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ON CUMULATIVE DAMAGE AND RELIABILITY OF COMPONENTS

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SANTA MONICA, CALIFORNIA

Contract No. F33615-67-C-1865
Project No. 7071

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AEROSPACE RESEARCH LABORATORIES
OFFICE OF AEROSPACE RESEARCH
UNITED STATES AIR FORCE
WRIGHT-PATTERSON AIR FORCE BASE, OHIO

FOREWORD

This report was prepared for the Applied Mathematics Research Laboratory, Aerospace Research Laboratories, Wright-Patterson Air Force Base, by Dr. V. K. Murthy and Dr. B. P. Lientz, System Development Corporation, under Contract F33-615-67-C-1865. In this report the authors develop the concepts of cumulative damage for general reliability and renewal models.

ABSTRACT

Let T, X(t), and C denote the time to failure, the accumulated damage by time t, and the "critical" damage. Let F be the distribution function of T. Let "E" stand for the event of the component undergoing damage and $\{t_n\}$ denote the sequence of intervals of time between successive occurrences of "E". Let $T_n = \sum_{i=1}^{n} t_i$ and Y_i denote the amount of damage experienced at time T_n. Assume $\{t_n, Y_n\}$ is a sequence of independent, identically distributed variables with distribution function H(t, y), so that $\{t_n\}$ and $\{Y_n\}$ are renewal processes. The inequality $F(t) \leq H(t, \infty)$ is obtained with equality if and only if $C = 0$. For "E" a Poisson process, sufficient conditions are given for F to be IHR and DMR. The classes of distribution functions are considered with the topology of complete convergence. Empirical estimates for F from observing occurrences of "E" are given.

TABLE OF CONTENTS

1.0 SUMMARY

Let T, X(t), and C denote the time of failure, the accumulated damage by time t, and the "critical" damage the component can withstand, respectively. Let "E" stand for the event of the component undergoing damage and $\{t_n\}$ denote the sequence of intervals of time between successive occurrences of "E". Let $T_n = \sum_{i=1}^{n} t_i$ and let y_n denote the amount of damage experienced at time T_n. Assume $\{t_n, y_n\}$ is a sequence of independent, identically distributed random variables with distribution function H(t, y), so that $\{t_n\}$ and $\{y_n\}$ are renewal processes with $X(t) = \sum_{n=1}^{N_t} y_n$, where N_t is the number of events by time t with respect to $\{t_n\}$. If G(t, x) is the distribution function of X(t), then F, the distribution function of T, can be expressed in terms of G and H. The inequality $F(t) \leq H(t, \infty)$ is obtained, with equality holding if and only if $C \equiv 0$. For "E" a Poisson process, sufficient conditions are given for F to be IHR and DMR. The classes of distribution functions are considered with the topology of complete convergence. Empirical estimates for F from observed occurrences of "E" are given. The asymptotic properties of F are examined. Generalizations are made to several types of multicomponent structures.

2.0 INTRODUCTION

In what follows, the word "device" is used to denote the given piece of equipment or hardware under consideration. At time $t = 0$, say, the device is put to use. As time passes by, the device steadily wears out until it fails or no longer performs the functions it should. Let T denote the time to failure. Let $X(t)$ denote the accumulated wear out or damage the device has suffered by time t. $X(t)$ is a random function of the argument t while T is a random variable. Let C denote the "critical" damage the device can stand. C is considered to be a constant without loss of generality. The inequality

$$X(t) > C \tag{2.1}$$

is now the event that by time t the accumulated damage the device has suffered is greater than the critical damage; in other words the device has failed by time t. Thus, (2.1) is the same event as $T \leq t$.

Let

$$P[X(t) \leq x] = G(t, x) \tag{2.2}$$

and

$$P[T \leq t] = F(t). \tag{2.3}$$

Then

$$F(t) = 1 - G(t, C) \tag{2.4}$$

Equation (2.4) expresses the fundamental relationship between the accumulated damage X(t) and the time to failure T. Figure I illustrates these concepts.

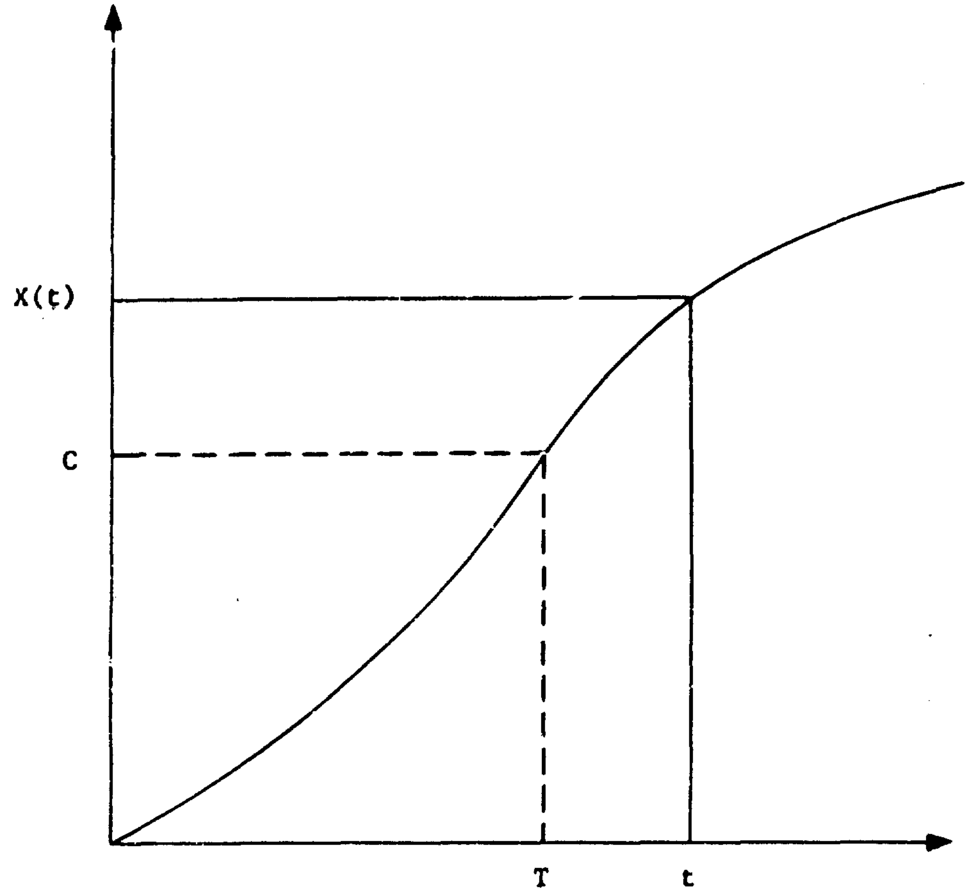

Figure I

3.0 RENEWAL PROCESS DAMAGE MODEL

Let "E" stand for the event of the device undergoing damage. Let $\{t_i\}$ $i = 1, 2, \ldots,$ denote the sequence of intervals of time between successive occurrences of "E". Let

$$T_n = t_1 + t_2 + \ldots + t_n, \; n = 1, 2, \ldots, \tag{3.1}$$

be the actual time instants corresponding to the occurrence of "E". In other words, $T_1, T_2, T_3, \ldots,$ are the successive instants of time when the device suffers damage. Let y_n denote the amount of damage the device experiences at time T_n, $n = 1, 2, \ldots\ldots$ We will now assume that $\{t_n, y_n\}$ $n = 1, 2, \ldots,$ is a sequence of independently, identically distributed random variables with the same distribution function $H(t,y)$. Clearly now $\{t_n\}$, $n = 1, 2, \ldots,$ and $\{y_n\}$, $n = 1, 2, \ldots,$ are both ordinary renewal processes. Under this set-up the cumulative damage $X(t)$ suffered by the device by time t is given by

$$X(t) = \sum_{n=1}^{N_t} y_n \tag{3.2}$$

where N_t is the number of events up to the specified time t with respect to the renewal process $\{t_n\}$. For the cumulative damage process $X(t)$ given by (3.2) the corresponding distribution is given by

$$G(t,x) = P[X(t) \le x] = P\left[\sum_{n=1}^{N_t} y_n < x\right] = 1 - H(t,\infty)$$

$$+ \int_{-\infty}^{\infty} \int_{0}^{t} G(t - t_1, x - x_1)\, dH_{t_1,x_1}(t_1,x_1) \tag{3.3}$$

because $1 - H(t, \infty)$ denotes the probability that there is no event in the specified interval; that $X(t) = 0 < x$; in the remaining mutually exclusive cases, there is some first event in the interval and the last term on the right-hand side of (3.3) gives this probability.

If we use the fundamental relationship (2.4), the underlying law of failures associated with the cumulative damage process (3.2) is given by

$$(3.4) \quad F(t) = H(t,\infty) - \int_{-\infty}^{\infty} \int_{0}^{t} G(t - t_1, C - x_1) dH_{t_1, x_1}(t_1, x_1).$$

Poisson Process for the Events "E"

We will now specialize (3.3) and (3.4) for the case of a Poisson process for the events "E" and discover some basic properties of the exponential law of failures. In this case

$$G(t, x) = P\left[\sum_{n=1}^{N_t} y_n \le x\right]$$

$$(3.5) \qquad = \sum_{n=0}^{\infty} P[N_t = n]\, P[y_1 + y_2 + \dots + y_n < x \mid N_t = n]$$

Since $H(\infty, x)$ denote the distribution of a y_1, we let $H^{(n)}(\infty, x)$ denote the n-fold convolution of $H(\infty, x)$ with itself. Also, since N_t is the number of events in the interval $(0, t)$ according to a Poisson process, say, with parameter λ

(3.6) $$P[N_t = n] = \frac{e^{-\lambda t}(\lambda t)^n}{n!}.$$

Hence

(3.7) $$G(t,x) = \sum_{n=0}^{\infty} \frac{e^{-\lambda t}(\lambda t)^n}{n!} H^{(n)}(\infty, x)$$

where

(3.8) $$H^{(0)}(\infty, x) = \begin{cases} 0 & \text{for } x < 0 \\ 1 & \text{for } x \geq 0, \end{cases}$$
$$H^{(1)}(\infty, x) = H(\infty, x),$$

and for $n > 1$

$$H^{(n)}(\infty, x) = \int_{-\infty}^{\infty} H^{(n-1)}(\infty, x - z)\, dH(\infty, z)$$

Combining (3.7) and (3.8) we can write

(3.9) $$G(t, x) = e^{-\lambda t} + \sum_{n=1}^{\infty} \frac{e^{-\lambda t}(\lambda t)^n}{n!} H^{(n)}(\infty, x)$$

From (2.4) and (3.9) we discover that in this case

$$F(t) = (1 - e^{-\lambda t}) - \sum_{n=1}^{\infty} \frac{e^{-\lambda t}(\lambda t)^n}{n!} H^{(n)}(\infty, C) \tag{3.10}$$

Equation (3.10) is a special case of equation (3.4) when the events "E" form a Poisson process. We will not interpret the fundamental implications of equations (3.4) and (3.10).

Let D stand for the event that by time t the device has experienced damage. Now

$$P(D) = \quad t_1 \le t) = H(t, \infty). \tag{3.11}$$

Let A stand for the event that the device failed by time t and $\bar{A}$ its complement. Clearly

$$D = (DA) \cup (D\bar{A})$$

$$= A \cup (D\bar{A}), \text{ since } A \subset D.$$

Hence,

$$P(D) = P(A) + P(D\bar{A}) \tag{3.12}$$

but

$$P(A) = F(t) \tag{3.13}$$

and

$$(3.14)\qquad P(D\bar{A}) = \int_{-\infty}^{\infty} \int_{0}^{t} G(t - t_1, C - x_1)\, dH_{t_1, x_1}(t_1, x_1)$$

Thus equation (3.4) is clearly identical with equation (3.12).

We discover from (3.4) in the general case that

$$(3.15)\qquad F(t) \leq H(t, \infty),$$

the equality sign holding if and only if

$$(3.16)\qquad C = 0.$$

The condition (3.16) that the critical damage C is zero is equivalent to saying that the cumulative damage process X(t) is a constant independent of t, i.e., the device does not deteriorate due to damage or wear out. In this case the event "E" of the device undergoing damage between t and t + dt is synonymous with the event of the device failing between t and t + dt.

There has been a popular misconception that the exponential law of failures is characterized by the situation that the cumulative damage is independent of time. Equations (3.10), (3.15), and (3.16) now establish that when the cumulative damage process is constant in time, the underlying law of failures is identical with the distribution of the intervals between consecutive occurrences of "E" and is exponential only when the events "E" are Poisson.

4.0 EMPIRICAL ESTIMATION OF F

Let $H_k(.,.)$ denote the two dimensional empirical distribution function for $H(.,.)$. That is, $H_k(t,x) = 1/n$ {number of observations (T_i, X_i) with $T_i < t$ and $X_i < x$}.

This will then produce an F_k where

$$F_k(t) = H_k(t,\infty) - \int_{-\infty}^{+\infty} \int_{0}^{t} G_k \quad (t - t_1, C - x_1) \, dH_k \, (t_1, x_1).$$

4.1 Theorem

Let G be a continuous function with respect to the usual product topology on E^2. Then $F_k(t) \to F(t)$ completely, or, in the sense of Feller, properly.

Proof. One has

$$(4.1.1) \qquad |F_k(t) - F(t)| \leq |H_k(t, \infty) - H(t, \infty)|$$
$$+ \left| \int_{-\infty}^{+\infty} \int_{0}^{t} G_k(t - t_1, C - x_1) \, dH_k(t_1, x_1) - \int_{-\infty}^{+\infty} \int_{0}^{t} G(t - t_1, C - x_1) \, dH(t_1, x_1) \right|.$$

The first term in (4.1.1) tends to 0 since H_k is the empirical distribution function for H. For the second term one has

$$(4.1.2) \quad \left| \int_{\infty}^{+\infty} \int_{0}^{t} G_k(t - t_1, C - x_1)\, dH_k(t_1, x_1) - \int_{-\infty}^{+\infty} \int_{0}^{t} G(t - t_1, C - x_1)\, dH(t_1, x_1) \right|$$

$$\leq \left| \int_{\infty}^{+\infty} \int_{0}^{t} G(t - t_1, C - x_1)\, dH_k(t_1, x_1) - \int_{-\infty}^{+\infty} \int_{0}^{t} G(t - t_1, C - x_1)\, dH(t_1, x_1) \right|$$

$$+ \left| \int_{-\infty}^{+\infty} \int_{0}^{t} G_k(t - t_1, C - x_1) - G(t - t_1, C - x_1)\, dH_k(t_1, x_1) \right|.$$

The first summand in (4.1.2) tends to 0 as $k \to \infty$ by the two dimensional version of the Helly-Bray theorem. The second term in (4.1.2) is bounded by

$$(4.1.3) \quad \int_{-\infty}^{+\infty} \int_{0}^{t} \left| G_k(t - t_1, C - x_1) - G(t - t_1, C - x_1) \right| dH_k(t_1, x_1).$$

To show this tends to 0 and k tends to ∞ we first consider

$$\psi_k(a, t) = \int_{-a}^{a} \int_{0}^{t} g_k(t - t_1, C - x_1)\, dH_k(t_1, x_1)$$

where

$$g_k(t - t_1, C - x_1) = |G_k(t - t_1, C - x_1) - G(t - t_1, C - x_1)|.$$

Expressing the finite multiple integral as the limit of Riemann-Stieljies sums and interchanging limits one obtains

$$\psi_k (a, t) \to 0 \text{ as } k \to \infty.$$

Hence,

$$\lim_{a \to \infty} \lim_{k \to \infty} \psi_k (a, t) = 0$$

and since g_k is bounded and $H_k(.,.)$ is a distribution function, we obtain

$$\lim_{k \to \infty} \lim_{a \to \infty} \psi_k (a, t) = \lim_{k \to \infty} \psi_k (\infty, t) = 0.$$

In the particular case for the Poisson process one defines $G_k(t, x)$ by

(4.1.4) $$G_k(t,x) = \sum_{n=0}^{\infty} e^{-\lambda t} \frac{(\lambda t)^n}{n!} H_k^{(n)} (\infty, x)$$

where $H_k(.,.)$ is the two dimensional empirical distribution function.

4.2 Theorem

With G_k as in (4.1.4) and G as in (3.3), $G_k(.,.)$ converges completely to $G(.,.)$.

Proof. One first observes the following inequality

$$(4.2.1) \quad |G_k(t, x) - G(t, x)| \le \sum_{n=0}^{\infty} e^{-\lambda t} \frac{(\lambda t)^n}{n!} \left| H_k^{(n)}(\infty, x) - H^{(n)}(\infty, x) \right|$$

For the right-hand side we employ Lemma 2, p. 252 of [1] which states that for $\mathcal{H}_k$ and $\mathcal{H}$ as the respective operators of H_k and H, and u a continuous function on E^1 the inequality

$$\| \mathcal{H}_k^{(n)}(u) - \mathcal{H}^{(n)}(u) \| \le n \| \mathcal{H}_k(u) - \mathcal{H}(u) \|$$

holds. If we let $k \to \infty$ it follows that $H_k^{(n)} \to H^{(n)}$ by theorem 1, p. 249 of [1]. Now by letting $k \to \infty$ in (4.2.1), since the limit can be brought inside the summation sign, it can be seen that the right hand side tends to 0.

5.0 CONDITIONS FOR DMR AND IHR

In this section we assume $\{t_i\}$ forms a Poisson process and derive sufficient conditions for F, defined in (3.10), to be DMR (decreasing mean residual life) and IHR (increasing hazard rate). The section is concluded with an example whose derived F is IHR.

Set $p_n = H^{(n)}(\infty, C)$. Then one has the following theorem.

5.1 Theorem

A sufficient condition for F to be DMR is for the following inequalities to be satisfied for all N > M for some **positive integer M.**

$$(5.1.1) \quad \sum_{j=0}^{k} \binom{k}{j} p_j \left(\sum_{i=k-j+2}^{N} p_i \right) \leq \sum_{j=k-N+1}^{k} \binom{k}{j} p_{j+1} \left(\sum_{i=k-j+1}^{N} p_i \right)$$

for $0 \leq k \leq N-1$.

$$(5.1.2) \quad \sum_{j=k-N+2}^{N} \binom{k}{j} p_j \left(\sum_{i=k-j+2}^{N} p_i \right) \leq \sum_{j=k-N+1}^{N-1} \binom{k}{j} p_{j+1} \left(\sum_{i=k-j+1}^{N} p_i \right)$$

for $N \leq k \leq 2N-2$.

Proof. The quantity we wish to consider is

$$(5.1.3) \quad \psi(t) = \frac{\int_t^{\infty} (1-F(x))dx}{1-F(t)}$$

for $t > 0$.

To show this is non-increasing in t, we evaluate $\psi(t)$ by substituting the expression for F in (3.10) and obtain

$$\psi(t) = \frac{\sum_{n=0}^{\infty} \frac{p_n}{\lambda} e^{-\lambda t} \sum_{j=0}^{n} \frac{(\lambda t)^j}{j!}}{e^{-\lambda t} \sum_{n=0}^{\infty} \frac{(\lambda t)^n}{n!} p_n} . \tag{5.1.4}$$

Without loss of generality we can assume $\lambda=1$. Differentiating ψ with respect to t one has $\psi'(t) < 0$ if and only if

$$\left(\sum_{n=0}^{\infty} \frac{p_n t^n}{n!}\right)\left(\sum_{n=2}^{\infty} p_n \sum_{j=0}^{n-2} \frac{t^j}{j!}\right) \le \left(\sum_{n=1}^{\infty} p_n \sum_{j=0}^{n-1} \frac{t^j}{j!}\right)\left(\sum_{n=0}^{\infty} p_{n+1} \frac{t^n}{n!}\right) \tag{5.1.5}$$

Now restrict the summations to finite summations for $N > 2$. One has

$$\left(\sum_{n=0}^{N} \frac{p_n t^n}{n!}\right)\left(\sum_{n=2}^{N} p_n \sum_{j=0}^{n-2} \frac{t^j}{j!}\right) \le \left(\sum_{n=1}^{N} p_n \sum_{j=0}^{n-1} \frac{t^j}{j!}\right) \sum_{n=0}^{N-1} p_{n+1} \frac{t^n}{n!} . \tag{5.1.6}$$

Considering the coefficients of the polynomials one has for the left-hand side of (5.1.6)

$$(5.1.7) \qquad \sum_{k=0}^{N-1} \sum_{j=0}^{k} \frac{p_j}{j!} \left(\sum_{i=k-j+2}^{N} \frac{p_i}{(k-j)!} \right) t^k$$

$$+ \sum_{k=N}^{2N-2} \sum_{j=k-N+2}^{N} \frac{p_j}{j!} \left(\sum_{i=k-j+2}^{N} \frac{p_i}{(k-j)!} \right) t^k.$$

For the right-hand side one has

$$(5.1.8) \qquad \sum_{k=0}^{N-1} \sum_{j=0}^{k} \frac{p_{j+1}}{j!} \left(\sum_{j=k-j+1}^{N} \frac{p_i}{(k-j)!} \right) t^k$$

$$+ \sum_{k=N}^{2N-2} \sum_{j=k-N+1}^{N} \frac{p_{j+1}}{j!} \left(\sum_{j=k-j+1}^{N} \frac{p_i}{(k-j)!} \right) t^k.$$

Substituting the expressions in (5.1.7) and (5.1.8) into (5.1.6) and comparing coefficients one obtains (5.1.1) and (5.1.2).

5.2 Theorem

A sufficient condition for F to be IHR is for the following to hold for $N \geq M$ for some $M \geq 1$

$$(5.2.1) \qquad \sum_{j=0}^{k} \binom{k}{j} p_{j+1} \; p_{k-j+1} \geq \sum_{j=0}^{k} \binom{k}{j} \; p_j \; p_{k-j+2}$$

for $0 \leq k \leq N-1$.

and

$$(5.2.2) \qquad \sum_{j=k-N+1}^{N-1} \binom{k}{j} p_{j+1} \; p_{k-j+1} \geq \sum_{j=k-N+2}^{N} \binom{k}{j} p_j \; p_{k-j+2}$$

for $N < k < 2N-2$.

Proof. We wish to show

$$\frac{F(t+x) - F(t)}{1 - F(t)} \text{ is nondecreasing in } t \text{ for } x > 0.$$

Substituting for F one has

$$(5.2.3) \qquad \psi_x(t) = 1 - \frac{e^{-\lambda x} \sum_{n=0}^{\infty} \frac{(\lambda(t+x))^n}{n!} p_n}{\sum_{n=0}^{\infty} \frac{(\lambda t)^n}{n!} p_n}$$

As in (5.1), we first can set $\lambda=1$ and then simplify by finding $e^x (1-\psi_x(t))$ which is

$$(5.2.4) \qquad \phi_x(t) = e^x(1-\psi_x(t)) = \frac{\sum_{n=0}^{\infty} \frac{(t+x)^n}{n!} p_n}{\sum_{n=0}^{\infty} \frac{t^n}{n!} p_n} .$$

We wish to derive conditions on $\{p_n\}$ so that the expression in (5.2.4) will be nonincreasing. Differentiating $\phi_x(t)$ with respect to t one has $\phi'_x(t) \leq 0$ if and only if

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} p_n \left(\sum_{n=0}^{\infty} \frac{(t+x)^n}{n!} p_{n+1} \right) \le \sum_{n=0}^{\infty} \frac{(t+x)^n}{n!} p_n \left(\sum_{n=0}^{\infty} \frac{t^n}{n!} p_{n+1} \right)$$

or, equivalently

(5.2.5) $$\frac{\sum_{n=0}^{\infty} \frac{t^n}{n!} p_n}{\sum_{n=0}^{\infty} \frac{t^n}{n!} p_{n+1}} \le \frac{\sum_{n=0}^{\infty} \frac{(t+x)^n}{n!} p_n}{\sum_{n=0}^{\infty} \frac{(t+x)^n}{n!} p_{n+1}} .$$

Since we are considering this expression for any $x > 0$, (5.2.5) will hold if and only if

$$\mu(t) = \frac{\sum_{n=0}^{\infty} \frac{t^n}{n!} p_n}{\sum_{n=0}^{\infty} \frac{t^n}{n!} p_{n+1}}$$

is nondecreasing in t. Differentiating $\mu(t)$ one has $\mu'(t) \ge 0$ if and only if

$$\left(\sum_{n=0}^{\infty} \frac{t^n}{n!} p_{n+1} \right)^2 \ge \left(\sum_{n=0}^{\infty} \frac{t^n}{n!} p_n \right) \left(\sum_{n=0}^{\infty} \frac{t^n}{n!} p_{n+2} \right) .$$

Restricting to finite summations, say N, one has

$$(5.2.6)\qquad \left(\sum_{n=0}^{N-1} \frac{t^n}{n!} p_{n+1}\right)^2 \le \left(\sum_{n=0}^{N} \frac{t^n}{n!} p_n\right)\left(\sum_{n=0}^{N-2} \frac{t^n}{n!} p_{n+2}\right)$$

The polynomial on the left-hand side of (5.2.6) is

$$(5.2.7)\qquad \sum_{k=0}^{N-1} \sum_{j=0}^{k} \frac{1}{j!(k-j)!} p_{j+1} p_{k-j+1} t^k +$$

$$\sum_{k=N}^{2N-2} \sum_{j=k-N+1}^{N-1} \frac{1}{j!(k-j)!} p_{j+1} p_{k-j+1} t^k$$

and the polynomial on the right side of (5.2.6) is

$$(5.2.8)\qquad \sum_{k=0}^{N-1} \left[\sum_{j=0}^{k} \frac{1}{j!(k-j)!} p_j p_{k-j+2}\right] t^k$$

$$+ \sum_{k=N}^{2N-2} \left[\sum_{j=k-N+2}^{N} \frac{1}{j!(k-j)!} p_j p_{k-j+2}\right] t^k .$$

Comparing respective coefficients in (5.2.7) and (5.2.8) one obtains the statement of the theorem.

5.3 Example

An example of a distribution function F satisfying (5.1.1), (5.1.2), (5.2.1), and (5.2.2). Assume $\{y_j\}$ are independent, ident ally distributed rando variables with common distribution function exponential with, say, parameter 1. Then p_n is given by

$$(5.3.1) \qquad p_n(x) = P\left[\sum_{j=1}^{n} y_j \leq x\right] \qquad 1-e^{-x} \left[\sum_{j=0}^{n-1} \frac{x^j}{j!}\right]$$

That the expressions in the statements of the theorems 5.1 and 5.2 are satisfied can be seen by direct substitution.

6.0 ASYMPTOTIC PROPERTIES OF CUMULATIVE DAMAGE X(t)

We have from (3.3) for the distribution function of cumulative damage $X(t) = \sum_{n=1}^{N_t} Y_n$, the following

(6.1)
$$G(t, x) = P[X(t) \le x] = P\left[\sum_{n=1}^{N_t} Y_n \le x\right]$$

$$= 1 - H(t, \infty) + \int_{-\infty}^{\infty} \int_{0}^{t} G(t - t_1, x - x_1)\, d_{t_1, x_1} H(t_1, x_1)$$

For the renewal process $\{t_n, y_n\}$, $n = 1, 2, \ldots$, with common distribution $H(t, y)$, let

$$\mu_r = E\left(t_n^{\,r}\right)$$

$$\nu_r = E\left(y_n^{\,r}\right)$$

$$\mu_{ij} = E\left(t_n^{\,i} y_n^{\,j}\right)$$

Define the transform

(6.2)
$$H^*(s, \theta) = \int_0^{\infty} \int_{-\infty}^{\infty} e^{-st + i\theta y}\, d_{t,y} H(t, y)$$

Applying the transform (6.2) to both sides of equation (6.1) we obtain

(6.3) $$G^{*}(s, \theta) = 1 - H^{*}(s, 0) + G^{*}(s, \theta)\, H^{*}(s, \theta)$$

Hence

(6.4) $$G^{*}(s, \theta) = \frac{1 - H^{*}(s, 0)}{1 - H^{*}(s, \theta)}$$

Noting that the left hand side of equation (6.4) is the characteristic function of the distribution function $G^{*}(s, y)$ and that for the real part of s greater than zero, the right hand side is differentiable with respect to θ, we obtain that the first absolute moment of $G^{*}(s, y)$ is finite and

(6.5) $$i \int_{-\infty}^{\infty} y \, d_y G^{*}(s, y) = \left. \frac{(1 - H^{*}(s, 0)) H^{*'}(s, \theta)}{[1 - H^{*}(s, \theta)]^{2}} \right|_{\theta = 0}$$

$$= \frac{i\, R^{*}(s)}{1 - H^{*}(s, 0)} ,$$

where the symbol prime denotes differentiation with respect to θ and

(6.6) $$R^{*}(s) = \left. \frac{1}{i} H^{*'}(s, \theta) \right|_{\theta = 0}$$

If μ_1, μ_2 and μ_{11} are finite, we obtain from Murthy [2] that

$$(6.7) \qquad \int_{-\infty}^{\infty} y \, d_y G^*(s,y) = \frac{R^*(s)}{1 - H^*(s,0)}$$

$$= \frac{\nu_1}{\mu_1 s} + \frac{\nu_1 \mu_2}{2\mu_1^2} \quad \frac{\mu_{11}}{\mu_1} + 0(1), \text{ as } s \to +0$$

Since the left-hand side of equation (6.7) is the Laplace-Stieltjes transform (L-S.T) of $E(X(t))$, we obtain that

$$(6.8) \qquad E(X(t)) = t \frac{\nu_1}{\mu_1} + \frac{\nu_1 \mu_2}{2\mu_1^2} - \frac{\mu_{11}}{\mu_1} + 0(1), \text{ as } t \to \infty.$$

Similarly

$$(6.9) \qquad -\int_{-\infty}^{\infty} y^2 \, d_y G^*(s, y) = \frac{H^{*''}(s,\theta)\Big|_{\theta = 0}}{1 - H^*(s, 0)} + 2\frac{\left[H^{*'}(s, \theta)\Big|_{\theta = 0}\right]^2}{[1 - H^*(s, 0)]^2}$$

$$= -\frac{Q^*(s)}{1 - H^*(s, 0)} + 2\frac{[iR^*(s)]^2}{[1 - H^*(s, 0)]^2},$$

where

$$Q^*(s) = -H^{*''}(s, \theta)\Big|_{\theta = 0}$$ is the L-S.T

of

$$Q(t) = \int_{-\infty}^{\infty} y^2 \, d_y H(t, y) .$$

Hence

$$(6.10) \qquad \int_{-\infty}^{\infty} y^2 \, d_y G^*(s, y) = \frac{Q^*(s)}{1 - H^*(s, 0)} + \frac{2[R^*(s)]^2}{[1 - H^*(s, 0)]^2}$$

If ν_2, μ_3, μ_{21} and μ_{12} are finite, we obtain after a straightforward calculation that

$$(6.11) \qquad E\left(X^2(t)\right) = \frac{\nu_1^2}{\mu_1^2} t^2 + t \left(\frac{2\mu_2 \nu_1^2}{\mu_1^3} - \frac{4\nu_1 \mu_{11}}{\mu_1^2} + \frac{\nu_2}{\mu_1} \right)$$

$$+ \frac{3\nu_1^2 \mu_2^2}{2\mu_1^4} - \frac{2\nu_1^2 \mu_3}{3\mu_1^3} + \frac{2\mu_{11}^2}{\mu_1^2} + \frac{2\nu_1 \mu_{21}}{\mu_1^2}$$

$$- \frac{4\nu_1 \mu_{11} \mu_2}{\mu_1^3} + \frac{\mu_2 \nu_2}{2\mu_1^2} - \frac{\mu_{12}}{\mu_1} + o(1), \text{ as } t \to \infty.$$

Combining (6.8) and (6.11) we finally discover that

$$\text{(6.12)} \qquad \mathrm{Var}\,(X(t)) = t\left(\frac{\nu_2}{\mu_1} + \frac{\nu_1^2\mu_2}{\mu_1^3} - \frac{2\nu_1\mu_{11}}{\mu_1^2}\right)$$

$$+ \frac{5\nu_1^2\mu_2^2}{4\mu_1^4} - \frac{2\nu_1^2\mu_3}{3\mu_1^3} - \frac{3\nu_1\mu_{11}\mu_2}{\mu_1^3} + \frac{\mu_{11}^2}{\mu_1^2}$$

$$+ \frac{2\nu_1\mu_{21}}{\mu_1^2} + \frac{\mu_2\nu_2}{2\mu_1^2} - \frac{\mu_{12}}{\mu_1} + O(1), \text{ as } t \to \infty.$$

7.0 EXTENSIONS TO MULTICOMPONENT STRUCTURES

In the previous section the damage y_n was considered a scalar. In the case of a general structure consisting of m components, at each occurrence of E, say, at time $T_n = t_1 + t_2 \dots + t_n$, the damage the structure suffers can be denoted by an m dimensional random vector.

$$y_n = \begin{pmatrix} y_{1n} \\ y_{2n} \\ \cdot \\ \cdot \\ y_{mn} \end{pmatrix}$$

The assumption in this case is that the sequence $\{t_n, y_n\}$, n = 1, 2, ... is a renewal process with a common (m+1) dimensional distribution given by

(7.2) $$P\{t_n < t, y_n < y\} = H(t, y_1, y_2, \dots, y_m).$$

The corresponding m dimensional damage process is then

(7.3) $$\begin{pmatrix} X_1(t) \\ X_2(t) \\ \cdot \\ \cdot \\ X_m(t) \end{pmatrix} = \begin{pmatrix} \sum_{n=1}^{N_t} y_{1n} \\ \sum_{n=1}^{N_t} y_{2n} \\ \cdot \\ \cdot \\ \sum_{n=1}^{N_t} y_{mn} \end{pmatrix}$$

Now the m dimensional joint distribution of life lengths of the m components is given by

(7.4) $$F(t_1, t_2, \dots t_m) = P[X_1(t_1) > C_1, \dots X_m(t_m) > C_m],,$$

where C_i, $i = 1, 2, \dots m$ is the critical threshold for the i^{th} component of the structure.

Using procedures similar to the single component situation, we can obtain explicitl; $F(t_1, t_2, \dots, t_m)$ in terms of $H(t, y_1, y_2, \dots, y_m)$ and its convolutions evaluated at the critical threshold. Also, the mean vector and the variance-covariance matrix of the m dimensional cumulative damage process $X(t_m)$ can be easily evaluated.

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13. ABSTRACT

Let T, X(t), and C denote the time to failure, the accumulated damage by time t, and the "critical" damage. Let F be the distribution function of T. Let "E" stand for the event of the component undergoing damage and $\{t_n\}$ denote the sequence of intervals of time between successive occurrences of "E". Let $T_n = \sum_{i=1}^{n} t_i$ and Y_i denote the amount of damage experienced at time T_n. Assume $\{t_n, Y_n\}$ is a sequence of independent, indentically distributed variable with distribution function $H(t,y)$, so that $\{t_n\}$ and $\{Y_n\}$ are renewal processes. The inequality $F(t) \leq H(t,\infty)$ is obtained with equality if and only if $C=0$. For "E", a Poisson process, sufficient conditions are given for F, to be IHR and DMR. The classes of distribution functions are considered with the topology of complete convergence. Empirical estimates for F from observing occurrences of "E" are given.

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