Division of Engineering Mechanics

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ELASTIC-PLASTIC DEFORMATION AT FINITE STRAINS

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Abstract

In some circumstances, elastic-plastic deformation occurs in which
both components of strain are finite. Such situations fall outside the
scope of classical plasticity theory which assumes either infinitesimal
strains or plastic-rigid theory for large strains. The present theory
modifies the kinematics to include finite elastic and plastic strain
components. For situations requiring this generalization, dilatational
influences are usually significant including thermo-mechanical coupling.
This is introduced through the consideration of two coupled thermodynamic
systems: one comprising thermo-elasticity at finite strain and the other
the irreversible process of dissipation and absorption of plastic work.
The present paper generalizes a previous theory to permit arbitrary
deformation histories.

Introduction

Fig. 1 shows the stress-strain curve in tension or shear for a
ductile metal such as aluminum. Plastic deformation sets in at A , and
the material work hardens along ABE . Unloading from B is elastic
along BC , and the elastic strain at B , CD   in magnitude, is recov-
ered on unloading down to zero stress at C . The remaining strain OC
is the plastic strain associated with loading to the point B . The
magnitude of the elastic strain CD is equal to the yield stress divided
by the elastic modulus and is of the order 10^{-3} . Plastic flow can

continue along BE to strains of the order unity. Plasticity theory incorporates these features by considering two limiting situations which cover many practical circumstances. If strains are large, of the order unity, as in many metal-forming problems, the elastic component of strain is considered negligible in comparison and plastic-rigid theory is utilized. Since plasticity is an incremental or flow type phenomenon, strain rates based on the velocity field appear in the stress-deformation relation, which with the equations of motion and boundary conditions determines the velocity field at each instant of time, and hence the total deformation by integration. Elastic-plastic theory has been developed to consider situations in which elastic and plastic components of strain are of the same order, that is about 10^{-3} as stated above, so that infinitesimal strain analysis can be used. There are however situations of plastic flow under high pressure, such as can occur in explosive forming, in which the elastic and plastic components of strain are each finite, so that neither limit covered in classical plasticity theory applies [1][*]. A theory incorporating finite strain for both elastic and plastic components is presented below, which, in addition to its appropriateness for high pressure loading problems, provides a uniformly valid theory encompassing the two limits covered by classical theory.

As discussed in the next section, the kinematics of finite strain invalidates the usual assumption that the total strain is the sum of elastic and plastic components. This assumption arises in infinitesimal

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[*]Numbers in square brackets refer to the references at the end of the paper.

strain analysis because of linearity and in the one dimensional case depicted in Fig. 1. Finite elastic strains are usually predominently dilatational, since increase of elastic shear strain components beyond the elastic limit ($\sim 10^{-3}$) produces plastic flow. The finite dilatation introduces significant thermo-mechanical coupling effects which must be included to provide a satisfactory theory. With plastic flow this demands the inclusion of irreversibility.

Kinematics

As discussed in [2], the kinematics of elastic-plastic deformation can be analysed on the basis of the configurations depicted in Fig. 2. Using a single set of rectangular Cartesian coordinates (x_1, x_2, x_3) , particles in an undeformed body at a uniform base temperature θ_o have positions represented by the column vector $\underset{\sim}{x}^1$. In the deformed state at time t , following elastic-plastic deformation, they occupy the positions $\underset{\sim}{x}^3$, according to the mapping

$$\underset{\sim}{x}^3 = \underset{\sim}{x}^3(\underset{\sim}{x}^1,t) \tag{1}$$

If each element of the body is unstressed and reduced to the base temperature, the configuration $\underset{\sim}{x}^2$ results, according to the mapping

$$\underset{\sim}{x}^2 = \underset{\sim}{x}^2(\underset{\sim}{x}^1,t) \tag{2}$$

If a body is reduced to the uniform base temperature θ_o and the surface tractions are removed, a residual stress distribution will normally remain following non-homogeneous plastic flow. To remove this the body will usually have to be considered dissected into small elements and the corresponding mapping, (2), will be discontinuous. The configuration $\underset{\sim}{x}^2$ at uniform base temperature and zero stress comprises the pure plastic

deformation, since thermal expansion and elastic strain components are both zero.

As discussed in [2], local deformation in the neighborhood of each particle at time t can be conveniently expressed in terms of the deformation gradient matrix:

$$\underset{\sim}{F} = \partial x_i^3 / \partial x_j^1 \tag{3}$$

For the case when the mapping (2) is continuous and differentiable, the plastic deformation from $\underset{\sim}{x}^1$ to $\underset{\sim}{x}^2$ can be expressed by the deformation gradient matrix:

$$\underset{\sim}{F}^p = \partial x_i^2 / \partial x_j^1 \tag{4}$$

and the thermo-elastic deformation from $\underset{\sim}{x}^2$ to $\underset{\sim}{x}^3$ by:

$$\underset{\sim}{F}^e = \partial x_i^3 / \partial x_j^2 \tag{5}$$

The chain rule for partial differentiation then gives the matrix product relation:

$$\underset{\sim}{F} = \underset{\sim}{F}^e \, \underset{\sim}{F}^p \tag{6}$$

In the case of a discontinuous mapping (2), the deformation gradients (4) and (5) reduce to matrices of local linear mappings in the limit of small material elements, which also satisfy (6) but not the continuity conditions for partial derivatives. The analysis of elastic-plastic deformation is not essentially modified by this circumstance.

The matrix product (6) provides the relation between elastic, plastic and total deformation valid for finite strains. It replaces the usual assumption that the total strain is the sum of elastic and plastic components:

$$\epsilon_{ij} = \epsilon_{ij}^{e} + \epsilon_{ij}^{p} \qquad (7)$$

That the latter is not valid at finite strain is clear since the plastic displacement

$$\underset{\sim}{u}^{p} = \underset{\sim}{x}^{2} - \underset{\sim}{x}^{1}$$

and elastic displacement

$$\underset{\sim}{u}^{e} = \underset{\sim}{x}^{3} - \underset{\sim}{x}^{2}$$

are additive:

$$\underset{\sim}{u} = \underset{\sim}{u}^{e} + \underset{\sim}{u}^{p} \qquad (8)$$

where

$$\underset{\sim}{u} = \underset{\sim}{x}^{3} - \underset{\sim}{x}^{1}$$

The finite strain components are non-linear expressions in the displacements, and hence will not be additive. Infinitesimal strain components are linear in the displacements so that (7) will hold in that case. From Fig. 1 it is clear that (7) also holds in that one-component case.

Nonlinearity is not, however, the most significant feature differentiating (6) and (7). The matrix product (6) is not in general commutative, which corresponds to the fact that the plastic deformation $\underset{\sim}{F}^{p}$ is a functional which represents the past history of plastic flow that has already occurred, while the elastic deformation matrix $\underset{\sim}{F}^{e}$ is a function of the current stress and pre-multiplies $\underset{\sim}{F}^{p}$. For finite strain the elastic and plastic deformations are coupled through (6) in a much more involved manner than that corresponding to the addition (7).

The un-stressing process relating the configurations $\underset{\sim}{x}^{3}$ and $\underset{\sim}{x}^{2}$ is not uniquely defined since an arbitrary rigid body rotation can be superposed on $\underset{\sim}{x}^{2}$ and still leave that configuration unstressed. Since in general the $\underset{\sim}{x}^{2}$ configuration is not continuous, such an arbitrary rotation can differ for each element of the body.

Since we shall assume that the constitutive relation for elastic-plastic deformation contains the deformation history through F^p and F^e only, and does not, for example, include strain gradient or size effects, we need consider only homogeneous states of stress and deformation in formulating the constitutive law. Unloading the surface of a body so deformed would then reduce the stresses to zero at uniform temperature, and no questions of a discontinuous configuration x^2 arise.

Work Hardening

Plastic flow is an extremely complicated phenomenon, and a theory which attempts to include all its aspects is likely to be prohibitively difficult to either formulate or apply. We shall therefore develop an approach based on the classical theory of isotropic work hardening generalized to include additional features which finite strain analysis appears to require. Other features not considered in this approach, such as a Bauschinger effect or the localized modification of the yield surface due to plastic flow, could be incorporated as they are in developments of classical plasticity theory.

Our analysis will follow the approach to isotropic work hardening ably expounded by Hill ([3] Chap. II) for strain increments with linear elastic law. The system of yield surfaces associated with continued work hardening consists of a one parameter family of non-intersecting surfaces in stress space. Plastic flow generating a stress corresponding to a point on one of these surfaces, determines that surface as the current yield surface until further plastic flow takes place. This yield surface is independent of the history of loading as long as the stress has not previously exceeded this yield value. The yield condition is

found to be effectively independent of superposed hydrostatic pressure, as is the hardening expressed by the stress-strain relation when stress deviator components are plotted. The yield limit as plastic flow develops is found to be a function of the expenditure of work associated with the plastic strain. Since superposed hydrostatic pressure is found not to influence the yield limit or the work hardening phenomenon, it follows that no volume change occurs in plastic deformation, for otherwise the superposed hydrostatic pressure would contribute to the work of plastic flow and hence influence plasticity. This deduction is found to be in good agreement with experiment. At finite elastic strains, thermo-mechanical coupling influences become significant, so that temperature variation must be considered in the theory, and thermo-elastic theory utilized for the elastic component of deformation. For consistency, thermo-mechanical coupling influences must also be incorporated in the plasticity relations.

To formulate a theory on the basis of the features mentioned above, one must separate the work stored in elastic strain energy from the total work expended in plastic flow. For the infinitesimal strain case such separation follows at once using (7). The rate of expenditure of work on the material per unit volume is given by :

$$\dot{w} = \sigma_{ij} \, \dot{\epsilon}_{ij} = \sigma_{ij} (\dot{\epsilon}^e_{ij} + \dot{\epsilon}^p_{ij}) \tag{9}$$

clearly

$$\dot{w}^e = \sigma_{ij} \, \dot{\epsilon}^e_{ij}$$

gives the rate of expenditure of work stored in elastic deformation, while

$$\dot{w}^P = \sigma_{ij}\, \dot{\epsilon}^P_{ij}$$

gives the rate of increase of plastic work which governs the work hardening phenomenon. An analogous separation must be achieved when (6) replaces (7) for the finite deformation case. This was effected in [2] only for restricted deformations in which principal directions remained fixed in the body throughout the deforming process.

Consider a body deforming through the configuration $x^3(x^1, t)$ shown in Fig. 2. The rate of work expended in deforming the material of the body, that is the total rate of work less the rate of generation of kinetic energy, is given by (see, for example [4] p. 87)

$$\dot{W} = \int_{V^3} \sigma_{ij} \frac{\partial v^3_i}{\partial x^3_j}\, dV^3 \tag{10}$$

where v^3_i is the particle velocity, and V^3 the volume in the configuration x^3 . It will prove convenient to use matrix notation and write

$$\underset{\sim}{T} = \sigma_{ij} \tag{11}$$

and

$$\frac{\partial v^3_i}{\partial x^3_j} = \frac{\partial v^3_i}{\partial x^1_k} \frac{\partial x^1_k}{\partial x^3_j} = \dot{\underset{\sim}{F}}\, \underset{\sim}{F}^{-1} = \underset{\sim}{L} \tag{12}$$

$\underset{\sim}{F}$ being defined in (3). $\dot{\underset{\sim}{F}}$ represents time differentiation at constant x^1 , and hence convected differentiation at a particular particle, since the body is at rest in the initial configuration x^1 . In order to

determine the work expended on a specified element of material, it is
convenient to change the variables to integration over V_1^1 the body
configuration in $\underset{\sim}{x}^1$. Equation (10) then becomes:

$$\dot{W} = \int_{V^1} tr(\underset{\sim}{T} \ \underset{\sim}{L}) \ det(\underset{\sim}{F}) \ dV^1 \tag{13}$$

where "tr" stands for the trace of the matrix argument (the sum of
diagonal terms) and "det" for determinant. Det($\underset{\sim}{F}$) is the Jacobean of
the transformation of variables for the volume integral. Substituting
(6) for $\underset{\sim}{F}$ in (13) and (12), and using the product formula for differen-
tiation of a matrix product, (13) becomes:

$$\dot{W} = \int_{V^1} tr[\underset{\sim}{T}(\dot{\underset{\sim}{F}}^e\underset{\sim}{F}^p +\underset{\sim}{F}^e\dot{\underset{\sim}{F}}^p)\underset{\sim}{F}^{p^{-1}} \underset{\sim}{F}^{e^{-1}}] \ det(\underset{\sim}{F}^e)det(\underset{\sim}{F}^p) \ dV^1 \tag{14}$$

Zero volume change in plastic flow determines:

$$det(\underset{\sim}{F}^p) = 1 \tag{15}$$

and hence W divides into the two components which will be tentatively
labeled elastic and plastic work rates:

$$\dot{W}^e = \int_{V^1} tr[\underset{\sim}{T}\dot{\underset{\sim}{F}}^e \ \underset{\sim}{F}^{e^{-1}}] \ det(\underset{\sim}{F}^e) \ dV^1 \tag{16}$$

and

$$\dot{W}^p = \int_{V^1} tr[\underset{\sim}{T}\underset{\sim}{F}^e \ \dot{\underset{\sim}{F}}^p \ \underset{\sim}{F}^{p^{-1}} \ \underset{\sim}{F}^{e^{-1}}] \ det(\underset{\sim}{F}^e) \ dV^1 \tag{17}$$

According to the kinematics discussed in connection with Fig. 2,
$\underset{\sim}{F}^e(\underset{\sim}{x}^1,t)$ represents the deformation gradient associated with elastic strain
of the plastically deformed body at each instant. Thus (16) has the form of
the rate of storage of elastic strain energy in an elastic body taken through
the deformation history $\underset{\sim}{F}^e(x^1,t)$. However, care must be exercised in

transferring this interpretation to the elastic-plastic case, since the unstressed state x^2 is continuously deforming plastically, whereas normally for purely elastic deformation the unstressed state is a fixed undistorting configuration of the body. Since the elastic strain energy depends on the elastic deformation gradient and the elastic characteristics of the material, (16) might not represent the rate of increase of elastic strain energy if the elastic characteristics are changing due to the continuing plastic flow. For example, the elastic strain energy could change at constant F^e in that case. Thus the interpretation of (16) and (17) depends crucially on the coupling between elastic and plastic deformations.

We have already seen that characteristics of plastic flow are insensitive to hydrostatic pressure and so to the elastic dilatation. Correspondingly, elastic characteristics are insensitive to plastic flow. The usual influence assumed of plastic flow is on the yield limit with the elastic constants remaining unchanged. Hill ([3] p. 24) states that appreciable anisotropy only appears at large strains. Indeed, stress analysis is usually based on standard elastic constants even though the components may have been cold-worked. Moreover, to our knowledge, little information exists on the modification of elastic constants due to plastic flow. This state of affairs is consistent with our knowledge of the physical basis of plasticity associated with the generation and migration of dislocations. Even in a highly plastically deformed material the basic crystal structure is retained with disturbance from the regular lattice of only a very small proportion of the atoms. Thus the elastic constants and heat capacity are not appreciably influenced by plastic flow. We will therefore assume the elastic characteristics to be governed by an invariant isotropic thermoelastic law.

Thus the stress T in the configuration x^3 will depend only on F^e and temperature θ , since plastic deformation and rigid-body rotation of the unstressed state x^2 , will not modify the stress for an isotropic elastic material with invariant elastic properties. The rate of investment of elastic energy in the thermo-elastic component of the system will thus be the same as that for a purely elastic body deformed by F^e at tempera-ture θ from a fixed undistorted state. Equation (16) thus represents the rate of expenditure of thermo-elastic energy invested in the body mechanically. The elastic properties can be conveniently expressed in terms of the Helmholtz free energy per unit mass $\psi(C^e,\theta)$, where $C^e = F^{e^T} F^e$ (superscript T denotes transpose of the matrix), and the stress and entropy in the thermo-elastic system are then given by ([5] p. 302 and 309, or [6] p. 199):

$$T = 2\rho_o F^e \frac{\partial \psi}{\partial C^e} F^{e^T}/\det(F^e) \qquad (18)$$

$$S^e = - \frac{\partial \psi}{\partial \theta} \qquad (19)$$

where ρ_o is the initial density, and S^e the entropy. Examples of forms of the free energy function ψ are given in [2] and [7].

It is interesting to note that the lack of coupling between the laws of elastic and plastic deformation described above is also manifest in the incompressible property of plastic flow. For a change in density of the unstressed state would imply modified elastic characteristics.

Since (16) gives the rate of elastic working, (17) must represent the mechanical work being expended in plastic flow. This appears to exhibit involved coupling with the elastic deformation F^e , but this can be largely eliminated as shown below. Since we shall be concerned with stress-strain

relations, as mentioned earlier, we can without loss assume homogeneous stress and deformation distributions and replace the volume integral in (17) by $\dot{w}^P$, the rate of plastic work per unit undeformed volume.

$$\dot{w}^P = tr(\underset{\sim}{T}\underset{\sim}{F}^e \underset{\sim}{\dot{F}}^P \underset{\sim}{F}^{P^{-1}} \underset{\sim}{F}^{e^{-1}}) \ det(\underset{\sim}{F}^e) \tag{20}$$

It was pointed out in the discussion on kinematics that the state $\underset{\sim}{x}^2$ is not uniquely defined since de-stressing can involve arbitrary rotation. Analogously with (12), the term $\underset{\sim}{\dot{F}}^P \underset{\sim}{F}^{P^{-1}}$ represents the velocity gradient of plastic flow in $\underset{\sim}{x}^2$, and its symmetric part gives the velocity strain, often loosely called the rate of strain, in $\underset{\sim}{x}^2$. When $\underset{\sim}{x}^2$ is arbitrarily rotated, the work associated with this rate of plastic strain can clearly be completely changed. For example, stretch in the direction of a tensile stress could become lateral stretch by a 90° rotation of $\underset{\sim}{x}^2$, with consequent drastic modification of the plastic work term. Since we assume the material to be isotropic, the plastic strain rate tensor will be parallel to the stress tensor, and the work relation as it actually takes place will be achieved by considering $\underset{\sim}{x}^2$ to be defined by de-stressing without rotation. This special choice of $\underset{\sim}{F}^e$, denoted by $\underset{\sim}{\overline{F}}^e$, will then be symmetric and represent pure stretch in the directions of the principal axes of stress, in conformity with the isotropic elastic assumption. The state $\underset{\sim}{x}^2$ is then uniquely prescribed. The availability of this choice for $\underset{\sim}{F}^e$ can be seen analytically since the polar decomposition theorem ([5] p.52) permits any matrix of the form $\underset{\sim}{F}^e$ to be expressed as:

$$\underset{\sim}{F}^e = \underset{\sim}{V}^e \underset{\sim}{R}^e \tag{21}$$

where $\underset{\sim}{V}^e$ is symmetric and $\underset{\sim}{R}^e$ a rotation matrix. Thus (6) can be written

$$\underset{\sim}{F} = \underset{\sim}{F}^e\underset{\sim}{F}^p = \underset{\sim}{V}^e\underset{\sim}{R}^e\underset{\sim}{F}^p = \underset{\sim}{V}^e(\underset{\sim}{R}^e\underset{\sim}{F}^p) = \overline{\underset{\sim}{F}}^e\overline{\underset{\sim}{F}}^p \tag{22}$$

where $\overline{\underset{\sim}{F}}^e = \underset{\sim}{V}^e$ and $\overline{\underset{\sim}{F}}^p = \underset{\sim}{R}^e\underset{\sim}{F}^p$.

Since the trace of a product of matrices is not influenced by cyclic permutation of the elements as is clear from the subscript representation, (20) can be written for the special choice $\overline{\underset{\sim}{x}}^2$:

$$\dot{w}^p = tr(\overline{\underset{\sim}{F}}^{e^{-1}} \underset{\sim}{T}\overline{\underset{\sim}{F}}^e\dot{\overline{\underset{\sim}{F}}}^p\overline{\underset{\sim}{F}}^{p^{-1}}) \, det(\overline{\underset{\sim}{F}}^e) \tag{23}$$

The trace is a scalar invariant of the tensor arguments, so that change of axes does not modify its value. Selecting axes parallel to the directions of principal stress, $\underset{\sim}{T}$ and $\underset{\sim}{F}^e$ become diagonal matrices which are commutative, so that F^e and $F^{e^{-1}}$ cancel, and (23) reduces to

$$\dot{w}^p = tr(\underset{\sim}{T} \, \dot{\overline{\underset{\sim}{F}}}^p \, \overline{\underset{\sim}{F}}^{p^{-1}}) \, det(\overline{\underset{\sim}{F}}^e) \tag{24}$$

which retains this form on a transformation back to the original coordinates. This can alternatively be written

$$\dot{w}^p = \sigma_{ij} \frac{\partial \overline{v}_i^2}{\partial \overline{x}_j^2} \, det(\overline{\underset{\sim}{F}}^e) \tag{25}$$

where $\overline{v}_i^2$ is the velocity of plastic straining in the configuration $\overline{\underset{\sim}{x}}^2$. Thus the plastic rate of work term takes on the simple form of work done by the stress T on the plastic rate of deformation with a scalar coupling term to the elastic deformation : $det(\underset{\sim}{F}^e)$. It is now seen that the result is identical with that discussed in [2] in which the analysis was

limited to total deformation, throughout the loading, having principal directions fixed in the body, for which all matrices considered were thus diagonal, so that the matrix contribution of F^e cancelled directly.

In retrospect, since the rate of total work (13) and the rate of elastic work (16) do not depend on the particular choice of x^2, nor can the rate of plastic work. Substituting (21) for F^e, and permuting the elements in the trace, (17) becomes

$$\dot{W}^p = \int_{V^1} tr(V^{e^{-1}} \underset{\sim}{T} V^e R^e \dot{F}^p F^{p^{-1}} R^{e^{-1}}) \ det \ (F^e) \ dV^1 \qquad (26)$$

For isotropic elastic behavior the rotation R^e prior to the pure stretch V^e in (21) does not affect the stress T which is parallel to the pure stretch tensor V^e. Thus the $V^{e^{-1}}$ and V^e terms cancel. The R^e, $R^{e^{-1}}$ terms simply represent rotation of the plastic velocity gradient tensor, or equivalently velocity strain since T is symmetric. Thus, (26) tells us that, for arbitrary x^2, we must rotate the plastic velocity strain term by the elastic rotation to determine the plastic rate of work, and this is equivalent to making the special selection x^{-2}.

Having determined the rate of plastic work, we can formulate an isotropic work hardening law analogous to that described by Hill ([3] Chap. II). However, since thermal effects can be significant in the present study, the influence of varying temperature on the yield limit should be included, and we thus need to generalize Prager's analysis for the rigid-plastic case [8]. This was done in [2] for the special case of principal directions fixed in the body.

The rate of plastic work invested per unit initial volume, (24), can be written:

$$\dot{w}^p = tr[(\underset{\sim}{T} \ det \ \overline{F}^e) \ \dot{F}^p F^{p^{-1}}] \qquad (27)$$

which corresponds to

$$\dot{w}^P = tr[\underset{\sim}{T}\dot{\underset{\sim}{F}}^P\underset{\sim}{F}^{P^{-1}}] \tag{28}$$

in the infinitesimal strain case. $det(\underset{\sim}{F}^e)$ is a scalar quantity equal to ρ_o/ρ , which is less than unity under hydrostatic pressure. Since

$$\dot{\underset{\sim}{F}}^P\underset{\sim}{F}^{P^{-1}} = \partial v_i^2/\partial x_j^2 \tag{29}$$

and the divergence of v_i^2 is zero because of the incompressibility of plastic flow,

$$tr(\dot{\underset{\sim}{F}}^P\underset{\sim}{F}^{P^{-1}}) = 0 \tag{30}$$

and thus $\underset{\sim}{T}$ can be replaced by its deviator without modifying (27) or (28). Thus hydrostatic pressure only influences the plastic work through the scalar factor $det(\underset{\sim}{F}^e)$, which is a term expressing geometrical non-linearity associated with reduction in scale of the initially unit volume element and hence also of the areas over which the stress acts and the velocity through which the forces are working. As mentioned above, the work-hardening produced by plastic flow is not appreciably modified by the existence of superposed hydrostatic pressure, and physically it is associated with the generation of increased dislocation density. The production of a prescribed dislocation density associated with a prescribed state of hardening is likely to require more plastic work under high pressure rather than less, since the small coupling existing relates to a slight increase of volume due to disturbance of the regular crystal lattice. To avoid the contradiction of less work according to (27) under hydrostatic pressure, we suggest that the yield condition for finite elastic strain should take the form

$$f[\underset{\sim}{T} (det \overline{\underset{\sim}{F}}^e)] = c \tag{31}$$

where f takes on one of the usual forms, such as Mises or Tresca, but could be a more general function of J_2 and J_3, the second and third invariants of the argument deviator, as discussed in [3]. Note that $\det(F^e)$ is a scalar multiplier which does not alter the orientation of the tensor argument. The yield condition (31) reduces to classical yield conditions in the case of infinitesimal elastic strain for which $\det(\underset{\sim}{F}^e)$ can be taken equal to unity. The static experiments carried out to date from which the form of (31) might have been tested, are not sufficiently precise to achieve this in view of the closeness of $\det(\underset{\sim}{F}^e)$ to unity.

For a work hardening material subject to temperature changes, c in (31) has been considered to be a function of w^p and the temperature θ [2]. However, it is known that increasing the temperature increases the mobility of dislocations and hence permits the generation of increased dislocation density, and so of hardening, with a smaller expenditure of plastic work. Hence a functional rather than a function of w^p is needed in the expression for c in (31), which represents the hardening, and the form

$$c = c\left[\int_o^t \alpha(\theta) \; \dot{w}^p \; d_\tau, \theta \right] \tag{32}$$

suggests itself. The term $\alpha(\theta)$ will be an increasing function of θ to achieve the desired temperature - dislocation mobility influence. Relation (31) with changing c according to (32) will yield the desired set of similar yield surfaces when f is an isotropic homogeneous function of its argument components, as, for example, is the Mises condition.

Following Prager [8] it is revealing to write the yield condition (31), (32) in the form:

$$f(\underset{\sim}{Q}) - c(\varphi,\theta) \leq 0 \tag{33}$$

where $\underset{\sim}{Q}$ stands for

$$\underset{\sim}{Q} = \underset{\sim}{T}(\det \overline{\underset{\sim}{F}}^e) \tag{34}$$

and φ represents the functional

$$\varphi = \int_0^t \alpha(\theta) \, \dot{w}^P \, d\tau \tag{35}$$

The equality sign in (33) denotes plastic flow or neutral loading, and the inequality sign, an elastic state. For continuing plastic flow, differentiation of (33) with the equality sign gives

$$\frac{\partial f}{\partial \underset{\sim}{Q}} \dot{\underset{\sim}{Q}} - \frac{\partial c}{\partial \varphi} \dot{\varphi} - \frac{\partial c}{\partial \theta} \dot{\theta} = 0 \tag{36}$$

The positive nature of plastic work and its hardening effect determines

$$\frac{\partial c}{\partial \varphi} \dot{\varphi} > 0$$

so that continued plastic flow is associated with

$$\frac{\partial f}{\partial \underset{\sim}{Q}} \dot{\underset{\sim}{Q}} - \frac{\partial c}{\partial \theta} \dot{\theta} > 0 \tag{37}$$

and this becomes the loading condition. Note that in the isothermal case this simply corresponds to the usual condition $\dot{f} > 0$.

In order to define the law governing rate of plastic flow, it is convenient to work in terms of the plastic velocity strain, often loosely called the rate of strain:

$$\overline{\underset{\sim}{D}}^P = (\overline{\underset{\sim}{L}}^P + \overline{\underset{\sim}{L}}^{P^T})/2 = \left(\frac{\partial \overline{v}_i^2}{\partial \overline{x}_j^2} + \frac{\partial \overline{v}_j^2}{\partial \overline{x}_i^2}\right)/2 \; : \; \overline{\underset{\sim}{L}}^P = \dot{\overline{\underset{\sim}{F}}}^P \overline{\underset{\sim}{F}}^{P^{-1}} \tag{38}$$

Because $\underset{\sim}{T}$ is symmetric, (27) can be written:

$$\dot{w}^P = \mathrm{tr}[(\underset{\sim}{T} \det \overline{\underset{\sim}{F}}^e) \, \overline{\underset{\sim}{D}}^P] \tag{39}$$

Whereas this relation combined with (35) and (36) was sufficient to determine $\overline{\underset{\sim}{D}}^P$ in the case of one dimensional strain [2], since only one non-zero component remained having introduced incompressibility, in general we need the concept of plastic potential to determine the several components. This takes the form [8]

$$\overline{\underset{\sim}{D}}^P = k \, \frac{\partial f}{\partial \underset{\sim}{Q}} \, (\frac{\partial f}{\partial \underset{\sim}{Q}} \, \dot{\underset{\sim}{Q}} - \frac{\partial c}{\partial \theta} \, \dot{\theta}) \tag{40}$$

where k is a function of the current state to be determined. The term in the parentheses gives the rate of loading in analogy with (37), and the normality condition of the strain rate components and the yield surface according to the tensor term $\partial f/\partial \underset{\sim}{Q}$, leads to a maximum rate of work principal and a uniqueness of solution argument as for classical plasticity.

Since the equality sign applies in (33) we can substitute (40) into (36), using (39) and obtain:

$$\frac{\partial f}{\partial \underset{\sim}{Q}} \, \dot{\underset{\sim}{Q}} - \frac{\partial c}{\partial \theta} \, \dot{\theta} = \frac{\partial c}{\partial \varphi} \, \alpha(\theta) \, \operatorname{tr}[\underset{\sim}{Q} \, k \, \frac{\partial f}{\partial \underset{\sim}{Q}} \, (\frac{\partial f}{\partial \underset{\sim}{Q}} \, \dot{\underset{\sim}{Q}} - \frac{\partial c}{\partial \theta} \, \dot{\theta})] \tag{41}$$

The scalar quantities k and $(\frac{\partial f}{\partial \underset{\sim}{Q}} \, \dot{\underset{\sim}{Q}} - \frac{\partial c}{\partial \theta} \, \dot{\theta})$ factorize from the trace expression, the latter simplifies by use of Euler's theorem since f is homogeneous in $\underset{\sim}{Q}$, and (41) reduces to

$$1 = \frac{\partial c}{\partial \varphi} \, k \, \alpha \, n \, f \tag{42}$$

where n is the order of $\underset{\sim}{Q}$ in $f(Q)$, and (40) becomes

$$\overline{\underset{\sim}{D}}^P = \frac{\partial f}{\partial \underset{\sim}{Q}} \, (\frac{\partial f}{\partial \underset{\sim}{Q}} \, \dot{\underset{\sim}{Q}} - \frac{\partial c}{\partial \theta} \, \dot{\theta})/\alpha \, n \, f \, \frac{\partial c}{\partial \varphi} \tag{43}$$

This relation is equivalent to that for isothermal work hardening given by Hill ([3] p. 38) and is seen not to be appreciably complicated by the inclusion of finite strain, non-linear elasticity and thermo-mechanical coupling.

The Thermodynamic System

The analysis presented already, permits the determination of deformation given the stress history apart from the temperature variation. To determine this, thermo-mechanical coupling in plastic flow must be considered in addition to that included in the thermo-elastic theory (18) and (19). The approach follows directly that used in the fixed principal directions case [2]. We have seen that the elastic and plastic phenomena are only loosely coupled, and that thermo-elastic properties depend on the basic crystal structure whereas plastic flow depends on generation and migration of dislocations which leave most of the crystal structure intact. We thus base the thermodynamics on two interacting systems i) reversible thermoelasticity governed by (18) and (19) and the corresponding specific heat relations and ii) irreversible plastic flow, comprising some dissipation of the plastic work to heat which appears as a source in the thermo-elastic system and hence as an increase in the elastic entropy in adiabatic loading. The randomness of the dislocation distribution also leads to an increase in entropy associated with plastic distortion. A strain energy field surrounds each dislocation which absorbs some of the work of plastic flow as the dislocation density grows. For the specific thermo-mechanical coupling process generated by plastic flow, we adopt that measured by Taylor and Farren [9] and Taylor and Quinney [10], in which temperature rise was observed in elastic-plastic tests which exceeded that expected from thermo-elastic theory by an amount equivalent to a heat source equal to $\gamma \dot{w}^p$ per unit initial volume, where γ is a factor which varies slowly from about 0.9 to unity with increasing plastic flow. The rest of the plastic work rate, $(1-\gamma)\dot{w}^p$, is considered to be stored in the developing

dislocation system. Thus the internal energy per unit mass associated with thermo-elastic deformation, E^e , and that stored in the dislocation distribution E^p , are, for adiabatic loading, determined by

$$\rho_o \dot{E}^e = \dot{w}^e + \gamma \dot{w}^P \qquad (44)$$

and

$$\rho_o \dot{E}^P = (1-\gamma) \dot{w}^P \qquad (45)$$

Growth of elastic entropy S^e arises due to the mechanical energy dissipated in plastic flow, and is given by

$$\theta \rho_o \dot{S}^e = \gamma \dot{w}^P \qquad (46)$$

We shall consider γ to take on the constant value 0.9. These relations assume continuous variations of deformation. In the case of shock-wave loading, involving discontinuous changes of stress and strain, additional increase in entropy occurs over and above that given by (46).

Combination of the contributions developed in this and the previous section determines the total stress-deformation relation. Equation (18) for elastic strain, (33) and (43) for plastic flow and (19), (44) and (46) for the coupling between these components.

Discussion

The rather concise structure of the theory developed in this paper arises because of the comparatively loose coupling between the elastic and plastic phenomena. The coupling needed to achieve simplicity and the retention of the basic structure of classical theory was determined by the expression for plastic work rate (27) which led to the appropriate yield relation (31). The introduction of the influence of plastic flow on elastic properties would clearly lead to a more elaborate theory, but fortunately

these effects are not dominant.

With the present development in mind, it is interesting to review the presentation by Hill ([2] Chap. II). He carefully limits (7) to increments since he is concerned with finite plastic deformation and infinitesimal elastic strains. His strain increment relation corresponds to our velocity strain relation from (12)

$$\underset{\sim}{D} = (\underset{\sim}{L} + \underset{\sim}{L}^T)/2 \tag{47}$$

where

$$\underset{\sim}{L} = \dot{\underset{\sim}{F}}^e \underset{\sim}{F}^{e^{-1}} + \underset{\sim}{F}^e \dot{\underset{\sim}{F}}^p \underset{\sim}{F}^{p^{-1}} \underset{\sim}{F}^{e^{-1}} \tag{48}$$

Clearly the elastic and plastic velocity strains components

$$\underset{\sim}{L}^e = \dot{\underset{\sim}{F}}^e \underset{\sim}{F}^{e^{-1}} \;, \; \underset{\sim}{L}^p = \dot{\underset{\sim}{F}}^p \underset{\sim}{F}^{p^{-1}} \tag{49}$$

are not additive to give the total velocity strain, but for infinitesimal elastic strains $\underset{\sim}{F}^e \sim \underset{\sim}{I}$, the unit matrix, and additivity applies to some order of approximation. For the fixed principal directions case, the matrices in (48) are diagonal and the $\underset{\sim}{F}^e$ terms cancel, leading to additivity, as utilized in [2].

Acknowledgements

References

1. E. H. Lee and D. T. Liu, "Finite Strain Elastic-Plastic Theory", Proc. IUTAM Symp. on Irreversible Aspects of Continuum Mechanics, Springer Verlag, Vienna, 1968.

2. E. H. Lee and D. T. Liu, Jl. Appl. Phys., 38, 19-27, 1967.

3. R. Hill "The Mathematical Theory of Plasticity", Clarendon Press, Oxford, 1950.

4. W. Prager, "Introduction to Mechanics of Continua", Ginn and Co., Boston, 1961.

5. C. Truesdell and W. Noll, "The Non-linear Field Theories of Mechanics", Handbuch der Physik, v. III/3, Springer-Verlag, Berlin, 1965.

6. C. E. Pearson, "Theoretical Elasticity", Harvard University Press, Cambridge, Mass., 1959.

7. E. H. Lee and T. Wierzbicki, Jl. Appl. Mech. v. 34, 931-036, 1967.

8. W. Prager, Koninkl. Ned. Akad. Wetenschap., Proc. 61, p. 176, 1958.

9. W. S. Farren and G. I. Taylor, Proc. Roy. Soc. v. 107, 422-51, 1925.

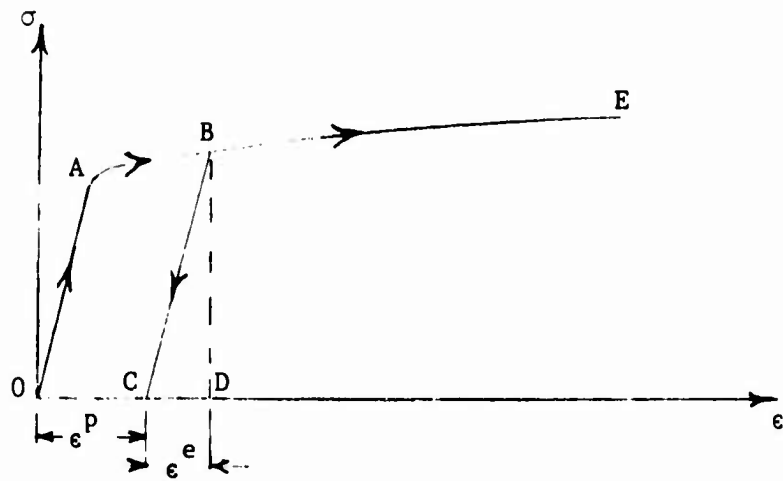10. H. Quinney and G. I. Taylor, Proc. Roy. Soc. v. 143, 307-26, 1934.
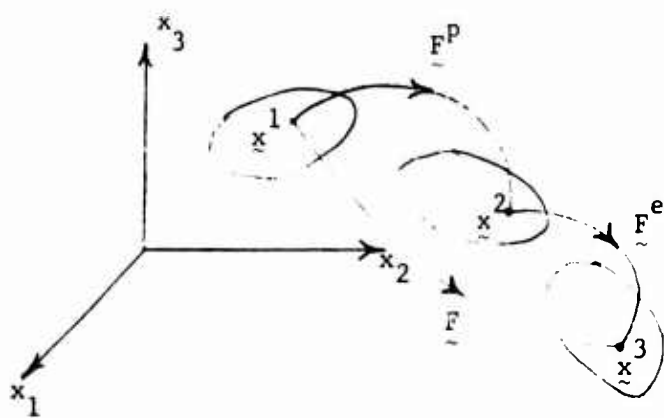
Fig. 1.   Stress-strain curve for a ductile metal

Fig. 2.   The kinematics of elastic-plastic deformation

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13. ABSTRACT

In some circumstances, elastic-plastic deformation occurs in which both components of strain are finite. Such situations fall outside the scope of classical plasticity theory which assumes either infinitesimal strains or plastic-rigid theory for large strains. The present theory modifies the kinematics to include finite elastic and plastic strain components. For situations requiring this generalization, dilatational influences are usually significant including thermo-mechanical coupling. This is introduced through the consideration of two coupled thermodynamic systems: one comprising thermo-elasticity at finite strain and the other the irreversible process of dissipation and absorption of plastic work. The present paper generalizes a previous theory to permit arbitrary deformation histories.

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