THEMIS SIGNAL ANALYSIS STATISTICS RESEARCH PROGRAM

*

A NOTE ON THE EVALUATION OF A MULTIVARIATE

NORMAL INTEGRAL BY THE METHOD OF DAS

by

J. T. WEBSTER

Technical Report No. 15
Department of Statistics THEMIS Contract

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Department of Statistics
Southern Methodist University
Dallas, Texas 75222

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1.   Introduction

Das (1956) presents a method of evaluating the integral

$$I = \int_{a_1}^{\infty} \cdots \int_{a_n}^{\infty} f(x_1, x_2, \cdots, x_n)\, dx_1 dx_2 \cdots dx_n$$

(where $f(x_1, x_2, \cdots, x_n)$ is the joint multivariate normal density function
with zero means and nonsingular variance-covariance matrix Σ) through the
combining of $n + k$ independent normal variables with zero means and unit
variances. Later Marsaglia (1963) shows that this is a special case of a
convolution formula. The complexity of implementing the solution is highly
dependent upon the size of k and Marsaglia (1963) notes that k equal to
n minus the multiplicity of the smallest latent root of Σ can always be
achieved. This note investigates properties of Σ that will allow smaller
values of k .

2.   The equivalent expression

As a method for evaluating I , Das (1956) considers two row rectors
$\underline{y}' = (y_1, y_2, \cdots, y_n)$ and $\underline{z}' = (z_1, z_2, \cdots, z_k)$ all of whose elements
are normally and independently distributed with zero means and unit variances.
The problem is to choose a positive constant c and an $n \times k$ real matrix B
such that

$$\Sigma = c^2 I_n + BB' \quad ; \qquad\qquad 1)$$

for then $\underline{x}' = (x_1, x_2, \cdots, x_n)$ can be expressed as

$$x = c\underline{y} - B\underline{z} \ .$$

I can now be expressed as

$$I = Pr(x_1 \geq a_1, x_2 \geq a_2, \cdots, x_n \geq a_n)$$

$$= Pr\left[y_1 \geq \left(a_1 + \sum_{j=1}^{k} b_{1j}z_j\right)/c \ , \ y_2 \geq \left(a_2 + \sum_{j=1}^{k} b_{2j}z_j\right)/c \ , \ \cdots \right.$$

$$\left. y_n \geq \left(a_n + \sum_{j=1}^{k} b_{nj}z_j\right)/c\right]$$

$$= (2\Pi)^{-k/2}\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \prod_{i=1}^{n} P\left[\left(a_i + \sum_{j=1}^{k} b_{ij}z_j\right)/c\right] \exp\left(-\underline{z}'\underline{z}/2\right) \prod_{j=1}^{k} dz_j \ . \quad 2)$$

where $P[a] = Pr(y_1 \geq a)$. It is evident that a small k is advantageous.

The slight change in 1)

$$\underline{\underline{\Sigma}} = C^2 + BB' \ , \qquad\qquad\qquad\qquad 3)$$

where C is a diagonal matrix with positive diagonal elements c_i , results in

$$\underline{x} = C\underline{y} - B\underline{z}$$

and 2) of the form

$$I = (2\Pi)^{-k/2}\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \prod_{i=1}^{n} P\left[\left(a_i + \sum_{j=1}^{k} b_{ij}z_j\right)/c_i\right] \exp\left(-\underline{z}'\underline{z}/2\right) \prod_{j=1}^{k} dz_j \ . \quad 4)$$

3. Latent vector properties

If $\underline{\underline{\Sigma}}$ can be expressed in the form 3) and $\underline{w}_\ell$ is a latent vector

of $\underset{\sim}{\xi}$ then

$$\lambda_\ell \underline{w}_\ell = \underset{\sim}{\xi} \, \underline{w}_\ell$$

$$= C^2 \underline{w}_\ell + BB'\underline{w}$$

$$= C^2 \underline{w}_\ell + \sum_{j=1}^{k} \underline{b}_j \underline{b}_j' \underline{w}_\ell \qquad\qquad 5)$$

where $\underline{b}_j$ is the jth column of B . From expression 5)

$$\lambda_\ell w_{i\ell} = c_i^2 w_{i\ell} + \sum_{j=1}^{k} \theta_{j\ell} b_{ij} \qquad\qquad 6)$$

where $w_{i\ell}$ is the ith element of $\underline{w}_\ell$ and $\theta_{j\ell} = \underline{b}_j'\underline{w}_\ell$. If the c_i^2 are not all equal the n equations of 6) are parametric equations for a k dimension variety in n space of order two, V_k^2 ; a brief discussion of this may be found in Kendall (1956), page 5. That is given a vector $\underline{w}_\ell$, algebraic operations to eliminate λ_ℓ and the $\theta_{j\ell}$ will result in n-k second degree homogeneous equations in the $w_{i\ell}$. If the c_i^2 are all equal say to c then 6) can be written as

$$w_{i\ell} = \sum_{j=1}^{k} \theta_{j\ell}^* b_{ij} \quad . \qquad\qquad 7)$$

where $\theta_{j\ell}^* = \underline{b}_j'\underline{w}_\ell / (\lambda_\ell - c^2)$, which are the parametric equations of a k dimension flat, S_k .

Conversely if the latent vectors of $\underset{\sim}{\xi}$ all fall in a V_k^2 of the form of 6) with

$$\theta_{j\ell} \underline{w}'\underline{b}_j' \geq 0 \text{ and } \lambda_\ell \geq c_i^2 > 0 \qquad\qquad 8)$$

then $\underset{\sim}{\xi}$ can be written in the form 3) . If the latent vectors of $\underset{\sim}{\xi}$ all

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fall in a S_k with

$$\theta^*_{j\ell} \underline{w}'_{\ell}\underline{b}_j \geq 0 \qquad\qquad\qquad 9)$$

then it can be written in the form 1) .

Thus the smallest value of k for evaluating I through form 4) is determined by the smaller of the minimum dimension of the V^2_k's of the form 6) satisfying conditions 8) and the minimum dimension of the S_k satisfying the condition 9) that contain the latent vectors of $\pounds$.

Alternative conditions can be given for the special case of $k = 1$.
6) becomes

$$\lambda_\ell w_{i\ell} = c^2_i w_{i\ell} + \theta_\ell b_i$$

or

$$\frac{1}{w_{i\ell}} = -\frac{1}{\theta_\ell}\frac{c^2_i}{b_i} + \frac{\lambda_\ell}{\theta_\ell}\frac{1}{b_i} \qquad\qquad 10)$$

which is the parametric expression of a plane, S_2 , passing through the origin. Thus if the $\underline{s}'_\ell = (1/w_{1\ell}, 1/w_{2\ell}, \cdots, 1/w_{n\ell})$ all fall in a plane with parametric conditions 10) satisfying

$$(\lambda_\ell - c^2_i)\,\theta_\ell\,\underline{w}'_\ell\underline{b} > 0$$

only one auxiliary variable is necessary. For the case of $c^2_i = c^2$ for all i , $n - 1$ of the λ_ℓ equal c^2 and their corresponding θ_ℓ are zero. That is the minimum latent root of $\pounds$ has multiplicity $k - 1$.

4. Summary

In general the minimum number of auxiliary variables necessary to evaluate a multivariate normal integral through the method of Das (1956)

can be determined by the dimensions of the second order varieties with parametric form 6) and the dimensions of the flats that contain the latent vectors of the variance-covariance matrix.

This problem could also be looked at as determining the diagonal matrix D such that $D \Sigma D$ has maximum multiplicity of its smallest latent root. Then Marsaglia's (1963) solution suffices. It is interesting to note that the difficulty of the solution, the dimension of the integration in I , is affected by change of scales of the normal variables.

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