AD

AD 680256

CR-0484-1

SOME STRONG WAVE PROBLEMS IN AN ELASTIC MATERIAL WITH MICROSTRUCTURE

Ъy

C. W. Bert B. E. Cummings

June 1968



60

GENERAL RESEARCH CORPORATION

P. O. BOX 3587, SANTA BARBARA, CALIFORNIA 93105

This document has been approved for public release and sale; its distribution is unlimited.

> Reproduced by the CLEARINGHOUSE for Federal Scientific & Technical Information Springfield Va 22151

DISCLAIMER NOTICE

THIS DOCUMENT IS BEST QUALITY PRACTICABLE. THE COPY FURNISHED TO DTIC CONTAINED A SIGNIFICANT NUMBER OF PAGES WHICH DO NOT REPRODUCE LEGIBLY.

; :

ACKNOWLEDGMENT

"

This analysis presented here was obtained in the course of research carried out on Contract DAAD 05-67-C-0484 under the technical supervision of Dr. C. M. Glass of the Ballistic Research Laboratories, Aberdeen Proving Ground, Maryland.

> This document has been approved for public release and sale; its distribution is unlimited.

> > i

ABSTRACT

The historical development of the foundations of thermomechanics is reviewed, with emphasis placed upon the critical contribution of each investigator. The theories are discussed in terms of their relevance to real materials in the important high-velocity regime which lies between the low and intermediate velocity regime (which involves essentially isothermal elastic and plastic wave propagation) on one hand and the ultrahighvelocity regime (in which a hydrodynamic description of the material behavior is adequate) on the other. Various facets of high-velocity wave propagation in various materials involve elastic, plastic, viscous, and thermal effects on a macroscopic scale and perhaps on a microscopic scale. Evaluations are made from the standpoint of thermodynamics as well as classical mechanics.

Using fundamental principles of thermodynamics and mechanics, thermomechanically coupled linear energy and constitutive equations are derived for the elastic solid with microunit cells discussed by Mindlin.

Equations of motion for the microstructure are found for the large amplitude case where Eulerian and Lagrangian formulations are not identical.

iii

CONTENTS

. .

•

States and the second and the second second

- Aller Contraction

anime and

1

SECTION		· ····································	PAGE
	ACKM	OWLEDGMENT	i
	ABST	RACT	iii
I	REVI	EW OF PREVIOUS WORK	1
	Α.	Introduction	1
	в.	Thermoelasticity	2
	с.	Thermoplasticity	4
	D.	Thermoviscoelasticity	6
•	E.	Wave Propagation	7
II	COUPLED THERMOELASTICITY THEORY FOR A LINEAR SOLID WITH MICROSTRUCTURE		
	Α.	Introduction	12
	Β.	Conservation of Energy	14
	с.	Derivation of Constitutive Equations	20
	D.	Relation to Other Theories	24
III	DILA	TION WAVES INCLUDING THE EFFECTS OF MICROSTRUCTURE	25
	Α.	Introduction	25
	Β.	Definition of an Ideal Elastic Solid with Microstructure for Shock Wave Studies	25
	с.	Planar Wave in Micromedia	28
APPENDIX	THER RELA ELAS	MODYNAMICALLY VALID THERMOELASTIC CONSTITUTIVE TIONS CONSIDERING TEMPERATURE DEPENDENCY OF THE TIC COEFFICIENTS AND STRESS DEPENDENCY OF THE	
	THER	MAL-EXPANSION COEFFICIENT	35
	REFE	RENCES	43
	DIST	RIBUTION LIST	54

.

v

I. REVIEW OF PREVIOUS WORK

A. INTRODUCTION

In the past decade, a very large body of experimental data on the behavior of various solids under strong shock loading has been obtained by means of a variety of high-pressure dynamic loading techniques.^{1,2,3} Since most of the original experimental work was conducted at very high pressures (say, three or more orders of magnitude above the ultimate strength) at which it has been recognized that strength effects are not important, the analyses developed to describe this work mathematically were based on hydrodynamics of dense, inviscid fluids. More recent effort has been devoted to hypervelocity impact (HVI), i.e., the impact of a small body with a massive body at relative velocities exceeding the acoustic velocities of both. Since HVI involves creation of craters of finite size, it must depend upon the strength properties of the materials involved. Most of the attempts to analyze HVI have involved either minor modification of hydrodynamic theory or correlation of highly simplified phenomenological models with experimental results.

An example of the inadequacy of the hydrodynamic approach for the high-velocity regime has been reported by Curran.⁴ The results of his plate-slap experiments on 2024 aluminum-alloy plates at an impact velocity of 1.9 mm/µs indicated that attenuation begins approximately 80-percent sooner and is approximately twice as severe as predicted by hydrodynamic theory. An analysis able to predict this greater attenuation accurately is highly desired.

Until very recently, thermodynamic aspects of high-velocity wave propagation have been highly nerlected. Although the theory of linear elastic materials with thermomechanical coupling (through the energy equation) dates back to 1837 with Duhammel's work,⁵ there have been few new major accomplishments in thermomechanics until about ten years ago. Since the topic of linear thermoelasticity without thermomechanical coupling has been treated extensively in Boley and Weiner's textbook,⁶ it is omitted here.

B. THERMOELASTICITY

の利用である

Although the energy equation and constitutive equations for a classical linear elastic solid with thermomechanical coupling were originated by Duhammel in 1837, surprisingly little work has been done in this area. Here the term "classical" means one without microstructural effects and with elastic properties independent of temperature. Most of the analyses reported in the literature are concerned with the response (thermal and mechanical) due to sudden or cyclic heating but no mechanical loading. It appears that more analysis should be devoted to problems involving purely mechanical loading, with particular emphasis on determining the loading rate at which the thermomechanical coupling effects become important. On a strictly intuitive basis, it is believed that in the regime in which thermomechanical coupling effects become important, the use of a finite description of strain, rather than the linear infinitesimal one, is necessary.

One of the most interesting anomalies in the behavior of a classical, linear elastic solid with thermomechanical coupling has been discussed by Deresiewicz:⁷ His analysis predicted that the phase velocity of plane elastic waves approach the adiabatic value at <u>low</u> frequencies and the isothermal value at <u>high</u> frequencies. Deresiewicz pointed out that this is in disagreement with the Laplace-Rayleigh thermomechanical theory of gases, which Deresiewicz claimed is erroneous.

Although various theories of elasticity, ⁸⁻¹⁰ taking into consideration doublet stresses (doublet forces/area) in addition to classical stresses (forces/area), have been proposed to explain such phenomena as size effects in materials and crystal-lattice-type wave dispersion; only recently has attention been given to the thermodynamic aspects of such solids.¹¹⁻¹⁵

In all of the theories of thermoelasticity known to the authors, the temperature dependence of the elastic constants is neglected, even though a recent thermomechanical theory of elastoplastic waves in solids

considers an effect of the same order of magnitude: the effoct of temperature on the yield strength.¹⁵ Rosenfield and Averbach,¹⁶ in an analysis which is on a rather weak mathematical and thermodynamic basis, indicated that the temperature dependence of the elastic coefficients is related to the stress dependence of the thermal-expansion coefficient a so follows:

$$\alpha = \alpha_0 + \alpha_1 \sigma \tag{1}$$

where

$$\alpha_1 = -E^{-2} dE/dT$$
 (2)

where α_0 is the coefficient of thermal (linear) expansion at zero stress, α_1 is the change in this coefficient per unit of normal stress, σ is the normal stress applied in the direction in which α is measured, E is Young's modulus, and T is temperature. Rosenfield and Averbach conducted experiments which confirmed the linear dependence of α with stress, as predicted by Eq. 1. However, the discrepancy between the value of the proportionality constant α_1 predicted by their theory, Eq. 2, and that measured varied from 6 to 100 percent. An analysis which is on a more sound thermodynamic and mathematical basis than that of Ref. 16 is given in the appendix. Unfortunately the new theory requires additional experimental data which are not normally available, namely the variation of α_1 with temperature.

Following a procedure suggested in Boley and Weiner's,⁶ Fillon¹⁷ formulated a theory of thermoelasticity for physically nonlinear elastic solids. He considered the strains as independent thermodynamic variables ("fluxes") and the stresses as dependent thermodynamic variables ("forces"); thus, he utilized the Helmholtz free energy function (also called the Gibbs work function). It appears that he might just as well have considered the stresses as independent variables and the strains as dependent variables, and then utilized the Gibbs thermodynamic potential.¹⁸

Vakulenko¹⁹ has considered stress as the independent thermodynamic variable in his work in thermoplasticity, and Burridge and Knopoff²⁰ have introduced a formulation of mechanical energy, using stress as the independent variable, which is valid for prestressed and inelastic media as well as nonprestressed elastic media. This approach is used in the appendix to considerably simplify the derivation of the stress dependency of the thermal-expansion coefficient.

Another type of nonlinearity, namely geometric nonlinearity (i.e., finite deformations), has been discussed very ably by Lee and Liu,²¹ and some of the underlying thermodynamics aspects have been put on a sound mathematical basis by Coleman and Noll²² and Wang and Bowen.²³

C. THERMOPLASTICITY

The quantitative experiemnts of Farren and Taylor²⁴ in 1925 form the basis for thermoplasticity. They found that approximately 90 percent of the work done in plastic deformation is dissipated in the form of heat and the remaining 10 percent is irreversibly stored due to permanent distortion of the crystal lattice.

Some of the work in thermoplasticity has been largely of a philosophical nature.²⁵⁻²⁷ The first attempts to put the subject on a quantitative basis were made independently by Freudenthal,²⁸ Prager,²⁹ Ziegler,³⁰ and Vakulenko.^{31,19} Their work has been discussed critically by Naghdi,³² who considered their work strictly exploratory in nature. His primary objection was that their theories are uncoupled, i.e., they do not contain a heat conduction term. He also questioned their choice of state variables and generalization of Onsager's principle to nonlinear phenomena.

Grigorian³³ took the viewpoint that, although thermodynamic concepts are of limited usefulness in constructing the mechan_cal parts of constitutive equations, one should check the thermodynamic consequences of a tentative constitutive relation in order to make the mathematical model

thermodynamically complete. (This point is illustrated in the consideration in the appendix.) It is interesting to note that Grigorian recognized that some of the plastic work goes into increasing the internal energy due to changes in structure of the material.

Foster³⁴ devoted attention to relaxation-spectrum effects. In particular, he noted that there are different relaxation spectra for loading (increasing load) and unloading (decreasing load).

Assuming that all of the plastic wor'. is dissipated in the form of heat, Dillon^{35} formulated a theory of thermoplasticity with thermomechanical coupling for small strains and Backman³⁶ formulated an analogous theory for finite strains. However, the latter work has been criticized and improved upon by Lee and Liu.²¹

To date the most complete mathematical models for dynamic thermoplasticity analyses are those of Grigorian³⁷ (with emphasis on soils) and Lee and Wierzbicki¹⁵ (with particular reference to metals).

As mentioned previously, Lee and Wierzbicki¹⁵ included the effect of temperature on the yield strength, but neglected the thermodynamically related effect of stress on the thermal-expansion coefficient, α . In their experiments, Rosenfield and Averbach¹⁶ measured this effect for stresses well into the plastic range. They found that as soon as the elastic limit is exceeded, α increases with stress at a greater slope \therefore an in the elastic range. However, the <u>slope</u> decreases with increasing stress, eventually becoming zero, and finally negative. When they removed the stress after each successive loading, they found that the permanent changes in α also increased, then leveled off, and finally decreased at higher applied-stress levels. This appears to be closely related to the observations of Farren and Taylor²⁴ that the percentages of the total plastic work which goes into permanent distortion of the wttice varies with applied strain. In view of these considerations, it

is conjectured that the effect of the stress dependency of the thermalexpansion coefficient would be greatest for short-duration pulses in which the loading is removed before there is sufficient time for very large plastic strains to be reached. However, as in the case of the same effect in the elastic range, quantitative inclusion of this effect in a theory must await experimental measurements of it over a range of temperatures ranging upward from room temperature.

D. THERMOVISCOELASTICITY

The theory of thermoviscoelasticity has strong thermodynamic bases, thanks to extensive research by Biot, $^{38-40}$ Chu, $^{41-42}$ Schapery, $^{43-45}$ and Coleman. Other research on the formulation of constitutive equations for thermoviscoelasticity has been carried out by Eringen, 49 Cowin, 50 Chudnovskii, 51 and Leigh. 52

Various aspects of the nonlinear theory of viscoelasticity have been treated in recent work by Koh and Eringen,⁵³ Kline,⁵⁴ and Valanis.⁵⁵ Recently Eringen⁵⁶ has formulated a theory of microstructural viscoelasticity including thermodynamic considerations.

The literature which has appeared in the past decade on dynamic thermoviscoelastic (DTVE) problems is too voluminous to mention in detail here. For a basic treatment of the subject, reference is made to Boley and Weiner's book.⁶ Most of the DTVE analyses found in the literature neglect thermomechanical coupling. An example of one of the few DTVE analyses in which this coupling is considered is found in Ref. 57.

It appears that the possibility of the type of thermomechanical instability discussed by Berg⁵⁸ should be investigated further, especially for high-intensity waves. He pointed out that the softening of a body due to heating allows external loads to deliver more power to the body at an increasing rate.

E. WAVE PROPAGATION

There are two fundamentally different types of waves:

- Simple, continuous waves, which can be mathematically described by a Fourier series of harmonic waves. These waves are of small amplitude and are often called <u>acoustic waves</u>.
- 2. Discontinuous waves which involve a discontinuous jump in some physical quantity. The order, n, of a discontinuous wave is the order of the time derivative of the displacement which is discontinuous. Thus, a <u>shock wave</u>, which involves a jump in velocity, is a first-order wave and is usually the most severe.^{*} Higher order waves involve jumps in acceleration (acceleration waves) and in higher time derivatives.

The classical treatment, as used by Rayleigh,⁵⁹ is adequate for handling acoustic waves. The literature on the propagation of such waves in elastic and viscoelastic solids is too voluminous to mention here except to refer to two texts^{60,61} and three review papers devoted to this topic.⁵²⁻⁶⁴

The common metallic alloys used in load-carrying applications, namely steel, aluminum and most titanium alloys, behave macroscopically as isotropic materials at least in the elastic range. However, since they are really aggregates of many highly anisotropic crystals, they behave anisotropically on a microscale. This has been the motivation for various theoretical $^{65-68}$ and experimental studies 69 of acoustic-wave propagation in single crystals. A few metals and alloys of increasing engineering importance, namely beryllium, certain titanium alloys, zirconium, and uranium, are anisotropic even on a macroscale. Furthermore, filamentary composites behave anisotropically on a macroscale.

An alternative, but equivalent, definition of the order n of a discontinuity is one plus the order of the space derivative of the strain which is discontinuous. Thus, a shock wave, which involves a jump in strain, i. a first-order wave.

The development of theories of solids with microstructure^{10,13,14} now permits a mathemacical treatment of solids which are isotropic on a macroscale and yet anisotropic on a microscale. This opens a new avenue of improved correlation between theory and experimental studies of acoustic wave propagation. However, to date, studies along this line have been limited to theoretical studies of relatively simple systems.^{10,70-72}

The method of characteristics, which is based on the concept of a characteristic surface or discontinuity surface appears to have been developed at the turn of the century by Hadamard⁷³ for the study of the propagation of discontinuities in hydrodynamics. The method has found wide use in fluid mechanics, including gas dynamics and magnetofluidmech-anics as well as hydrodynamics.

Apparently the first application of the method of characteristics to solids behaving nonhydrodynamically is due to Thomas. He was originally concerned with plastic solids⁷⁴ but later generalized his compatibility conditions.⁷⁵⁻⁷⁶ Hill⁷⁷ discussed stability and uniqueness of acceleration waves in general media with emphasis on incompressible and elastoplastic media. Acceleration waves in anisotropic elastic solids were analyzed by Howard,⁷⁸ who was concerned with transversely isotropic media and Nariboli⁷⁹ who treated general anisotropic elastic media. Nariboli's approach is interesting in that he combined the singular surface concept of Thomas with the ray theory of Courant and Hilbert to obtain an equation which upon integration predicts the growth or decay of the discontinuities. Very recently Lur'e⁸⁰ has analyzed the propagation of discontinuities in continuation with microstructure.

Recently a number of investigators⁸¹⁻⁸⁵ have investigated nonlinear aspects of acceleration waves and higher-order waves in nonlinear elastic materials. Some of the general results of these studies are:

1. When the strain has a discontinuity, the derivative of the strain also must have a discontinuity.⁸² Thus, shock waves

(to be discussed later) are accompanied by acceleration waves although the converse situation does not necessarily hold.

- 2. Varley and Dunwoody⁸³ found that the linear theory of irrational waves has an error which increases with time. Their nonlinear theory predicts that either a shock forms or the material "forgets" the details of the disturbance in a <u>finite</u> time. Green's theory⁸⁴ predicts that acceleration waves either become infinite in finite time (i.e., a shock forms) or decay to zero in <u>infinite</u> time, with the exception of transverse waves normal to a shock.
- 3. Discontinuities of higher order than the second propagate with constant strength 84 and at the same velocity as the acceleration wave.

In their book,⁸⁶ Truesdell and Noll presented a review of acceleration waves in generalized elastic media. Recently considerable attention has been devoted to acceleration waves in viscoelastic media.⁸⁷⁻⁹³ Of these, the work of Varley is especially important because the material may be generally anisotropic and nonhomogeneous.

The problem of the propagation of discontinuities into a prestressed medium is important in connection with propagation of unloading waves (if the duration of loading is sufficiently long compared to the transit time of the reflected wave), since then the medium is essentially prestressed by the loading wave. For small deformations (i.e., acoustic waves), this problem was treated by Biot.⁹⁴ The case of acoustic waves but large deformation due to the prestress have been considered in Ref. 95-97. Finally, the propagation of finite discontinuities in prestressed media has been treated by Green.⁹⁸

Although the foregoing analyses employ thermodynamics in the formulation of the constitutive relations, they do not consider thermomechanical coupling. This has been treated in two recent papers by

Chadwick and Powdrill⁹⁹ for the linear thermoelastic case and by Fine¹⁰⁰ for the thermoelastoplastic case.

In the hydrodynamic formulation of the shock wave problem, the major physical effect considered in the constitutive equation is compressibility, i.e., dilatational (volumetric) waves. Thus, shear waves, in fact any shear effects whatsoever, are not considered at all. As has been pointed out by Fyfe et al, 101 this does not mean that the hydrodynamic theory cannot predict <u>some</u> physical aspects of a shock wave even at high velocities (as opposed to hypervelocities). An example of an early attempt to combine shear effects with dilatational effects is contained in Ref. 102.

It should be pointed out that shock waves are not limited to compressible media. In fact, recent analyses by $Chu^{103-104}$ and by Collins¹⁰⁵ have been concerned with the propagation of shock waves in <u>incompressible isotropic elastic media</u>.^{*} In such media, only nondilatational shock waves can occur.

Recently Bland,¹⁰⁶⁻¹⁰⁷ Green,⁸⁴ and Coleman, Gurtin, and Herrera,⁸⁸ considered plane shock waves in <u>compressible</u> isotropic elastic media. Although these investigators considered certain thermodynamic aspects, they did not consider them to the depth considered by Dewey¹⁰⁸ and by Lee and Wierzbicki.¹⁵ The case of plane shock waves in compressible isotropic thermoelastic media was recently treated by Chadwick and Powdrill.¹⁰⁹ Their analysis is interesting in that they claim that their analysis leads to an infinite family of shock wave solutions which range thermodynamically from isothermal to adiabatic. They claim that they know of no thermodynamic or mechanical principle which enables selection of the "correct" solution, although they suggest the possibility of using the theory of nonlinear hyperbolic partial differential equations or the irreversibility principle of thermodynamics (second law). The present writers suggest that a stability investigation might be in order here, i.e., it is unlikely that all of the possible solutions are stable.

For an isotropic linearly elastic solid, incompressibility requires that Poisson's ratio be equal to 1/2. This value is closely approached in certain rubber-type materials, but not in metallic materials.

In Ref. 106, Bland found that two kinds of shock waves, dilatational and nondilatational, are possible in a compressible isotropic elastic solid. However, it is interesting to note that he concluded that nondilatational shock waves cannot propagate into a compressible hyperelastic solid at rest in its reference state. To the present writers, this implies that nondilatational shock waves can occur only during unloading, since then the material, being already loaded, is not at its reference state. In Ref. 107, Bland showed that in many solids (including simple elastic materials), the only kind of shock waves which are stable and admissible thermodynamically are tensile dilatational shock waves (in contrast to compressive dilatational shock waves in fluids). Although plane compressive shock waves in elastoplastic media have been studied extensively, Bland claimed that they are possible only in media in which the stressstrain curve is concave upward in the plastic range.

To study the shock structure (i.e. the shape of the shock wave front), it is necessary to include viscous dissipative effects. Recently Bland¹¹⁰ has performed such an analysis for plane waves in a simple viscoelastic solid. He showed that all monotonic wave profiles ultimately adopt the same constant wave profile, and he calculated the profile width and the time formation of the profile.

In this brief review, a number of related special topics should at least be mentioned:

- Disintegration or pulverization of material behind a shock front, applicable to certain soils.
- Changes of phase such as melting¹¹² due to shock wave heating, shock-induced metallurgical transformations,¹¹³ and shock hardening of metals.¹¹⁴
- Interactions between acoustic waves and shock waves,¹¹⁵ reflection and refraction of shock waves at free surfaces and interfaces, and spallation fracture criteria.^{116,117}
- Experimental techniques for generating shock waves and measuring various characteristics of them.¹⁻³

II. COUPLED THERMOFLASTICITY THEORY FOR A LINEAR SOLID WITH MICROSTRUCTURE

A. INTRODUCTION

The use of high-pressure dynamic loading techniques in studying the behavior of solids under strong shock loading has produced a very large body of experimental data in recent years.³ Most of the original experimental work was associated with very high pressures (three or more orders of magnitude above the ultimate strength of the material) at which it has been recognized that strength effects are not important. Thus, the theory ^{*} developed to accompany this work was based on hydrodynamics of dense fluids. Later interest turned to hypervelocity impact of a swall body on a massive body. This problem involved creation of craters of finite size and consequently involved high-pressure waves that must necessarily depend on the strength properties of the materials involved. The natural extension of hydrodynamic theory to this problem has been by the most expedient addition of special effects to existing computer codes (programs) and in correlation of simple phenomenological models with experimental results.

There is a well recognized need for an attack on the problem of propagation of high-pressure waves form a fundamental thermomechanical point of view.¹⁵ This effort is underway in several areas. The effort reported here is concerned with developing an applicable coupled thermoelastic theory that will take into consideration the microstructure of an elastic solid.

12

or any second and a second state of the second

Ч.

[&]quot;The use of "theory" here is in its most general sense, since most of what has been done in the past is in the nature of computer solutions to very involved nonlinear flow problems and might properly be called computer experiments on mathematical simulations.

The coupled thermomechanical theory (thermal and mechanical effects coupled in energy and constitutive equations) is well established for the macroscopic description of solid media. The elastic case is due to Duhammel⁵ in 1837, the viscoelastic case to Biot⁴⁰ in 1955, and the viscoplastic case to Dillon³⁵ in 1963.

Experimental observation of such mechanical phenomena as size effects and crystal-lattice type wave dispersion effects has led to various micromechanical theories. Of these, perhaps the most notable is the couple-stress theory, which in addition to ordinary stresses (forces per unit area) also considers couples per unit area. This theory was originated by Voigt⁸ in 1887 and was considerably developed by the Cosserat brothers⁹ approximately sixty years ago. In the past ten years, this theory has been developed further by various investigators too numerous to mention here.

Mindlin¹⁰ noted that couple stresses are actually only the antisymetric part of a more general tensor which he called the double stress tensor; thus, he originated a theory which he called linear elasticity with microstructure. He also pointed out that hic theory is mathematically equivalent to a linearized version of Ericksen and Truesdell's theory of deformable directors.¹¹⁸ In the latter theory, the mechanical behavior of a body at a given point is assumed to depend upon not only the deformation of the point but also the deformation of an oriented "director" located at the point. If the director is assumed to be rigid, the equations reduce to those of the Voigt-Cosserat couple-stress theory.

Upon completion of the present work, the work of Green, Rivlin, and Naghdi¹¹⁹⁻¹²¹ came to the attention of the authors. In Refs. 119 and 121 they derived an energy equation for a simple multipolar material, but they did not deal with constitutive equations. In Ref. 120 they extended their previous work to include multipolar deformation fields. Their work differs from ours in that they started with an entirely different set of

hypotheses in their theory. We believe that our derivation on the basis of a microelement is more satisfactory from a physical viewpoint. Also, they did not include initial stresses nor anisotropic behavior in their constitutive equation.

In the work reported here, the usual material constitutive equations are replaced by equations derived for the specific case of shock wave loading. This is consistent with the current interest in constitutive equations that have been experimentally determined from plate slap and other strong shock wave techniques.

B. CONSERVATION OF ENERGY

Using Mindlin's expression 10 for the kinetic energy density, the kinetic energy T of a macrovolume V can be rewritten as follows.

$$T = \frac{1}{2} \int_{V} \rho \dot{u}_{j} \dot{u}_{j} dV + \frac{1}{6} \int_{V} \rho' d^{2}_{\ell j} \dot{\psi}_{\ell k} \dot{\psi}_{jk} dV$$
(3a)

where

$$\rho = \rho_{1} + \rho$$

$$d_{\ell j}^{2} = d_{p} d_{q} \left(\delta_{p 1} \delta_{q 1} \ell_{\ell 1} \ell_{j 1} + \delta_{p 2} \delta_{q 2} \ell_{\ell 2} \ell_{j 2} + \delta_{p 3} \delta_{q 3} \ell_{\ell 3} \ell_{j 3} \right)$$
(3c)

where

and a dot denotes differentiation with respect to time. The first term on the right-hand side of Eq. (3a) is the classical kinetic energy expression and the second term represents the contribution of the microelement.

Mindlin's¹⁰ expression for the rate of work W done on the macrovolume V by the external forces (surface and body types) is equivalent to the form:

$$W = \int_{A} t_{j} \dot{u}_{j} dA + \int_{V} f_{j} \dot{u}_{j} dV + \int_{A} T_{jk} \dot{\psi}_{jk} dA + \int_{V} F_{jk} \dot{\psi}_{jk} dV \quad (4)$$

where

t_i = surface force per unit area

f_i = body force per unit macrovolume

 T_{jk} = surface force doublet per unit area with moment arm normal to axis X_j and acting in the X_k direction F_{jk} = body force doublet per unit volume with same subscript notation as for T_{jk}

A = area

The terms on the right-hand side represent the work done by the respective macro and micro surface and body forces.

The internal energy U^{*} contained in both macroelemonts and microelements can be expressed as

$$U^{*} = \int_{V} \rho_{M} U_{M} \, dV + \int_{V} \rho' U' \, dV = \text{total energy} - \text{kinetic energy}$$
(5)

where U_{M} , U' are the internal energies per unit mass for the macroelement and microelement, respectively.

Considering a closed system consisting of both macroelements and microelements, thermal energy is transferred out of the system at the rate

$$C = \int_{A} q \, dA \tag{6}$$

where q is the outward heat flux. Conservation of mechani al energy requires

$$W - Q = \frac{d}{dt} (T + U^{*})$$
⁽⁷⁾

The traction conditions at the boundary of the macroelement are:

$$t_{j} = n_{i}(\overline{\sigma}_{ij} + \hat{\sigma}_{ij})$$
(8a)

$$T_{jk} = n_{i}(\bar{\mu}_{ijk} + \hat{\mu}_{ijk})$$
(8b)

where

dinasion and the submitted with the set of the state of the state of the set of the set of the set of the set of the set

 n_i = direction cosines of the outer normal to the surface $\bar{\sigma}_{ij}, \hat{\sigma}_{ij}$ = components of the respective symmetric and antisymmetric parts of the ordinary stress vector (force/ area)

Application of the divergence theorem and Eqs. (8) yields the following result from Eq. (4):

$$W = \int_{V} \left[(\bar{\sigma}_{ij} + \hat{\sigma}_{ij}) \dot{u}_{j} \right]_{,i} dV + \int_{V} \left[(\bar{\mu}_{ijk} + \hat{\mu}_{ijk}) \dot{\psi}_{j\ell} \right]_{,i} dV$$
$$+ \int_{V} f_{j} \dot{u}_{j} dV + \int_{V} F_{jk} \dot{\psi}_{jk} dV \qquad (9)$$

where

 $[]_{i} = \partial []/\partial X_{i}$

The heat transfer can be rewritten in terms of a heat flux vector as follows:

$$Q = \int_{A} q_{j} n_{j} dA = \int_{V} q_{j,j} dV$$
 (10)

The differentiation on the right-hand side of Eq. (5) can be brought under the integral signs appearing in Eqs. (1) and (3) with the following results:

$$\frac{d}{dt} (T + U^{*}) = \int_{V} \rho \ddot{u}_{j} \dot{u}_{j} dV + \frac{1}{3} \int_{V} \rho' d^{2}_{\ell j} \ddot{\psi}_{\ell k} \dot{\psi}_{j k} dV$$

$$+ \int_{V} (\rho_{M} \dot{U}_{M} + \rho' \dot{U}') dV \qquad (11.)$$

This interchange of the order of differentiation and integration can be justified by assuming small displacements.⁶ However, it can also be justified for arbitrary displacements by considering conservation of mass.¹²² Consistent with the assumptions of linear elastedynamics the densities are treated as constants.

The quantity $\dot{u}_{j,i}$ may be written in terms of its symmetric and antisymmetric parts as follows:

$$\dot{u}_{j,i} = \frac{1}{2} (\dot{u}_{i,j} + \dot{u}_{j,i}) - \frac{1}{2} (\dot{u}_{i,j} - \dot{u}_{j,i}) = \dot{\epsilon}_{ij} - \dot{\omega}_{ij}$$
 (12a)

where ε_{ij} and ω_{ij} are the macrostrain and macrorotation, respectively, given by the following relations:

$$\varepsilon_{ij} = \frac{1}{2} (u_{i,j} + u_{j,i}), \quad \omega_{ij} = \frac{1}{2} (u_{i,j} - u_{j,i})$$
(12b)

The relative deformation γ_{ij} is defined as the difference between the macrodisplacement gradient and the microdeformation, i.e.

$$\gamma_{ij} = u_{j,i} - \psi_{ij}$$
(13a)

From Eqs. (10a) and (11), the following relationship is obtained:

$$u_{j,i} = \varepsilon_{ij} - \omega_{ij} = \psi_{ij} + \gamma_{ij}$$
(13b)

Then the first term in the expression for W, Eq. (7), can be rewritten as follows:

$$\int_{V} \left[(\bar{\sigma}_{ij,i} + \hat{\sigma}_{ij,i}) \dot{u}_{j} + \bar{\sigma}_{ij} \dot{\epsilon}_{ij} + \hat{\sigma}_{ij} (\dot{\psi}_{ij} + \dot{\gamma}_{ij}) \right] dV \qquad (14)$$

where the integrand term $-\overline{\sigma}_{ij}\omega_{ij}$ has been omitted because it is a product (of a symmetric tensor and an antisymmetric tensor) which must be zero.

Incorporating relations (9-11, 12) into (7) and rearranging terms and subscripts yields:

$$\begin{split} &\int_{V} (\bar{\sigma}_{ij,i} + \hat{\sigma}_{ij,i} + f_{j} - \rho \ddot{u}_{j}) \dot{u}_{j} dV \\ &+ \int_{V} \left[\bar{\mu}_{ijk,i} + \hat{\mu}_{ijk,i} + \hat{\sigma}_{jk} + F_{jk} - \frac{1}{3} \rho' d_{\ell j}^{2} \ddot{\psi}_{k_{\ell}} \right] \dot{\psi}_{jk} dV \\ &+ \int_{V} \left[\bar{\sigma}_{ij} \dot{\epsilon}_{ij} + \hat{\sigma}_{jk} \dot{\gamma}_{jk} + (\bar{\mu}_{ijk} + \hat{\mu}_{ijk}) \dot{\psi}_{jk,i} - q_{j,j} - \rho_{M} \dot{U}_{M} - \rho' \dot{U}' \right] dV = 0 \end{split}$$

$$(15)$$

The first two integrals appearing in Eq. (15) vanish by virtue of the following translational and rotational equations of motion:

$$\bar{\sigma}_{ij,i} + \hat{\sigma}_{ij,i} + f_j = \rho u_j$$
(16)

$$\bar{\mu}_{ijk,i} + \hat{\mu}_{ijk,i} + \hat{\sigma}_{jk} + F_{jk} = \frac{1}{3} \rho' d_{k\ell}^2 \bar{\psi}_{k\ell}$$
(17)

Thus, the only integral remaining in Eq. (15) is the last one, which then must be equal to the right-hand side, namely zero. Furthermore, for this integral to be zero for arbitrarily small regions, its integrand must be zero. Thus,

$$\bar{\sigma}_{ij}\dot{\epsilon}_{ij} + \hat{\sigma}_{jk}\dot{\gamma}_{jk} + (\bar{\mu}_{ijk} + \hat{\mu}_{ijk})\dot{\psi}_{jk,i} - q_{j,j} = \rho\dot{U}$$
(18)

where

$$\rho U = U_M + \rho^* U^*$$

The microstrain gradient κ_{ijk} is defined as follows:

$$\kappa_{ijk} = \psi_{jk,i} \tag{19}$$

Then Eq. (18) can be rewritten in the following form:

$$\vec{\sigma}_{ij} \cdot \hat{\epsilon}_{jk} + \hat{\sigma}_{jk} \cdot \hat{\gamma}_{jk} + \vec{\mu}_{ijk} \cdot \hat{\vec{\kappa}}_{ijk} + \hat{\mu}_{ijk} \cdot \hat{\vec{\kappa}}_{ijk} - q_{j,j} = \rho \cdot \vec{U}$$
(20)

Equation (18) is the general energy equation for an elastic medium with microstructure. When microstructural effects are absent $(\bar{\mu}_{ijk} = \hat{\mu}_{ijk} = \psi_{kl} \equiv 0)$, Eq. (17) requires that the antisymmetric part of the ordinary stress tensor also vanish $(\hat{\sigma}_{jk} \equiv 0)$ and Eq. (20) reduces to the well-known energy equation of a classical elastic solid.⁵

It is of interest to note that when all of the stress (σ and μ), strain (ε , γ) and strain gradient (κ) components have the same sign, the presence of the antisymmetric ordinary stresses and both the symmetric and antisymmetric parts of the doublet stresses represent <u>energy sinks</u> in the system.

It is especially important to note that even though Eq. (20) is coupled in the sense that it involves terms containing σ_{ij} , μ_{ijk} , q, and U, there are no cross-energy terms such as $\hat{\sigma_{ij}}_{ijk}$ or $\tilde{\mu}_{ijk}\hat{\epsilon}_{ij}$. This considerably reduces the number of coefficients which are required in the constitutive equations.

C. DERIVATION OF CONSTITUTIVE EQUATIONS

Using Caratheodory's statement of the second law of thermodynamics, Boley and Weiner 6 derived the following relations for a locally reversible thermodynamic process:

$$A_{a}\dot{a}_{s} + \rho T \dot{\eta} = \dot{U}$$
(21a)

$$B_{t}\dot{\beta}_{t} - q_{j,j} = \rho T \dot{\eta}$$
(21b)

where

 α_s , β_t = nondissipative and dissipative deformation variables A_s , B_t = state functions ρ = density T = absolute temperature η = entropy density function

Here we neglect viscous effects because it is believed that shock wave deformation occurs so rapidly that there is insufficient time for viscous forces to develop; thus, $B_t \dot{\beta}_t = 0$. Then Eq. (21b) yields:

$$\mathbf{I}_{\mathbf{i},\mathbf{j}} = -\rho \mathbf{T} \mathbf{n} \tag{22}$$

^{*} According to Coleman and Mizel¹²³ the assumption of microscopic reversibility is unnecessary.

Taking ε_{ij} , γ_{jk} , κ_{ijk} as state variables, we must consider $\overline{\sigma}_{ij}$, $\hat{\alpha}_{jk}$, μ_{ijk} as state functions:

$$\bar{\sigma}_{ij} = \bar{\sigma}_{ij}(\epsilon_{ij}, \gamma_{jk}, \kappa_{ijk}, T)$$

$$\hat{\sigma}_{jk} = \hat{\sigma}_{jk}(\epsilon_{ij}, \gamma_{jk}, \kappa_{ijk}, T)$$

$$\mu_{ijk} = \mu_{ijk}(\epsilon_{ij}, \gamma_{jk}, \kappa_{ijk}, T)$$

$$\bar{\sigma}_{ij}\dot{\epsilon}_{ij} + \hat{\sigma}_{ij}\dot{\gamma}_{ij} + \mu_{ijk}\kappa_{ijk} + \rho T \dot{\eta} = \rho \dot{U}$$
(23)

A free-energy function ϕ is defined by the following equation:

$$\phi(\varepsilon_{ij}, \gamma_{jk}, \kappa_{ijk}, T) = U(\varepsilon_{ij}, \gamma_{jk}, \kappa_{ijk}, T) - T\eta(\varepsilon_{ij}, \gamma_{jk}, T)$$
(24)

Then

$$\dot{\phi} = \dot{U} - T\dot{\eta} - \dot{T}\eta$$
(25)

But since

$$\phi = \phi(\varepsilon_{ij}, \gamma_{jk}, \kappa_{ijk}, T)$$

$$\dot{\phi} = \frac{\partial \phi}{\partial \varepsilon_{ij}} \dot{\varepsilon}_{ij} + \frac{\partial \phi}{\partial \gamma_{jk}} \dot{\gamma}_{jk} + \frac{\partial \phi}{\partial \kappa_{ijk}} \dot{\kappa}_{ijk} + \frac{\partial \phi}{\partial T} \dot{T}$$
(26)

Equating from Eqs. (25) and (26):

$$\dot{\mathbf{U}} - \mathbf{T}\dot{\mathbf{n}} - \dot{\mathbf{T}}\mathbf{n} = \frac{\partial \phi}{\partial \epsilon_{\mathbf{ij}}} \dot{\epsilon}_{\mathbf{ij}} + \frac{\partial \phi}{\partial \gamma_{\mathbf{jk}}} \dot{\gamma_{\mathbf{jk}}} + \frac{\partial \phi}{\partial \kappa_{\mathbf{ijk}}} \dot{\kappa}_{\mathbf{ijk}} + \frac{\partial \phi}{\partial \mathbf{T}} \dot{\mathbf{T}}$$
(27)

Thus, since it has been assumed that the other quantities are independent of \dot{T} , comparison of Eqs. (26) and (27) shows that

$$\eta = -\frac{\partial \phi}{\partial T}$$
(28)

$$\dot{\mathbf{U}} = \dot{\mathbf{n}}\mathbf{T} + \frac{\partial\phi}{\partial\varepsilon}\dot{\varepsilon}_{\mathbf{i}\mathbf{j}}\dot{\varepsilon}_{\mathbf{i}\mathbf{j}} + \frac{\partial\phi}{\partial\gamma}\dot{\gamma}_{\mathbf{j}\mathbf{k}}\dot{\gamma}_{\mathbf{j}\mathbf{k}} + \frac{\partial\phi}{\partial\kappa}\dot{\kappa}_{\mathbf{i}\mathbf{j}\mathbf{k}}\dot{\kappa}_{\mathbf{i}\mathbf{j}\mathbf{k}}$$
(29)

Putting Eq. (29) into Eq. (23) gives

$$\tilde{\sigma}_{ij} \dot{\epsilon}_{ij} + \hat{\sigma}_{jk} \dot{\gamma}_{jk} + \mu_{ijk} \dot{\kappa}_{ijk} = \rho \left(\frac{\partial \phi}{\partial \epsilon}_{ij} \dot{\epsilon}_{ij} + \frac{\partial \phi}{\partial \gamma}_{jk} \dot{\phi}_{jk} + \frac{\partial \phi}{\partial \kappa}_{ijk} \dot{\kappa}_{ijk} \right)$$
(30)

Since it has been assumed that ϕ is independent of $\dot{\epsilon}_{ij}$, $\dot{\gamma}_{ij}$, and $\dot{\kappa}_{ijk}$, the following additional equations must be satisfied:

 $\bar{\sigma}_{ij} = \rho \frac{\partial \phi}{\partial \epsilon_{ij}}, \ \hat{\sigma}_{jk} = \rho \frac{\partial \phi}{\partial \gamma_{jk}}, \ \mu_{ijk} = \rho \frac{\partial \phi}{\partial \kappa_{ijk}}$ (31a) (31b) (31c)

Equation (31a) has often been used as the thermodynamic basis for the symmetry of the elastic coefficients in the generalized Hooke's law for a general anisotropic elastic solid. In analogous fashion, Eq. (31b) can now be used to establish the symmetry of the microstructural elastic coefficients in general (anisotropic). It is of fundamental importance to note that Eqs. (31) show that thermodynamically there can be no coupling of macrostrains and microstrains <u>in the constitutive equations</u>. Needless to say, this simplifies the constitutive equations considerably.

To establish the form of the macro and micro constitutive equations more specifically, the free-energy function can be assumed to be a polynomial in the macrostrain components ϵ_{ij} , the relative strain components γ_{ij} , the macrostrain-gradient components κ_{ijk} , and the temperature T. For a linear material, defined as one having linear constitutive relations, the highest degree polynomial required is of second degree. Thus,

$$\rho \phi = a + b_{ij} \varepsilon_{ij} + d_{ijkl} \varepsilon_{ij} \varepsilon_{kl} + g_{ij} \varepsilon_{ij}^{T}$$
$$+ b_{jk}^{*} \gamma_{jk} + d_{jk}^{*} \gamma_{jk} \gamma_{m} + g_{jk}^{*} \gamma_{jk}^{T}$$
$$+ B_{ijl} \varepsilon_{ijl} + D_{ijklmn} \varepsilon_{ijk} \varepsilon_{lmn} + G_{ijk} \varepsilon_{ijk}^{T}$$
(32)

and

In view of Eqs. (31) and (32), the following constitutive equations can be written:

$$\bar{\sigma}_{ij} = b_{ij} + d_{ijij}\varepsilon_{ij} + d_{ijkl}\varepsilon_{kl} + g_{ij}T; \ kl \neq ij$$
(33)

$$\hat{\sigma}_{ij} = b_{jk}^{\star} + d_{jkjk}^{\star}\gamma_{jk} + d_{jk\ell m}^{\star}\gamma_{\ell m} + g_{jk}^{\star}T; \ \ell m \neq jk$$
(34)

$${}^{\mu}_{ijk} = {}^{B}_{ijk} + {}^{D}_{ijkijk}{}^{\kappa}_{ijk} + {}^{D}_{ijk\ell mn}{}^{\kappa}_{\ell mn} + {}^{G}_{ijk}{}^{T}; \ \ell mn \neq ijk$$
(35)

There is a total of forty independent parameters involved in a general three-dimensional thermoelastic problem involving microstructure. These are temperature plus thirty-nine mechanical deformation parameters:

Six
$$\varepsilon_{ij}$$
 (since $\varepsilon_{ji} = \varepsilon_{ij}$)
Six γ_{jk} (since $\gamma_{kj} = -\gamma_{jk}$)
Twenty-seven κ_{ijk}

(Mindlin⁸ stated that there are forty-two mechanical deformation parameters, but he had not determined that $\gamma_{kj} = -\gamma_{jk}$.)

Coefficient Array	Type ^(a)	Total No. of Non-Zero Coefficients	No. of Independent Coefficients
Ъ	IM	6	б
d	AM	36	21
g	Т	3(b)	3(b)
b*	IM	6	3
d*	AM	36	. 15
g*	Т	6	3
B	IM	27	27
D	AM	729	729
G	Т	27	27
	Tota	L IM <u>39</u>	36
	Tota	1 AM 801	765
	Tota	1 T 36	33
	Gran	d Total 876	834

The arrays of coefficients have the following sizes:

(a) IM = initial mechanical, AM Ξ active mechanical, T Ξ thermal expansion
(b) This relies on the well-established fact that there are no thermal shear strains in linear theory.

The total number of independent active mechanical coefficients listed in the above tabulation (765) is considerably smaller than the 1764 claimed by Mindlin,¹⁰ although it is still a formidable number. The advantage of including anisotropic behavior both on a macroscopic and on a microscopic scale is that it permits handling of any material. However, many metals and alloys of engineering importance (steel, aluminum, some titanium) are macroscopically isotropic and microscopically anisotropic depending upon the crystalline structure.

Recently Gurtin^{124,125} proved a theorem stating that the Clausius-Duhem inequality requires that every elastic material be "simple," i.e., have stress, free energy, and entropy at a material point dependent at most on deformation gradient and temperature at that point. However, Gurtin did not include the possibility of any doublet stress effects in his work. Thus, the present work is not inconsistent with his theorem; rather it is beyond the scope of his theorem.

D. RELATION TO OTHER THEORIES

It may be possible to relate the present theory to the solid-state theory of the Grüneisen parameter.¹²⁶⁻¹²⁸ For example, Gilgarry¹²⁸ has argued that the Dugdale-MacDonald corrections¹²⁷ for finite strain are not valid; yet many experiments indicate better agreement with the Dugdale-MacDonald predictions.

It has been shown that there is a direct correspondence between doublet stresses and a continuous distribution of dislocations.^{129,130} This may provide a basis for relating the present work to various theories of dislocational damping.^{131,132}

The present theory has no provisions for phase and state transformations, which sometimes take place in certain crystalline materials under strong shock conditions.³

III. DILATION WAVES INCLUDING THE EFFECTS OF MICROSTRUCTURE

A. INTRODUCTION

The accumulation of a number of years of experimental work with shock waves in solids is reported in Ref. 133. That work has led to the conclusion that any proper explanation of shock wave phenomena in solids must include the effects of a rotation and deformation of microscopic regions even when the input loading is a longitudinal pulse. These conclusions provide sufficient reason for the investigation of a theory of "microstructural" deformation in shock wave processes. A necessary requirement for such an investigation is the development of appropriate constitutive equations as done above.

These constitutive equations are used here with the equations of motion for plane waves.

B. DEFINITION OF AN IDEAL ELASTIC SOLID WITH MICROSTRUCTURE FOR SHOCK WAVE STUDIES

The definition of an "ideal" continuum involves the assumption that a state variable (vector) exists and that a reversible path exists between any two possible conditions of the state vector. The state vector for an ideal elastic solid with microstructure must include the temperature, T , and information about either the state of stress or strain in both conventional macrostructure and the additional microstructure. Constitutive equations have been developed which involve the state variables ε_{ij} , γ_{jk} , and κ_{ijk} (macrostrain, relative deformation between make of microstructure media, and the microstrain gradient), and the state functions:

$$\bar{\sigma}_{ij}(\epsilon_{ij}, \gamma_{jk}, \kappa_{ijk}, T) = symmetric macrostress (36)$$

$$\hat{\sigma}_{ik}(\epsilon_{ij}, \gamma_{jk}, \kappa_{ijk}, T) = unsymmetric macrostress$$
 (37)

$$i_{jk}(\epsilon_{ij}, \gamma_{jk}, \kappa_{ijk}, T) = doublet stress.$$
 (38)

This development is valid whenever the kinematics of the physical problem make Eulerian and Lagrangian coordinates interchangeable. In particular, the assumption is made that

$$\frac{\partial \mathbf{u}_{j}}{\partial a_{i}} \simeq \frac{\partial \mathbf{u}_{j}}{\partial \mathbf{x}_{i}}$$
(39)

where $u_j = x_j - a_j$ is the displacement of the particle, x_j is the Eulerian coordinate and a_j is the Lagrangian coordinate. And ε similar assumption applies to derivatives in the microstructure. In the problems where the absolute value of displacement gradients is small, i.e., where

$$\left|\frac{\partial \mathbf{u}_{\mathbf{j}}}{\partial \mathbf{a}_{\mathbf{j}}}\right| << 1 \tag{40}$$

this is a valid approximation. However, we wish to deal with shock waves in the macrostructure where this approximation could be considered invalid. The validity of the assumption in reference to the microstructure also needs to be considered. The size of the microstructure elements has been related to the spacing <f dislocations in Ref. 133. In Ref. 134, a typical value for the density of dislocations in a crystalline structure is suggested as $10^9/\text{cm}^2$ before shock wave loading and $10^{11}/\text{cm}^2$ after loading.

This would lead to typical sell sizes of 3.10^{-5} cm/sell edge before and 3.10^{-6} cm/sell edge after shock wave loading. The same reference also suggests that the shock front thickness would typically be of the order of a phonon mean free path or from 10^{-5} cm to 10^{-7} cm, which is comparable to the sell size. This leaves the speculation that the gradient is <u>not</u> <u>small</u> in the microstructure with some support. Consequently it will be necessary to reexamine the formulation of the microstructural representation.

and

The Eulerian equations of motion for an elastic solid with microstructure are $^{10}\,$

$$\frac{\partial}{\partial x_{i}} (\bar{\sigma}_{ij} + \hat{\sigma}_{ij}) + \rho f_{j} = \rho u_{j}$$
(41a)

$$\frac{\partial}{\partial x_{i}} \left(\bar{\mu}_{ijk} + \hat{\mu}_{ijk} \right) + \hat{\sigma}_{jk} + \rho F_{jk} = \frac{1}{3} \rho' d_{k\ell}^{2} \bar{\psi}_{k\ell}$$
(41b)

These equations are converted to Lagrangian form by substitution of the Lagrangian variables.

$$\frac{\partial}{\partial a_{i}} \left(\overline{\sigma}_{i}^{(L)} \right) + \hat{\sigma}_{ij}^{(L)} + \rho_{o} f_{j}^{(L)} = \rho_{o}^{u} u_{j}$$
(42a)

$$\frac{\partial}{\partial a_{i}} \left(\overline{\mu}_{ijk}^{(L)} \right) + \hat{\mu}_{ijk}^{(L)} + \hat{\sigma}_{jk}^{(L)} + \rho'_{o} F_{jk}^{(L)} = \frac{1}{3} \rho'_{o} \left(\frac{I}{k\ell} \right)^{2} \psi_{k\ell}$$
(42b)

The Eulerian symmetric stress tensor σ_{ij} gives rise to a nonsymmetric Lagrangian stress tensor $\sigma_{ij}^{(L)}$. In order to recapture a symmetric stress tensor, the Langrangian tensor may be transformed to the Kirchhoff stress tensor, and Eq. (7a) becomes

$$\frac{\partial}{\partial a_{i}} \left[\left(\overline{\sigma}_{ik}^{(K)} + \widehat{\sigma}_{ik}^{(K)} \right) \frac{\partial x_{j}}{\partial a_{k}} \right] + \rho_{o} f_{j}^{(L)} = \rho_{o} u_{j}$$

In order to simplify the notation, we let $f_j^{(L)} \equiv f_j$ and define

$$\sigma^{(K)} = S$$

.....

so that the macro material equation of motion in Lagrangian coordinates becomes

$$\frac{\partial}{\partial a_{i}} \left[(\bar{s}_{ik} + \hat{s}_{ik}) \frac{\partial x_{j}}{\partial a_{k}} \right] + \rho_{o} f_{j} = \rho_{o} u_{j}$$
(43a)

Use may be made of the definition of μ_{ijk} as a gradient of the scalar potential energy of an elastic solid to obtain an equivalent "Kirchhoff" doublet stress in the following form

$$\mu_{\mathbf{ijk}}^{(K)} = \left(\frac{\rho_{o}}{\rho}\right) \frac{\partial \mathbf{a}_{\mathbf{i}}^{\prime}}{\partial \mathbf{x}_{\delta}^{\prime}} \frac{\partial \mathbf{a}_{\mathbf{j}}^{\prime}}{\partial \mathbf{x}_{\delta}^{\prime}} \frac{\partial \mathbf{a}_{\mathbf{j}}^{\prime}}{\partial \mathbf{x}_{\varepsilon}^{\prime}} \frac{\partial \mathbf{a}_{\mathbf{k}}^{\prime}}{\partial \mathbf{x}_{\varepsilon}^{\prime}} \mu_{\gamma\delta\varepsilon}$$

While the relation between the "Kirchhoff" and "Lagrangian" forms of the doublet stress is

$$\mu_{i\ell m} \begin{pmatrix} K \\ \partial a_{i}' \\ \partial a_{i}' \\ \partial a_{i}' \\ \partial a_{k}' \\ \partial a_{i}' \\ \partial a_{k}' \\ \partial a_{i}' \\ \partial a_{k}' \\ \partial a_{$$

The desired form of the micro material equation of motion is obtained as

$$\frac{\partial}{\partial a_{i}} \left[\left(\widetilde{M}_{i\ell m} + \widehat{M}_{i\ell m} \right) \frac{\partial x_{j}'}{\partial a_{\ell}'} \frac{\partial x_{k}'}{\partial a_{m}'} \right] + \widehat{S}_{jk} \frac{\partial x_{j}}{\partial a_{k}} + \rho_{\delta}F_{jk} = \frac{1}{3} \rho_{o}' (d_{k\ell})^{2} \widetilde{\psi}_{k\ell}$$
(43b)
with the simplifying notation $\mu^{(K)} = M$, $d^{(L)} = d$ and $F_{jk}^{(L)} \equiv F_{jk}$

Equations (43a) and (43b) are the equations of motion for use with plane wave phenomena.

C. PLANAR WAVE IN MICROMEDIA

In a macromedia the planar wave is defined by requiring all spacial derivatives to be zero except for those in the direction of the wave propagation, i.e. $\frac{\partial}{\partial a_1}$ () $\neq 0$, $\frac{\partial}{\partial a_4}$ () = 0, i = 2,3 . The same idea may be extended to the micromedia, in which case $\frac{\partial}{\partial a_1'}$ () $\neq 0$, $\frac{\partial}{\partial a_1'}$ () = 0, i = 2,3 . When this is done, Eqs. (43a) and (43b) take the form

$$\frac{\partial}{\partial a} \left[(\bar{S} + \hat{S}) \frac{\partial x}{\partial a} \right] + \rho_0 f = \rho_0 u$$
(44a)

$$\frac{\partial}{\partial a} \left[M \left(\frac{\partial x}{\partial a} \right)^2 \right] + \hat{S} \frac{\partial x}{\partial a} + \rho_0 F = \frac{\rho_0 d^2}{3} \ddot{\psi}$$
(44b)

If we introduce the notion that $\frac{\partial u}{\partial a}$ is a measure of the wave strength 107 (or shock strength), then we may give a meaning to $\frac{\partial x}{\partial a}$ in the following way: Let $\lambda = \frac{\partial u}{\partial a}$ and recall that u = x - a, so that $\frac{\partial x}{\partial a} = 1 + \lambda$, then Eqs. (44a) and (44b) become (for f = F = 0)

$$\frac{\partial}{\partial a} \left[(\bar{S} + \hat{S})(1 + \lambda) \right] = \rho_0^{"}$$
(45a)

$$(1+\lambda)\hat{S} = \frac{\rho_0 d^2}{3} \ddot{\psi} - \frac{\partial}{\partial a} [M(1+\Psi)^2]$$
(45b)

Again following Bland, 107 the stresses are replaced by derivatives of the potential energy function, which we will call W. So that *

$$\overline{S} = \frac{\partial W}{\partial \varepsilon} \left(\text{and } \varepsilon = \lambda + \frac{1}{2} \lambda^2 \right)$$
$$M = \frac{\partial W}{\partial \kappa}$$

In addition, note that

$$\frac{\partial W}{\partial \lambda} = \frac{\partial W}{\partial \varepsilon} \frac{\partial \varepsilon}{\partial \lambda} = \frac{\partial W}{\partial \varepsilon} (1 + \lambda)$$

and

$$\frac{\partial}{\partial a} \left(\frac{\partial W}{\partial \lambda} \right) = \frac{\partial^2 W}{\partial \lambda^2} \frac{\partial \lambda}{\partial a} = \frac{\partial^2 W}{\partial \lambda^2} \frac{\partial^2 u}{\partial a^2}$$

then Eq. (10a) becomes

$$\frac{\partial^2 W}{\partial \lambda^2} \cdot \frac{\partial^2 u}{\partial a^2} + \frac{\partial}{\partial a} \left[(1 + \lambda) \hat{S} \right] = \rho_0 u$$
(46a)

In order to make a similar elimination of M, one needs to differentiate the last term in Eq. (45b).

* For the present we will not alter S.

Equation (45b) with M replaced by $\frac{\partial W}{\partial \kappa}$ and carrying out the differentiation gives

$$(1 \div \lambda)\hat{s} = \frac{\rho_0 d^2}{3} \ddot{\Psi} - \left[(1 + \Psi)^2 \frac{\partial^2 W}{\partial \kappa^2} \frac{\partial \kappa}{\partial a} + \frac{\partial W}{\partial \kappa^2} (1 + \Psi) \frac{\partial \Psi}{\partial a} \right]$$

But

$$\kappa = \frac{\partial \Psi}{\partial a}$$

so that

$$\frac{\partial \kappa}{\partial a} = \frac{\partial^2 \Psi}{\partial a^2}$$

Then

$$(1 + \lambda)\hat{s} = \frac{\rho_0 d^2}{3} \ddot{\psi} - \frac{\partial^2 W}{\partial \kappa^2} \frac{\partial^2 \psi}{\partial a^2} - \left[(2 + \psi)\psi \frac{\partial^2 \psi}{\partial a^2} \frac{\partial^2 W}{\partial \kappa^2} + 2\psi \frac{\partial \psi}{\partial a} \frac{\partial W}{\partial \kappa} \right]$$
(46b)

It is clear that if Eq. (46b) is introduced into Eq. (46a) a very complicated partial differential equation results. In order to find any acceptable simplification of these equations, we wish to know what physical facts can be used to simplify the mathematics of the problem at this point?

Up to the end of the elastic regime, the wave equation (obtained by $\hat{s} = 0$ in Eq. (46a)) gives a very good approximation to the experimental obser stions. This seems to be true even when a large number of dislocations are present (i.e., when a microstructure is present).

くたちというという

We will appeal to these observations as an argument for setting $\hat{s} = 0$ and see if its consequence gives any meaningful results. To do this we must study the simultaneous equations:

$$\rho_{0}\ddot{u} - \frac{\partial^{2}W}{\partial\lambda^{2}}\frac{\partial^{2}u}{\partial\lambda^{2}} = 0$$
(47a)

$$\frac{\rho_{o}d^{2}}{3}\ddot{\Psi} - \frac{\partial^{2}W}{\partial\kappa^{2}}\frac{\partial^{2}\Psi}{\partiala^{2}} - 2\frac{\partial W}{\partial\kappa}\frac{\partial\Psi}{\partiala} = (2 + \Psi)\Psi\frac{\partial^{2}\Psi}{\partiala^{2}}\frac{\partial^{2}W}{\partial\kappa^{2}} + 2\Psi\frac{\partial\Psi}{\partiala}\frac{\partialW}{\partial\kappa}$$
(47b)

and the constitutive equation

$$W = A + B\lambda + C\lambda^{2} + D\lambda T + F\kappa + G\kappa^{2} + H\kappa T \qquad (47c)^{*}$$

with proper boundary and initial conditions. For now we take D = H = A = 0. Then the behavior of Eq. (47a) is well known and Eq. (47b) becomes the equation of interest. Substituting W into Eq. (47b) leads to:

$$\frac{\rho_{o}d^{2}}{3} \ddot{\Psi} - 2G \frac{\partial^{2}\Psi}{\partial a^{2}} - 2(F) \frac{\partial\Psi}{\partial a} = 2G \left(\frac{\partial\Psi}{\partial a}\right)$$

$$+ 2(2 + \Psi)\Psi \frac{\partial^{2}\Psi}{\partial a^{2}}G + 2\Psi \frac{\partial\Psi}{\partial a}\left(F + 2G \frac{\partial\Psi}{\partial a}\right)$$
(48a)

The right hand of Eq. (48a) is highly nonlinear. In order to see if anything can be done simply, we will arbitrarily set all nonlinear terms to zero (the validity of this step must be evaluated). So doing gives Eq. (48b)

$$\frac{\rho_o d^2}{3} \ddot{\Psi} - 2G \frac{\partial^2 \Psi}{\partial a^2} - 2F \frac{\partial \Psi}{\partial a} = 0$$
(48b)

* Equation (32).

In order to make the study of Eq. (48b) easier, nondimensional variables are introduced in the form:

$$\Psi = \frac{1}{2} \frac{G^2}{F} \phi$$
$$a = \frac{G}{F} \zeta$$
$$t = \frac{d}{F} \sqrt{\frac{G\rho_0}{6}}$$

with the result

$$\phi_{\zeta\zeta} - \phi_{\tau\tau} - \phi_{\tau} = 0 \tag{49}$$

Equation (49) has the form of the telegraph equation with space and time variables having their roles transposed. A substitution of the form $\phi = \phi e^{-\zeta/2}$ transforms Eq. (49) into

$$\phi_{\zeta\zeta} = \phi_{\tau\tau} + \frac{1}{4} \phi$$
 (50)

and a formal solution with initial conditions of $\phi_{\zeta}(\zeta,0) = P(\zeta), \phi(\zeta,0) = 0$ yields

$$\phi(\zeta,\tau) = e^{-\zeta/2} \int_{0}^{\zeta} J_{o}\left(\frac{i}{2} \sqrt{\zeta^{2}\rho^{2}}\right) Q(\rho,0) d\rho$$
 (51a)

where

and the second life

$$Q(\rho,\tau) = \frac{1}{2} \left[P(\rho + \tau) + P(\rho - \tau) \right]$$
(51b)

But $P(\rho + \tau)$ corresponds to $\tau < 0$ and does not exist. And since the initial velocity $P(\zeta)$ is applied only at the surface $\zeta = 0$, it can be represented as a delta function $P(\zeta) = \delta(\zeta)$ so that Eqs. (51) give

$$\phi(\zeta,\tau) = e^{-\zeta/2} \frac{1}{2} \int_{0}^{\zeta} J_{o}\left(\frac{i}{2} - \rho^{2}\right) \delta(\rho - \tau) d\rho$$
(52)

The solution for $\tau \ge \zeta \ge 0$ is

والمتعادية والمعادية والمعالية والمعادية والمعالية والمع

$$\phi(\zeta,\tau) = \frac{1}{2} e^{-\zeta/2} J_{o} \left(\frac{1}{2} \sqrt{\tau^{2} - \zeta^{2}}\right)$$
(53)

and for $\tau < \zeta$ the signal has not yet arrived at ζ , so $\phi = 0$. Returning Eq. (53) to dimensional notation yields

$$\frac{\partial u'}{\partial a'} = \frac{1}{2} \frac{G^2}{F} e^{-F/2G} a J_o \left(\frac{1}{2} \sqrt{\frac{6F^2}{G\rho_o d^2} t^2 - \frac{F^2}{G^2} a^2} \right)$$
(54)

for an initial microstrain velocity

$$\frac{\partial}{\partial t} \frac{\partial u'}{\partial a'} = \sqrt{\frac{3G^3}{2\rho_0}} \frac{1}{Fd}$$
(55)

corresponding to $\phi_{\tau} = 1$.

This result corresponds to an oscillating decay of microstrain as a function of position along the space axis.

If F = 0 in the constitutive equation (Eq. (47c)), then Eq. (48b) becomes

$$\frac{\rho_0 d^2}{3} \ddot{\Psi} - 2G \frac{\partial^2 \Psi}{\partial a^2} = 0$$

This is a standard wave equation and the microstrain may take any form that is a solution to a wave equation. Now if either of these cases obtains, then a space oscillation of microstrain is a possible elastic solution. Since there has been experimental evidence of a space oscillation in strain hardness after shock loading, ¹³³ it is tempting to speculate on whether or not these oscillating solutions can be related

to the strain hardness observations. If the loading takes place so rapidly that the deformation extends its elastic range on loading, there may be some correlation between these results and those for which oscillation of the strain hardening has been observed.

And the second second second second

APPENDIX

THERMODYNAMICALLY VALID THERMOELASTIC CONSTITUTIVE RELATIONS CONSIDERING TEMPERATURE DEPENDENCY OF THE ELASTIC COEFFICIENTS AND STRESS DEPENDENCY OF THE THERMAL-EXPANSION COEFFICIENT

There are two ways to formulate the thermodynamics of solids. Traditionally, it has been assumed that deformation gradients (strains) and temperature are the thermodynamic fluxes (i.e., the independent variables, also called state variables), while stress was considered as the thermodynamic force (i.e., the dependent variable, also called the state function). However, in some instances ^{19,135} stress and temperature were considered as the independent variables and strain as the dependent variable. In his recent paper on thermodynamics of strained solids, Kestin²⁷ stated explicitly that either strain or stress could be used as the independent variable. The latter formulation is believed to be more desirable for the following reasons:

- Prestress (i.e., initial stress) is included in a much more natural fashion, as discussed by Burridge and Knopoff.²⁰ Prestress must be considered in problems involving unloading waves.
- 2. The definitions of the elastic and thermal-expansion coefficients permit a much simpler derivation of the interrelationship between temperature dependency of the elastic coefficients and stress dependency of the thermal-expansion coefficient.
- 3. As shown by Westergaard, it is easier to deal with nonlinear stress-strain relations by complementary energy as a function of stress rather than strain energy as a function of strain.
- 4. The yield criterion (or plastic potential) in plasticity theory is a function of stress not strain.

In view of these considerations, especially the second one, the formulation having stress and temperature as the independent variables will be used here.

The energy equation is as follows:

$$\varepsilon_{ij}\dot{\sigma}_{ij} + Tn = \dot{U}$$
 (A1)

where ϵ_{ij} is the strain tensor, σ_{ij} is the stress tensor, T is absolute temperature, n is the entropy per unit-reference-state volume, U is the internal energy per unit-reference-state volume, and a dot denotes a derivative with respect to time.

The free-energy function is now defined as follows:

$$\phi(\sigma_{ij}, T) = U(\sigma_{ij}, T) - Tn(\sigma_{ij}, T)$$
(A2)

Putting Eq. (A2) into Eq. (A1) gives

$$(\varepsilon_{ij} - \partial\phi/\partial\sigma_{ij})\dot{\sigma}_{ij} - (\eta + \partial\phi/\partial T)\dot{T} = 0$$
(A3)

Since thermodynamically 6,124,125 $_{\phi},\,\eta$, and ϵ_{ij} are independent of $\dot{\sigma}_{ij}$ and $\dot{T},$

$$\varepsilon_{ij} = \frac{\partial \phi}{\partial \sigma_{ij}}$$
, $\eta = -\frac{\partial \phi}{\partial T}$ (A4) (A5)

Onsager's reciprocal relation implies sufficient continuity of ϕ that the order of partial differentiation can be interchanged. Thus, the following equality must be satisfied:

Since the Lagrangian formulation is used, this unit-reference-state volume basis is equivalent to using a unit-mass basis, but it has the advantage of eliminating the symbol for reference-state density in numerous subsequent equations.

$$\frac{\partial}{\partial \sigma_{ij}} \left| \begin{array}{c} \frac{\partial}{\partial T} \left(\frac{\partial \phi}{\partial \sigma_{ij}} \right) \right|_{T} = \frac{\partial}{\partial T} \left| \begin{array}{c} \frac{\partial}{\partial \sigma_{ij}} \left(\frac{\partial \phi}{\partial \sigma_{ij}} \right) \right|_{\sigma_{ij}}$$

or

$$\frac{\partial}{\partial \sigma_{ij}} \left| \frac{\partial \varepsilon_{ij}}{\partial T} \right|_{T} = \frac{\partial}{\partial T} \left| \frac{\partial \varepsilon_{ij}}{\partial \sigma_{ij}} \right|_{\sigma_{ij}}$$
(A6)

To obtain constitutive equations which are linear in stress, the free-energy function is assumed to have the following form:

$$\phi = B_1(T)J_1^2 + B_2(T)J_2 + C_1(J_1,T)(J_1)(T - T_0) + F(T)$$
 (A7)

where T_0 is the temperature at the relaxation state (i.e., the temperature at which the material is free of strain), B_1 and B_2 are temperature-dependent elastic coefficients, C_1 is the usual thermal-expansion coefficient, F is an arbitrary function of temperature only (detailed form selected to represent the nonlinear dependence of specific heat at constant volume on temperature) and J_1 and J_2 are stress invariants defined as follows:

$$J_{1} \stackrel{\Delta}{=} \sigma_{ii}, \ J_{2} \stackrel{\Delta}{=} (1/2) \delta^{ij}_{\ell m} \sigma_{i\ell} \sigma_{jm}$$
(A8)

where i, j, ℓ , m = 1, 2, 3 and δ_m^{ij} is the generalized Kronecker delta given by:

 $\delta_{\ell m}^{ij} = \begin{cases} +1 \text{ if } i,j \text{ form an even permutation of } \ell,m \\ -1 \text{ if } i,j \text{ form an odd permutation of } \ell,m \\ 0 \text{ if } i,j \text{ do not form a permutation of } \ell,m \end{cases}$

The form of the C_1 term in Eq. (A7) implies the existence of a relaxation state, a point which has been questioned by Eckart,²⁵ who offered no specific alternative to use in constructing constitutive relations.

In view of Eq. (A4)

$$\varepsilon_{ij} = (\partial \phi / \partial J_1) (\partial J_1 / \partial \sigma_{ij}) + (\partial \phi / \partial J_2) (\partial J_2 / \partial \sigma_{ij})$$
(A9)

Differentiation of Eq. (A8) yields

$$\partial J_1 / \partial \sigma_{ij} = \delta_{ij}, \ \delta J_2 / \partial \sigma_{ij} = \delta_{ij} J_1 - \sigma_{ij}$$
 (A10)

where δ_{ij} is the simple Kronecker delta (takes values of unity for i = j, zero for $i \neq j$).

Inserting Eqs. (A7) and (A10) into Eq. (A9), one obtained

$$\varepsilon_{ij} = 2B_1(T)J_1\delta_{ij} + B_2(T)(\delta_{ij}J_1 - \sigma_{ij}) + \delta_{ij}(T - T_0)\partial(C_1J_1)/\partial J_1$$
(A11)

or

$$\varepsilon_{11} = 2B_1(T)\sigma_{11} + [2B_1(T) + B_2(T)](\sigma_{22} + \sigma_{33}) + (T - T_0)\partial(C_1J_1)/\partial\sigma_{11}$$
(A12)

(Similar relations for $\varepsilon_{22}^{}$ and $\varepsilon_{33}^{}$).

and

$$\varepsilon_{12} = -B_2(T)\sigma_{12} \tag{A13}$$

(similar relations for ε_{23} and ε_{31}).

Comparing Eqs. (A12) and (A13) with those of classical elasticity theory, we find

$$2B_1 \rightarrow E^{-1}$$
, where E is Young's modulus
 $2B_1 + B_2 \rightarrow -\nu/E$, where ν is Poisson's ratio
 $\partial(C_1J_1)/\partial J_1 \rightarrow \alpha$, lineal coefficient of thermal expansion

To check whether the Onsager requirement, represented by Eq. (A6) is satisifed, the mixed partial derivatives are taken as follows:

$$\frac{\partial}{\partial \sigma_{ij}} \left(\frac{\partial \varepsilon_{ij}}{\partial T} \right) = 2 \frac{d}{d} \frac{B_1}{T} \delta_{ij} + \frac{d}{d} \frac{B_2}{T} (\delta_{ij} - 1) + \delta_{ij} \frac{\partial^2}{\partial J_1 \partial T} [(T - T_0) \partial (C_1 J_1) / \partial J_1]$$
$$= \frac{\partial}{\partial T} \left(\frac{\partial \varepsilon_{ij}}{\partial \sigma_{ij}} \right)$$
(A14)

Equation (A14) shows that Eq. (A6) is satisfied without placing any restrictions whatsoever on the form of the constitutive functions $B_1(T)$, $B_2(T)$, $C_1(J_1,T)$, and F(T).

The specific entropy is computed from Eqs. (A5) and (A7) with the following result:

$$n = -J_{1}^{2} d B_{1}/d T - J_{2} d B_{2}/d T - J_{1}^{2}[(T - T_{0})C_{1}]/\partial T - d F/d T$$
(A15)

Since there can be no entropy production during continuous deformation of a perfectly elastic material, the specific entropy must be independent of the stresses. Thus, from Eq. (A15):

$$d B_{2}/d T = 0$$
 (A16)

$$J_{1} d B_{2}/d T + \partial [(T - T_{0})C_{1}]/\partial T = 0$$
 (A17)

It can be shown that B_2 is the reciprocal of the shear modulus G = E/2(1 + v); thus, Eq. (A16) implies that the shear modulus should be independent of temperature. However, it is an experimentally observed

fact that the shear modulus <u>is</u> temperature dependent. This can be accommodated quite easily by adding a term of the following form to the freeenergy function ϕ in Eq. (A7).

$$C_2(J_2,T)(J_2)(T - T_0)$$

In the stress-strain equation, Eq. (All), this adds a term of the form

$$(\delta_{ij}J - \sigma_{ij})(T - T_0)\partial(C_2J_2)/\partial J_2$$

which is analogous to the deviatoric-thermal-expansion effect considered theoretically and experimentally by Dillon.¹⁷ Addition of this term does not prevent satisfaction of Eq. (A6).

To obtain some useful information from Eq. (A17), use is made of the following relation observ d in experiments by Rosenfield and Averbach: 16

$$C_1(\sigma_{kk}, T) = C_{10}(T) + C_{11}(T)\sigma_{kk}$$

Rosenfield and Averbach used only uniaxal loading and measured C'_{11} only in the direction of that loading. However, it seems intuitively apparent that a stress in direction 1 could affect the thermal expansion in direction 1 only, i.e., since the loading is directional, its effect on C_1 should also be directional. This probably could be proved by functional analysis. However, the present theory is an isotropic one; thus, the preceding equation must be rewritten as its isotropic equivalent:

$$C_1(J_1,t) = C_{10}(t) + C_{11}(t)J_1$$
 (A18)

Then Equation (A17) leads to

$$J_{1} dB_{1}/dT Td[(T - T_{0})C_{10}]/dT$$

+
$$J_{1} d[(T - T_{0})C_{11}]/dT = 0$$
 (A19)

Again invoking independence of stress, Eq. (A19) implies:

$$d[(T - T_{0})C_{10}]/dT = 0$$
 (A20)

and

$$dB_{1}/dT + d[(T - T_{0})C_{11}]/dT = 0$$
 (A21)

Integration of Eq. (A20) yields:

$$(T - T_{o})C_{10}(T) = Const.$$
 (A22)

Equation (A22) does not appear to be realistic; this suggests that further work on this topic is required.

Equation (A21) expresses an interrelationship between B_1 and C_{11} . Since the experiments of Rosenfield and Averbach were conducted at room temperature only, the validity of Eq. (A21) cannot be checked with experimental data at present. Table 1 lists some sets of specific forms for C_{11} and B_1 which satisfy Eq. (A21).

Since $B_1 = (2E)^{-1}$, Eq. (2) in the body of the report can be put in the following form:

$$C_{11} = 2 \ dB_1 / dT$$
 (A23)

Thus, Eq. (A23) corresponds to a g value of 2, where g is defined as follows:

$$g \stackrel{\Delta}{=} C_{11}^{/(dB_1/dt)}$$
(A24)

It is noted that the <u>form</u> of Eq. (A23) is identical to Cases 2-4 in Table 1, even though the coefficient g differs considerably. It is



TABLE 1

* K, k, p are constants

noted that for temperatures near to the strain-free temperature (t_0) , Cases 2-4 predict g \approx -1. However, Rosenfield and Averbach's experiments for five metals gave a value of g within 3% of 2 (although individual variations for specific metals differed from 2 by 6% to 100%).

Apparently Rosenfield and Averbach made a mathematical error in applying Eq. (A6), and in doing so, they obtained the interrelationship Eq. (2). This is contrary to current thinking in thermodynamics that Onsager's relation should not place numerical restrictions on the constitutive equation coefficients, but rather should be used only in determining the general <u>form</u> of the constitutive equations. Nevertheless, it is puzzling that Eq. (A23) agrees with experimental results better than with the values of Cases 2-4 in Table 1.

REFERENCES

- M. H. Rice, R. G. McQueen, and J. M. Walsh, <u>Compression of Solids by</u> <u>Strong Shock Waves</u>, Solid State Physics, Vol. 6, Seitz, F. and D. Turnbull, Editors, Academic Press, Inc., New York, 1958, pp. 1-63.
- 2. I. C. Skidmore, <u>An Introduction to Shock Waves in Solids</u>, Applied Materials Research, Vol. 4, 1965, pp. 131-147.
- L. V. Al'tshuler, <u>Use of Shock Waves in High Pressure Physics</u>, Soviet Physics Uspekhi, Vol. 8, 1965, pp. 52-91.
- 4. D. R. Curran, <u>Nonhydrodynamic Attenuation of Shock Waves in Aluminum</u>, Journal of Applied Physics, Vol. 34, 1963, pp. 2677-2685.
- 5. J.M.C. Duhammel, <u>Second mémoire sur les phénomènes thermomécaniques</u>, Journal de l'école polytechnique, Vol. 15, No. 25, 1837, pp. 1-57.
- 6. B. A. Boley, and J. H. Weiner, <u>Theory of Thermal Stresses</u>, John Wiley and Sons, Inc., New York, 1960.
- 7. H. Deresiewicz, <u>Plane Waves in a Thermoelastic Solid</u>, Journal of the Acoustical Soc. of America, Vol. 29, 1957, pp. 204-209.
- 8. W. Voigt, <u>Theoretische Studien über die Elasticitätsverhältnisse</u> der Krystalle, Abh. Ges. Wissenschaften Göttingen, Vol. 34, 1887.
- 9. E. and F. Cosserat, <u>Theorie des Corps Déformables</u>, <u>Hermann</u>, <u>Paris</u>, France, 1909.
- R. D. Mindlin, <u>Micro-structure in Linear Elasticity</u>, International Journal of Solids and Structures, Vol. 1, 1965, pp. 51-78.
- 11. A. E. Green, and R. S. Rivlin, <u>Simple Force and Stress Multipoles</u>, Archive for Rational Mechanics and Analysis, Vol. 16, 1964, pp. 325-353.
- A. E. Green, and R. S. Rivlin, <u>Multipolar Continuum Mechanics</u>, Archive for Rational Mechanics and Analysis, Vol. 17, 1964, pp. 113-147.
- A. C. Eringen, and E. S. Suhubi, <u>Nonlinear Theory of Simple Micro-Elastic Solids-I</u>, Internat. Journal of Engineering Science, Vol. 2, 1964, pp. 189-202.

- E. S. Suhubi and A. C. Eringen, <u>Nonlinear Theory of Simple Micro-Elastic Solids-II</u>, Internat. Journal of Engineering Science, Vol. 2, 1964, pp. 389-403.
- 15. E. H. Lee, and T. Wierzbicki, <u>Analysis of the Propagation of Plane</u> <u>Elastic-Plastic Waves at Finite Strain</u>, American Soc. of Mechanical Engineers, Paper No. 67-APM-36, June 1967.
- A. R. Rosenfield, and B. L. Averbach, <u>Effect of Stress on the</u> <u>Expansion Coefficient</u>, Journal of Applied Physics, Vol. 27, 1956, pp. 154-156.
- W. Dillon, Jr., <u>A Nonlinear Thermoelasticity Theory</u>, Journal of the Mechanics and Physics of Solids, Vol. 10, 1962, pp. 123-132.
- Y. C. Fung, <u>Foundations of Solid Mechanics</u>, Prentice-Hall, Inc., Englewood Cliffs, N.J., 1965, pp. 348-349.
- 19. A. A. Vakulenko, <u>A Thermodynamic Investigation into the Relation</u> <u>Between Stress and Deformation in Isotropic Elastoplastic Media</u>, <u>Doklady Akademii Nauk SSSR</u>, Vol. 126, 1958, pp. 736-739; English Translation, Soviet Physics-Doklady, Vol. 4, 1959, pp. 697-700.
- R. Burridge, and L. Knopoff, <u>The Effect of Initial Stress on Elastic</u> <u>Energy Calculations</u>, Bulletin of the Seismological Soc. of America, Vol. 56, 1966, pp. 421-424.
- 21. E. H. Lee, and D. T. Liu, <u>Finite-Strain Elastic-Plastic Theory with</u> <u>Application to Plane-Wave Analysis</u>, Journal of Applied Physics, Vol. 38, 1967, pp. 19-27.
- B. D. Coleman, and W. Noll, <u>Material Symmetry and Thermostatic</u> <u>Inequalities in Finite Elastic Deformation</u>, Archive for Rational Mechanics and Analysis, Vol. 15, 1964, pp. 87-111.
- C. C. Wang, and R. M. Bowen, <u>On the Thermodynamics of Non-Linear</u> <u>Materials with Quasi-Elastic Response</u>, Archive for Rational Mechanics and Analysis, Vol. 22, 1966, pp. 79-99.
- 24. W. S. Farren, and G. I. Taylor, <u>The Heat Developed During Plastic</u> <u>Extension of Metals</u>, Proc. of the Royal Society, London, Vol. 107A, 1925, pp. 422-451.
- C. Eckart, <u>The Thermodynamics of Irreversible Processes. IV The</u> <u>Theory of Elasticity and Plasticity</u>, Physical Review, 2nd Series, Vol. 73, 1948, pp. 573-382.

- Bridgman, P.W., <u>The Thermodynamics of Plastic Deformation and</u> <u>Generalized Entropy</u>, Reviews of Modern Physics, Vol. 22, 1950, pp. 56-63.
- Xestin, J., <u>Application of the Principles of Thermodynamics to</u> the Description of Deformed <u>Materials</u>, Mechanika Teoretyczna i Stosowana, Vol. 5, 1967, pp. 147-179.
- Freudenthal, A.M., <u>The Mathematical Theories of the Inelastic</u> <u>Continuum</u>, Handbuch der Physik, Vol. VI, Flugge, S., Editor, Springer-Verlag, Berlin, 1958.
- 29. Prager, W., <u>Non-Isothermal Plastic Deformation</u>, Proc. Nederland. Konj. Akad. Wetenchappen, Vol. 61B, 1958, pp. 176-192.
- Ziegler, H., <u>An Attempt to Generalize Onsager's Principle, and Its</u> <u>Significance for Rheological Problems</u>, Zeitschrift für angewandte Mathematik und Physik, Vol. 9, 1958, pp. 748-763.
- 31. Vakulenko, A.A., <u>The Connections Between Stresses and Strains in</u> <u>Inelastic Media</u>, Doklady Akademii Nauk SSSR, Vol. 118, 1957, pp. 665-668; English Translation, Soviet Physics-Doklady, Vol. 3, 1958, pp. 193-196.
- 32. Naghdi, P.M., <u>Stress-Strain Relat ons in Plasticity and Thermo-plasticity</u>, Plasticity (Proc. of he 2nd Symposium on Naval Structural Mechanics), Lee, E.H. and Symonds, P.S., Editors, Pergamon Press, New York, 1960, pp. 121-169.
- 33. Grigorian, S.S., <u>On Some Special P oblems of Thermodynamics of Continuous Media</u>, Prikladnaia Matenatika i Mekhanika, Vol. 24, 1960, pp. 651-662; English Translation, Journal of Applied Mathematics and Mechanics, Vol. 24, 1960, pp. 971-986.
- 34. Fastov, N.S., <u>The Thermodynamics of Irreversible Processes in</u> <u>Plastic Deformation</u>, Doklady Akademii Nauk SSSR, Vol. 130, 1960, p. 541; English Translation, Soviet Physics-Doklady, Vol. 5, 1960, pp. 160-163.
- 35. Dillon, O.W., Jr., <u>Coupled Thermoplasticity</u>, Journal of Mechanics and Physics of Solids, Vol 11, 1963, pp. 21-23.
- Backman, M.E., Form for the Relation Between Stress and Finite Elastic and Plastic Strains Under Impulsive Loading, Journal of Applied Physics, Vol. 35, 1964, pp. 2524-2533.
- 37. Grigorian, S.S., <u>On Basic Concepts in Soil Dynamics</u>, Prikladnaia Matematika i Mekhanika, Vol. 24, 1960, pp. 1057-1072; English Translation, Journal of Applied Mathematics and Mechanics, Vol. 24, 1960, pp. 1604-1627.

- Biot, M.A., <u>Theory of Stress-Strain Relations in Anisotropic</u> <u>Viscoelasticity and Relaxation Phenomena</u>, Journal of Applied Physics, Vol. 25, 1954, pp. 1385-1391.
- Biot, M.A., <u>Variational Principles In Irreversible Thermodynamics</u>, with Application to Viscoelasticity, Physical Review, 2nd Series, Vol. 97, 1955, pp. 1463-1469.
- Biot, M.A., <u>Linear Thermodynamics and the Mechanics of Solids</u>, Proc. 3rd U.S. National Congress of Applied Mechanics, ASME, 1958, pp. 1-18.
- Chu, B.-T., <u>Thermodynamics of Elastic and Some Visco-Elastic Solids</u> and Non-Linear Thermoelasticity, Brown University, Division of Applied Math., Contract No. Nonr 562(20), Tech. Report No. 1, July 1957.
- Chu, B.-T., <u>Deformation and Thermal Behavior of Linear Viscoelas-</u> <u>tic Materials, I and II</u>, Brown University, Division of Applied Mathematics, Contract No. Nord-17838, Tech. Report Nos. 1 and 2, 1958.
- Schapery, R.A., <u>Irreversible Thermodynamics and Variational Prin-</u> <u>ciples with Applications to Viscoelasticity</u>, Aero. Research Laboratories (Wright-Patterson AFB, Ohio), Report No. ARL-62-418, August 1962.
- 44. Schapery, R.A., <u>Application of Thermodynamics to Thermomechanical</u>, <u>Fracture, and Birefringent Phenomena in Viscoelastic Media</u>, Journal of Applied Physics, Vol. 35, 1964, pp. 1451-1465.
- 45. Schapery, R.A., <u>A Theory of Non-Linear Thermoviscoelasticity Based</u> on Irreversible Thermodynamics, Proc. 5th U.S. National Congress of Applied Mechanics, ASME, 1966, pp. 511-530.
- Coleman, B.D. and Noll, W., <u>The Thermodynamics of Elastic Materials</u> with Heat Conduction and Viscosity, Archive for Rational Mechanics and Analysis, Vol. 13, 1963, pp. 167-178.
- 47. Coleman, B.D., <u>Thermodynamics of Materials with Memory</u>, Archive for Rational Mechanics and Analysis, Vol. 17, 1964, pp. 1-46.
- Coleman, B.D., <u>On Thermodynamics, Strain Impulses, and Visco-</u> <u>elasticity, Archive for Rational Mechanics and Analysis</u>, Vol. 17, 1964, pp. 230-254.
- 49. Eringen, A.C., <u>Irreversible Thermodynamics and Continuum Mechanics</u>, Physical Review, 2nd Series, Vol. 117, 1950, pp. 1174-1183.

- 50. Cowin, S.C., <u>Constitutive Equations for Thermoviscoelasticity</u>, Pennsylvania State University, Department of Engineering Mechanics, AFOSR TN 1868, October 1961 (DDC AD-270542).
- Chudnovskii, A.I., <u>On Noniscthermal Deformation of the Viscoelastic</u> <u>Medium</u>, Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, 1966, No. 3, pp. 84-89.
- Leigh, D.C., <u>On the Thermodynamics and Approximation Theory of</u> <u>Viscoelastic Materials</u>, Proc. 5th U.S. National Congress of Applied Mechanics, ASME, 1966, pp. 481-492.
- Koh, S.L. and Eringen, A.C., <u>On the Foundations of Non-Linear</u> <u>Thermoviscoelasticity</u>, Internat. Journal of Engineering Science, Vol. 1, 1963, pp. 199-229.
- 54. Kline, K.A., <u>A Nonlinear Theory of Thermo-Viscoelasticity for</u> <u>Materials with Memory</u>, Ph.D. Dissertation, University of Minnesota, 1965. (University Microfilms, Inc., No. 65-7892, Xerox copy \$5.40.)
- 55. Valanis, K.C., <u>Thermodynamics of Large Viscoelastic Deformations</u>, Journal of Mathematics and Physics, Vol. 45, 1966, pp. 197-212.
- 56. Eringen, A.C., <u>Linear Theory of Micropolar Viscoelasticity</u>, Internat. Journal of Engineering Science, Vol. 5, 1967, pp. 191-204.
- 57. Wolosewick, R.M. and Gratch, S., <u>Transient Response in a Visco-elastic Material with Temperature-Dependent Properties and Thermo-mechanical Coupling</u>, Journal of Applied Mechanics, Vol. 32 (Trans. ASME, Vol. 87 E), 1965, pp. 620-622.
- Berg, C.A., <u>On Thermomechanical Instability in Viscoelastic Struc-</u> <u>tures</u>, Internat. Journal of Mechanical Sciences, Vol. 8, 1966, pp. 411-419.
- 59. Strutt, J.W. (Lord Rayleigh), <u>Theory of Sound</u>, Vol. I, American Edition, Dover Publications, Inc., New York, 1945.
- Lindsay, R.B., <u>Mechanical Radiation</u>, McGraw-Hill Book Company, Inc., 1960.
- 61. Kolsky, H., <u>Stress Waves in Solids</u>, revised edition, Dover Publications, Inc., New York, 1964.
- Davies, R.M., "Stress Waves in Solids," <u>Applied Mechanics Reviews</u>, January 1953; revised version, <u>Applied Mechanics Surveys</u>, Abramson, H.N., et al, editors, Spartan Books, Washington, D.C., 1966, pp. 803-807.

- Miklowitz, J., "Elastic Wave Propagation," <u>Applied Mechanics Reviews</u>, December 1960; revised version, <u>Applied Mechanics Surveys</u>, Abramson, H.N., et al, editors, Spartan Bocks, Washington, D.C., 1966, pp. 809-339.
- 64. Kolsky, H., "The Propagation of Stress Waves in Viscoelastic Solids," <u>Applied Mechanics Reviews</u>, September 1958; revised version <u>Applied</u> <u>Mechanics Surveys</u>, Abramson, H.N. et al, editors, Spartan Books, Washington, D.C., 1966, pp. 841-846.
- Waterman, P.C., "Orientation Dependence of Elastic Waves in Single Crystals," <u>Physical Review</u>, 2nd. series, Vol. 113, 1959, pp. 1240-1253.
- 66. Musgrave, M.J.P., "On the Propagation of Elastic Waves in Aeolotropic Media," <u>Proc. of the Royal Soc.</u> (London), Vol. 226A, 1954, pp. 339-366.
- 67. Musgrave, M.J.P., "Elastic Waves in Anisotropic Media," <u>Progress in</u> <u>Solid Mechanics</u>, Vol. 2, Sne'don, I.N. and Hill, R., editors, North Holland Publishing Company, Amsterdam, Holland, 1961, pp. 61-85.
- Fedorov, F.I., "A General Approximate Theory of Elastic Waves in Crystals," <u>Doklady Akademii Nauk</u>, Vol. 155, 1964, pp. 792-794; English translation, <u>Soviet Physics - Doklady</u>, Vol. 9, 1964, pp. 254-256.
- 69. Smith, R.T. and Stephens, R.W.B., "Efforts of Anisotropy on Ultrasonic Propagation in Solids," <u>Progress in Applied Materials Research</u>, Vol. 5, Stanford, E.G., Fearon, J.H., and McGonnagle, W.J., editors, Temple Press Books Ltd., London, 1964, pp. 39-64.
- 70. Kunin, I.A., "Model of an Elastic Medium of Simple Structure with Three Dimensional Dispersion," <u>Prikladnaia Matematika i Mekhanika</u>, Vol. 30, 1966, pp. 542-550; English translation, <u>Journal of Applied</u> <u>Mathematics and Mechanics</u>, Vol. 30, 1967, pp. 642-652.
- 71. Kunin, I.A., "Theory of Elasticity with Spatial Dispersion. One-Dimensional Complex Structure," <u>Prikladnaia Matematika i Mekhanika</u>, Vol. 30, 1966, pp. 866-874; English translation, <u>Journal of Applied</u> <u>Mathematics and Mechanics</u>, Vol. 30, 1967, pp. 1025-1034.
- 72. Eringen, A.C. and Parfitt, V.R., "Reflection of Plane Waves from the Flat Boundary of a Micropolar Elastic Halfspace," General Technology Corp., Report, July 1966. (NASA Document N66-34668).
- 73. Hadamard, J., <u>Lecons sur la propagation des ondes et les equations</u> <u>de l'hydrodynamique</u>, Hermann aris, 1903.

になりたたけのないでなったのではないのでは、そのではいろい

- 74. Thomas, T.Y., "On the Propagation of Weak Discontinuities in Perfectly Plastic Solids," <u>Journal of Mathematics and Mechanics</u>, Vol. 6, 1957, pp. 67-85.
- 75. Thomas, T.Y., "Extended Compatibility Conditions for the Study of Surfaces of Discontinuity in Continum Mechanics," <u>Journal of Mathe-</u> <u>matics and Mechanics</u>, Vol. 6, 1957, pp. 311-312.
- 76. Thomas, T.Y., "The General Theory of Compatibility Conditions," <u>Internat. Journal of Engineering Science</u>, Vol. 4, 1966, pp. 207-233.
- 77. Hill R., "Acceleration Waves in Solids," <u>Journal of Mechanics and</u> <u>Physics of Solids</u>, Vol. 10, 1962, pp. 1-16.
- 78. Howard, I.C., "Finite Simple Waves in a Compressible Transversely Isotropic Elastic Solid," <u>Quarterly Journal of Mechanics and</u> <u>Applied Mathematics</u>, Vol. 19, 1966, pp. 329-341.
- 79. Nariboli, G.A., "Wave Propagation in Arisotropic Elasticity," Journal of Mathematical Analysis and Applications, Vol. 16, 1966, pp. 108-122.
- Lur'e, M.V., "Application of a Variational Principle to Investigate Discontinuities in a Continuum," <u>Prikladnaia Matematika i Mekhanika</u>, Vol. 30, 1966, pp. 747-753; English translation, <u>Journal of Applied</u> <u>Mathematics and Mechanics</u>, Vol. 30, 1966, pp. 886-893.
- Bernstein, B., "Conditions for Second-Order Waves in Hypo-Elasticity," <u>Trans. of the Scc. of Rheology</u>, Vol. 6, 1962, pp. 263-273.
- Fine, A.D. and Shield, R.T., "Second Order Effects on the Propagation of Elastic Waves," <u>Internat. Journal of Solids and Structures</u>, Vol. 2, 1966, pp. 605-620.
- Varley, E. and Dunwoody, J., "The Effect of Non-Linearity at an Acceleration Wave," <u>Journal of the Mechanics and Physics of Solids</u>, Vol. 13, 1965, pp. 17-28.
- 84. Green, W.A., "The Growth of Plane Discontinuities Propagating into a Homogeneously Deformed Elastic Material," <u>Archive for Rational</u> <u>Mechanics and Analysis</u>, Vol. 16, 1964, pp. 79-88.
- 85. Balaba, M., "Acceleration Waves in Non-Linear Elastic-Plastic Media," Ph.D. dissertation, University of California, Berkeley, Calif., 1966. (University Microfilms, Inc., No. 66-8266, Xerox copy \$6.60.)
- Truesdell, C. and Null, W., <u>The Non-Linear Field Theories of Mech-anics</u>, Vol. III/3, Handbuch der Physik, Flügge, S., editor, Springer-Verlag, Berlin, 1965, pp. 267-272.

- Chen, P.J., "Acceleration Waves in Rheological Materials," Ph.D. dissertation, University of Washington, Seattle, Wash., 1965. (University Microfilms, Inc., No. 65-11, 445, Xerox copy \$4.80.)
- Coleman, B.D., Gurtin, M.E., and Herrara, I., "Waves in Materials with Memory. I. The Velocity of One-Dimensional Shock and Acceleration Waves," <u>Archive for Rational Mechanics and Analysis</u>, Vol. 19, 1965, pp. 1-19.
- 89. Coleman, B.D. and Gurtin, M.E., 'Waves in Materials with Memory, II. On the Growth and Decay of One-Dimensional Acceleration Waves," <u>Archive for Rational Mechanics and Analysis</u>, Vol. 19, 1965, pp. 239-265.
- Coleman, B.D. and Gurtin, M.E., "Waves in Materials with Memory. III. Thermodynamic Influences on the Growth and Decay of Acceleration Waves," <u>Archive for Rational Mechanics and Analysis</u>, Vol. 19, 1965, pp. 266-298.
- 91. Coleman, B.D., Greenberg, J.M., and Gurtin, M.E., "Waves in Materials with Memory. V. On the Amplitude of Acceleration Waves and Mild Discontinuities," <u>Archive for Rational Mechanics and Analysis</u>, Vol. 22, 1966, pp. 333-354.
- 92. Dunwoody, J. and N.I., "Acceleration Waves in Visco-Elastic Media of Maxwell-Zaremba Type," <u>Internat. Journal of Engineering Science</u>, Vol. 3, 1965, pp. 417-427.
- 93. Varley, E., <u>Acceleration Fronts in Viscoelastic Materials</u>, Archive for Rational Mechanics and Analysis, Vol. 19, 1965, pp. 215-225.
- 94. Biot, M.A., <u>The Influence of Initial Stress on Elastic Waves</u>, Journal of Applied Physics, Vol. ii, 1940, pp. 522-530.
- 95. Hayes, M. and Rivlin, R.S., <u>Propagation of a Plane Wave in an Iso-</u> <u>tropic Elastic Material Subjected to Pure Homogeneous Deformation</u>, Archive for Rational Mechanics and Analysis, Vol. 8, 1961, pp. 15-22.
- Hayes, M., <u>Wave Propagation and Uniqueness in Pre-Stressed Elastic</u> <u>Solids</u>, Proc. of the Royal Soc. (London), Vol. 274A, 1963, pp. 500-506.
- 97. Toupin, R.A. and Bernstein, B., <u>Sound Waves in Deformed Perfectly</u> <u>Elastic Materials; Acoustoelastic Effect</u>, Journal of the Acoustical Soc. of America, Vol. 33, 1961, pp. 216-223.
- 98. Green, W.A., <u>The Growth of Plane Discontinuities Propagating into</u> <u>a Homogeneously Deformed Elastic Material</u>, Archive for Retional <u>Mechanics and Analysis</u>, Vol. 16, 1964, pp. 79-88, Vol. 19, 1965, pp. 20-23.

99. Chadwick, P. and Powdrill, B., <u>Singular Surfaces in Linear Thermo-elasticity</u>, Internat. Journal of Engineering Science, Vol. 3, 1965, p. 561.

1

- 100. Fine, A.D., <u>Continuity Conditions at Wave Fronts in Coupled Thermo-plasticity</u>, Quarterly of Applied Mathematics, Vol. 25, 1967, pp. 121-128.
- 101. Fyfe, I.M., Ensminger, R.R., and Swift, R., The Effects of Shear on Stress Wave Propagation, Air Force Weapons Laboratory, AFWL-TR-65-12, August 1965, DDC AD-469553.
- 102. Herrmann, W., Jones, A.H. and Percy, J.H., <u>The Inclusion of Material Strength in Hydrodynamic Calculations</u>, Air Force Special Weapons Center, AFSWC-TDR-63-12, April 1963, DDC AD-410386.
- 103. Chu, B.-T., <u>Finite Amplitude Waves in Incompressible Perfectly</u> <u>Elastic Materials</u>, Journal of Mechanics and Physics of Solids, Vol. 12, 1964, pp. 45-57.
- 104. Chu, B.-T., <u>Transverse Shock Waves in Incompressible Elastic Solids</u>, Journal of Mechanics and Physics of Solids, Vol. 15, 1967, pp. 1-14.
- 105. Collins, W.D., <u>One-Dimensional Non-Linear Wave Propagation in In-</u> <u>compressible Elastic Materials</u>, Quarterly Journal of Mechanics and Applied Mathematics, Vol. 19, 1966, pp. 259-328.
- 106. Bland, D.R., <u>On Shock Waves in Hyperelastic Media</u>, Second-Order Effects in Elasticity, Plasticity, and Fluid Dynamics (IUTAM Symposium, Haifa, Jsrael, 1962), Reiner, M. and Abir, D., editors, Pergamon Press, Oxford, England, 1964, pp. 93-108.
- 107. Bland, D.R., <u>Dilational Waves and Shocks in Large Displacement Isen-</u> tropic Dynamic Elasticity, Journal of Mechanics and Physics of Solids, Vol. 12, 1964, pp. 245-267.
- 108. Dewey, J., Strong Shocks and Stress-Strains in Solids, Part I, Ballistic Research Laboratories (Aberdeen Proving Ground, Maryland), Report No. 1074, April 1959, DDC AD-219376.
- 109. Chadwick, P. and Powdrill, B., <u>Application of the Laplace Transform</u> <u>Method to Wave Motions Involving Strong Discontinuities</u>, Proc. of the Cambridge Philosophical Soc., Vol. 60, 1964, pp. 313-324.
- 110. Bland, D.R., <u>On Shock Structure in a Solid</u>, Journal of the Institute of Mathematics and Its Applications, Vol. 1, 1965, pp. 56-75.

Ł

- 111. Aliev, Kh. M., <u>Destructive Shock Waves in Frangible Media</u>, Doklady Akademii Nauk SSSR, Vol. 151, 1963, pp. 80-83; English Translation, Soviet Physics-Doklady, Vol. 8, 1964, pp. 673-675.
- 112. Urlin, V.D. and Ivanov, A.A., <u>Melting Under Shock-Wave Compression</u>, Doklady Akademii Nauk SSSR, Vol. 149, 1963, pp. 1303-1306; English Translation, Soviet Physics-Doklady, Vol. 8, 1963, pp. 380-382.
- 113. Smith, C.S. and Dowler, C., <u>Further Metallographic Studies on</u> <u>Metals After Explosive Shock</u>, Response of Metals to High Velocity Deformation, Shewmon, P.G. and Vackay, V.F., editors, Interscience Publishers, New York, 1961, pp. 309-341.
- 114. Kozorezov, K.I. and Mirkin, L.I., <u>Hardening of Metals by Shock-Wave Cumulation</u>, Doklady Akademii Nauk SSSR, Vol. 171, 1966, pp. 324-326; English Translation, Soviet Physics-Dcklady, Vol. 11, 1967, pp. 982-983.
- 115. Gourley, M.G.F., <u>Propagation and Interaction of Elastic Waves and</u> <u>Shock Waves in Solids</u>, Ph.D. Dissertation, University of Texas, 1959. (University Microfilms, Inc., No. 59-6709, Xerox Copy \$4.00.)
- 116. Rinehart, J.S. and Pearson, J., <u>Behavior of Metals Under Impulsive</u> <u>Leads</u>, American Society for Metals, Cleveland, Ohio, 1954; Reprint, Dover Publications, Inc., 1965.
- 117. Goldsmith, W., Impact, Edward Arnold, Ltd., London, 1960.
- 118. J.L. Ericksen and C. Truesdell, "Exact Theory of Stress and Strain in Rods and Shells," <u>Archive for Rational Mechanics and Analysis</u>, Vol. 1, pp. 295-323 (1958).
- 119. A.E. Green and R.S. Rivlin, "Simple Force and Stress Multipoles," <u>Archive for Rational Mechanics and Analysis</u>, Vol. 16, pp. 325-353 (1964).
- 120. A.E. Green and R.S. Rivlin, "Multipolar Continuum Mechanics," <u>Archive for Rational Mechanics and Analysis</u>, Vol. 17, pp. 113-147 (1964).
- 121. A.E. Green, P.M. Naghdi, and R.S. Rivlin, "Directors and Multipolar Displacements in Continuum Mechanics," International Journal of Engineering Science, Vol. 2, pp. 611-620 (1965).
- 122. J. H. Weiner and B.A. Boley, <u>Basic Concepts of the Thermodynamics</u> of <u>Continuous Media</u>, Wright Air Development Center (Wright-Patterson Air Force Base, Ohio), WADC TR 57-288 (April 1958).
- 123. B.D. Coleman and V.J. Mizel, "Existence of Caloric Equations of State in Thermodynamics," Journal of Chemical Physics, Vol. 40, pp. 1116-1125 (1964).

- 124. Gurtin, M.E., <u>Thermodynamics and the Possibility of Spatial Inter-action in Elastic Materials</u>, Archive for Rational Mechanics and Analysis, Vol. 19, 1965, pp. 339-352.
- 125. Gurtin, M.E., <u>On the Thermodynamics of Elastic Materials</u>, Journal of Mathematical Analysis and Applications, Vol. 18, 1967, pp. 38-44.
- 126. J.C. Slater, <u>Introduction to Chemical Physics</u>, McGraw-Hill Book Company, Inc., New York, N. Y. (1939).
- 127. J.S. Dugdale and D.K.C. MacDonald, "The Thermal Expansion of Solids," Physical Review, 2nd Series, Vol. 89, pp. 832-834 (1953).
- 128. J.J. Gilgarry, "Grüneisen Parameter for a Solid Under Finite Strain," Physical Review, 2nd Series, Vol. 102, pp. 331-340 (1956).
- 129. B.A. Bilby, "Continuous Distributions of Dislocations," <u>Progress</u> <u>in Solid Mechanics</u>, Vol. 1, pp. 329-398, edited by I.N. Sneddon and R. Hill, North Holland Publishing Company, Amsterdam, Holland (1960).
- 130. E. Kröner, "On the Physical Reality of Torque Stresses in Continuum Mechanics," <u>International Journal of Engineering Science</u>, Vol. 1, pp. 261-278 (1963).
- 131. J.D. Eshelby, "Dislocations as a Cause of Mechanical Damping in Metals," Proc. of the Royal Soc. (London) Vol. 197A, pp. 396-416 (1949).
- 132. A. Granato and K. Lucke, "Theory of Mechanical Damping Due to Dislocations," Journal of Applied Physics, Vol. 27, pp. 583-593 (1956).
- 133. C.M. Glass, et al., <u>Spall and Fracture of Metals Under Impulse</u> <u>Loads</u>, Final Report, Contract MIPR-61-SWKR-2, BRL-Aberdeen Proving Ground, Maryland, January 1965.
- 134. R.S. Davis and K.A. Jackson, "On the Deformation Associated with Compression Shocks in Crystalline Solids," <u>Proceedings of the</u> <u>Second Symposium on Naval Structural Mechanics</u>, Pergamon Press, New York, 1960.
- 135. Nye, J.F., <u>Physical Properties of Crystals</u>, <u>Their Representation by</u> <u>Tensors and Matrices</u>, Clarendon Press, Oxford, 1957, p. 173.

UNCLASSIFIED						
Security Classification						
DOCUMENT CONT	ROL DATA - R	& D				
(Security classification of fifts, body of abstract and indexing 1. ORIGINATING ACTIVITY (Corporate author)	annotation must be	28. REPORT SE	CURITY CLASSIFICATION			
GENERAL RESEARCH COPPORATION		UNC	CLASSIFIED			
P.O. Box 3587, Santa Barbara, Californ:	La 03105	26. GROUP	. GROUP			
3. REPORT TITLE		L				
Some Strong Wave Problems in an Elasti	c Material v	with Micro	structure			
4. DESCRIPTIVE 1.OTES (Type of report and inclusive dates) Final Technical Penart 19 Mars 10	67 through	17 Tuno 10	160			
5. AUTHOR(S) (First name, middle initial, last name)	507 Enrough	17 June 1	700			
C. H. Bont and D. E. Cumulass						
C. w. bert and B. E. Cummings						
6. REPORT DATE	TAL TOTAL NO. O	FPAGES	76. NO. OF REES			
June 1968	58		135			
84. CONTRACT OR GRANT NO.	S. ORIGINATOR	S REPORT NUME	ER(5)			
DAAD 05-67-C-0484	CR-0484-	-1				
с.	95. OTHER REPORT NO(S) (Any other numbers that may be assign					
	ane reportj					
d.			······			
This document has been approved for pub	lic release	and sale.	ite distribution			
is unlimited.		and bare,	its distribution			
11. SUPPLEMENTARY NOTES	12. SPONSORING	U. S.				
	Aberdeen Research & Lo. Topmont Conter					
	Aberdee	n Provins (1_0.1.1. kd. 21005			
13. ABSTRACT						
The historical development of the foundat	tions of the	ermomechani	lcs is reviewed, with			
emphasis placed upon the critical contrib	oution of ea	ach investi	gator. The theories			
are discussed in terms of their relevance	e to real ma	aterials in	the important high-			
velocity regime which lies between the lo	ow and inter	mediate ve	locity regime (which			
involves essentially isothermal elastic a	and plastic	wave propa	agation) on one hand			
material behavior is adequate) on the oth	len a nyuroù Ner. Varioù	iynamic des 1s facets d	of high-velocity wave			
propagation in various materials involve	elastic, pl	lastic, vie	scous, and thermal			
effects on a macroscopic scale and perhap	os on a mici	coscopic so	ale. Evaluations			
are made from the standpoint of thermodyn	namics as we	ell as clas	ssical mechanics.			
Using fundamental principles of thermodyn	namics and m	nechanics.	thermomechanically			
coupled linear energy and constitutive ed	quations are	e derived f	for the elastic solid			
with micro unit cells discussed by Mindli	in.					
Equations of motion for the microstructur	re are found	for the 1	large amplitude case			
where Eulerian and Lagrangian formulation	ns are not i	Identical.	(
			1			
			/			
DD 100 1473		INCL	ASSTRTED			
	<u> </u>	Security	Classification			

ż

UNCLASSIFIED

ź

Man Aller Man and a state

ġ.

Security Classification

KEY WORDS				<u> </u>	
	RULE		ROLE	WT	ROLE
Elastic Solid With Microstructure					
ish Velocity Moyee					
rue Amplitude Nova Reportions		l			1
(mouse from the name equations ((1			
TERRET CE OF DELVER SER THE OSTERVER	í.	í			ļ
		}			
		1			
			ł		
	1		I		
				Í	
		l			
			Í	ĺ	
	1				
	ł				
		-		1	
Í	1			ł	
		1			
	1				
	1				
			ļ		
	1				
	[1	
			1	(
	1				
	Ì			1	
	}				
	1		}		
		1			
1			ļ		
					ĺ
	1				
	UNCLASSIFIED				
_	S	ecurity C	lassificat	tion	

÷t

₀**₽**.

ʻ-,