x}-\underline{y}\| \tag{1-125}$$

If $\underline{x}$ approaches $\underline{y}$, the right side tends to zero. Thus, $\|\underline{x}\|$ is indeed a continuous function on H, whence

$$\|h_0\| = \lim_{n \to \infty} \|x_n\| = \delta \tag{1-126}$$

To show that h_0 is unique, let $\|h_0\| = \|f\| = \delta$, then the inequality (1-120) yields:

$$\begin{aligned} \|h_0 - f\|^2 &\leq 0 \\ h_0 &= f \end{aligned} \tag{1-127}$$

The importance of this theorem is problems of application is self-evident. Whenever we have a closed convex set X in a Hilbert space, we can find a unique element of smallest norm. If points in X portray the performance of a physical system then h_0 describes the most "<u>efficient performance</u>" among all $x \in X$. Many problems of optimal control fall in this category. The challenge to the reader is to identify such a clear mathematical model in the heart of the tedious technical literature of application where no main path is visible.

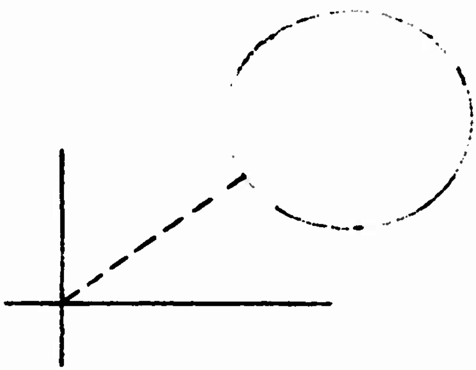

Fig. 1-9a The element of minimal norm is unique

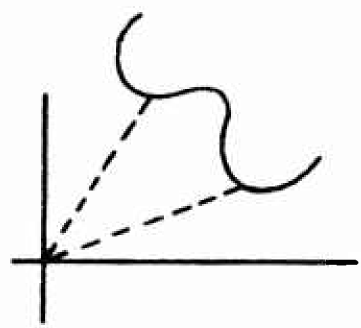

Fig. 1-9b The elements of minimal norms.

Chapter 2

LINEAR OPERATORS IN HILBERT SPACE

2-1. Introduction

In the context of engineering application, the material of the preceding two chapters offers a representation and an approximation theory for signals in a Hilbert space. In a similar vein, the material of this chapter is directed toward a representation theory for linear processing systems of a Hilbert signal-space.

The theory of operators is a powerful tool of functional analysis with broad applications to problems of engineering and physics. For example, in control theory, wave propagation and in quantum mechanics one is constantly faced with the problem of determining critical frequencies of a physical system which amounts to the determination of eigenfunctions of certain differential or integral equations. In most applications, of this type the concept of linear operator plays a very important role. Take for instance the familiar integral transformation $f(x) \to g(x)$

$$g(x) = \int_a^b k(x,y)\, f(y)\, dy \tag{2-1}$$

The correspondence between functions $g(x)$ and $f(x)$ can be denoted by the compact notation

$$\underset{\sim}{g} = A\, \underset{\sim}{f} \tag{2-2}$$

In this format one can readily appreciate that the solution to the equation

$$f(x) = \lambda \int_a^b k(x,y)\, f(y)\, dy \tag{2-3}$$

where $f(x)$ is an unknown function is coincident with solving an eigenvalue problem, since

$$\xi = \lambda A \xi$$
$$\frac{1}{\lambda} \xi = A \xi \qquad (2\text{-}4)$$

Likewise, the solution to the vector differential equation

$$\dot{x} = A x \qquad x(0) = c \qquad (2\text{-}5)$$

where A is a constant n x n matrix, was found to be

$$x = e^{At} c \qquad (2\text{-}6)$$

we have discussed in detail, that the solution vector can be obtained from the initial condition vector c by applying the linear operator e^{At}.

Some of the basic properties of linear operators on finite-dimensional vector spaces were discussed in Chapter 3 of TR-65-394. The information provided there was adequate for an introductory treatment of transformation of elements of a finite dimensional Euclidean space. More general and interesting results may be derived, however, if the content of that chapter is extended to linear operators in Hilbert and Banach spaces. The fulfillment of this aim is the object of the present chapter.

2-2. Functionals and Operators

By direct analogy with the functions defined in a finite-dimensional vector space, one can define scalar and vector functions over the elements of a linear space of infinite dimension. The first category of these functions is commonly referred to as functionals and the second type as operators.

Let D be a subset of the space X, to each point $\xi \in D$, one may attach as scalar valued function $\phi(\xi)$. This is a functional whose domain is D.

The definition of an operator in an infinite-dimensional space is essentially the same as in the case of a finite-dimensional space. Let X and Y be two linear spaces, and consider the set $D \subset X$; to every element $x \in S$, by a suitable rule, we associate one element $y \in Y$. The relationship between x and y may be designated by

$$y = A\, x \tag{2-7}$$

The set D is referred to as the domain of definition of the operator A. The set D_A of all corresponding image points y is called the range or the domain of values of the operator A. If the domain of definition covers the entire space X, and if $D_A \subset X$, then it is said that the operator A maps X into itself, or A is an operation on X.

In the following we shall concentrate on linear operators acting on linear normed spaces in general, and on Hilbert spaces for the most part. In this respect, first we need to define the linear operator, and then fully grasp the meaning of the two independent concepts, continuity, and boundedness for linear operators of a metric space.

2-3. Linear Operators

The definition of a linear operator mapping an infinite-dimensional real linear space X to Y stands essentially the same as in the case of a real finite dimensional space. There is one important difference however that in the finite-dimensional case the requirement of continuity does not present itself explicitly. In the infinite-dimensional case an operator is said to be linear if it is additive and continuous as described below.

a) Additivity. For every pair x_1, $x_2 \in X$ we require:

$$A(x_1 + x_2) = A\, x_1 + A\, x_2 \tag{2-8}$$

b) Continuity. If a sequence $x_n \in X$ converges to $x \in X$, then we require that the sequence $A\, x_n \in Y$ converges to $A\, x \in Y$ in the sense of the convergence in Y.

Based on properties a) and b) one can show that a linear operator is also homogeneous*, i.e., for every real number λ, $A(\lambda,\underline{x}) = \lambda\, A\, \underline{x}$.

For any arbitrary additive operator we have:

$$A(\underline{0}) = A(\underline{0} + \underline{0}) = A(\underline{0}) + A(\underline{0}) = \underline{0} \tag{2-9}$$

$$A(\underline{x}) + A(-\underline{x}) = A(\underline{x}-\underline{x}) = \underline{0}\ , \quad A(-\underline{x}) = -A(\underline{x}) \tag{2-10}$$

If X and Y are complex spaces, the additivity as described in Eq. (2-8) is insufficient. One has to add the requirement.

$$A(i\ \underline{x}) = i\ A(\underline{x})\ , \quad i = \sqrt{-1} \tag{2-11}$$

We shall state without proofs that a) every additive and homogeneous operator in any finite-dimensional normed space is a linear operator. b) Every linear operator is homogeneous. (See for instance B.Z. Vulikh: Introduction to Functional Analyses, English translation by I.N. Sneddon, Pergamon Press, New York 1963, Chapter 8).

2-4. Continuity, Boundedness, and Norm of Linear Operators

a) Continuity. Continuity of an operator in a metric space has a simple meaning. Its concept is based on the idea of "closeness" of two vectors $A\underline{x}$ and $A\underline{y}$ whenever $\underline{x}$ and $\underline{y}$ are close to each other in the metric space. We let a point $\underline{x}$ of the metric space E vary within a sphere S of arbitrary small radius δ centered at x_o,

$$\{S:\ \|x - x_o\| < \delta\} \tag{2-12}$$

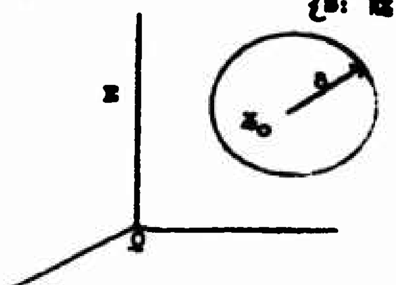

Fig. 2-1a

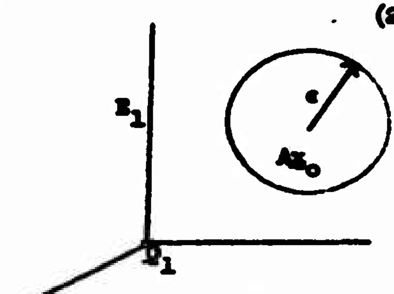

Fig. 2-1b

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* In general it is easier to check the homogeneity of an operator directly than by means of additivity.

A is said to be a continuous operator at x_o if the image of the sphere S by A remains within an arbitrarily small sphere S_a of the space E_1

$$\left\{S_a:\ \|Ax - Ax_o\| < \epsilon\right\} \tag{2-13}$$

In other words, if for any arbitrary $\epsilon > 0$ there exists a $\delta > 0$ such that the images of all points of S remain within S_a.

An operator continuous in a domain should possess this property for all points in that domain.

An additive operator has an additional interesting property that if it is continuous at any single point $x_o \in X$ then it will be continuous for the whole space E; and therefore it is a linear operator. In fact, let x be the limit of a sequence $\{x_n,\}$ then $\{x_n - x + x_o\}$ approaches x_o for $n \to \infty$. Therefore;

$$A(x_n - x + x_o) \to Ax_o \tag{2-14}$$

$$Ax_n - Ax + Ax_o \to Ax_o \tag{2-15}$$

$$Ax_n \to Ax$$

b) <u>Boundedness</u>. An operator A defined over a metric space E is said to be bounded if there exists a constant $M \geq 0$ such that

$$\|Ax\| \leq M \|x\| \quad \text{for all } x \in E \tag{2-16}$$

A geometric "reminder" of the concept of boundedness is sketched in Fig. 2-2. The norm of the image point Ax of any point x with a finite norm cannot become arbitrarily large. In other words, the image of any specified sphere $\|x\| = k$ must be contained within the sphere $\|Ax\| \leq Mk$ of the E_1-space. The distance between any two points x and y of E_1 should satisfy the inequality

$$\|Ax - Ay\| = \|A(x-y)\| \leq M\|x-y\| \tag{2-17}$$

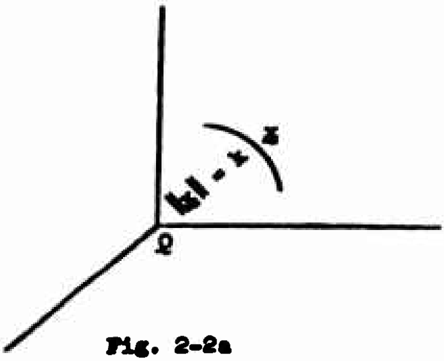

Fig. 2-2a

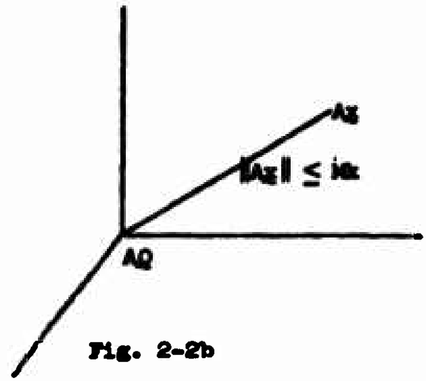

Fig. 2-2b

An operator need not be bounded. For instance, the operation of differentiation is linear but it may generate unbounded results. Likewise, an operator need not be continuous. For additive operators, however, it can be shown that the definition of continuity and boundedness become equivalent.

If an additive operator is bounded, then it is easy to show that it is also continuous. In fact, consider a sequence of vectors $\{x_1, x_2, \ldots, x_n, \ldots\}$ of the space E converging to x. For the corresponding sequence of images, we can write:

$$\|Ax - Ax_n\| = \|A(x-x_n)\| \leq M\ \|x-x_n\| \qquad (2\text{-}18)$$

where M is the constant appearing in Eq. 2-16. Thus, the sequence $\{Ax_1, Ax_2, \ldots, Ax_n, \ldots\}$ converges to Ax as $n \to \infty$. The converse is also true; a continuous additive operator is bounded. (The proof of this statement is basically simple and is available in most texts on the subject.) The essence of the proof is that when a vector x is changed continuously, the vector Ax varies also continuously, and a suitable constant $M \geq 0$ may be found.

c) <u>Norm of a Linear Operator</u>. The foregoing material suggests a natural definition for the "norm" of a bounded linear operator A in a metric space E. The norm of an operator A is the smallest number M satisfying the inequality (2-18). From this definition, it follows that for a linear bounded operator and any arbitrary vector x, we have

$$\|Ax\| \leq \|A\| \, \|x\| \tag{2-19}$$

Another equivalent definition of $\|A\|$ is as follows:

We may map the closed unit sphere of the space E by the linear operator A, and search for the supremum of the "length" of the image point $\|Ax\|$ in E_1. The norm of the operator A is denoted by $\|A\|$ and defined by

$$\|A\| = \sup \|Ax\| \qquad \text{for } \|x\| \leq 1 \tag{2-20}$$

2-5. <u>The Space of Linear Operators</u>

We have already shown that the linear operators defined over a linear vector space of finite dimensions themselves form a linear vector space. The null operator is the zero element of this space. The sum of two linear operators and the product of a linear operator by a scalar are well defined. Thus the linearity of the space of the linear operator is easily established. The introduction of a norm for the operator allows us to state:

<u>Theorem</u>

The space of linear operators defined over a normed linear space is a normed linear space.

<u>Proof</u>

The validity of axioms of a metric space follows directly from 1, 2, 3 below.

1. In the first place, we have associated a norm with every element of the space of linear operators, i.e., a non-negative number

$$\|A\| = \sup \|Ax\| \geq 0 \qquad \text{for } \|x\| \leq 1 \tag{2-21}$$

This number is equal to zero if and only if A is the null operator.

2. $\|\lambda A\| = \sup \|\lambda A x\| = |\lambda| \sup \|Ax\| = |\lambda| \; \|A\|$ (2-22))

for $\|x\| \leq 1$

3. $\|A+B\| = \sup\|Ax+Bx\| \leq \sup \|Ax\| + \sup\|Bx\| = \|A\| + \|B\|$ (2-23)

for $\|x\| \leq 1$

In addition to the above theorem, one can derive the following simple inequality for the norm of the product of two linear operators $A = A_2A_1$

$$\|Ax\| = \|A_2(A_1x)\| \leq \|A_2\| \; \|A_1x\| \leq \|A_2\| \; \|A_1\| \; \|x\| \tag{2-24}$$

That is,

$$\|A_2A_1\| \leq \|A_2\| \; \|A_1\| \tag{2-25}$$

The identity operator plays the role of unity.

Evidently, the operation of multiplication of linear operators is associative and distributive, i.e.,

$$A_1(A_2A_3) = (A_1A_2)A_3 \tag{2-26}$$

$$A_3(A_1+A_2) = A_3A_1 + A_3A_2 \tag{2-27}$$

$$(A_1+A_2)A_3 = A_1A_3 + A_2A_3 \tag{2-28}$$

If operators $A_1 = A_2 = A$, then the product A_1A_2 may be denoted by A^2. In this manner we can define powers of an operator A and note that for any positive integer n

$$\|A^n\| \leq \|A\|^n \tag{2-29}$$

<u>Theorem</u>

The space of linear operators which transform the normed space E into a complete normed space E_1 is itself a Banach space.

<u>Proof</u>

One has to show that the linear space of operators is a complete space. If $\{A_1, A_2, \ldots, A_m, \ldots, A_n \ldots\}$ is a converging sequence of linear operators, that is $\|A_n - A_m\| \longrightarrow 0$ $(n,m \longrightarrow \infty)$, then for any arbitrary element x of E the corresponding sequence of images will also converge

$$\|A_n x - A_m x\| = \|(A_n - A_m)\, x\| \leq \|A_n - A_m\|\, \|x\| \longrightarrow 0 \qquad (2\text{-}30)$$

$$n,m \longrightarrow \infty$$

Thus, the sequence $\{Ax\}$ is a converging sequence for every x which converges to Ax. Since the normed space E_1 is complete, then the limit of any sequence in E_1 will be contained in E_1; but it is not obvious that A also belongs to the space of linear operators.

The following three steps are required for completing the proof.

a) A is additive

b) A is bounded

c) $A = \lim\limits_{n \longrightarrow \infty} A_n$

A proof may be derived by the interested reader. (See B.Z. Vulikh, Chapter 8, or M. Davis, A first course in Functional Analysis, Gordon and Breach Co., New York 1966, pages 49-50).

2-6. The Inverse Operator

In order to give a precise definition of the inverse operator, consider the operator A which maps a Banach space E into a Banach space E_1.

$$y = Ax,\ x \in E,\ y \in E_1 \qquad (2\text{-}31)$$

A is said to be an invertible operator, if for every element $y \in E_1$ The equation $y = Ax$ has a unique solution x in E. The operator which represents this correspondence is referred to as the inverse operator A and is denoted by A^{-1}. When A has an inverse, we can formally write

$$\underset{\sim}{x} = A^{-1}\underset{\sim}{y}$$

$$A\underset{\sim}{x} = A\,A^{-1}\underset{\sim}{y} = \underset{\sim}{y} \qquad (2\text{-}32)$$

The following theorem can be proven without difficulty:

Theorem

If A is a bounded linear operator, whose inverse A^{-1} exists, then A^{-1} is also bounded.

The proof for this theorem is rather long.

It requires several preliminary steps and more space than we can afford. The interested reader is referred to standard mathematical texts, for instance Kolmogorov-Fomin, Chapter III.

To give an example, note that the operator $Ax = \int_0^t x(s)\,ds$ on C[0,1] to C[0,1] is bounded, but $A^{-1}\,y = \frac{d}{dt}\,y(t)$ is unbounded for a certain subset of continuous functions. The inverse operator does not exists for all points of the space.

Likewise the Sturm-Liouville operator

$$A\underset{\sim}{x} = \frac{d}{dt}\left\{f(t)\ \frac{dx}{dt}\right\} + g(t)\,x$$

which is defined on the subspace of twice continuously differentiable elements of C[0,1] is unbounded. Its inverse however is a bounded linear operator for all C[0,1] (Green's function)

$$A^{-1}\,\underset{\sim}{y} = \int_0^1 G(t,s)\ y(s)\ ds$$

In dealing with operators we frequently need to apply the following important theorem which allows a power series expansion of the inverse (I-A) in terms of powers of A.

Theorem

If A is an operator, with $\|A\| < 1$, mapping a Banach space E into itself, and I the identity operator, then the inverse of the operator I-A can be written as

$$(I-A)^{-1} = \sum_{k=0}^{\infty} A^k \tag{2-33}$$

Proof

The proof, in essence, is similar to the elementary proof for convergence of the scalar series

$$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k \quad \text{for } |x| < 1 \tag{2-34}$$

Here, we consider the transformation $(I-A)\underline{x} = \underline{y}$, and set forth to find its solution by means of iterations:

$$\underline{x}_{n+1} - A\underline{x}_n = \underline{y} \qquad n = 1,2,\ldots \tag{2-35}$$

With $\underline{x}_1 = \underline{y}$ this procedure yields:

$$\begin{aligned} \underline{x}_2 &= \underline{y} + A\underline{x}_1 = \underline{y}\ A\underline{y} \\ \underline{x}_3 &= \underline{y} + A\underline{y} + A^2\underline{y} \\ &\cdots\cdots\cdots\cdots \\ \underline{x}_n &= \underline{y} + A\underline{y} + A^2\underline{y} + \ldots + A^{n-1}\underline{y} \end{aligned} \tag{2-36}$$

The key to the convergence of this sequence is the fact that if $\|A\| \leq q < 1$, then $\|A^n\| \leq q^n$. Therefore, as $n \longrightarrow \infty$, $\underline{x}_n$ tends to the unique solution of $\underline{x} = \underline{y} + A\underline{x}$, that is,

$$\underline{x} = \sum_{k=0}^{\infty} A^k\underline{y} \tag{2-37}$$

Thus $\underline{x}$ satisfies $\underline{x} = \underline{y} + A\underline{x}$, and we have shown elsewhere (section on contraction mapping) that this equation has a unique solution; whence

$$(I-A)^{-1}\underline{y} = \sum_{k=0}^{\infty} A^k \underline{y} \tag{2-38}$$

Example 2-1.

Is the differential operator A on the set of continuous functions in [0,1] a bounded operator? The norm of x(t) is taken as $\max|x(t)|$ in [0,1].

Solution:

$$A\underline{x} = \frac{d}{dt}[x(t)]$$

Consider a sequence of points on the unit sphere:

$$\underline{x}_n = \sqrt{2} \sin n\pi t \qquad n = 1,2,\ldots$$

$$A\underline{x}_n = \sqrt{2}\, n\pi \cos n\pi t$$

$\|A\underline{x}_n\|$ increases without bound as $n \longrightarrow \infty$, thus the operator A is unbounded. Note that A is additive but not continuous, as it is not defined everywhere in the space C[0,1].

Example 2-2.

Consider a mapping of the linear space C[0,1] of all functions x(t) continuous in the interval [0,1]. a) Show that the operator A defined by the integral below is a linear operator from C to C.

$$y(t) = \int_0^1 k(s,t)\, x(t)\, dt$$

where the kernel k(s,t) is a function continuous in the square

$$0 \leq s \leq 1 \quad , \quad 0 \leq t \leq 1$$

b) Show that

$$\|A\| = \max_{0 \leq s \leq 1} \int_0^1 |K(s,t)|\, dt = M$$

Solution:

a) Evidently A maps C into itself. Moreover,

$$A(\underline{x}+\underline{y}) = A\underline{x} + A\underline{y}$$

The continuity of the operator can be proved by considering a sequence of vectors $\underline{x}_n$ converging to $\underline{x}$; and the convergence of the corresponding sequence of images. The convergence here is taken in the sense of uniform convergence, whence we can take the limit under the integration sign.

b) The proof is slightly more complicated than a) and can be found for instance in Kantorovitch-Akilov's Functional Analysis in Normed Spaces page 108-109.

Example 2-3. Determine the norm of the following operators:

a) Zero operator, b) identity operator, c) similarity operator, $A\underline{x} = \lambda\underline{x}$

Solution:

a) $\|A\| = 0$,

b) $\|A\| = 1$,

c) $\|A\| = |\lambda|$

Example 2-4. (operators of the normal form). Let $\{\underline{e}\}$ be a complete orthonormal system for a Hilbert space H of countable dimension, and $\{\lambda_1, \lambda_2, \ldots, \lambda_n, \ldots\}$ a bounded sequence of real numbers with $|\lambda_k| \leq C$. For every $x = \sum_{i=1}^{\infty} x_i\, \underline{e}_i$ of H we define the following operation:

$$A\underline{x} = \sum_{j=1}^{\infty} \lambda_j\, x_j\, \underline{e}_j$$

a) Is A a linear operator?

b) Find $\|A\|$.

Solution. a) A is a well defined operator on H. Its linearity is an immediate consequence of the properties of additivity and homogeneity.

b) Let C be the upper bound of the sequence $|\lambda_n| = C$, and let

$$\|x\|^2 = \sum_{i=1}^{\infty} x_i^2 = 1, \text{ then}$$

$$\|Ax\|^2 = \sum_{i=1}^{\infty} \lambda_i^2 x_i^2 \leq C^2$$

whence $\|A\| \leq C$

On the other hand

$$\|A\| \geq \sup \|Ae_n\| = \sup \|\lambda_n e_n\| = \sup \|\lambda_n\| = C$$

Consequently $\|A\| = C$

2-7. Approximate Solution of Functional Equations

Consider the linear operator A defined over a normed space. Frequently, we wish to solve a functional equation of the type $Ax = y$ or $x = Ax + y$ for a given y, but the inverse operator A^{-1} is not known. This is for example the case when it is desired to solve a set of algebraic equations but the inverse operator A^{-1} (inverse matrix) is not readily available; or when an exact solution to an integral or differential equation is hard to derive. According to a method suggested by L.V. Kantrovich, A^{-1} may be approximated in the following way.

a) $$x = Ax + y$$

Let A be a linear operator mapping a Banach space X into itself, I the identity operator of X, and A_o an approximant to A, then

$$(I-A)x = y \qquad (I-A_o)x_o = y$$

The success of the method depends on the choice of a suitable A_o such that we can assume that $A-A_o = \Delta$ is relatively small, that is, $\|\Delta\| \ \|(I-A)^{-1}\| < 1$, and if $(I-A)^{-1}$ exists. Under these conditions it follows that $(I-A_o)^{-1}$ also will exist and:

$$x_o = (I-A_o)^{-1}y = (I-A_o)^{-1}(I-A)x = (I-A_o)^{-1} \left[(I-A_o)+(A_o-A)\right]x$$

$$= (I-A_o)^{-1}(I-A_o)x + (I-A_o)^{-1}(A_o-A)x$$

$$x_o - x = (I-A_o)^{-1}(A_o-A)x$$

$$\|x_o\| \leq \|(I-A_o)^{-1}\| \ \|A_o-A\| \ \|x\| \leq (I-A_o)^{-1} \ \|\Delta\| \ \|x\| \qquad (2\text{-}39)$$

Thus if A_o is a approximant for A, then for any y the functional equation $(I-A_o)x_o = y$ will yield an approximate solution x_o for x. This inequality however does not explicitly indicate a bound for the error. If A_o is chosen such that:

$$\|\Delta\| \ \|(I-A_o)^{-1}\| \leq q < 1$$

Then we find:

$$\|x_o-x\| \leq q \ \|x\| \leq q(\|x - x_o\| + \|x_o\|) = q\|x-x_o\| + q\|x_o\|$$

or

$$\|x_o-x\| \leq \frac{q}{1-q} \ \|x_o\|.$$

b) $y = Ax$

Let A_o be an approximation to A such that A_o^{-1} can be found more easily. Let $\Delta = A-A_o$, and assume that $\|\Delta\| \ \|A_o^{-1}\| < 1$. For a given y we may approximate x by the element:

$$x_o = A_o^{-1}y \qquad (2\text{-}40)$$

The error vector $x - x_o$ satisfies the inequality:

$$\|x-x_o\| = \|A^{-1}y-A_o^{-1}y\| \|A^{-1}-A_o^{-1}\| \; \|y\| \tag{2-41}$$

It can be shown as an exercise (for instance by analogy with the scalar case) that under the above hypothesis

$$\|A^{-1}-A_o^{-1}\| < \frac{\|A_o^{-1}\|}{1-\|A_o^{-1}\| \; \|\Delta\|} \tag{2-42}$$

In spite of the fact that A^{-1} is not known, this inequality provides a direct upper estimate for the norm of error, by writing

$$\|x-x_o\| \leq \|A^{-1}-A_o^{-1}\| \; \|y\| < \frac{\|A_o^{-1}\| \; \|y\|}{1 - \|A_o^{-1}\| \; \|\Delta\|} \tag{2-43}$$

An example of the application of Kantorovich method is to be found in solving Fredholm's integral equation, by replacing its kernel with a degenerate kernel.

2-8. Representation of Linear Operators in A Hilbert Space

In Chapter 3 we discussed the general form of linear operators defined over finite-dimensional vector spaces. In this section we make a generalization of that material for linear operators on Hilbert and Banach spaces. Here we shall investigate whether linear operators on Hilbert space also admit a matrix representation similar to the representation of Sec. 3-8 of TR-65-399.

Consider a separable Hilbert space with an arbitrary complete set of orthonormal elements $\{e_1, e_2, \ldots e_k, \ldots\}$. With respect to this set, any vector $x \in H$ can be uniquely represented as

$$x = \sum_{k=1}^{\infty} x_k e_k \tag{2-44}$$

where

$$x_k = (\underline{x}, \underline{e}_k) \tag{2-45}$$

A linear (continuous) operator A maps $\underline{x}$ to $\underline{y}$ such that

$$\underline{y} = A\underline{x} = \sum_{k=1}^{\infty} x_k \, A\underline{e}_k \tag{2-46}$$

The image of $\underline{e}_k$ under A may be specified by its Fourier coefficients, i.e.,

$$A\underline{e}_k = \sum_{j=1}^{\infty} a_{kj} \, \underline{e}_j \qquad k = 1,2,\ldots \tag{2-47}$$

Therefore, the Fourier coefficients of $\underline{y}$ in its representation with respect to the $\underline{e}$ basis are

$$\underline{y} = \sum_{j=1}^{\infty} y_j \underline{e}_j = \sum_{k=1}^{\infty} y_k x_k \left(\sum_{j=1}^{\infty} a_{kj} \underline{e}_j \right) \tag{2-48}$$

whence

$$y_j = \sum_{k=1}^{\infty} a_{kj} \, x_k \qquad j=1,2,\ldots \tag{2-49}$$

In this manner, the vector $\underline{y} = A\underline{x}$ is completely specified through an infinite matrix:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1k} & \cdots \\ a_{21} & a_{22} & \cdots & a_{2k} & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{j1} & a_{j2} & \cdots & a_{jk} & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix} \tag{2-50}$$

There is an important difference between representation theorem of linear operators in the case of Hilbert space (and B-spaces) of infinite and finite dimensions. In contrast with the finite dimensional case, not all infinite matrices represent linear operations on some Hilbert space.*

2-9. Adjoint Operator

In Sec. (4-8) of TR-65-399 it was point out that with every linear operator A defined on a real Euclidean space of finite-dimension R^n, we can uniquely associate an operator A^* in a manner that for any pairs of vectors $\underline{x}$ and $\underline{y}$ of that space

$$(A^*\underline{x},\underline{y}) = (\underline{x},A\underline{y}) \tag{2-51}$$

The operator A^* is called the adjoint of the operator A. In any orthogonal basis, the matrix representing A is the transpose of the matrix of A^*. In particular, it may occur that the matrix of A is a symmetric matrix, i.e., $a_{ik} = a_{ki}$. In such a situation we have $A = A^*$, whence

$$(A\underline{x},\underline{y}) = (\underline{x},A\underline{y}) \tag{2-52}$$

When this relation is satisfied, the operator A is said to be self-adjoint.

When the space is a complex Euclidean space C^n, the above defining equation remains valid; but the elements of A and A^* are generally complex numbers. In the special case where $a_{ik} = \bar{a}_{ki}$ then A is identical with its transpose conjugate; that is, $A = A^*$ is self-adjoint. A self-adjoint transformation $A^* = A$ defined on C^n is represented by a Hermitian matrix.

The above definitions remain essentially valid for Hilbert and Banach spaces. Let A be a linear operator on H, whose range is also in H. According

---

*The necessary and sufficient conditions which the matrix in Eq. 2-50 must satisfy are given for instance in L.V. Kantorovich and G.P. Akilov, Functional Analysis in Normed Spaces, The Macmillan Co., New York 1964.

to Chapter 3, for each pair of vectors of H, the quantity $(A\underset{\sim}{x},\underset{\sim}{y})$ defines a linear functional on H. We consider the equation

$$(A\underset{\sim}{x},\underset{\sim}{y}) = (\underset{\sim}{x},\underset{\sim}{y}^*) \qquad (2\text{-}53a)$$

For every pair of $\underset{\sim}{x}$ and $\underset{\sim}{y}$, this equation provides a unique $\underset{\sim}{y}^* \in H$. When $\underset{\sim}{y}$ is changed, $\underset{\sim}{y}^*$ will be changed also. Let us denote their relationships by $y^* = A^*\underset{\sim}{y}$, or

$$(A\underset{\sim}{x},\underset{\sim}{y}) = (\underset{\sim}{x},A^*\underset{\sim}{y}) \qquad (2\text{-}53b)$$

This relation characterizes a unique operator A^* on H with range in H, referred to as the adjoint of A. It can be shown without difficulty (see Liusternik and V. Sobolev Sec. 23.) that the adjoint operator A^* of a linear operator A defined on H is indeed a linear operator and $\|A^*\| = \|A\|$.

The following properties of adjoint operators are easily verified. If A and B are linear operators over a Hilbert space H, and λ a scalar, then

$$(A+B)^* = A^* + B^* \qquad (2\text{-}54)$$

$$(\lambda A)^* = \bar{\lambda}A^* \qquad (2\text{-}55)$$

$$(B\ A)^* = A^*B^* \qquad (2\text{-}56)$$

$$(A^{-1})^* = (A^*)^{-1} \text{ if } A^{-1} \text{ exists} \qquad (2\text{-}57)$$

$$I^* = I \qquad (2\text{-}58)$$

Normal Operator. A linear operator A is said to be normal if it commutes with its adjoint, i.e.,

$$AA^* = A^*A \qquad (2\text{-}59)$$

Normal operators of a Hilbert space satisfy the relation

$$\|A\underset{\sim}{x}\| = \|A^*\underset{\sim}{x}\| \text{ , } \underset{\sim}{x} \in H \qquad (2\text{-}60)$$

An arbitrary self-adjoint operator is clearly a normal operator.

Unitary Operator. This is a subclass of normal operators satisfying the additional requirement

$$AA^* = A^*A = I \tag{2-61}$$

It is to be noted that for unitary operators Eq. 2-60 reduces to

$$\|A\underset{\sim}{x}\| = \|\underset{\sim}{x}\|, \text{ since } (A\underset{\sim}{x}, A\underset{\sim}{x}) = (\underset{\sim}{x}, A^*A\underset{\sim}{x}) = (\underset{\sim}{x}, \underset{\sim}{x})$$

A rotation in Hilbert space is an example of a unitary linear operator. The rotation operator A maps the space into itself while preserving the norm, that is: $\|A\underset{\sim}{x}\| = \|\underset{\sim}{x}\|$.

As an application of the foregoing mateial, consider the familiar linear integral operator of $L^2[a,b]$,

$$y(s) = \int_a^b K(s,t)\, x(t)\, dt \tag{2-62}$$

where K(s,t) is a continuous kernel function, whence

$$\int_a^b \int_a^b |K(s,t)|^2\, ds\, dt < +\infty \tag{2-63}$$

In our current notation, this integral equation is written as $\underset{\sim}{y} = A\underset{\sim}{x}$. The adjoint operator A^* now may be introduced by $\underset{\sim}{y}^* = A^*\underset{\sim}{y}$. We require that the equation (2-64) remains satisfied. It is not difficult to show that this requirement is equivalent to choosing the integral transformation below with the new kernel as:

$$y^*(s) = \int_a^b K^*(s,t)\, y(t)\, dt \tag{2-64}$$

$$K^*(s,t) = \overline{K(t,s)} \tag{2-65}$$

In order to verify whether this is the true adjoint of the linear operator under consideration, we write:

$$(A\underset{\sim}{x},\underset{\sim}{y}) = \int_a^b \left[\int_a^b K(s,t)\; x(t)\; dt\right] \overline{y(s)}\; ds \tag{2-66}$$

$$= \int_a^b x(t) \left[\int_a^b K(s,t)\; \overline{y(s)}\; ds\right]$$

$$(\underset{\sim}{x},\underset{\sim}{y}^*) = \int_a^b x(t)\; \overline{y^*(t)}\; dt \tag{2-67}$$

The equality of the two quantities implies that

$$\overline{y^*(t)} = \int_a^b K(s,t)\; \overline{y(s)}\; ds \tag{2-68}$$

$$y^*(t) = \int_a^b \overline{K(s,t)}\; y(s)\; ds \tag{2-69}$$

A change of variables yields

$$y^*(s) = \int_a^b \overline{K(t,s)}\; y(t)\; dt \tag{2-70}$$

A comparison with Eq. (2-52) completes the validity of the statement.

The above linear integral operator is self-adjoint if

$$\overline{K(s,t)} = K(t,s) \tag{2-71}$$

When the kernel is a real function s and t, then the requirement for self-adjoint will reduce to that of the symmetry of the kernel, i.e., $K(s,t) = K(t,s)$.

2-10. Positive Operators. A self-adjoint operator A is said to be a positive operator if for any $x \in H$, $(Ax,x) \geq 0$. If $(Ax,x) > 0$ for every non-zero vector of H, then A is said to be a positive-definite operator.

An arbitrary positive operator A has a unique (self-adjoint) positive square root B, that is, $B^2 = A$, $B > 0$. An operator A_1 is said to be greater than an operator A_2 if $A_1 - A_2$ is a positive operator. The following statements can be directly established:

1. For arbitrary operator A, we have $AA^* \geq 0$, $A^*A \geq 0$. This is in view of

$$(A^*Ax,x) = (Ax,Ax) \geq 0 \tag{2-72}$$

2. For arbitrary A, $A^2 \geq 0$.

3. If $A \geq 0$, then for any positive integer n, $A^n \geq 0$.

2-11. Symmetric Operator.

In common mathematical terminology the term symmetric operator applies to an operator A which is additive, homogeneous, and satisfies

$$(Ax,y) = (x,Ay) \tag{2-73}$$

Based on this definition, a symmetric operator needs not be bounded. Thus, a self-adjoint operator is a bounded symmetric operator. The class of self-adjoint operators is a subset of the wider class of symmetric operators.

Examples of applications of symmetric operators occur in the study of Sturm-Liouville equations:

$$-\frac{d}{dt}\left(p(t)\frac{dx}{dt}\right) - q(t)\,x(t) = \lambda\ \ x(t) \tag{2-74}$$

where

$p(t)$ is a continuous function $C_1[a,b]$

$q(t)$ is a continuous function $C[a,b]$

λ is a real parameter and the boundary conditions can be arbitrary. We search for the set S of functions $x(t)$ which are twice differentiable and belong to $L^2[a,b]$. By rewriting the left hand side of this equation as an operator A acting on $\underset{\sim}{x}$ we find:

$$A\underset{\sim}{x} = \lambda \underset{\sim}{x} \tag{2-75}$$

The operator A is additive and homogeneous but not continuous, thus unbounded. In order to show the symmetry of A, we calculate:

$$(A\underset{\sim}{x},\underset{\sim}{y}) = - \int_a^b [(px')' + qx]\, y\, dt\ ,\ \underset{\sim}{x},\underset{\sim}{y} \in S$$

$$= - \int_a^b (px')'\, y\, dt - \int_a^b qx\, y\, dt \tag{2-76}$$

In view of the assumed bounda·· conditions we find:

$$\int_a^b (px')'\, y\, dt = (px')\, y \Big|_a^b \int_a^b (px')\, y'\, dt = -\int_a^b x'(py')\, dt \tag{2-77}$$

$$= -x(py') \Big|_a^b + \int_a^b x(py')'dt = \int_a^b x(py')'dt$$

whence

$$(A\underset{\sim}{x},\ \underset{\sim}{y}) = - \int_a^b x[py')' + q\, y]\, dt = (\underset{\sim}{x},\ A\underset{\sim}{y}) \tag{2-78}$$

This establishes the symmetry of the Sturm-Liouville operator. The importance of this result in solving boundary value problems of this general type is due to the fact that symmetric operators possess a number of simple properties. The most interesting property of these operators pertains to the class of positive operators and can be outlined in the following statement.

If A is a symmetric operator and $(A\underset{\sim}{x}, \underset{\sim}{x}) \geq 0$ for all $\underset{\sim}{x}$ in some linear subspace of a Hilbert space, then all its eigenvalues are non-negative.

For proof, let λ be an eigenvalue of A and $\underset{\sim}{x}$ a corresponding eigenfunction, and observe that:

$$(A\underset{\sim}{x}, \underset{\sim}{x}) = (\lambda\underset{\sim}{x}, \underset{\sim}{x}) = \lambda(\underset{\sim}{x}, \underset{\sim}{x}) \geq 0 \tag{2-79}$$

2-12. Projection Operator

We have discussed in full detail the projection of a point $\underset{\sim}{x}$ of a Hilbert space H on a subspace $S \subset H$. Let $\underset{\sim}{x}_0$ be the unique point called the projection of $\underset{\sim}{x}$ on S. This relationship between $\underset{\sim}{x}$ and $\underset{\sim}{x}_0$ may be denoted by $P\underset{\sim}{x}$ where P stands for "projection operator" or "projector". The following properties of a projection operator of a Hilbert space are easily verified.

1. $P\underset{\sim}{x} \perp \underset{\sim}{x} - P\underset{\sim}{x}$

2. $\underset{\sim}{x} \in S$ and $P\underset{\sim}{x} = \underset{\sim}{x}$ are equivalent statements.

3. $\underset{\sim}{x} \perp S$ and $P\underset{\sim}{x} = \underset{\sim}{0}$ are equivalent statements.

4. $\|\underset{\sim}{x}\|^2 = \|P\underset{\sim}{x}\|^2 + \|\underset{\sim}{x}-P\underset{\sim}{x}\|^2$.

5. $\|P\underset{\sim}{x}\| \leq \|x\|$. This inequality follows from 4, above, and implies that P is a bounded operator.

6. $\|P\| = 1$ unless S is reduced to null element.

To prove the latter assertion, let $x \in S$ with $\|x\| = 1$. Whence $\|P\| \leq \|Px\| = \|x\| = 1$. Since $\|P\| = \sup \|Px\|$ for $\|x\| \leq 1$, we must have in this case $\|P\| \geq 1$. On the other hand, assertion 5 along with the definition of the norm of an operator implies that $\|P\| \leq 1$. Therefore, we conclude that $\|P\| = 1$.

7. Projectors of Hilbert space are positive operators; that is, $P(x,x) \geq 0$. In fact, we will show that

$$(Px,x) = (P^2x,x) \geq 0 \tag{2-80}$$

The following statement is an interesting theorem about projection operators of a Hilbert space.

<u>Theorem</u>

The necessary and sufficient conditions for a linear operator P to be a projection operator in a Hilbert space are a) the operator P be self-adjoint, and b) P be an idempotent operator; that is, $P^2 = P$.

<u>Proof</u>

The necessity of a) and b) will be considered first. Let S be a subspace of H, x and y arbitrary points of H, and P a projector from H to S. Consider vector x', y'

$$x - Px = x' \tag{2-81a}$$

$$y - Py = y' \tag{2-81b}$$

whence

$$x' \perp S, \; y' \perp S \tag{2-82}$$

$$\begin{aligned} (Px,y) &= (x-x', \; y' + Py) \\ &= (x-x', \; Py) = (x, \; Py) \end{aligned} \tag{2-83}$$

For part b), we note that for any $x \in H$

$$P(Px) = P(x-x') = Px - Px' = Px \tag{2-84}$$

To show the sufficiency of requirements a) and b), let S be the set of all elements $\underline{x}$ of H such that their images by the linear operator P satisfy the relation $P\underline{x} = \underline{x}$. One must show that S is a subspace. To this end, any element $\underline{x} \in S$ can be conveniently written as

$$\underline{x} = P\underline{x} + (\underline{x} - P\underline{x}) \tag{2-85}$$

Note that by assumption $P(P\underline{x}) = P\underline{x}$, therefore $P\underline{x} \in S$, and $\underline{x} - P\underline{x} \in S$. Moreover, for any two arbitrary points $\underline{x}, \underline{y} \in S$ we have

$$(\underline{x} - P\underline{x}, \underline{y}) = (\underline{x}, \underline{y}) - (P\underline{x}, \underline{y}) = (\underline{x}, \underline{y}) - (\underline{x}, P\underline{y}) = 0 \tag{2-86}$$

That is, $\underline{x} - P\underline{x} \perp S$.

2-13. Completely Continuous Operators

There is a natural class of linear operators in a Hilbert space which is very similar in its behavior to the class of linear transformation in a finite-dimensional space. These are referred to as <u>completely continuous operators</u>. A bounded linear operator A on H, in order to be completely continuous, must satisfy the following property: If $\{x_n\}$ is any bounded sequence of vectors (that is, there exists some $k > 0$ such that for all n, $\|x_n\| \leq k$), the sequence $\{A\, x_n\}$ must contain at least one convergent subsequence.

The definition of completely continuous operator encompasses the concept of the so-called "compact" sets. Thus, the above definition can be replaced by an equivalent statement using the concept of compactness.*

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*A set S contained in a metric space X is said to be compact, if from any infinite sequence of points $\{x_n\} \in S$ it is possible to select a subsequence $\{x_{n1}, x_{n2}, \ldots, x_{nk}, \ldots\}$ $n_1 < n_k < \ldots$, convergent in X to some limit. (The limit may or may not belong to S.) An additive operator defined over a metric space X is said to be completely continuous, if the image of any bounded set of X is a compact set in the image space Y.

In a finite-dimensional space every bounded linear operator is completely continuous since it maps bounded sets. But this is not necessarily true in infinite-dimensional spaces. Take, for instance, the identity operator in ℓ^2 space. The sequence of an orthonormal basis $\{\underline{e}_n\}$ is certainly a bounded sequence since $\|\underline{e}_n\| = 1$ for all n. But the sequence $I\underline{e}_n$ contains no convergent subsequence. In fact

$$\|\underline{e}_n - \underline{e}_m\| = 2 \quad \text{for } n \neq m \tag{2-87}$$

The identity operator of an infinite-dimensional Hilbert space is not a completely continuous operator. Completely continuous operators along with self-adjoint operators are the simplest and the most common type of linear operators in a Hilbert space. The relatively complete results available in problems of applications are generally pertinent to this category. In particular, if a completely continuous operator of a Hilbert space is also self-adjoint, the structure of the operator will resemble a generalization of the symmetric matrices representing finite-dimensional self-adjoint operators. The broad class of linear integral operators with continuous kernel on $0 \leq s, t \leq 1$.

$$\underline{y}(s) = \int_a^b K(s,t)\, \underline{x}(t)\, dt \tag{2-88}$$

offers a most common example of completely continuous operators on the space $L^2[0,1]$. For an introductory treatment of completely continuous operators, theorems, and related proof see Kolmogorov and Fomin, Chapter IV, or other standard texts.

2-14. Completely Continuous Self-Adjoint Operators

In this section, we state some of the properties of completely continuous self-adjoint operators A in H.

As we know, an eigenvalue of A is a number λ such that there exists a non-zero element x_0 satisfying

$$Ax_0 = \lambda x_0 \tag{2-89}$$

Any element x, satisfying this equation is referred to as an eigenvector or eigenelement. The totality of eigenvectors associated with an eigenvalue λ form an eigensubspace which will be denoted by H_λ. The subspace H' of H is said to be an <u>invariant space</u> of the self-adjoint operator A, if for $x \in H'$ we have also $Ax \in H'$.

As an extension of the material of Chapter 4, of TR-65-399 the following assertions for every element of H and every completely continuous self-adjoint operator A are maintained.

1. The expression (Ax,x) is real.

$$(Ax,x) = (x,Ax) = \overline{(Ax,x)} \tag{2-90}$$

2. The eigenvalues of A are real.

$$\lambda = \frac{(Ax,x)}{(x,x)} \tag{2-91}$$

3. Eigensubspaces $H_{\lambda 1}$ and $H_{\lambda 2}$ corresponding to distinct non-zero eigenvalues are orthogonal. Let $x \in H_{\lambda 1}$, $y \in H_{\lambda 2}$

$$Ax = \lambda_1 x \qquad Ay = \lambda_2 y \tag{2-92}$$

Then

$$(x,y) = \frac{1}{\lambda_1} (Ax,y) = \frac{1}{\lambda_1} (x,Ay) = \frac{\lambda 2}{\lambda 1} (x,y) \tag{2-93}$$

Whence

$$(x,y) = 0, \; x \perp y \tag{2-94}$$

4. The operator A has at least one eigenvalue.

5. The operator A has at most a denumerable set of eigenvalues.

6. The spectrum of a self-adjoint operator A lies entirely in the interval [m,M] of the real axis, where

$$\begin{aligned} M &= \sup\ (A\underline{x},\underline{x}) \qquad \text{for } \|\underline{x}\| = 1 \\ m &= \inf\ (A\underline{x},\underline{x}) \qquad \text{for } \|\underline{x}\| = 1 \end{aligned} \tag{2-95}$$

(For the proof of these latter statements, see standard texts.)

The main motivation for the study of completely continuous self-adjoint operators stems from the fact they are the natural extension of linear transformations of a finite-dimensional Euclidean space. As we know, finite-dimensional Euclidean spaces are complete and separable - and all linear operators on such spaces are completely continuous. A most essential property of completely continuous self-adjoint operators is expressed by the following fundamental theorem due to D. Hilbert:

<u>Theorem</u>:

In a complete separable Hilbert space every completely continuous self-adjoint operator possesses a complete orthogonal system of eigenvectors. If the proof of this theorem requires more space than what is available at present. In view of this theorem one can visualize the particular simplicity for handling problems, one may search for a complete system of eigenvectors from which a complete set of orthonormal coordinates $\underline{e}_i$ may be constructed. Thus, every point $\underline{x} \in H$ will have an image representable as:

$$A\,\underline{x} = \sum_{i=1}^{\infty} \lambda_i (\underline{x}_i\ \underline{e}_i)\ \underline{e}_i$$

Where $\lambda_i (i=1,2,\ldots)$ are eigenvalues of the operator A, $\underline{e}_i (i = 1,2,\ldots)$ the corresponding eigenvectors, and

$$(\underline{e}_i,\underline{e}_j) = \delta_{ij} \qquad (i,j = 1,2,\ldots)$$

$$\delta_{ii} = 1 \qquad \delta_{ij} = 0 \quad \text{when } i \neq j.$$

We will show in the chapter on integral equations that due to the availability of orthogonal coordinate systems in the function space, the solution to integral equations with completely continuous self-adjoint kernels admits a very simple form.

*<u>Spectrum of a self-adjoint operator</u>

Consider the equation

$$(A - \lambda I)\,\underline{x} = \underline{y} \tag{2-96}$$

where λ and $\underline{y}$ are given, $\underline{x}$ is unknown, A is a self-adjoint operator, and λ any arbitrary complex number. Let us assume that for a certain value of λ, the operator $R_\lambda = (A - \lambda I)^{-1}$ exists. Then R_λ is called to as the <u>resolvent</u> of Eq. 2-96, and for this value of λ and any arbitrary $\underline{y}$ the equation (2-96) has the unique solution $\underline{x} = R_\lambda\,\underline{y}$.

Those λ for which Eq. 2-96 has a unique solution for all $\underline{y}$ are referred to as <u>regular values</u> of the operator A. Any non trivial solution of the homogeneous Eq. 2-96, i.e., $\underline{y} \neq \underline{0}$ is an eigen element of A for this eigenvalue λ. The totality of non-regular values of λ is called <u>the spectrum</u> of the operator A. In particular all the eigenvalues belong to the spectrum.

Example 2-4.

a) Show that the (real) operator

$$A\,\underline{x} = tx(t) \qquad \underline{x} \in L^2[0,1]$$

$$0 \leq t \leq 1$$

is self-adjoint.

b) Show that $A \geq 0$

Solution:

a) Clearly A is an additive operator. The image of any point $\underset{\sim}{x} \in L^2$ is also a point in L^2. Moreover A is bounded since

$$\|A\underset{\sim}{x}\| \le \|\underset{\sim}{x}\|$$

Thus A is a linear operator. In order to show that A is self-adjoint we write:

$$(A\underset{\sim}{x},\underset{\sim}{y}) = \int_0^1 t\ x(t)\ y(t)\ dt$$

$$(\underset{\sim}{x},A^*y) = \int_0^1 x(t) \cdot t(y(t)\ dt$$

b) $(A\underset{\sim}{x},\underset{\sim}{x}) = \int_0^1 t[x(t)]^2\ dt \ge 0$

It can be also shown by Cauchy-Buniakovski's inequality that for $\|\underset{\sim}{x}\| = 1$:

$$0 \le (A\underset{\sim}{x},\underset{\sim}{x}) \le 1$$

Example 9-5. In the function space $L^2[0,1]$ an operator A is defined by

$$A\ \underset{\sim}{f} = t\ f(t)$$

a) Show that A is a positive operatory b) determine the square root of A.

Solution. a) For two arbitrary functions f(t) and g(t) we have

$$(A\underset{\sim}{f},\underset{\sim}{g}) = \int_0^1 t\ f(t)\ g(t)\ dt = (f,A^*\underset{\sim}{y})$$

Hence

$A^*\ \underset{\sim}{g} = t\ g(t)$, that is A is self-adjoint. We note also that $(A\underset{\sim}{f},\ \underset{\sim}{f}) \ge 0$; consequently A is a positive operator.

b) The square root of A is defined by B:

$$B\,\underset{\sim}{f} = \sqrt{t}\;\; f(t)$$

Example 9-6. Consider the operator

$$A\underset{\sim}{x} = \int_0^t x(s)\, ds$$

defined over the Banach space of real continuous functions $C[0,1]$.

a) Is A bounded? b) Is A^{-1} bounded?

Solution: a) yes

b) $A^{-1}\underset{\sim}{y} = \frac{d}{dt}\, y(t)$

A^{-1} is not bounded.

Example 9-7. Show that the (real) linear operator

$$A\underset{\sim}{x} = tx(t) \qquad \underset{\sim}{x} \in L^2[0,1]$$

$$0 \leq t \leq \text{.}$$

is self-adjoint.

Solution:

$$(A\underset{\sim}{x},\underset{\sim}{y}) = t\; x(t)\; y(t)$$

$$(\underset{\sim}{x},\; A^{*}y) = x(t) \cdot t(y(t)$$

Example 2-8. Determine the eigenvalues and eigenfunctions of the operator

$$A = \frac{-d^2\; x(t)}{dt} \quad \text{with the periodic boundary conditions}$$

$$x(0) = x(2\pi) \qquad , \qquad x'(0) = x'(2\pi)$$

Solution: A is a linear symmetric operator on the subspace $C_2[0,2\pi]$. The determination of eigenvalues amounts to solving

$$x'' + \lambda x = 0$$

or

$$x = c_1 e^{\sqrt{-\lambda}\, t} + c_2 e^{\sqrt{-\lambda}\, t}$$

There are three possibilities to be examined:

a) $\lambda < 0$

The specified boundary conditions requires $c_1 = c_2 = 0$, whence there can be no negative eigenvalues.

b) $\lambda = 0$

In this case the solution to the differential equation reduces to $x = c_1 + c_2 t$. The boundary conditions impose $c_2 = 0$. Therefore, $\lambda = 0$ is an eigenvalue, and all functions $x = c_1$ in $[0,2\pi]$ are associated eigenfunctions.

c) $\lambda > 0$

In this case one finds that the set of integers $\left\{\sqrt{\lambda} = 1,2,3,\ldots\right\}$ are eigenvalues. With each eigenvalue $\lambda_k = k^2$, we can associate an eigenspace of dimension two spanned by sin kt and cos kt.

Example 2-9.

$$-x'' = t$$

$$x(0) = x(\pi) = 0$$

Solution. The operator $A = -\frac{d^2}{dt^2}$ acting on the subspace $C_2[0,\pi]$ satisfying the specified boundary conditions is a linear symmetric operator. The eigenvalues of A maybe found from the equations:

$$-x'' - \lambda x = 0$$

$$x(0) = x(\pi) = 0$$

These are $\lambda_n = n^2$, $(n = 1,2,\ldots)$. The set of eigenfunctions $\underline{f}_n = \sin nt$, $n = 1,2,\ldots$ forms a complete set for the space under considerations. This allows us to write:

$$x(t) = \sum_{n=1}^{\infty} \frac{1}{n^2} \frac{(t, \underline{f}_n)}{(\underline{f}_n, \underline{f}_n)} \underline{f}_n$$

$$(\underline{x}, f_n) = \int_0^{\pi} t \sin nt \, dt = (-1)^{n+1} \pi/n$$

$$(\underline{f}_n, \underline{f}_n) = \pi/2$$

$$x(t) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{2}{n^3} \sin nt$$

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13. ABSTRACT

The vast and rapid advancement in telecommunications, computers, controls, and aerospace science has necessitated major changes in our basic understanding of the theory of electrical signals and processing systems. There is strong evidence that today's engineer needs to extend and to modernize his analytical techniques. The latest fundamental analytical approach for the study of signals and systems seems to have its roots in the mathematics of Functional Analaysis.

This report contains a bird's-eye view of the elements of Hilbert spaces and their associated linear operators. The first chapter of the report gives an exposition of the most essential properties of Hilbert spaces. The second chapter presents the elements of linear operators acting on such spaces.

The report is addressed to engineers and scientists interested in the theory of signals and systems. The applications of the theory will be undertaken in a separate report.

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