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LARGE-SAMPLE PROPERTIES OF LEAST-SQUARES ESTIMATORS
OF HARMONIC COMPONENTS IN A TIME SERIES WITH STATIONARY
RESIDUALS. I. INDEPENDENT RESIDUALS

BY
A. M. WALKER

TECHNICAL REPORT NO. 23
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Department of

STANFORD, CALIFORNIA



Statistics

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DEPARTMENT OF STATISTICS

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1. Introduction

Let $\{X_t, t = 0, \pm 1, \pm 2, \dots\}$ be a discrete parameter time series generated by a model of the form

$$X_t = m_t + Y_t \quad (1.1)$$

where

$$m_t = E(X_t) = \sum_{r=1}^q (A_r \cos \omega_r t + B_r \sin \omega_r t) \quad , \quad (1.2)$$

with $0 \leq \omega_r \leq \pi$ (which involves no loss of generality), and

$$Y_t = \sum_{u=0}^{\infty} g_u(\underline{\theta}) \epsilon_{t-u} \quad , \quad (1.3)$$

the ϵ_t being distributed identically and independently each with mean zero and finite variance $E(\epsilon_t^2) = v$, and the $g_u(\underline{\theta})$ being specified functions of an unknown vector-valued parameter $\underline{\theta} = (\theta_1, \theta_2, \dots, \theta_p)$ such that $\sum_{u=0}^{\infty} g_u^2(\underline{\theta}) < \infty$; to avoid indeterminacy we take $g_0(\underline{\theta}) \equiv 1$. $\{X_t\}$ thus has a systematic component consisting of the sum of q simple harmonic components with different frequencies ω_r and a residual or 'noise' component which is a completely stationary series having spectral density

$$f(\omega, \underline{\theta}) = (v/2\pi) \left| \sum_{u=0}^{\infty} g_u(\underline{\theta}) e^{i\omega u} \right|^2 \quad , \quad (1.4)$$

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and is usually called a linear process (see, for example, Hannan, 1960, p. 33).

Suppose that the values of the other parameters in the model, namely A_r, B_r, ω_r ($1 \leq r \leq q$) and v , are also unknown. We then have a fairly general type of 'hidden periodicities' model, the term 'hidden periodicity' denoting a harmonic component whose frequency as well as its amplitude and phase is unknown. The restricted model obtained by taking the residual component to consist of 'white noise', that is, $g_u(\underline{\theta}) = 0$ for $u \geq 1$, so that (1.3) just becomes $Y_t = \epsilon_t$ and the parameter $\underline{\theta}$ disappears, has been quite widely used in connection with physical and economic phenomena, though in recent years it has become less popular because of the realisation that analyses based on this can be very misleading if the 'white noise' assumption is not a good approximation.

The problem of estimating the parameters in the model (1.1) from data consisting of n observations $X^{(n)} = (X_1, X_2, \dots, X_n)$, and of determining the approximate distribution of the estimators for large n was dealt with by Whittle (1952). He used a method of estimation which was approximately equivalent to an application of the principle of least squares, becoming approximately the method of maximum-likelihood when ϵ_t has a normal distribution so that $\{X_t\}$ becomes a normal or Gaussian process. By means of heuristic arguments he obtained the following results.

(1) The estimator $(\hat{A}_r, \hat{B}_r, \hat{\omega}_r)$ is asymptotically ($n \rightarrow \infty$) normal with mean $\underline{0}$ and covariance matrix

$$\frac{2\text{vg}(\omega, \underline{\theta})}{n} \begin{pmatrix} 1 & 0 & \frac{1}{2} n B_r \\ 0 & 1 & -\frac{1}{2} n A_r \\ \frac{1}{2} n B_r & -\frac{1}{2} n A_r & \frac{1}{3} n^2 (A_r^2 + B_r^2) \end{pmatrix}^{-1}, \quad (1.5)$$

where

$$g(\omega, \underline{\theta}) = \left| \sum_{u=0}^{\infty} g_u(\underline{\theta}) e^{i\omega u} \right|^2. \quad (1.6)$$

See Whittle, 1952, p. 53, equation (4.14) and 1954, p. 224, equation (11). Note that the change in sign of the first two elements in the last row and last column in (1.5) is due to Whittle having interchanged A_r and B_r in (1.2), and that his computation of the bottom diagonal element is incorrect, the numerical factor being $1/3$, not $1/6$.

(2) $(\hat{A}_1, \hat{B}_1, \hat{\omega}_1)$, $(\hat{A}_2, \hat{B}_2, \hat{\omega}_2)$, ..., $(\hat{A}_q, \hat{B}_q, \hat{\omega}_q)$ and $\hat{\underline{\theta}}$, the estimator of $\underline{\theta}$, are all mutually asymptotically uncorrelated.

(3) $\hat{\underline{\theta}}$ is asymptotically normal with mean $\underline{0}$ and covariance matrix $(nW)^{-1}$, where the elements of W are given by

$$\begin{aligned} W_{ij} &= \frac{1}{4\pi} \int_{-\pi}^{\pi} \frac{\partial \log g(\omega, \underline{\theta})}{\partial \theta_i} \frac{\partial \log g(\omega, \underline{\theta})}{\partial \theta_j} d\omega \\ &= \frac{1}{2} \left(\text{constant term in expansion of } \frac{\partial}{\partial \theta_i} \left\{ \log \left| \sum_{u=0}^{\infty} g_u(\underline{\theta}) z^u \right|^2 \right\} \right. \\ &\quad \left. \times \frac{\partial}{\partial \theta_j} \left\{ \log \left| \sum_{u=0}^{\infty} g_u(\underline{\theta}) z^u \right|^2 \right\} \text{ on the unit circle} \right). \end{aligned} \quad (1.7)$$

See Whittle, 1952, p.49, equation (3.7), 1954, p.214, equation (7).

The estimators are obtained by minimising the expression

$$\begin{aligned}
 \sum_{|s| \leq n-1} \alpha_s(\underline{\theta}) C_s - 2n^{-1} \sum_{r=1}^q h(\omega_r, \underline{\theta}) \sum_{t=1}^n X_t (A_r \cos \omega_r t + B_r \sin \omega_r t) \\
 + \frac{1}{2} \sum_{r=1}^q h(\omega_r, \underline{\theta}) (A_r^2 + B_r^2) \quad , \quad (1.8)
 \end{aligned}$$

where $C_s = n^{-1} \sum_{t=1}^{n-|s|} X_t X_{t+|s|}$ ($0 \leq s \leq n-1$) are the sample covariances (with divisor n), $h(\omega, \underline{\theta}) = \{g(\omega, \underline{\theta})\}^{-1}$, and

$$\alpha_s(\underline{\theta}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i\omega s} h(\omega, \underline{\theta}) d\omega \quad . \quad (1.9)$$

Compare Whittle, 1952, p.50, equation (4.5) or 1954, p.223, equation (7). These, apart from a factor n , are the same as (1.8) when the expression for $\alpha_s(\underline{\theta})$ given by (1.9) is substituted in the latter.

Our object will be to present proofs of these results stated precisely as limit theorems, under conditions which should be satisfied in nearly all applications. These follow the usual pattern, whereby consistency of the estimators is first established, and then the mean value theorem applied to obtain asymptotic normality with the aid of a central limit theorem. However, great care needs to be taken with the details, especially in the consistency part of the proof. It might be thought that a simple modification of the argument used in a previous paper (Walker, 1964) to give a rigorous proof of the result (3) above when no harmonic components are present would suffice, but the presence of the unknown frequencies makes the situation a great deal more complicated. In the 'white noise' case, $X_t = m_t + \epsilon_t$, the complications, although still quite troublesome, are substantially reduced. We therefore deal first with this simpler situation in the present paper. The general model with residuals generated by a linear process will be

considered in a subsequent paper. Appreciable simplification is also achieved when there is only one harmonic component present. For this reason, we shall take $q=1$ in (1.2) in the proof that follows in §§2-4, and then indicate the modifications required when $q>1$ in §5, the final section of the paper.

2. Consistency of the estimator of angular frequency.

For $q=1$ it will be convenient to make a slight change of notation, and write

$$m_t = A \cos \omega t + B \sin \omega t \quad (2.1)$$

In the 'white noise' case, $h(\omega, \theta) = 1$, and so (1.8) reduces to

$$C_0 = 2n^{-1} \sum_{t=1}^n X_t (A \cos \omega t + B \sin \omega t) + \frac{1}{2} (A^2 + B^2) \quad (2.2)$$

Minimisation of (2.2) with respect to A , B , and ω then gives estimators

$$\hat{A}_n = \frac{2}{n} \sum_{t=1}^n X_t \cos \hat{\omega}_n t, \quad \hat{B}_n = \frac{2}{n} \sum_{t=1}^n X_t \sin \hat{\omega}_n t, \quad (2.3)$$

and $\hat{\omega}_n$ such that

$$I_n(\hat{\omega}_n) = \max_{0 \leq \omega \leq \pi} I_n(\omega), \quad (2.4)$$

where

$$I_n(\omega) = \frac{2}{n} \left| \sum_{t=1}^n X_t e^{i\omega t} \right|^2, \quad (2.5)$$

the usual definition of the periodogram intensity function (see, for example, Hannan, 1960, p.52). The suffices n have been added to emphasise that the estimators depend on n ; strictly we should write $\hat{A}_n(X^{(n)})$ etc., but the omission of the argument $X^{(n)}$ will cause no ambiguity.

Now (2.2) is approximately equal to $n^{-1} S(A,B,\omega)$, where

$$S(A,B,\omega) = \sum_{t=1}^n (X_t - A \cos \omega t - B \sin \omega t)^2 \quad (2.6)$$

is the residual sum of squares, provided that ω is not near 0 or π .

For

$$\begin{aligned} \sum_{t=1}^n (A \cos \omega t + B \sin \omega t)^2 - \frac{1}{2} (A^2 + B^2) \\ = \frac{1}{2} \sum_{t=1}^n \{ (A^2 - B^2) \cos 2\omega t + 2AB \sin 2\omega t \} \end{aligned}$$

and

$$\sum_{t=1}^n (\cos 2\omega t + i \sin 2\omega t) = e^{i\omega(n+1)} \left(\frac{\sin n\omega}{\sin \omega} \right) \text{ is } O(1)$$

if ω lies in a closed interval contained in $[0, \pi]$. Thus our estimation procedure should be a reasonable one if the true value of ω , ω_0 say, is not 0 or π .

From now on we therefore make the assumption

$$\omega_0 \neq 0 \text{ or } \pi, \quad (2.7)$$

which is quite a mild restriction. (2.7) would not be needed if we were to determine $\hat{A}_n, \hat{B}_n, \hat{\omega}_n$ by minimising $S(A,B,\omega)$ exactly, but this leads to much less pleasant estimation equations which are extremely inconvenient for theoretical investigation of properties of the estimators.

When ϵ_t is normal, the log-likelihood function is of course

$$L_n(A,B,\omega,v) = -\frac{1}{2} n \log 2\pi v - \frac{S(A,B,\omega)}{2v} \quad (2.8)$$

Thus \hat{A}_n , \hat{B}_n , $\hat{\omega}_n$ are approximately maximum-likelihood estimators, being obtained by maximising a modified log-likelihood function given by replacing $n^{-1} S(A, B, \omega)$ in (2.8) by (2.2).

Since $\hat{\omega}_n$ is determined by maximising $I_n(\omega)$ it is natural to show first that this is consistent. We shall in fact obtain a much stronger result, without which it is not clear how to establish consistency of \hat{A}_n and \hat{B}_n .

Theorem 1. Let $m_t = A_0 \cos \omega_0 t + B_0 \sin \omega_0 t$ where $\omega_0 \neq 0$ or π . Then if $\hat{\omega}_n$ is such that $I_n(\hat{\omega}_n) = \max_{0 \leq \omega \leq \pi} I_n(\omega)$,

$$\hat{\omega}_n - \omega_0 = o_p(n^{-1}), \quad n \rightarrow \infty, \quad (2.9)$$

so that in particular $\hat{\omega}_n$ is consistent.

Proof

From (2.5) we have

$$\frac{1}{2} n I_n(\omega) = \left| \sum_{t=1}^n e^{i\omega t} (A_0 \cos \omega_0 t + B_0 \sin \omega_0 t + \epsilon_t) \right|^2. \quad (2.10)$$

Write

$$A_0 \cos \omega_0 t + B_0 \sin \omega_0 t = D_0 e^{i\omega_0 t} + D_0^* e^{-i\omega_0 t}$$

where $D_0 = \frac{1}{2} (A_0 - iB_0)$, $D_0^* = \frac{1}{2} (A_0 + iB_0)$. Then (2.10) gives

$$\frac{1}{2} n I_n(\omega) = \left| D_0 M_n(\omega + \omega_0) + D_0^* M_n(\omega - \omega_0) + \sum_{t=1}^n \epsilon_t e^{i\omega t} \right|^2, \quad (2.11)$$

where

$$M_n(u) = \sum_{t=1}^n e^{iut} = e^{\frac{1}{2}i(n+1)u} \left(\frac{\sin \frac{1}{2} nu}{\sin \frac{1}{2} u} \right), \quad 0 < u < 2\pi$$

$$= n, \quad u = 0 \text{ or } 2\pi$$

(2.12)

that is,

$$\frac{1}{2} n I_n(\omega) = \left| \sum_{t=1}^n \epsilon_t e^{i\omega t} \right|^2 + 2\Re \left\{ \left(\sum_{t=1}^n \epsilon_t e^{i\omega t} \right) (D_0 M_n(\omega + \omega_0) + D_0^* M_n(\omega - \omega_0)) \right\}$$

$$+ |D_0 M_n(\omega + \omega_0) + D_0^* M_n(\omega - \omega_0)|^2.$$

(2.13)

When $\omega = \omega_0$, (2.13) is dominated by the term

$$|D_0^* M_n(0)|^2 = \frac{1}{4} n^2 (A_0^2 + B_0^2).$$

In fact, since the real and imaginary parts of $\sum_{t=1}^n \epsilon_t e^{i\omega_0 t}$ each have variance $\frac{1}{2} n\sigma^2 + o(1)$, so that

$$\sum_{t=1}^n \epsilon_t e^{i\omega_0 t} = o_p(n^{1/2}), \quad \text{and} \quad M_n(2\omega_0) = o_p(1)$$

because of the condition $\omega_0 \neq 0$ or π , we see that

$$\frac{1}{2} n I_n(\omega_0) = \frac{1}{4} n^2 (A_0^2 + B_0^2) + o_p(n^{3/2}),$$

or

$$I_n(\omega_0) = \frac{1}{2} n (A_0^2 + B_0^2) + o_p(n^{1/2}). \quad (2.14)$$

We now obtain an estimate of $\max_{|\omega - \omega_0| \geq n^{-1}\delta} I_n(\omega)$, where δ can be arbitrarily small. For this we require the inequality

$$\mathbb{E} \left\{ \max_{0 \leq \omega \leq \pi} \left| \sum_{t=1}^n \epsilon_t e^{i\omega t} \right|^2 \right\} \leq v \{ n+2 \int_1^n x^{1/2} dx \} . \quad (2.15)$$

This follows since

$$\begin{aligned} \left| \sum_{t=1}^n \epsilon_t e^{i\omega t} \right|^2 &= \sum_{|s| \leq n-1} e^{i\omega s} \sum_{t=1}^{n-|s|} \epsilon_t \epsilon_{t+|s|} \\ &\leq \sum_{|s| \leq n-1} \left| \sum_{t=1}^{n-|s|} \epsilon_t \epsilon_{t+|s|} \right| , \end{aligned}$$

whose expectation does not exceed

$$\mathbb{E} \left(\sum_{t=1}^n \epsilon_t^2 \right) + 2 \sum_{s=1}^{n-1} \mathbb{E} \left\{ \left(\sum_{t=1}^{n-s} \epsilon_t \epsilon_{t+s} \right)^2 \right\}^{1/2} = v \{ n+2 \sum_{s=1}^{n-1} (n-s)^{1/2} \} .$$

(Compare Walker, 1965, p.112, equation (29)). From (2.15) we have

$$\max_{0 \leq \omega \leq \pi} \left| \sum_{t=1}^n \epsilon_t e^{i\omega t} \right|^2 = o_p(n^{3/2}) . \quad (2.16)$$

Using (2.16) and

$$\max_{0 \leq \omega \leq \pi} |M_n(\omega + \omega_0)| = o(1) , \quad \max_{0 \leq \omega \leq \pi} |M_n(\omega - \omega_0)| = n ,$$

we see from (2.13) that

$$\max_{0 \leq \omega \leq \pi} \left| \frac{1}{2} n I_n(\omega) - \left| \frac{D_n^* M_n(\omega - \omega_0)}{O_n} \right|^2 \right| = o_p(n^{3/2}) + o_p(n^{7/4}) + o(n) ,$$

giving

$$\max_{0 \leq \omega \leq \pi} \left| I_n(\omega) - \frac{1}{2} n^{-1} (A_0^2 + B_0^2) |M_n(\omega - \omega_0)|^2 \right| = o_p(n^{3/4}) . \quad (2.17)$$

If we were to add the assumption that $E(|\epsilon_t|^r) < \infty$ for some $r > 4$, we could use a much more powerful, but by no means elementary, result of Whittle (1959, p.180, equation (44)), according to which the factor $n^{3/2}$ in (2.16) can be reduced to $n \log n$, so that (2.17) becomes $O_p\{(n \log n)^{1/2}\}$, but this is not necessary.

Now the function $|M_n(u)|^2 = (\sin^2 \frac{1}{2} nu) / (\sin^2 \frac{1}{2} u)$, $0 < u < 2\pi$ decreases steadily from its absolute maximum of n^2 at $u = 0$ to a minimum of zero at $u = 2\pi/n$. For the derivative of $\log |M_n(u)|^2$ is

$$n \cot \frac{1}{2} nu - \cot \frac{1}{2} u = (2/u) \{\psi(nu) - \psi(u)\},$$

where $\psi(x) = x \cot x$, and

$$\psi'(x) = \frac{1}{2} \operatorname{cosec}^2 x (\sin 2x - 2x) < 0 \quad \text{when } x > 0.$$

Hence for any prescribed δ , such that $(\sin \frac{1}{2} \delta / \frac{1}{2} \delta)^2 > 1/\pi^2$,

$$\max_{|\omega - \omega_0| \geq n^{-1}\delta} |M_n(\omega - \omega_0)|^2 = \frac{\sin^2 \frac{1}{2} \delta}{\sin^2 \frac{1}{2} n^{-1}\delta} \quad (2.18)$$

when n is sufficiently large, since further local maxima of this function must be less than $\operatorname{cosec}^2 \frac{\pi}{n}$. It follows from (2.17) that

$$\begin{aligned} \max_{|\omega - \omega_0| \geq n^{-1}\delta} I_n(\omega) &\leq \frac{1}{2} n^{-1} (A_0^2 + B_0^2) \left(\frac{\sin^2 \frac{1}{2} \delta}{\sin^2 \frac{1}{2} n^{-1}\delta} \right) + O_p(n^{3/4}) \\ &= \frac{1}{2} n (A_0^2 + B_0^2) \left(\frac{\sin \frac{1}{2} \delta}{\frac{1}{2} \delta} \right)^2 \left(\frac{\frac{1}{2} n^{-1}\delta}{\sin \frac{1}{2} n^{-1}\delta} \right)^2 + O_p(n^{3/4}) \end{aligned} \quad (2.19)$$

if $n > n_0(\delta)$, say. Hence

$$p \lim_{n \rightarrow \infty} \{n^{-1} \max_{|\omega - \omega_0| \geq n^{-1}\delta} I_n(\omega)\} = \frac{1}{2} (A_0^2 + B_0^2) \left(\frac{\sin \frac{1}{2} \delta}{\frac{1}{2} \delta} \right)^2$$

$< \frac{1}{2} (A_0^2 + B_0^2) = p \lim_{n \rightarrow \infty} \{n^{-1} I_n(\omega_0)\}$, from (2.14). This implies

$$\lim_{n \rightarrow \infty} P\{ \max_{|\omega - \omega_0| \geq n^{-1}\delta} I_n(\omega) < I_n(\omega_0) \} = 1 ,$$

and therefore

$$\lim_{n \rightarrow \infty} P\{n|\hat{\omega}_n - \omega_0| < \delta\} = 1 . \quad (2.20)$$

As δ can be arbitrarily small, (2.20) is equivalent to $\hat{\omega}_n - \omega_0 = o_p(n^{-1})$.

3. Consistency of estimators of coefficients of sine and cosine terms.

We had $\hat{A}_n = \frac{2}{n} \sum_{t=1}^n X_t \cos \hat{\omega}_n t$, $\hat{B}_n = \frac{2}{n} \sum_{t=1}^n X_t \sin \hat{\omega}_n t$. (2.3)

Theorem 2. Under the conditions of Theorem 1,

$$p \lim_{n \rightarrow \infty} \hat{A}_n = A_0 \quad \text{and} \quad p \lim_{n \rightarrow \infty} \hat{B}_n = B_0. \quad (3.1)$$

Proof

From the definition (2.3),

$$\hat{A}_n + i \hat{B}_n = \frac{2}{n} \sum_{t=1}^n (D_0 e^{i\omega_0 t} + D_0^* e^{-i\omega_0 t} + \epsilon_t) e^{i\hat{\omega}_n t},$$

and so

$$(\hat{A}_n - A_0) + i(\hat{B}_n - B_0) = \frac{2}{n} \{D_0 M_n(\hat{\omega}_n + \omega_0) + D_0^* (M_n(\hat{\omega}_n - \omega_0) - n) + \sum_{t=1}^n \epsilon_t e^{i\hat{\omega}_n t}\}.$$

Thus

$$\begin{aligned} & |(\hat{A}_n - A_0) + i(\hat{B}_n - B_0)| \\ & \leq \frac{2|D_0|}{n} \{ |M_n(\hat{\omega}_n + \omega_0)| + |M_n(\hat{\omega}_n - \omega_0) - n| \} + \frac{2}{n} \left| \sum_{t=1}^n \epsilon_t e^{i\hat{\omega}_n t} \right|. \end{aligned} \quad (3.2)$$

Since $|\hat{\omega}_n - \omega_0| < \delta$ is equivalent to $2\omega_0 - \delta < \hat{\omega}_n + \omega_0 < 2\omega_0 + \delta$, consistency of $\hat{\omega}_n$ clearly gives

$$p \lim_{n \rightarrow \infty} n^{-1} |M_n(\hat{\omega}_n + \omega_0)| = 0. \quad (3.3)$$

Also

$$\begin{aligned}
 M_n(\hat{\omega}_n - \omega_0) - n &= M_n(\hat{\omega}_n - \omega_0) - M_n(0) \\
 &= (\hat{\omega}_n - \omega_0) M'_n(\omega_n^*)
 \end{aligned} \tag{3.4}$$

where $\omega_n^* \in (\omega_0, \hat{\omega}_n)$, by the mean value theorem. But for all ω ,

$$|M'_n(\omega)| = \left| \sum_{t=1}^n t e^{i\omega t} \right| < n^2 .$$

Hence

$$\begin{aligned}
 |n^{-1}(M_n(\hat{\omega}_n - \omega_0) - n)| &< n|\hat{\omega}_n - \omega_0| \\
 &= o_p(1) \quad \text{by Theorem 1 ,}
 \end{aligned}$$

that is,

$$p \lim_{n \rightarrow \infty} |n^{-1}(M_n(\hat{\omega}_n - \omega_0) - n)| = 0 . \tag{3.5}$$

Finally, from (2.16), $|\sum_{t=1}^n \epsilon_t e^{i\hat{\omega}_n t}| = o_p(n^{3/4})$, and so

$$p \lim_{n \rightarrow \infty} \frac{2}{n} \left| \sum_{t=1}^n \epsilon_t e^{i\hat{\omega}_n t} \right| = 0 . \tag{3.6}$$

(3.3), (3.5), (3.6) thus give

$$p \lim_{n \rightarrow \infty} |(\hat{A}_n - A_0) + i(\hat{B}_n - B_0)| = 0 .$$

4. Asymptotic distribution of the estimators.

Define

$$U_n(A, B, \omega) = \sum_{t=1}^n X_t^2 - 2 \sum_{t=1}^n X_t (A \cos \omega t + B \sin \omega t) + \frac{1}{2} n(A^2 + B^2) \quad , \quad (4.1)$$

which is equal to the approximation to the residual sum of squares that is minimised to obtain the estimators $\hat{\omega}_n, \hat{A}_n, \hat{B}_n$. Writing

$$\frac{\partial U_n}{\partial A} = (U_n)_{A_0}, \quad \frac{\partial^2 U_n}{\partial A \partial \omega} = (U_n)_{A\omega}, \quad \text{etc, we have, by the mean value theorem,}$$

$$(U_n)_{A_0} = (U_n)_{A_n^* A_n^*} (A_0 - \hat{A}_n) + (U_n)_{A_n^* B_n^*} (B_0 - \hat{B}_n) + (U_n)_{A_n^* \omega_n^*} (\omega_0 - \hat{\omega}_n) \quad , \quad (4.2)$$

$$(U_n)_{B_0} = (U_n)_{A_n^* B_n^*} (A_0 - \hat{A}_n) + (U_n)_{B_n^* B_n^*} (B_0 - \hat{B}_n) + (U_n)_{B_n^* \omega_n^*} (\omega_0 - \hat{\omega}_n) \quad , \quad (4.3)$$

and

$$(U_n)_{\omega_0} = (U_n)_{A_n^* \omega_n^*} (A_0 - \hat{A}_n) + (U_n)_{B_n^* \omega_n^*} (B_0 - \hat{B}_n) + (U_n)_{\omega_n^* \omega_n^*} (\omega_0 - \hat{\omega}_n) \quad , \quad (4.4)$$

where we use the generic notation $(A_n^*, B_n^*, \omega_n^*)$ for a point on the line joining (A_0, B_0, ω_0) and $(\hat{A}_n, \hat{B}_n, \hat{\omega}_n)$, so that

$$(A_n^*, B_n^*, \omega_n^*) = \lambda(A_0, B_0, \omega_0) + (1-\lambda)(\hat{A}_n, \hat{B}_n, \hat{\omega}_n) \quad , \quad (4.5)$$

where $0 < \lambda < 1$. The fact that the points $(A_n^*, B_n^*, \omega_n^*)$ in (4.2), (4.3) and (4.4) are different will cause no ambiguity.

Now

$$\begin{aligned}
 (U_n)_{A_0} &= n A_0 - 2 \sum_{t=1}^n X_t \cos \omega_0 t \\
 &= n A_0 - 2 \sum_{t=1}^n (A_0 \cos^2 \omega_0 t + B_0 \sin \omega_0 t \cos \omega_0 t + \epsilon_t \cos \omega_0 t) \quad (4.6) \\
 &= -2 \sum_{t=1}^n \epsilon_t \cos \omega_0 t + o(1) .
 \end{aligned}$$

Similarly

$$(U_n)_{B_0} = -2 \sum_{t=1}^n \epsilon_t \sin \omega_0 t + o(1) , \quad (4.7)$$

and

$$(U_n)_{\omega_0} = 2 \sum_{t=1}^n \epsilon_t t (A_0 \sin \omega_0 t - B_0 \cos \omega_0 t) + o(n) . \quad (4.8)$$

The sums in (4.6) - (4.8) are all of the form $\sum_{t=1}^n k_t \epsilon_t$, where

$$\lim_{n \rightarrow \infty} \max_{1 \leq t \leq n} |k_t| / \left(\sum_{t=1}^n k_t^2 \right)^{1/2} = 0 . \quad (4.9)$$

For example, with (4.8)

$$\begin{aligned}
 \sum_{t=1}^n k_t^2 &= 4 \sum_{t=1}^n t^2 \left\{ \frac{1}{2} A_0^2 (1 - \cos 2\omega_0 t) + \frac{1}{2} B_0^2 (1 + \cos 2\omega_0 t) - A_0 B_0 \sin 2\omega_0 t \right\} \\
 &= 2(A_0^2 + B_0^2) \sum_{t=1}^n t^2 + o(n^2) = \frac{2}{3} n^3 (A_0^2 + B_0^2) + n^2 .
 \end{aligned}$$

It follows that the central limit theorem will apply to these. For (4.9) implies the Lindeberg condition (see, for example, Rao, 1965, p.108),

since, writing

$$\eta_t = k_t \epsilon_t, \quad \sigma_n = \left(v \sum_{t=1}^n k_t^2 \right)^{1/2},$$

and G_t for the distribution function of η_t , we have

$$\sum_{t=1}^n \int_{|\eta| > \delta \sigma_n} \eta^2 dG_t(\eta) / \sigma_n^2 = \frac{1}{\sigma_n^2} \sum_{t=1}^n k_t^2 \int_{|\epsilon| > \delta \sigma_n / |k_t|} \epsilon^2 dF(\epsilon),$$

where F is the distribution function of ϵ_t , and so does not exceed $\int_{|\epsilon| > \delta \sigma_n / \max |k_t|} \epsilon^2 dF(\epsilon)$, which tends to zero when $n \rightarrow \infty$. Thus $n^{-1/2}(U_n)_{A_0}$, $n^{-1/2}(U_n)_{B_0}$, $n^{-3/2}(U_n)_{\omega_0}$ converge in law respectively to $N(0, 2v)$, $N(0, 2v)$, and $N(0, \frac{2}{3}(A_0^2 + B_0^2)v)$ when $n \rightarrow \infty$.

For the limiting joint distribution we consider the random variable $V_n(\lambda_1, \lambda_2, \lambda_3) = \lambda_1 n^{-1/2}(U_n)_{A_0} + \lambda_2 n^{-1/2}(U_n)_{B_0} + \lambda_3 n^{-3/2}(U_n)_{\omega_0}$, where the λ_i are arbitrary real numbers. Now

$$V_n = 2 \sum_{t=1}^n \epsilon_t \{ \lambda_3 n^{-3/2} (A_0 \sin \omega_0 t - B_0 \cos \omega_0 t) - n^{-1/2} (\lambda_1 \cos \omega_0 t + \lambda_2 \sin \omega_0 t) \} + o(n^{-1/2}), \quad (4.10)$$

and the sum in (4.10) is of the form $\sum_{t=1}^n k_{n,t} \epsilon_t$ where

$$\lim_{n \rightarrow \infty} \max_{1 \leq t \leq n} |k_{n,t}| / \left(\sum_{t=1}^n k_{n,t}^2 \right)^{1/2} = 0, \quad (4.11)$$

the numerator in (4.11) being $o(n^{-1/2})$ and the denominator $O(1)$.

Hence the central limit theorem will apply to this sum also by using the generalised Lindeberg condition (see, for example, Loève, 1960, p.295), which is implied by (4.11) in exactly the same way as the

ordinary Lindeberg condition is implied by (4.9). Now a straight-forward calculation shows that

$$\lim_{n \rightarrow \infty} \sum_{t=1}^n k_{n,t}^2 = 2(\lambda_1^2 + \lambda_2^2) + \frac{2}{3}(A_0^2 + B_0^2)\lambda_3^2 + 2B_0\lambda_1\lambda_3 - 2A_0\lambda_2\lambda_3 \quad (4.12)$$

Thus V_n converges in law to a normal distribution with mean zero and variance (4.12). Consequently, by using the equivalence of convergence in law and pointwise convergence of characteristic functions, (see, for example, Rao, 1965, p.103), we see that the joint distribution of $n^{-1/2}(U_n)_{A_0}$, $n^{-1/2}(U_n)_{B_0}$, $n^{-3/2}(U_n)_{\omega_0}$ converges in law to

$$N((0,0,0), 2vW) \quad (4.13)$$

where

$$W = \begin{pmatrix} 1 & 0 & \frac{1}{2} B_0 \\ 0 & 1 & -\frac{1}{2} A_0 \\ \frac{1}{2} B_0 & -\frac{1}{2} A_0 & \frac{1}{3}(A_0^2 + B_0^2) \end{pmatrix} \quad (4.14)$$

Next, we look at the behaviour of the second partial derivatives occurring on the right-hand sides of (4.2) - (4.4). Three of these require no analysis, as

$$(U_n)_{AA} = (U_n)_{BB} = n, \quad (U_n)_{AB} = 0 \quad (4.15)$$

Now $(U_n)_{A\omega} = 2 \sum_{t=1}^n X_t t \sin \omega t$, and so

$$\begin{aligned} (U_n)_{A^*\omega^*} &= 2 \sum_{t=1}^n (A_0 \cos \omega_0 t + B_0 \sin \omega_0 t + \epsilon_t) t \sin \omega^* t \\ &= B_0 \sum_{t=1}^n t \cos(\omega_n^* - \omega_0) t + 2 \sum_{t=1}^n \epsilon_t t \sin \omega_n^* t + o(n) \quad (4.16) \end{aligned}$$

From Theorem 1, $\hat{\omega}_n - \omega_0 = o_p(n^{-1})$, and so, from (4.5), we also have $\omega_n^* - \omega_0 = o_p(n^{-1})$. Hence

$$\begin{aligned} \sum_{t=1}^n t e^{i(\omega_n^* - \omega_0)t} &= M_n'(\omega_n^* - \omega_0) \\ &= M_n'(0) + (\omega_n^* - \omega_0) M_n''\{\lambda(\omega_n^* - \omega_0)\}, \quad 0 < \lambda < 1, \\ &= \frac{1}{2} n(n+1) + o_p(n^2), \end{aligned}$$

since $|M_n''(\omega)| \leq \sum_{t=1}^n t^2 \sim \frac{1}{3} n^3$ for all ω , and so

$$p \lim_{n \rightarrow \infty} n^{-2} \sum_{t=1}^n t \cos(\omega_n^* - \omega_0)t = \frac{1}{2}. \quad (4.17)$$

Also, employing an argument similar to that following equation (2.15) in §2, we have

$$\begin{aligned} \left| \sum_{t=1}^n \epsilon_t t e^{i\omega t} \right|^2 &= \sum_{|s| \leq n-1} e^{i\omega s} \sum_{t=1}^{n-|s|} \epsilon_t \epsilon_{t+|s|} t(t+|s|) \\ &\leq \sum_{|s| \leq n-1} \left| \sum_{t=1}^{n-|s|} \epsilon_t \epsilon_{t+|s|} t(t+|s|) \right|, \end{aligned}$$

so that

$$\begin{aligned} E\left\{ \max_{0 \leq \omega \leq \pi} \left| \sum_{t=1}^n \epsilon_t t e^{i\omega t} \right|^2 \right\} &\leq E\left(\sum_{t=1}^n t \epsilon_t^2 \right) + 2 \sum_{s=1}^{n-1} [E\left\{ \left(\sum_{t=1}^{n-s} \epsilon_t \epsilon_{t+s} t(t+s) \right)^2 \right\}]^{1/2} \\ &= \frac{1}{2} n(n+1) + 2 \sum_{s=1}^{n-1} \left\{ \sum_{t=1}^{n-s} t^2 (t+s)^2 \right\}^{1/2} \\ &< \frac{1}{2} n(n+1) + 2(n-1) n^{5/2} < 3n^{7/2}. \end{aligned} \quad (4.18)$$

((4.18) is a rather crude inequality, but it suffices here.) It follows from (4.18) that

$$\left| \sum_{t=1}^n \epsilon_t t \sin \hat{\omega}_n t \right| \leq \left\{ \max_{0 \leq \omega \leq \pi} \left| \sum_{t=1}^n \epsilon_t t e^{i\omega t} \right|^2 \right\}^{1/2} = o_p(n^{7/4}) \quad (4.19)$$

Hence, from (4.16), (4.17), and (4.19),

$$p \lim_{n \rightarrow \infty} n^{-2} (U_n)_{A^* \omega_n^*} = \frac{1}{2} B_0 \quad (4.20)$$

Similarly, since

$$\begin{aligned} (U_n)_{B^* \omega_n^*} &= -2 \sum_{t=1}^n (A_0 \cos \omega_0 t + B_0 \sin \omega_0 t + \epsilon_t) t \cos \omega_n^* t \\ &= -A_0 \sum_{t=1}^n t \cos(\omega_n^* - \omega_0) t - 2 \sum_{t=1}^n \epsilon_t t \cos \omega_n^* t + o(n) \quad , \end{aligned}$$

we obtain

$$p \lim_{n \rightarrow \infty} n^{-2} (U_n)_{B^* \omega_n^*} = -\frac{1}{2} A_0 \quad (4.21)$$

Finally,

$$\begin{aligned} (U_n)_{\omega_n^* \omega_n^*} &= 2 \sum_{t=1}^n X_t t^2 (A_n^* \cos \omega_n^* t + B_n^* \sin \omega_n^* t) \\ &= 2 \sum_{t=1}^n t^2 (A_0 \cos \omega_0 t + B_0 \sin \omega_0 t) (A_n^* \cos \omega_n^* t + B_n^* \sin \omega_n^* t) \\ &\quad + 2 A_n^* \sum_{t=1}^n \epsilon_t t^2 \cos \omega_n^* t + 2 B_n^* \sum_{t=1}^n \epsilon_t t^2 \sin \omega_n^* t \quad (4.22) \end{aligned}$$

We can show that

$$E\left\{ \max_{0 \leq \omega \leq \pi} \left| \sum_{t=1}^n \epsilon_t t^2 e^{i\omega t} \right|^2 \right\} = o(n^{11/2})$$

in exactly the same way as we obtained (4.18), and hence, (4.5) giving consistency of A_n^* and B_n^* , that the last two terms of (4.22) are each $O_p(n^{11/4})$. Also the first term of (4.22)

$$\begin{aligned} &= \sum_{t=1}^n t^2 \{A_0 A_n^* \cos(\omega_n^* - \omega_0)t + B_0 B_n^* \sin(\omega_n^* - \omega_0)t\} + O_p(n^2) \\ &= \frac{1}{3} n^3 (A_0^2 + B_0^2) + o_p(n^3) \end{aligned} \quad (4.23)$$

since

$$\begin{aligned} \sum_{t=1}^n t^2 e^{i(\omega_n^* - \omega_0)t} &= M_n''(\omega_n^* - \omega_0) \\ &= M_n''(0) + (\omega_n^* - \omega_0) M_n''[\lambda(\omega_n^* - \omega_0)], \quad 0 < \lambda < 1. \end{aligned}$$

Hence

$$p \lim_{n \rightarrow \infty} n^{-3} (U_n)_{\omega_n^* \omega_n^*} = \frac{1}{3} (A_0^2 + B_0^2). \quad (4.24)$$

Thus if

$$W_n^* = \begin{pmatrix} n^{-1}(U_n)_{A_n^* A_n^*} & n^{-1}(U_n)_{A_n^* B_n^*} & n^{-2}(U_n)_{A_n^* \omega_n^*} \\ n^{-1}(U_n)_{A_n^* B_n^*} & n^{-1}(U_n)_{B_n^* B_n^*} & n^{-2}(U_n)_{B_n^* \omega_n^*} \\ n^{-2}(U_n)_{A_n^* \omega_n^*} & n^{-2}(U_n)_{B_n^* \omega_n^*} & n^{-3}(U_n)_{\omega_n^* \omega_n^*} \end{pmatrix}, \quad (4.25)$$

we have, from (4.15), (4.20), (4.21) and (4.24),

$$p \lim_{n \rightarrow \infty} W_n^* = W. \quad (4.26)$$

We can now easily establish the following result.

Theorem 3. Under the conditions of Theorem 1,
 $\{n^{1/2}(\hat{A}_n - A_0), n^{1/2}(\hat{B}_n - B_0), n^{3/2}(\hat{\omega}_n - \omega_0)\}$ converges in law to

$$N(\underline{0}, 2vW^{-1}) \quad ; \quad (4.27)$$

when $n \rightarrow \infty$, where $\underline{0}$ denotes the row vector $(0,0,0)$.

Proof. Writing (4.2) as

$$\begin{aligned} n^{-1/2}(U_n)_{A_0} &= n^{-1}(U_n)_{A_n^* A_n^*} n^{1/2}(A_0 - \hat{A}_n) + n^{-1}(U_n)_{A_n^* B_n^*} n^{1/2}(B_0 - \hat{B}_n) \\ &\quad + n^{-2}(U_n)_{A_n^* \omega_n^*} n^{3/2}(\omega_0 - \hat{\omega}_n) \quad , \end{aligned}$$

and (4.3), (4.4) similarly, we see that

$$\begin{aligned} &\{n^{-1/2}(U_n)_{A_0}, n^{-1/2}(U_n)_{B_0}, n^{-3/2}(U_n)_{\omega_0}\} \\ &= - \{n^{1/2}(\hat{A}_n - A_0), n^{1/2}(\hat{B}_n - B_0), n^{3/2}(\hat{\omega}_n - \omega_0)\} W_n^* \quad . \end{aligned}$$

Thus

$$\begin{aligned} &\{n^{1/2}(\hat{A}_n - A_0), n^{1/2}(\hat{B}_n - B_0), n^{3/2}(\hat{\omega}_n - \omega_0)\} \\ &= \{n^{-1/2}(U_n)_{A_0}, n^{-1/2}(U_n)_{B_0}, n^{-3/2}(U_n)_{\omega_0}\} (W_n^*)^{-1} \quad . \quad (4.28) \end{aligned}$$

Hence, as $\{n^{-1/2}(U_n)_{A_0}, n^{-1/2}(U_n)_{B_0}, n^{-3/2}(U_n)_{\omega_0}\}$ converges in law to $N(\underline{0}, 2vW)$, equation (4.13), (4.27) follows from an obvious generalisation of an elementary limit theorem namely that if Y_n, Y are row vector-valued and Z_n matrix-valued random variables such that when $n \rightarrow \infty$,

$Y_n \xrightarrow{L} Y$, $p \lim Z_n = C$, then $Y_n Z_n \xrightarrow{L} YC$ (\xrightarrow{L} denoting convergence in law), whenever the products are defined. (Compare, for example, Rao, p.102 (xb)).

(4.27) is the rigorous statement of the result (1.5) for $q=1$ and $g(\omega, \theta) = 1$ as a limit theorem. The explicit formula for the inverse of W is

$$W^{-1} = \frac{1}{A_0^2 + B_0^2} \begin{pmatrix} A_0^2 + 4B_0^2 & -3A_0B_0 & -6B_0 \\ -3A_0B_0 & 4A_0^2 + B_0^2 & 6A_0 \\ -6B_0 & 6A_0 & 12 \end{pmatrix}. \quad (4.29)$$

An equivalent way of stating the result of Theorem 3 is therefore that the distribution of $(\hat{A}_n, \hat{B}_n, \hat{\omega}_n)$ is asymptotically normal with mean $(0,0,0)$ and covariance matrix

$$\frac{2v}{A_0^2 + B_0^2} \begin{pmatrix} n^{-1}(A_0^2 + 4B_0^2) & -3n^{-1}A_0B_0 & -6n^{-2}B_0 \\ -3n^{-1}A_0B_0 & n^{-1}(4A_0^2 + B_0^2) & 6n^{-2}A_0 \\ -6n^{-2}B_0 & 6n^{-2}A_0 & 12n^{-3} \end{pmatrix}. \quad (4.30)$$

The most notable feature of (4.30) is the very rapid decrease of the asymptotic variance of $\hat{\omega}_n$, $24v/\{n^3(A_0^2 + B_0^2)\}$, when n increases. In fact in maximum likelihood estimation it is very rare to obtain an asymptotic variance of order n^{-3} , although asymptotic variances of n^{-2} are not unusual. This phenomenon is due to the sharpness of the largest peak of the periodogram intensity function I_n . We might consequently expect the asymptotic variances of \hat{A}_n and \hat{B}_n to be the same as if the frequency ω_0 were known, but we see from (4.30) that

this is not so. For example, the asymptotic variance of \hat{A}_n is $2v(A_0^2 + 4B_0^2)/\{n(A_0^2 + B_0^2)\}$, which is greater than $2v/n$, the asymptotic variance of $\frac{2}{n} \sum_{t=1}^n X_t \cos \omega_0 t$, unless $B_0 = 0$.

5. The case of several harmonic components.

Suppose now that

$$m_t = E(X_t) = \sum_{r=1}^q (A_{r,0} \cos \omega_{r,0} t + B_{r,0} \sin \omega_{r,0} t) , \quad (5.1)$$

$A_{r,0}$, $B_{r,0}$ and $\omega_{r,0}$ denoting the true values of the parameters. The function corresponding to (2.2) whose minimisation yields estimators $\hat{A}_{r,n}$, $\hat{B}_{r,n}$, $\hat{\omega}_{r,n}$ ($1 \leq r \leq q$) then becomes

$$C_0 - 2n^{-1} \sum_{r=1}^q \sum_{t=1}^n X_t (A_r \cos \omega_r t + B_r \sin \omega_r t) + \frac{1}{2} \sum_{r=1}^q (A_r^2 + B_r^2) \dots \quad (5.2)$$

Thus

$$\hat{A}_{r,n} = \frac{2}{n} \sum_{t=1}^n X_t \cos \hat{\omega}_{r,n} t , \quad \hat{B}_{r,n} = \frac{2}{n} \sum_{t=1}^n X_t \sin \hat{\omega}_{r,n} t , \quad (5.3)$$

and if we write

$$\underline{\omega} = (\omega_1, \omega_2, \dots, \omega_q) , \quad \underline{\hat{\omega}}_n = (\hat{\omega}_{1,n}, \hat{\omega}_{2,n}, \dots, \hat{\omega}_{q,n}) ,$$

$$\varphi_n(\underline{\omega}) = \sum_{r=1}^q I_n(\omega_r) \quad (5.4)$$

is a maximum when $\underline{\omega} = \underline{\hat{\omega}}_n$.

Here, however, since terms of the form $A_r A_s \sum_{t=1}^n \cos \omega_r t \cos \omega_s t$ and $B_r B_s \sum_{t=1}^n \sin \omega_r t \sin \omega_s t$ have been dropped in obtaining (5.2) from the residual sum of squares

$$\sum_{t=1}^n \left\{ X_t - \sum_{r=1}^q (A_r \cos \omega_r t + B_r \sin \omega_r t) \right\}^2 ,$$

the maximisation of (5.4) cannot be unrestricted; some condition must

be imposed to keep the ω_r from being too close together and so prevent two angular frequency estimators from converging in probability to the same value. In fact unrestricted maximisation obviously makes the $\hat{\omega}_{r,n}$ all equal to the angular frequency for which I_n attains its absolute maximum, and this will converge in probability to the $\omega_{r,0}$ for which the corresponding amplitude $(A_{r,0}^2 + B_{r,0}^2)^{1/2}$ is largest. The required condition is

$$\lim_{n \rightarrow \infty} \min_{1 \leq r \neq s \leq q} n |\omega_r - \omega_s| = \infty. \quad (5.5)$$

We might therefore, for example, minimise (5.4) subject to

$$\min_{r \neq s} |\omega_r - \omega_s| = n^{-1/2}. \quad (5.6)$$

When (5.5) holds, then in the relevant domain, S_n say, in $\underline{\omega}$ space of the function φ_n , only q of the q^2 differences $\omega_s - \omega_{r,0}$ can be $O(n^{-1})$. If we label the components of the argument of φ so that these differences are $\omega_r - \omega_{r,0}$, we see that the behaviour of

$$\frac{1}{2} n \varphi_n(\underline{\omega}) = \sum_{r=1}^q \left| \sum_{s=1}^q \{ D_{s,0} M_n(\omega_r + \omega_{s,0}) + D_{s,0}^* M_n(\omega_r - \omega_{s,0}) \} + \sum_{t=1}^n \epsilon_t e^{i \omega_r t} \right|^2,$$

where

$$D_{s,0} = \frac{1}{2} (A_{s,0} - i B_{s,0}), \quad D_{s,0}^* = \frac{1}{2} (A_{s,0} + i B_{s,0})$$

is controlled by the sum of terms $|D_{r,0}^* M_n(\omega_r - \omega_{r,0})|^2$ when $\omega_r - \omega_{r,0}$ ($1 \leq r \leq q$) are small. In fact, we can show, just as in §2, that if we take a sequence of sets $\{S_n\}$ for which (5.5) holds, then

$$\max_{\underline{\omega} \in S_n} \left| \varphi_n(\underline{\omega}) - \frac{1}{2} n^{-1} \sum_{r=1}^q (A_{r,0}^2 + B_{r,0}^2) |M_n(\omega_r - \omega_{r,0})|^2 \right| = o_p(n^{3/4}) .$$

It will follow that if $R_{n,\delta} = \{\underline{\omega}; |\omega_r - \omega_{r,0}| \leq n^{-1}\delta, 1 \leq r \leq q\}$ is contained in S_n , which will be true for sufficiently large n , and $R_{n,\delta}^{(c)} = S_n - R_{n,\delta}$ denotes its complement with respect to S_n , then for sufficiently small δ ,

$$p \lim_{n \rightarrow \infty} \{n^{-1} \max_{\underline{\omega} \in R_{n,\delta}^{(c)}} \varphi_n(\underline{\omega})\} = \frac{1}{2} \sum_{r=1}^q (A_{r,0}^2 + B_{r,0}^2) \left(\frac{\sin \frac{1}{2} \delta}{\frac{1}{2} \delta} \right)^2 . \quad (5.7)$$

From (5.7) it is easily deduced that

$$p \lim_{n \rightarrow \infty} n(\hat{\omega}_{r,n} - \omega_{r,0}) = 1 . \quad (5.8)$$

Since φ_n is symmetrical in its q arguments, a means of determining which component of $\hat{\underline{\omega}}$ is associated with a particular frequency has to be found. We can, however, obtain this from the fact that

$$p \lim_{n \rightarrow \infty} n^{-1} I_n(\hat{\omega}_{r,n}) = \frac{1}{2} (A_{r,0}^2 + B_{r,0}^2) , \quad (5.9)$$

which is fairly readily demonstrated by using Taylor's Theorem and (5.8).

If therefore the $\omega_{r,0}$ are labelled so that

$A_{1,0}^2 + B_{1,0}^2 \geq A_{2,0}^2 + B_{2,0}^2 \geq \dots \geq A_{q,0}^2 + B_{q,0}^2$, then with probability tending to unity as $n \rightarrow \infty$,

$$I_n(\hat{\omega}_{1,n}) \geq I_n(\hat{\omega}_{2,n}) \geq \dots \geq I_n(\hat{\omega}_{q,n}) .$$

Thus if we determine the $\hat{\omega}_{r,n}$ as the q largest local maxima of the periodogram intensity subject to a separation condition satisfying (5.5)

these will, for sufficiently large n , almost certainly estimate the frequencies of the harmonic components arranged in descending order of magnitude.

We then deduce that

$$p \lim_{n \rightarrow \infty} \hat{A}_{r,n} = A_{r,0}, \quad p \lim_{n \rightarrow \infty} \hat{B}_{r,n} = B_{r,0}$$

by an argument of the type used in §3.

Finally, if we denote (5.2) multiplied by n by $U_n(\underline{A}, \underline{B}, \underline{\omega})$, where $\underline{A} = (A_1, A_2, \dots, A_q)$, $\underline{B} = (B_1, B_2, \dots, B_q)$, (compare equation (4.1)), we see that $U_n(\underline{A}, \underline{B}, \underline{\omega})$ is of the form

$$\sum_{t=1}^n X_t^2 + \sum_{r=1}^n f_{n,r}(A_r, B_r, \omega_r, \underline{X}^{(n)}) ,$$

so that applying the mean value theorem as in §4 gives us q sets of equations each of the form (4.2) - (4.4), namely

$$\begin{aligned} & \{n^{1/2}(\hat{A}_{r,n} - A_{r,0}), n^{1/2}(\hat{B}_{r,n} - B_{r,0}), n^{3/2}(\hat{\omega}_{r,n} - \omega_{r,0})\} \\ & = -\{n^{-1/2}(U_n)_{A_{r,0}}, n^{-1/2}(U_n)_{B_{r,0}}, n^{-3/2}(U_n)_{\omega_{r,0}}\} (W_{r,n}^*)^{-1}, \quad (5.10) \end{aligned}$$

in an obvious notation. From results of the type (4.6) - (4.8), for example,

$$(U_n)_{A_{r,0}} = -2 \sum_{t=1}^n \epsilon_t \cos \omega_{r,0} t + o(1) ,$$

and an application of the central limit theorem as in §4, it follows that the row vectors on the right-hand sides of (5.10) are asymptotically distributed independently as $N(\underline{0}, 2vW_r)$, where the matrix W_r is

obtained by replacing A_0, B_0 in the expression (4.14) for W respectively by $A_{r,0}, B_{r,0}$. Again as in §4, we can show that

$$p \lim_{n \rightarrow \infty} W_{r,n}^* = W_r,$$

and so we reach the conclusion that the row vectors on the left-hand sides of (5.10) are asymptotically distributed independently as $N(0, 2vW_r^{-1})$. This is the required generalisation of Theorem 3.

In practice the determination of the estimators $\hat{\omega}_{r,n}$ could be very troublesome because of the difficulty of maximising $\phi_n(\omega)$ subject to a restriction such as (5.6) and the awkward problem of the appropriate choice of an appropriate minimum separation to be used for a particular set of data would also arise. We shall not consider such questions here as our purpose is restricted to the rigorous derivation of asymptotic properties. We note, however, that an asymptotically equivalent procedure would be to determine $\hat{\omega}_{1,n}$ by maximising $I_n(\omega)$ unconditionally, then $\hat{\omega}_{2,n}$ by maximising unconditionally $|\sum_{n=1}^t (X_t - \hat{A}_{1,n} \cos \hat{\omega}_{1,n} t - \hat{B}_{1,n} \sin \hat{\omega}_{1,n} t) e^{i\omega t}|^2$, where $\hat{A}_{1,n}, \hat{B}_{1,n}$ are obtained from (5.3), and so on, $\hat{\omega}_{q,n}$ being finally determined by maximising

$$|\sum_{n=1}^t \{X_t - \sum_{r=1}^{q-1} (\hat{A}_{r,n} \cos \hat{\omega}_{r,n} t + \hat{B}_{r,n} \sin \hat{\omega}_{r,n} t)\} e^{i\omega t}|^2.$$

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13. ABSTRACT Let $\{X_t, t=0, \pm 1, \pm 2, \dots\}$ be a discrete parameter time series generated by a model such that $E(X_t)$ is the sum of a finite number of simple harmonic terms of the form $A \cos \omega t + B \sin \omega t$, and $X_t - E(X_t)$ is a moving average $\sum_{u=0}^{\infty} g_u(\theta) \epsilon_{t-u}$ of independently identically distributed random variables ϵ_t , whose weights $g_u(\theta)$ are specified functions of a vector-valued parameter θ . In 1952 P. Whittle proposed an approximate least-squares method of simultaneously estimating θ and the angular frequencies, sine and cosine coefficients of each harmonic term from observations (X_1, X_2, \dots, X_n) , and derived heuristically the asymptotic ($n \rightarrow \infty$) distribution of the estimators. This paper presents rigorous proofs of Whittle's statements concerning the asymptotic distribution, formulated precisely as limit theorems, for the simpler but important case where the moving average reduces to ϵ_t , so that the parameter θ disappears. Proofs for the general case will be given in a subsequent paper.		

14. KEY WORDS Stationary Time Series Hidden Periodicity Harmonic Analysis	LINK A		LINK B		LINK C	
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