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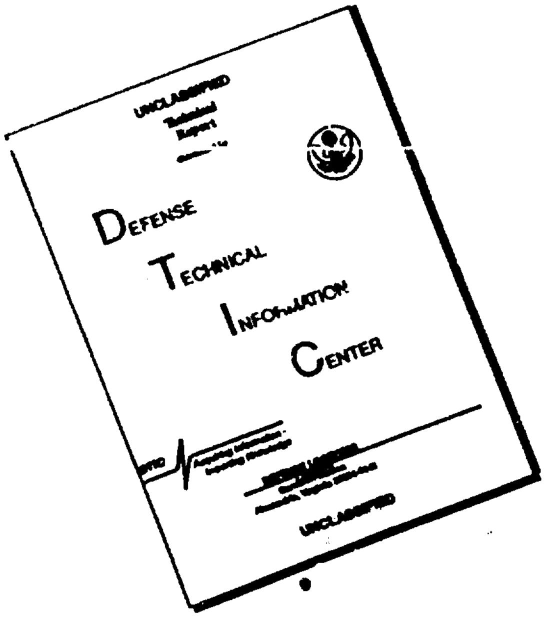

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SLIGHT ATMOSPHERIC VARIATIONS AND THE ADAPTATION OF METEOROLOGICAL FIELDS

by

A. S. Monin and A. M. Obukhov

On the basis of the solution of the problem of slight variations in a baroclinic atmosphere, a general classification of the main types of dynamic processes in the atmosphere (horizontal vorticity motions and gravitational and acoustic waves) is given. The work also gives the general form of the invariant, with which the critical stationary state of the atmosphere can be computed using arbitrary initial data, without analyzing the wave processes which cause reorganization of the fields. The "filtering" role of the quasistatic approximation is explained: it "filters out" internal acoustic waves and somewhat overrates the frequencies of gravitational waves. In particular, it is shown that only several minutes are required to establish quasistatic equilibrium in the atmosphere.

As is known, the complete system of the equations of atmospheric hydrodynamics is a time system of the fifth-order (it contains derivatives of three components of velocity, pressure, and density with respect to time). Accordingly, in order to solve the Cauchy problem for this system of equations one must know, at the initial moment of time, the fields of five meteorological elements. With arbitrary initial data, motions developing in the atmosphere can be separated into relatively slow ones (synoptic) and rapid ones (wave). The consideration of rapid wave motions essentially complicates the analysis of synoptic processes. Therefore, when investigating synoptic processes, the equations of atmospheric hydrodynamics are simplified so that the simplified equations will give an adequate enough description of the synoptic processes, but will not contain rapid wave motions among their solutions. Such approximations (quasistatic and quasigeostrophic approximations) lead to dynamic equations of the first time order, so that to describe the evolution of synoptic processes one need know the initial field of only one meteorological element (e.g., pressure or a stream function). The order of the dynamics equations is reduced (from fifth to first) through the exclusion - "filtering out" - of certain solutions of the initial system of equations (actually, rapid wave motions) by the above-mentioned simplifications. To prove the validity of such simplifications one must explain the nature of rapid wave motions.

Under actual conditions, rapid wave processes occur with small amplitudes, i.e., they are of the nature of slight variations. Accordingly, verification of the quasistatic and quasigeostrophic approximations requires study of the nature of slight atmospheric variations; the following results are obtained.

Rapid wave motions occur if static and geostrophic equilibrium is disrupted in a certain region of the atmosphere. The resultant waves are scattered, due to which the meteorological fields adjust to one another so that the atmosphere approaches the state of static and geostrophic equilibrium. This process is called the adaptation of meteorological fields. In the linear theory, the state of the static and geostrophic equilibrium is stationary and in order to distinguish this stationary state with arbitrary initial data we need only filter out the rapid wave motions.

The evolution of the adapted fields (i.e., those close to the state of static and geostrophic equilibrium) occurs as the result of non-linear effects, viz., absolute vorticity transfer and entropy. These non-linear effect continually create a tendency toward the destruction of static and geostrophic equilibrium. But here, due to the generation and scattering of rapid waves, adaptation continually occurs, and the meteorological fields remain close to the state of static and geostrophic equilibrium (adapted). The evolution of the adapted fields is also, strictly speaking, a synoptic process and when describing it, one must consider its cause - the non-linear effects - but in place of a description of continually occurring adaptation one may substitute the requirement that the meteorological fields remain adapted at all times. This requirement also provides the basis for the quasistatic and quasigeostrophic approximations.

A. M. Obukhov [1] first suggested the adaptation of meteorological fields as a method of verifying the quasigeostrophic approximation. In so doing, he examined the case of a barotropic atmosphere. Charney [2] made the preliminary analysis of the "filtering role" of the quasistatic and quasigeostrophic approximations. I. A. Kibel' [3] and A. S. Monin [4] described the processes of adaptation in a quasistatic baroclinic atmosphere. In this latter work an invariant was found which is a generalization of the "potential vorticity" for a quasistatic baroclinic atmosphere and it was established that in addition to the internal gravitational waves investigated in [3], in a baroclinic atmosphere more rapid "two-dimensional" gravitational waves may occur, which are analogous to the waves in a barotropic medium.

The question of the adaptation of meteorological fields to the state of quasistatic equilibrium and of the filtering role of the quasistatic approximation had not been studied in sufficient detail until now. The present work aims to fulfill this need and to give the most general formulation of the problem of the adaptation of meteorological fields in a baroclinic atmosphere.

The study of slight atmospheric variations is significant not only for describing the process of adaptation, but it is of independent interest in a number of problems of atmospheric physics, e.g., a certain family of slight variations is the basic object of study in atmospheric acoustics. Acoustic variations are of interest for atmospheric physics since acoustic waves, which form in the troposphere, penetrate into the upper layers of the atmosphere and are absorbed there, can transmit energy from the lower to the upper atmosphere. Internal gravitational waves arise, e.g., during the streamlining of mountains by an air stream (see the works of Lyra [6] and A. A. Dorodnitsyn [5]).

1. THE EQUATIONS OF SLIGHT ATMOSPHERIC VARIATIONS

We will examine the atmosphere as a liquid in which processes occur polytropically with the exponent of polytropy $\varkappa$. In other words, we will consider that when liquid particles move, the magnitude $\rho p^{-1/\varkappa}$ is preserved in them, where p and ρ are pressure and density. In a real atmosphere (outside the bounding layers) the processes occur adiabatically, so that $\varkappa = c_p/c_v$, which is equal to the ratio of the specific heat of air at constant pressure and constant volume. However, when interpreting certain formulas it will be convenient if we can examine polytropic processes with an arbitrary exponent.

As the basic equations of atmospheric dynamics, let us examine the equations of motion (disregarding dissipative factors*)), the equation of continuity, and the equation of the polytropic process

$$
\begin{aligned}
&\rho \frac{du}{dt} = -\frac{\partial p}{\partial x} + l\rho v, \\
&\rho \frac{dv}{dt} = -\frac{\partial p}{\partial y} - l\rho u, \\
&\rho \frac{dw}{dt} = -\frac{\partial p}{\partial z} - g\rho, \\
&\frac{\partial \rho}{\partial t} = -\left(\frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} + \frac{\partial \rho w}{\partial z}\right), \\
&\frac{dp}{dt} = \frac{\varkappa p}{\rho} \frac{d\rho}{dt}.
\end{aligned}
\tag{1}
$$

*) There would be no great difficulties in computing the dissipative forces, but the computations would become more unwieldy. When computing the dissipative forces in the form of the components $\nu\rho\Delta u$, $\nu\rho\Delta v$, and $\nu\rho\Delta w$ in the right-hand parts of the first three equations of (1) (ν is the viscosity coefficient), a basic change in the results given below would result in the appearance of damping factors $e^{-\nu k^2 t}$ in the formulas for the amplitudes of harmonic waves with wave number k.

Here t is time, u, v, and w are the velocity components along the axes of the cartesian coordinates x, y, and z (the z-axis is directed vertically upward). $\frac{d}{dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}$ is the symbol of the individual derivative, g the acceleration of gravity and ℓ the Coriolis parameter, from now on to be considered a constant*).

We will study slight variations which might arise in the atmosphere against the background of one of its basic states (whose characteristics will be designated by a line over the letters). The features of the slight variations will, naturally, depend on the properties of the basic state. In the present work we will not explain how atmospheric motion in its basic state affects the nature of slight variations and, in connection with this, we will use as the basic state the state of rest where pressure $\bar{p}$ and density $\bar{\rho}$ depend only on z and are connected by the statics equation

$$\frac{d\bar{p}}{dz} = -g\bar{\rho} \qquad (2)$$

The relationship between slight variations and properties of basic atmospheric motion can be explained in a separate paper. The conclusions of the present work relative to small-scale waves are also applicable in presence of basic atmospheric motions (provided the velocity field in the basic state does not contain small-scale inhomogeneities).

The distinguishing characteristics of the basic state of the atmosphere is the temperature profile $\bar{T}(z)$, instead of which it will be convenient to use the magnitude

$$c^2 = \frac{\varkappa\bar{p}}{\bar{\rho}} = \varkappa R\bar{T} \qquad (3)$$

which has the sense of the square of the speed of sound. Let us also introduce a special designation for the parameter of thermal stability

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*) Changes of ℓ with latitude make possible the appearance in the atmosphere of additional inertial variations - the so-called Rossby waves. These waves are of interest only when analyzing processes on a synoptic scale; computing them would introduce no essential changes into the results of the present work on gravitational and acoustic variations - they would only make the computations more unwieldy. A sufficiently detailed description of Rossby waves is given, e.g., in A. M. Iaglom's survey article [7].

$$\beta = (\varkappa-1)g + \frac{dc^2}{dz} = \varkappa R(\gamma_a - \gamma) \qquad (4)$$

Here R is the gas constant and γ_a and γ are the adiabatic and the actual vertical temperature gradients.

Linearizing equations (1) with respect to the state of rest we will get the equations of slight atmospheric variations in the form

$$\begin{aligned} \frac{\partial \bar{\rho} u}{\partial t} &= -\frac{\partial p}{\partial x} + l\bar{\rho} v, \\ \frac{\partial \bar{\rho} v}{\partial t} &= -\frac{\partial p}{\partial y} - l\bar{\rho} u, \\ \lambda \frac{\partial \bar{\rho} w}{\partial t} &= -\left(\frac{\partial p}{\partial z} + g\rho\right), \\ \frac{\partial \rho}{\partial t} &= -\left(\frac{\partial \bar{\rho} u}{\partial x} + \frac{\partial \bar{\rho} v}{\partial y} + \frac{\partial \bar{\rho} w}{\partial z}\right), \\ \frac{\partial p}{\partial t} &= -\beta\bar{\rho} w - c^2\left(\frac{\partial \bar{\rho} u}{\partial x} + \frac{\partial \bar{\rho} v}{\partial y} + \frac{\partial \bar{\rho} w}{\partial z}\right), \end{aligned} \qquad (5)$$

where u, v, w, p and ρ are the characteristics of slight variations (let us assume for them the same designations as for the initial meteorological elements). The parameter λ, equal to unity in the general case, is introduced here so that in the future it will be convenient to trace its influence on the slight variations of the vertical accelerations of particles. Actually, if we do not compute vertical accelerations, i.e., if we describe the slight variations in the quasistatic approximation, we should assume that $\lambda \to 0$.

There are no difficulties in obtaining the linearized equations (5) from the initial equations (1) except, perhaps, the last of these equations. Direct linearization of the last equation of (1) yields

$$\frac{\partial p}{\partial t} + w\frac{d\bar{p}}{dz} = c^2\left(\frac{\partial \rho}{\partial t} + w\frac{d\bar{\rho}}{dz}\right)$$

or

$$\frac{\partial p}{\partial t} = c^2 \frac{\partial \rho}{\partial t} + \bar{\rho} w\left(\frac{c^2}{\bar{\rho}}\frac{d\bar{\rho}}{dz} - \frac{1}{\bar{\rho}}\frac{d\bar{p}}{dz}\right). \qquad (6)$$

Using equations (2) - (4) as characteristics of the basic state of the atmosphere, let us transform the expression in parentheses in the right-hand part of this last equation as follows:

$$\frac{c^2}{\bar{\rho}}\frac{d\bar{\rho}}{dz} - \frac{1}{\bar{\rho}}\frac{d\bar{p}}{dz} = \frac{c^2}{\bar{\rho}}\frac{d}{dz}\frac{\varkappa\bar{p}}{c^2} + g = \frac{\varkappa}{\bar{\rho}}\frac{d\bar{p}}{dz} - \frac{\varkappa\bar{p}}{\bar{\rho}c^2}\frac{dc^2}{dz} + g = -\varkappa g - \frac{dc^2}{dz} + g = -\beta.$$

Substituting this value in (6) and eliminating $\partial\rho/\partial t$, we get the last equation of (5) by using the fourth equation of system (5).

In the future it will be convenient to introduce the new unknown magnitudes φ, ψ, and χ instead of the velocity components u, v and w, setting

$$\bar{\rho}u = \frac{\partial\varphi}{\partial x} - \frac{\partial\psi}{\partial y}, \quad \bar{\rho}v = \frac{\partial\varphi}{\partial y} + \frac{\partial\psi}{\partial x}, \quad \bar{\rho}w = \chi. \tag{7}$$

The relationships

$$\frac{\partial\bar{\rho}u}{\partial x} + \frac{\partial\bar{\rho}v}{\partial y} = \Delta\varphi, \quad \frac{\partial\bar{\rho}v}{\partial x} - \frac{\partial\bar{\rho}u}{\partial y} = \Delta\psi, \tag{8}$$

where $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ is the symbol for the two-dimensional Laplace operator, show that the velocity potential φ and the stream function ψ are determined by plane divergence and the vertical component of the relative vorticity of the velocity field, respectively. The magnitude χ has the sense of the vertical stream of mass.

Differentiating the first equation of system (5) with respect to x and the second with respect to y and adding the results we get

$$\Delta \frac{\partial\varphi}{\partial t} = l\Delta\psi - \Delta p .$$

Analogously, differentiating the second equation of system (5) with respect to x and subtracting the first equation, differentiated with respect to y, we get

$$\Delta \frac{\partial\psi}{\partial t} = -l\Delta\varphi$$

Under the natural requirement that functions φ, ψ, and p be regular to infinity, the sign of the Laplace operator in the last two equations can be eliminated. In addition, using the designations of (7) in the last three equations of (5) we may rewrite equations (5) in the form

$$\begin{aligned} \frac{\partial\psi}{\partial t} &= -l\varphi, \\ \frac{\partial\varphi}{\partial t} &= l\psi - p, \\ \frac{\partial p}{\partial t} &= -c^2\Delta\varphi - \beta\chi - c^2\frac{\partial\chi}{\partial z}, \\ \lambda\frac{\partial\chi}{\partial t} &= -\left(\frac{\partial p}{\partial z} + g\rho\right), \\ \frac{\partial\rho}{\partial t} &= -\Delta\varphi - \frac{\partial\chi}{\partial z} \end{aligned} \tag{9}$$

The equations of slight variations will be most convenient in this form for subsequent analysis. We will study the solution of these equations using arbitrary initial data. Then

$$t = 0: \quad \psi = \psi_0(x, y, z); \quad \varphi = \varphi_0(x, y, z); \quad \chi = \chi_0(x, y, z);$$
$$\rho = \rho_0(x, y, z); \quad p = p_0(x, y, z). \tag{10}$$

Since the system of equations (9) contains derivatives with respect to z and t of the second z-order, two boundary conditions with respect to z must be formulated to solve it. The natural boundary conditions are

$$\chi \to 0 \, (z \to 0), \qquad \chi \to 0 \, (z \to \infty). \tag{11}$$

The first of these conditions is obvious. The second indicates that the vertical stream of mass should revert to zero at the lower boundary of the atmosphere*).

2. THE STATIONARY SOLUTION AND THE INVARIANT

Equations (9) have a certain family of stationary solutions ψ_s, φ_s, p_s, χ_s, and ρ_s. For each of the stationary solutions the following relationships are fulfilled

$$\varphi_s \equiv 0, \quad \chi_s \equiv 0, \quad p_s \equiv l\psi_s, \quad \rho_s \equiv -\frac{l}{g}\frac{\partial \psi_s}{\partial z}, \tag{12}$$

so that stationary motions are horizontal and non-divergent and the formulas of geostrophic wind and the statics equations

$$u_s = -\frac{1}{l\rho}\frac{\partial p_s}{\partial y}, \quad v_s = \frac{1}{l\rho}\frac{\partial p_s}{\partial x}, \quad \frac{\partial p_s}{\partial z} = -g\rho_s. \tag{13}$$

are applicable to them.

Obviously, each such solution describes a certain stationary, horizontal vorticity motion. From (12) it is evident that each stationary solution can be determined completely by assigning one function $\psi_s(x, y, z)$.

It is not hard to see that any solution for which $\varphi \equiv 0$ is stationary. Actually, if $\varphi \equiv 0$, it follows from the first two equations of (9) that ψ and p are not functions of t (and are connected by the relationship $p = l\psi$). Then the third equation of (9) assumes the form

$$c^2 \frac{\partial \chi}{\partial z} + \beta \chi = 0$$

and the solution of χ for this equation with boundary conditions (11) is identical with zero. From the last two equations of (9) it follows that ρ is not a function of t and is connected with p by the relationship

$$\frac{\partial p}{\partial z} = -g\rho$$

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*) The question of the formulation of the condition at infinity deserves a more detailed examination. Let us note that when analyzing wave processes it turns out that the condition $\rho w \to 0$ is too weak and should be replaced by the stronger condition $w\sqrt{\rho} \to 0$ when $z \to \infty$.

The solution of equations (9) with initial data (10) will be stationary when, and only when, these initial data satisfy the relationship

$$\varphi_0 \equiv 0, \quad \chi_0 \equiv 0, \quad p_0 \equiv l\psi_0, \quad \rho_0 \equiv -\frac{l}{g}\frac{\partial \psi_0}{\partial z}. \tag{14}$$

It follows from (12) that this condition is necessary. To show that it suffices, let us differentiate the second and fourth equations of system (9) with respect to t and in the first part of the obtained equations exclude the derivatives $\partial\psi/\partial t$, $\partial p/\partial t$ and $\partial\rho/\partial t$, using the remaining equations of (9). As a result we get the following systems of fourth-order time equations for φ and χ:

$$\begin{gathered}\left(\frac{\partial^2}{\partial t^2} + l^2\right)\varphi = \beta\chi + c^2\frac{\partial \chi}{\partial z} + c^2\Delta\varphi, \\ \lambda\frac{\partial^2 \chi}{\partial t^2} = \frac{\partial}{\partial t}\left(\beta\chi + c^2\frac{\partial \chi}{\partial z} + c^2\Delta\varphi\right) + g\left(\frac{\partial \chi}{\partial z} + \Delta\varphi\right).\end{gathered} \tag{15}$$

The solutions of φ and χ for these equations with identical boundary conditions (11) is determined completely by the following initial data. When

$$t = 0: \; \varphi = \varphi_0; \quad \frac{\partial \varphi}{\partial t} = l\psi_0 - p_0; \quad \chi = \chi_0; \quad \lambda\frac{\partial \chi}{\partial t} = -\left(\frac{\partial p_0}{\partial z} + g\rho_0\right). \tag{16}$$

When conditions (14) are fulfilled, these initial data revert to zero and consequently, equations (15) have only the trivial solution $\varphi \equiv 0$ and $\chi \equiv 0$, but from the condition $\varphi \equiv 0$ it follows that the solution of equations (9) in this case will be stationary.

If the initial data (10) do not satisfy relationships (14), the solution of equations (9) with these initial data can be represented in the form of the sum of the stationary solution which can be determined by some function ψ_s and of the non-stationary solution of ψ', φ', p', χ', and ρ' which satisfies the initial data. When

$$\begin{gathered} t = 0: \; \psi' = \psi_0 - \psi_s; \quad \varphi' = \varphi_0; \quad p' = p_0 - l\psi_s; \quad \chi' = \chi_0; \\ \rho' = \rho_0 + \frac{l}{g}\frac{\partial \psi_s}{\partial z}. \end{gathered} \tag{17}$$

In order to find the function ψ_s which determines the stationary solution in this case, we may use the fact that equations (9) allow an invariant, i.e., some expression which does not change with time and which is a linear function of the unknown functions ψ, φ, p, χ, and ρ *). Let us use the following method to find this invariant. Let us introduce temporarily the new notations for the unknown functions

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*) Let us note that equations (9) also permit a certain integral invariant which is quadratic with respect to the unknown functions (the energy integral). Actually, using equations (5) we can convince ourselves that the following equality holds:

$$\frac{\partial}{\partial t}\left\{\bar\rho\frac{u^2 + v^2 + w^2}{2} + \frac{1}{2a\bar\rho}\left[p^2 + \frac{g}{\beta}(p - c^2\rho)^2\right]\right\} = -\left(\frac{\partial up}{\partial x} + \frac{\partial vp}{\partial y} + \frac{\partial wp}{\partial z}\right).$$

from which it follows that the magnitude [footnote continued next page]

$$y_1 = \psi,\ y_2 = \varphi,\ y_3 = p,\ y_4 = \chi,\ y_5 = \rho \tag{18}$$

and write equations (9) in the form

$$\frac{\partial y_i}{\partial t} = \sum_{k=1}^{5} A_{ik} y_k \quad (i = 1, 2, \ldots, 5), \tag{19}$$

where A_{ik} are, generally speaking, the differential operators

$$\| A_{ik} \| = \begin{Vmatrix} 0 & -l & 0 & 0 & 0 \\ l & 0 & -1 & 0 & 0 \\ 0 & -c^2\Delta & 0 & -\beta - c^2\frac{\partial}{\partial z} & 0 \\ 0 & 0 & -\frac{1}{\lambda}\frac{\partial}{\partial z} & 0 & -\frac{g}{\lambda} \\ 0 & -\Delta & 0 & -\frac{\partial}{\partial z} & 0 \end{Vmatrix} \tag{20}$$

We will seek the invariant in the form

$$J_1 = \sum_{i=1}^{5} \gamma_i y_i, \tag{21}$$

where γ_i, generally speaking, are linear operators independent of time t. Since $\partial J_1/\partial t$ should equal

$$\frac{\partial J_1}{\partial t} = \sum_{i=1}^{5} \gamma_i \frac{\partial y_i}{\partial t} = \sum_{k=1}^{5} \left(\sum_{i=1}^{5} \gamma_i A_{ik} \right) y_k \equiv 0,$$

the operators γ_i satisfy the system of equations

$$\sum_{i=1}^{5} \gamma_i A_{ik} \equiv 0, \tag{22}$$

connected with the system

$$\sum_{k=1}^{5} A_{ik} y_k = 0,$$

which determines the stationary solution $y_i = (y_i)_s$.

It is easy to solve equations (22) with respect to the operators γ_i in the given case. From the structure of the first and fifth columns of matrix (20) it is evident that we should set $\gamma_2 = \gamma_4 = 0$. In this case, the equation obtained using the third column is identically satisfied. Thus, only the

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$$H(S) = \int_0^\infty dz \iint \left\{ \bar{\rho} \frac{u^2 + v^2 + w^2}{2} + \frac{1}{2\kappa p} \left[p^2 + \frac{g}{\beta} (p - c^2 \rho)^2 \right] \right\} dx\, dy$$

is independent of time, if the total work of the pressure forces on the sides of a cylindrical container of base S is equal to zero. The expression in the braces is the density of the complete energy of the disturbances, and the second component in this expression is the density of potential energy. Thus, if the complete energy of the entire field of disturbances is finite, it does not change with time.

operators γ_1, γ_3, and γ_5 may differ from zero. For these operators, using the second and fourth columns of matrix (20), we get the equations

$$\gamma_1 l + \gamma_3 c^2 \Delta + \gamma_5 \Delta = 0, \quad \gamma_3\left(\beta + c^2 \frac{\partial}{\partial z}\right) + \gamma_5 \frac{\partial}{\partial z} = 0.$$

Setting $\gamma_1 = \Delta$ and seeking γ_3 and γ_5 in the form of linear functions of the operator $\partial/\partial z$, we get

$$\gamma_3 = l \frac{\partial}{\partial z} \frac{1}{\beta}, \quad \gamma_5 = -l \frac{\partial}{\partial z} \frac{c^2}{\beta} - l.$$

Substituting these values of the operators γ_i in (21) and returning to the old variables, we get the following invariant:

$$J_1 = \Delta\psi + l \frac{\partial}{\partial z} \frac{p - c^2 \rho}{\beta} - l\rho. \qquad (23)$$

This invariant can be called the potential vorticity [1, 4]. Let us note that from the third and fifth equations of (9) we get the equality

$$\frac{\partial}{\partial z}(p - c^2\rho) = -\beta\chi.$$

Writing this equality with $z = 0$ and remembering, according to (11), that the magnitude χ reverts to zero when $z = 0$, we get the additional invariant

$$J_2 = p^* - c^{*2}\rho^*, \qquad (24)$$

where the asterisks indicate values of the functions when $z = 0$. Let us note that J_1 and J_2 do not contain the parameter λ, i.e., they do not change form when transferring to a quasistatic approximation.

Let us examine the case where the initial conditions (10) satisfy conditions (14) everywhere with the exception of a certain limited region of space V. The non-stationary component of the solution of equations (9) under these initial conditions will be of the nature of waves, propagating with a certain finite velocity from region V, and the entire energy of these waves will be finite. Since this energy will be propagated throughout an ever increasing volume, the values of functions ψ', φ', p', χ' and ρ', which describe the non-stationary component, will tend toward zero when t approaches infinity, at every fixed point in space, and in this case a complete solution of equations (9) will tend toward their stationary component. From this it follows that the invariants J_1 and J_2 will coincide with their values for the stationary components of the solution and accordingly, for the non-stationary component, those invariants are equal to zero. The solutions of equations (9) for which $J_1 \equiv J_2 \equiv 0$ will be called wave solutions.

Thus, the solution of equations (9) with arbitrary initial data (10) can be represented in the form of a sum a) of the stationary solution, for which the invariants J_1 and J_2 are determined according to the initial data, and b) of the wave solution which satisfies the initial conditions (17). It remains only to show how we find the function ψ_s which determines the stationary solution, knowing the values of the invariants J_1 and J_2. For this, let us express J_1 and J_2 by ψ_s, using formulas (12). Here we get

$$\Delta\psi_s + l^2 \frac{\partial}{\partial z}\left(\frac{\psi_s}{\beta} + \frac{\psi_s}{g} + \frac{c^2}{\beta g}\frac{\partial\psi_s}{\partial z}\right) = J_1, \tag{25*)}$$

$$\left(l\psi_s + \frac{c^2 l}{g}\frac{\partial\psi_s}{\partial z}\right)_{z=0} = J_2. \tag{26}$$

Thus, the function ψ_s can be found as the solution of an inhomogeneous elliptical equation (25) with the inhomogeneous boundary condition (26) (as the second boundary condition it is sufficient to require that ψ_s be limited as z approaches infinity). In order to avoid inhomogeneity in the boundary condition here, we may write the invariants J_1 and J_2 in a single integral form. Thus, let us examine the following composite invariant

$$\begin{aligned}\bar\rho\Omega &= \int_0^\infty J_1\,dz + \frac{l}{\beta_0}J_2 + \frac{\bar\rho^*}{\kappa}\int_0^z \frac{\beta\,dz}{\bar\rho}\int_z^\infty J_1\,dz = \\ &= \int_0^\infty(\Delta\psi - lp)\,dz - \frac{\bar\rho^*}{\kappa}\int_0^z\left[l\,\frac{p - c^2\rho}{\bar p} - \frac{\beta}{\bar p}\int_z^\infty(\Delta\psi - lp)\,dz\right]dz.\end{aligned} \tag{27}$$

This invariant is the integral form of the potential vorticity.

In the quasistatic approximation, by excluding the magnitude $\rho = -\frac{1}{g}\frac{\partial p}{\partial z}$ from expression (27) and by switching from the integration variable z to the new variable $\zeta = \bar p/\bar p^*$, we can give (27) in the form

$$\frac{g\bar\rho\Omega}{\bar p} = \Delta\left(\int_0^1 \frac{\psi}{\bar p}\,d\zeta + \int_\zeta^1 \frac{a^2 d\zeta}{\zeta}\int_0^\zeta \frac{\psi}{\bar p}\,d\zeta\right) - l\,\frac{\bar p^*}{\bar p}\,\frac{p}{\bar p}, \tag{28}$$

where

$$a^2 = \frac{(\gamma_a - \gamma)R}{g}\,\frac{T}{T^*} = \frac{\kappa - 1}{\kappa}\left(\frac{\gamma_a - \gamma}{\gamma_a}\right)\frac{T}{T^*} \tag{29}$$

is the non-dimensional stability parameter. A. S. Monin [4] found the invariant (28) for a quasistatic baroclinic atmosphere.

Expressing invariant (27) by ψ_s using formulas (12), and setting $\psi_s/\bar\rho = \Psi$, we get the following equation with respect to Ψ:

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*) Replacing in this equation the magnitude c^2/β by its mean value in the troposphere, the components with the previous derivatives can be written in the form $\Delta\psi_s + \frac{l^2 c^2}{\beta g}\frac{\partial^2\psi_s}{\partial z^2}$, from which it is evident that the characteristic scales of the synoptic processes along the vertical and the horitonzal are most naturally determined such that their ratio is $cl/\sqrt{\beta g}$, which is approximately one one-hundredth.

$$\Delta\left(\int_0^t \Psi' dt + \int [illegible] \int_0^t \Psi' dt\right) - \frac{[illegible]}{\rho}\Psi' = \frac{[illegible]}{\rho}, \qquad (30)$$

where the right side can be determined from the initial data, using formula (27). The solution of this equation (in the case $a^2 \simeq$ const) is given in [4].

3. THE WAVE SOLUTION

After finding the function ψ_s which determines the stationary solution, from initial conditions (10) we can separate the part [the initial conditions (17)] which determines the wave solution. When finding this solution, one of equations (9) can be replaced by the condition that the potential vorticity (27) be equal to zero, so that the equations which describe the wave solution are fourth-order time equations. In principle, these equations make it possible to express ψ', p', and ρ' by φ and χ, without integrating with respect to time, while the magnitudes φ and χ are determined from equations (15) with initial conditions (16), not containing function ψ_s. Therefore, it is not necessary to know the stationary solution to find the wave solution.

Since the coefficients of equations (15) are independent of x, y and t, these equations have partial solutions in the form of harmonic waves with amplitudes which are functions of z

$$\varphi(x, y, z, t) = \Phi(z)e^{i(k_1x+k_2y-\sigma t)},$$
$$\chi(x, y, z, t) = X(z)e^{i(k_1x+k_2y-\sigma t)}, \qquad (31)$$

where k_1 and k_2 are the horizontal wave numbers which can be arbitrary, and σ the frequencies to be determined. Since the dynamic equations which we are examining are linear, and we are not considering either the influx or dissipation of energy in them, the energy of waves of type (31) cannot change with time. Accordingly, frequencies σ must be real.

The solution of equations (15) with arbitrary initial data can be represented in the form of the superposition of elementary wave solutions (31) with all possible values of k_1 and k_2. Therefore, the explanation of the nature of the elementary waves (31) makes it possible to draw definite conclusions as to the nature of any wave solution of the examined equations of hydrodynamics. In order to study waves (31) we have only to find their amplitudes $\Phi(z)$ and $X(z)$ and determine the frequencies σ (i.e., find the <u>frequency spectrum</u> of the elementary wave solutions). Substituting functions (31) in equations (15) we

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get the following equations for amplitudes $\Phi(z)$ and $X(z)$:

$$(l^2 + k^2c^2 - \sigma^2)\,\Phi = \beta X + c^2 \frac{dX}{dz},$$
$$(l^2 - \sigma^2)\left(c^2 \frac{d\Phi}{dz} + g\Phi\right) = (\beta g - \lambda c^2\sigma^2)\, X, \qquad (32)$$

where $$k^2 = k_1^2 + k_2^2.$$

Let us note first that these equations have a non-trivial partial solution in which $X \equiv 0$ (since this solution satisfies the boundary conditions it has a physical significance). The first equation of (32) shows that the frequencies corresponding to this solution are determined by the formula

$$\sigma^2 = l^2 + k^2c^2. \qquad (33)$$

According to the second equation of (32) the amplitude $\Phi(z)$ should satisfy the equation

$$c^2 \frac{d\Phi}{dz} + g\Phi = 0,$$

so that it has the form

$$\Phi(z) = \Phi^0 e^{-g\int_0^z \frac{dz}{c^2}} = \Phi^0 (\bar{p}/\bar{p}^0)^{1/\varkappa}. \qquad (34)$$

Evidently, the waves corresponding to this solution embrace the entire atmosphere at once, propagate only horizontally and do not cause vertical variations of air particles. In the quasistatic approximation ($\lambda = 0$), these waves retain their form. The maximum group velocity of motion of packets of similar waves (the velocity of the front of the wave) is equal to the speed of sound c. The indicated waves are completely analogous to the waves in a quasistatic barotropic atmosphere studied by A. M. Obukhov [1] when he examined the question of the adaptation of hydrodynamic fields. Such waves were singled out by A. S. Monin [4] for the case of a quasistatic baroclinic atmosphere. In the future, we will call these waves <u>two-dimensional</u>.

Turning to a study of the solutions for which $X \not\equiv 0$, for purposes of simplifying the computations we will limit ourselves to an examination of the case where the coefficients c^2 and β in equations (32) can be considered constant [this condition is accurate for an isothermal atmosphere in which $c^2 = \varkappa gH$ and $\beta = (\varkappa-1)g$, where $H = \bar{p}^*/g\bar{\rho}^*$ is the height of the uniform atmosphere]. In this case, each of the functions Φ and X satisfies the following equation, which stems from (32)

$$(l^2 - \sigma^2)\left(\frac{d^2F}{dz^2} + \frac{\beta + g}{c^2}\frac{dF}{dz} + \frac{\lambda\sigma^2}{c^2} F\right) - k^2\left(\frac{g\beta}{c^2} - \lambda\sigma^2\right) F = 0. \qquad (35)$$

This equation has a partial solution in the form [exp $(-M+im)z$], where M and m are real numbers. Substituting this function in (35), we get

$$(l^2-\sigma^2)\left[(-M+im)^2+\frac{\beta+g}{c^2}(-M+im)+\frac{\lambda\sigma^2}{c^2}\right]-k^2\left(\frac{g\beta}{c^2}-\lambda\sigma^2\right)=0.$$

Remembering that σ is real and equating the imaginary part of the obtained equality to zero, we get $M = (\beta+g)/2c^2$. Using this result and equating the real part of the equality to zero, we get the following relationship:

$$\frac{(\sigma^2-l^2)\left\{\lambda\sigma^2-\left[\frac{(\beta+g)^2}{4c^2}+m^2c^2\right]\right\}}{\sigma^2-\frac{g\beta}{c^2}}=k^2c^2. \tag{36}$$

Let us note that functions and χ of the form

$$\varphi,\chi\sim e^{-\frac{\beta+g}{2c^2}z+i(k_1x+k_2y+mz-\sigma t)}, \tag{37}$$

correspond to the examined partial solution of equation (35), i.e., plane harmonic waves with horizontal wave numbers k_1 and k_2 and a vertical wave number m, whose amplitudes decrease exponentially with height. These waves have physical significance only when $m \neq 0$, since when $m = 0$, no combination of such waves satisfies the boundary condition $\chi \to 0$ $(z \to \infty)$. Accordingly, the propagation rate of such waves always has a vertical component, i.e., these waves are essentially three-dimensional. We will call such waves internal waves.

Internal waves with any wave numbers can enter into the solution of the Cauchy problem for equations (15) with arbitrary initial data. Therefore, in (36) the magnitudes k and m must be considered arbitrary and then (36) is an equation with respect to σ. Noting that the left-hand part of equality (36) is not negative and considering that

$$l^2<\frac{g\beta}{c^2}<\frac{(\beta+g)^2}{4c^2},$$

we can easily convince ourselves that the roots of σ^2 in equation (36) can be found only within the intervals

$$l^2<\sigma^2<\frac{g\beta}{c^2},\quad \sigma^2>\frac{(\beta+g)^2}{4c^2\lambda}, \tag{38}$$

whereupon (due to the fact that the roots of σ^2 are continuous functions of k^2 and m^2), all points of these intervals are possible frequencies of internal waves. Thus, two families of internal waves can occur in the atmosphere, i.e., waves with frequencies from the first or from the second interval of (38). To analyze the nature of these waves, let us write the solutions of equation (36) for the square of the frequency σ^2 in the form

$$\sigma_\pm^2=\frac{c^2}{2}\left[k^2+\frac{l^2}{c^2}+\frac{m^2}{\lambda}+\frac{1}{\lambda}\left(\frac{\beta+g}{2c^2}\right)^2\right]\times$$
$$\times\left\{1+\sqrt{1-\frac{4g\beta}{\lambda c^2}\frac{k^2+\left[m^2+\left(\frac{\beta+g}{2c^2}\right)^2\right]\frac{l^2}{g\beta}}{\left[k^2+\frac{l^2}{c^2}+\frac{m^2}{\lambda}+\frac{1}{\lambda}\left(\frac{\beta+g}{2c^2}\right)^2\right]^2}}\right\}.$$

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$$\sigma_g^2 = \frac{c^2}{2}\left[k^2 + \frac{1}{L^2} + \frac{m^2}{\lambda} + \frac{1}{\lambda}\left(\frac{\beta+g}{2c^2}\right)^2\right] \times \tag{39}$$

$$\times\left\{1 - \sqrt{1 - \frac{4g\beta}{\lambda c^2}\frac{k^2 + \left[m^2 + \left(\frac{\beta-g}{2c^2}\right)^2\right]\frac{c^2}{g\beta L^2}}{\left[k^2 + \frac{1}{L^2} + \frac{m^2}{\lambda} + \frac{1}{\lambda}\left(\frac{\beta+g}{2c^2}\right)^2\right]^2}}\right\},$$

where $L = c/l$ is the characteristic scale of the horizontal movements of a compressible medium in a Coriolis force field, first introduced in [1].

From (39) it is evident that the influence of the Coriolis force is essential only with very small horizontal wave numbers k (i.e., for waves whose horizontal scales are not small compared with L). In order to explain the nature of waves with frequencies σ_a and σ_g, let us first digress from the effect of the Coriolis force (i.e., let us set $l = 0$ and accordingly, $L = \infty$). Let us assume that $\lambda = 1$ and examine the isothermal atmosphere in which $c^2 = \varkappa gH$ and $\beta = (\varkappa-1)g$. Here formulas (39) assume the form

$$\sigma_a^2 = \frac{\varkappa gH}{2}\left(k^2 + m^2 + \frac{1}{4H^2}\right)\left[1 + \sqrt{1 - \frac{4\frac{\varkappa-1}{\varkappa^2}\frac{k^2}{H^2}}{\left(k^2+m^2+\frac{1}{4H^2}\right)^2}}\right],$$

$$\sigma_g^2 = \frac{\varkappa gH}{2}\left(k^2 + m^2 + \frac{1}{4H^2}\right)\left[1 - \sqrt{1 - \frac{4\frac{\varkappa-1}{\varkappa^2}\frac{k^2}{H^2}}{\left(k^2+m^2+\frac{1}{4H^2}\right)^2}}\right]. \tag{40}$$

Let us remember that we are examining polytropic processes during which the magnitude $\rho p^{-1/\varkappa}$ is retained in liquid particles. In particular, when $\varkappa = \infty$, the value retained will be density ρ, so that this case corresponds to a non-compressible medium. When $\varkappa \to \infty$, all frequencies σ_a revert to infinity, i.e., the waves with these frequencies disappear. But in a non-compressible medium, only variations with frequencies $\bar{\sigma}_g$ may arise, determined from (40) within the boundary $\varkappa \to \infty$:

$$\bar{\sigma}_g^2 = \frac{g}{H}\frac{k^2}{k^2 + m^2 + \frac{1}{4H^2}}. \tag{41}$$

Evidently, waves with these frequencies are internal gravitational waves. A description of them can be found, e.g., in Prandtl's book [8]. Further, when $\varkappa = 1$, the value which is maintained will be temperature, so that this case corresponds to isothermal processes, with respect to which isothermal stratification is neutral. When $\varkappa \to 1$, all frequencies σ_g revert to zero, so that variations with these frequencies disappear (they convert into stationary vorticity motions). Thus, with isothermal processes in an isothermal medium, only variations with frequencies $\bar{\sigma}_a$ may occur, which can be determined from (40) within the limit $\varkappa \to 1$:

$$\sigma_a^2 = gH\left(k^2 + m^2 + \frac{1}{4H^2}\right). \tag{42}$$

Evidently, waves with these frequencies are acoustic. An analogous discussion shows that two-dimensional waves with frequencies (33) are acoustic.

Returning to the case of adiabatic processes during which $\varkappa \approx 1.4$, we may conclude that the frequencies σ_a determined by formulas (40) correspond to acoustic waves, distorted due to the absence of isothermy in neutral stratification, while frequencies σ_g correspond to gravitational waves distorted because of the compressibility of the air. An analogous interpretation can be applied to waves corresponding to frequencies (39) and containing an additional distortion due to the effect of the Coriolis force.

Assuming $\lambda = 1$ in formulas (39) and making use of the fact that with sufficiently large k and m, the subtrahend under the radical in these formulas is small (so that we may assume approximately that $\sqrt{1 - \varepsilon} \approx 1 - \frac{\varepsilon}{2}$), we get the following approximate formulas (which are the more accurate, the larger are k and m):

$$\sigma_a^2 \approx c^2\left[k^2 + \frac{1}{L^2} + m^2 + \left(\frac{\beta + g}{2c^2}\right)^2\right] > \frac{(\beta + g)^2}{4c^2},$$

$$\sigma_g^2 \approx \frac{g\beta}{c^2} \cdot \frac{k^2 + \left[m^2 + \left(\frac{\beta + g}{2c^2}\right)^2\right]\frac{c^4}{g\beta L^2}}{k^2 + \frac{1}{L^2} + m^2 + \left(\frac{\beta + g}{2c^2}\right)^2} < \frac{g\beta}{c^2}. \tag{43}$$

The inequalities shown here for σ_a^2 and σ_g^2 are strict with any k and m. Thus, waves with frequencies from the first interval of (38) are gravitational, those from the second interval are acoustic. In the case of an isothermal atmosphere, setting $k = 1.4$ and $H = 8$ km, we get the following numerical inequalities for the periods of acoustic and gravitational waves:

$$T_a = \frac{2\pi}{\sigma_a} < 300 \text{ sec}\,; \quad T_g = \frac{2\pi}{\sigma_g} > 330 \text{ sec}$$

In an atmosphere in which temperature drops with height, the interval between T_a and T_g increases.

The frequencies of slight variations, possible when using the quasistatic approximation, are obtained from frequencies (39) within the limit $\lambda \to 0$. Using this limited transition, all frequencies σ_g revert to infinity, i.e., the internal acoustic waves are completely filtered out. Here the group velocity of the vertical propagation of packets of internal acoustic waves $d\sigma_a/dm$ also reverts to infinity, so that in the quasistatic approximation the propagation rate of disturbances along the vertical appears infinitely large. When the quasistatic approximation is used, the only internal waves possible are the gravitational ones whose frequencies are determined by the formula

$$\sigma_4^2 = \frac{g\beta}{c^2}\cdot\frac{k^2+\left[m^2+\left(\frac{\beta+g}{2c^2}\right)^2\right]\frac{c^4}{g\beta L^2}}{m^2+\left(\frac{\beta+g}{2c^2}\right)^2}. \qquad (44)$$

Comparing this formula with the corresponding formula (43), we see that the frequencies of gravitational waves in the quasistatic approximation are too high, and they are the greater, the larger the horizontal wave number k; however, for the case $k \ll m$ (long waves), this change of σ remains insignificant.

Summing up the results of this analysis, we see that the slight variations which can occur in the atmosphere with arbitrary initial data break down into three classes: two-dimensional waves, internal acoustic waves and internal gravitational waves.

The role of the quasistatic approximation basically reduces to the filtering out of internal acoustic waves (a secondary effect is a certain distortion of internal gravitational waves). Therefore, when the internal acoustic waves, which arise due to the disruption of static equilibrium at the initial moment of time in a certain region of space V, scatter, we may consider that the atmosphere has basically already adjusted itself to the state of static equilibrium. The time of adaptation to the state of static equilibrium is of the same order of magnitude as the time it takes for the front of the internal acoustic waves to traverse the main part of the atmosphere. Since this front moves at the speed of sound c, the indicated time is only several minutes (in one minute a sound wave traverses a layer 20 km thick). After static equilibrium has been attained, the process of the adaptation of the atmosphere to the state of geostrophic equilibrium continues, whereupon, on an average, such a state is established throughout the atmosphere after the scattering of two-dimensional waves, while somewhat later, after the scattering of slow (see [4]) internal gravitational waves, geostrophic equilibrium is established at all altitudes. The rate of this latter process depends essentially on atmospheric stratification.

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