Division of Engineering
BROWN UNIVERSITY
PROVIDENCE, R. I.

ON THE KINEMATICS OF FINITE ELASTIC-PLASTIC DEFORMATION

L. B. FREUND

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by

L. B. Freund
Interdisciplinary Fellow in Engineering

BROWN UNIVERSITY
Division of Engineering
Providence, Rhode Island 02912

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ABSTRACT

The geometric and kinematic description of the motion of an elastic-plastic body undergoing finite deformation is considered. A configuration distinct from the initial and current configurations of the body is introduced, which is supposed to be the state of the body due to its plastic deformation alone, and also to be the reference configuration for elastic deformation. It is shown that the rate of deformation tensor may then be written as the sum of the rates of elastic and plastic deformation. Restrictions imposed by some physical assumptions are considered, and the form of some of the derived relations is given for two-dimensional deformations in Cartesian and cylindrical coordinates.

[redacted]

NOTATION

$a^i{}_K$, $b^i{}_\alpha$, ${}^o b^\alpha{}_K$	-	deformation tensors
${}_oB_{KL}$, $B_{\alpha\beta}$, C_{KL}	-	Green's tensors
D_{ij}	-	stretching
E_{KL}	-	strain
${}_oG_{KL}$, $G_{\alpha\beta}$, g_{ij}	-	metric tensor
${}_oG^{KL}$, $G^{\alpha\beta}$, g^{ij}	-	associated metric tensor
$g^i{}_\alpha$, $g^\alpha{}_K$, . . .	-	Euclidean shifters
J, J_o	-	measure of dilation
$p^K{}_i$, $q^K{}_\alpha$, ${}^o q^\alpha{}_i$	-	inverse deformation tensors
R_o, R, r	-	polar coordinates
S_o, S, s	-	arc length
t	-	time
X_o^K	-	material coordinate
X^α	-	reference coordinate
x^i	-	current coordinate
Y_o^K , Y^α , y^i	-	Cartesian coordinates
Z_o, Z, z	-	cylindrical coordinate
$\delta^K{}_L$, $\delta^i{}_j$	-	unit tensors
Γ^i_{jk}	-	Christoffel symbols
ε^{ijk}, ε_{KLM}	-	relative permutation tensors of weight +1 and -1, resp.
Θ_o, Θ, θ	-	cylindrical coordinates
cof ( )	-	cofactor matrix of ( )
comma $(\ )_{,K}$	-	covariant derivative
det ( )	-	determinant
dot $(\dot{\ })$	-	material time derivative
hat $(\hat{\ })$	-	physical component of a tensor
sym ( )	-	symmetric part of ( )
tr ( )	-	trace of ( )

INTRODUCTION

In recent years, interest in problems concerned with large stress, high speed loading of bodies of elastic-plastic material has been generated due to several important applications. Among these are penetration of armor plating by projectiles and the use of explosives in metal forming. Because of the large traction amplitudes and high loading rates involved, it has become necessary to construct constitutive models for such materials for the case of finite deformations. Before the existing theories of plasticity can be generalized to the case of finite deformation, however, certain questions about the geometric and kinematic description must be reconsidered. For example, the most obvious restruction imposed by introduction of finite deformation is that the elastic and plastic strains are no longer simply additive.

Even though consideration here is limited to discussion of kinematics and geometry, the terminology associated with a particular material, namely elastic-plastic, is used. This is because the work here is intended solely for applications involving such materials. As usual, the descriptions elastic and plastic deformation refer to the recoverable and irrecoverable parts, respectively, of a homogeneous deformation of an elastic-plastic body.

As is well known, the description of the motion of an elastic or viscoelastic body may be given by considering the relation between two configurations of the body, the initial and the current configurations. To discuss the motion of an elastic-plastic body it is found to be convenient to introduce a third, intermediate configuration. Thus, in addition to the initial and current configurations,we consider a varying intermediate configuration which is the state the body would be in due to its plastic deformation alone, and which is treated as the reference configuration for elastic deformation.

With the introduction of this intermediate state, it is shown that the total rate of deformation or stretching tensor can be written as the sum of the rates of elastic and plastic deformation. The significance of this is recognized when one considers the usual development of constitutive equations. An appropriate law of elasticity gives the rate of elastic deformation in terms of stress and stress rate. In addition, the rate of plastic deformation is prescribed in terms of stress by an appropriate flow rule. These two terms can then be added to yield the total rate of deformation in terms of stress.

The idea of introducing an intermediate configuration to discuss elastic-plastic deformation was apparently first put forth by Eckart [1] to discuss materials with rate effects. Related discussions concerned with kinematics or particular constitutive laws have been given by Backman [2], Eglit [3], Sedov [4], Truesdell [5], and Truesdell and Toupin [6]. The concepts were finally put to use by Lee and his colleagues, as discussed in a series of papers on the propagation of large amplitude, one-dimensional strain waves [7,8,9].

The object here is to generalize the kinematical concepts given by Lee et al. to the case of orthogonal curvilinear coordinates. To accomplish this, use is made of the standard tensor notation and the theory of double tensor fields, as presented by Ericksen [10]. For a discussion of the general theory it would, of course, be simpler and more reasonable to consider only Cartesian coordinates. We have particular applications in mind, however, which are best suited for curvilinear coordinate systems. The slight extra effort required here will hopefully be justified when the results are applied to these particular problems.

After the general geometric and kinematic relations have been derived, the forms of the assumptions of zero plastic volume change and infinitesimal shear deformations are investigated. Finally, brief remarks are made with regard to two-dimensional deformations in Cartesian and cylindrical coordinate systems.

GEOMETRIC AND KINEMATIC RELATIONS

Consider a simply-connected body of elastic-plastic material, prior to application of any loading, whose material particles may be located by a fixed Cartesian coordinate system. A fixed set of orthogonal curvilinear coordinates $\overset{o}{X}{}^K$ is then prescribed in the region occupied by the body, and each material particle is identified with the coordinate $\overset{o}{X}{}^K$ which it occupies, i.e., the $\overset{o}{X}{}^K$ are material coordinates. Since, in this configuration, the material coordinate system possesses an underlying Cartesian coordinate system the corresponding metric space[1] is Euclidean, and the distance dS_o between points X_o^K and $X_o^K + dX_o^K$ is given by

$$dS_o^2 = {}_oG_{KL}\, dX_o^K\, dX_o^L \quad . \tag{1}$$

The tensor ${}_oG_{KL}$ is the Euclidean metric tensor of the curvilinear coordinate system. If the system itself is rectilinear, this metric is just the identity tensor.

Suppose that now the body is subjected to (time-dependent) loading, resulting in a change of configuration and temperature distribution. It is assumed that the loading is severe enough so that part, and perhaps all, of the body undergoes some plastic deformation. The particle which was originally at X_o^K now moves to the place x^i. The deformation carrying this arbitrary particle from X_o^K to x^i is assumed to satisfy the axiom of continuity, as well as

[1]The word "space" is used here merely to denote the totality of points corresponding to all values of the coordinates within certain ranges. This is to be distinguished from "metric space," which is a space in which the concept of length has been introduced.

the principles of impenetrability and permanence of matter. The deformation is then a uniquely invertible topological mapping, and is described by the differentiable function

$$x^i = x^i(X_o^K, t) \quad , \tag{2}$$

which determines the current configuration of the body. Explicit dependence on time is included because the loading may be time-dependent or the material may be rate-dependent. The motion (2) prescribes the rule not only for the motion of an arbitrary particle, but also for the deformation of the neighborhood of the particle. Thus, if dX_o^K is a line element emanating from X_o^K, it is deformed into the line element dx^i emanating from x^i by the differential relation

$$dx^i = x^i_{,K} \, dX_o^K = a^i{}_K dX_o^K \quad . \tag{3}$$

The tensor $a^i{}_K$ is the deformation gradient.

Since x^i is also a point in the aforementioned space of orthogonal curvilinear coordinates, the metric tensor g_{ij}, which determines distance between x^i and $x^i + dx^i$ by

$$ds^2 = g_{ij} \, dx^i \, dx^j \quad , \tag{4}$$

has the same components as ${}_oG_{KL}$, except that they are evaluated at x^i rather than at X_o^K, i.e., if ${}_oG_{11} = F(X_o^K)$, then $g_{11} = F(x^i)$. The mathematical relation between ${}_oG_{KL}$ and g_{ij} can be given by introduction of the Euclidean shifter. If Y_o^K and y^i are the points in the underlying Cartesian system corresponding to X_o^K and x^i, then there exist invertible coordinate transformations $X_o^K = X_o^K(Y_o^L)$ and $x^i = x^i(y^j)$. The shifters $g^i{}_K$ and $g_i{}^K$ (from X_o^K to x^i) and $g^K{}_i$ and $g_K{}^i$ (from x^i to X_o^K) are then defined as

$$g^{i}{}_{K} = g_{K}{}^{i} = \delta^{j}{}_{L} \frac{\partial x^{i}}{\partial y^{j}} \frac{\partial Y_{o}^{L}}{\partial X_{o}^{K}} \quad , \quad g^{K}{}_{i} = g_{i}{}^{K} = \delta^{L}{}_{j} \frac{\partial X_{o}^{K}}{\partial Y_{o}^{L}} \frac{\partial y^{j}}{\partial x^{i}} \tag{5}$$

where $\delta^{i}{}_{L}$ and $\delta^{L}{}_{i}$ are the shifters for a rectilinear coordinate system and equal 1 or 0 according as the indices are the same or different. The relation between metrics at the two points is then

$$g_{ij} = {}_{o}G_{KL}\, g^{K}{}_{i} g^{L}{}_{j} \quad . \tag{6}$$

Relation (6) may be verified by direct substitution.

In an attempt to separate the elastic deformation from the plastic deformation, we follow Lee and Liu [7] in defining a particular intermediate state. This configuration is the one that the body would be in due to its plastic deformation only, that is, the stresses in the body which resulted in elastic deformation are completely relieved and the local temperature is reduced to its initial value. For a body which has undergone nonhomogeneous plastic deformation this configuration cannot, in general, be achieved; it is a conceptual configuration rather than a physical one. This is due to the fact that when the loads and temperature variations are removed from the body, after resulting in nonhomogeneous plastic deformation, a residual stress and associated elastic strain remain in the body. To reach the desired stress-free configuration, the body must be considered to be cut up into small elements. When a material element is removed from its neighboring elements, which have restrained it, the stress-free state is approached. The small, almost stress-free elements then no longer fit together to form a continuous body. As the size of the little elements approaches zero, the truly stress-free state is approached. Also, as the size of the elements diminishes, the aforementioned geometrical mismatch loses its interpretation. It is replaced by the assumption, however, that in the limit of elements with vanishing dimension the configuration of the body is the result of an incompatible deformation (in the sense of integrability of the deformation tensor) of a continuous body.[2] This intermediate state is called the reference configuration because it is the (time-varying) state of vanishing elastic deformation.

In an attempt to develop the desired relations with a minimum of confusion, notational rules to be followed in selecting indices for the various tensor kernels are introduced. Upper case Latin indices relate to the initial configuration, lower case Greek to the reference configuration, and lower case Latin to the current configuration.

Let X^{α} be the coordinate in the reference configuration of the material particle which was originally at X^K_o. The deformation from the initial to the reference state cannot be written as a differentiable function of material coordinate, as in (2), because of the assumption of incompatibility. Indeed, it is precisely the lack of such a deformation function that is implied by incompatibility. The plastic deformation of a line element dX^K_o emanating from X^K_o in the initial configuration into line element dX^{α} at X^{α} is defined by the deformation tensor ${}^{o}b^{\alpha}{}_{K}$ through

$$dX^K = {}^{o}b^{\alpha}{}_{K}\, dX^K_o \tag{7}$$

Again, because of the assumption of incompatibility, ${}^{o}b^{\alpha}{}_{K}$ is not a matrix of partial derivatives, regardless of the coordinate system being used.

Since the plastic deformation carries the particle with material coordinate X^K_o to the place X^{α}, and the total deformation (2) carries it to x^i, the

[2]The foregoing argument is not a mathematical one, but a physical one leading to mathematical assumptions.

elastic deformation carries it from X^α to x^i. This is a deformation from an incompatible configuration to the compatible current state and, therefore, the tensor describing this deformation is again not a matrix of partial derivatives, regardless of the coordinate system. The elastic deformation of the line element dX^α at X^α into the element dx^i at x^i is defined by the tensor $b^i_{\ \alpha}$ through the non-integrable relation

$$dx^i = b^i_{\ \alpha}\, dX^\alpha \quad . \tag{8}$$

Substituting dX^α from (7) into (8) there results

$$dx^i = b^i_{\ \alpha}\, {}^{o}b^\alpha_{\ K}\, dX^K_o \quad .$$

Comparing this with (3) and recalling the continuity assumption, we get a relation between deformation tensors

$$a^i_{\ K} = b^i_{\ \alpha}\, {}^{o}b^\alpha_{\ K} \quad . \tag{9}$$

Thus, even though the elastic and plastic parts of the deformation are defined by tensors which, in general, are not gradients, the total deformation is described by a tensor, each component of which is a partial derivative. A simple physical example of this phenomenon where incompatible deformations combine to yield a compatible one is easily devised. Consider a plate of elastic-plastic material, out of which we cut a piece of material, say of square shape. Loads are applied to, and then removed from, the cut out piece, resulting in homogeneous plastic deformation and no residual elastic deformation. The plate now resides in a configuration resulting from nonhomogeneous plastic deformation only. This state does not satisfy the continuity axiom, however, because we had to introduce the cut to achieve it. To restore the plate to a continuous configuration the cut out piece is welded back into the vacant square, so that initially adjacent particles are again adjacent, assuming this can be done without causing further plastic deformation. A residual elastic deformation will, in general, result. With this example in mind, an interpretation of (9) is that, even though nonhomogeneous plastic deformation alone leaves a body of elastic-plastic material in a physically impossible configuration, the body takes advantage of its elasticity to relieve this state so that it can reside in a continuous configuration.

The Green's tensors, which are the metric tensors for the various deformed configurations, may now be written in terms of the deformation tensors $a^i{}_K$, $b^i{}_\alpha$, ${}^o b^\alpha{}_K$. Consider first the total deformation. To express the current length of a line element ds in terms of the original element in the initial configuration dX^K_o, (3) is substituted into (4), yielding

$$ds^2 = C_{KL}\, dX^K_o\, dX^L_o \quad , \quad C_{KL} = g_{ij} a^i{}_K a^j{}_L \quad . \tag{10}$$

It is clear from (10) that C_{KL} is a metric tensor on our original point space. Furthermore, it can be shown to be a Euclidean metric, that is, to have a vanishing curvature tensor, by using the fact that $\partial a^i{}_K / \partial X^L_o = \partial a^i{}_L / \partial X^K_o$. In terms of material coordinates, the total strain E_{KL}, defined by

$$ds^2 - dS^2_o = 2E_{KL}\, dX^K_o\, dX^L_o \quad , \tag{11}$$

may be obtained from (1) and (10) as

$$2E_{KL} = C_{KL} - {}_oG_{KL} \quad . \tag{12}$$

As will subsequently be seen, the total strain cannot, without special interpretation, be written as the sum of elastic and plastic strain, as in the case of infinitesimal deformation. In a similar manner, the current length of a line element in terms of the reference configuration is given by

$$ds^2 = B_{\alpha\beta} dX^\alpha dX^\beta \quad , \quad B_{\alpha\beta} = g_{ij} b^i{}_\alpha b^i{}_\beta \quad , \tag{13}$$

and the length of a line element in the reference configuration dS in terms of the initial configuration is

$$dS^2 = {}_oB_{KL} dX_o^K \, dX_o^L \quad , \quad {}_oB_{KL} = G_{\alpha\beta} {}^ob^\alpha{}_K {}^ob^\beta{}_L \quad , \tag{14}$$

where $G_{\alpha\beta} = {}_oG_{KL} g^K{}_\alpha g^L{}_\beta$ and $g^K{}_\alpha$ is the appropriate Euclidean shifter. The two metric tensors are, in general, non-Euclidean metrics. The corresponding elastic and plastic strains are given, respectively, by

$$2E^e_{\alpha\beta} = B_{\alpha\beta} - G_{\alpha\beta} \quad , \quad 2E^p_{KL} = {}_oB_{KL} - {}_oG_{KL} \quad . \tag{15}$$

The fact that the elastic and plastic strains are not additive is readily demonstrated. By using relations (9), (10), (13), (14), (15) it can be shown that

$$C_{KL} = {}_oB_{KL} + 2E^e_{\alpha\beta} \, {}^ob^\alpha{}_K \, {}^ob^\beta{}_L \quad .$$

Subtracting ${}_oG_{KL}$ from both sides and making the substitution, which defines ${}^*b^\alpha{}_K$,

$${}^ob^\alpha{}_K = g^\alpha{}_K + {}^*b^\alpha{}_K$$

we obtain

$$E_{KL} = E^p_{KL} + E^e_{\alpha\beta} g^\alpha{}_K g^\beta{}_L + E^e_{\alpha\beta}(g^\alpha{}_K {}^*b^\beta{}_L + g^\beta{}_L {}^*b^\alpha{}_K + {}^*b^\alpha{}_K {}^*b^\beta{}_L) \ . \tag{16}$$

The left side of (16) is the total strain in terms of the material coordinates.

The first and second terms on the right side are the plastic and elastic strains, respectively, in terms of material coordinates. An additional term appears on the right side because initially the elastic strain is measured with respect to the varying reference configuration. Accordingly, the transformations indicated must be made to express elastic strains in terms of material coordinates. In the case of infinitesimal deformation, this additional term is of "higher order" than the others, the distinction between various configurations becomes negligible, and the total strain is simply the sum of elastic and plastic strains. Green and Naghdi [11] write the finite strain tensor as the sum of an elastic and a plastic part. Their elastic strain is different from ours, that is, it must include all but the first term on the right side of (16) and therefore must depend on the plastic deformation (as Green and Naghdi subsequently assume).

Consideration thus far has been limited to the geometrical properties of the deformation. Attention is now directed toward description of the kinematic features. The fundamental measure of rate of change of configuration will be taken to be the stretching tensor, which defines a measure of the rate of change of current length per unit current length. This tensor then defines a measure of instantaneous rates, and thus depends only on the current configuration.

To proceed it is necessary to introduce a few new tensors, related to those previously defined. These are the inverse deformation tensors $p^K{}_i$, $q^\alpha{}_i$, ${}^o q^K{}_\alpha$ which are defined by the systems of linear equations

$$p^L{}_i a^i{}_K = \delta^L{}_K \quad , \quad q^\beta{}_i b^i{}_\alpha = \delta^\beta{}_\alpha \quad , \quad {}^o q^L{}_\alpha {}^o b^\alpha{}_K = \delta^L{}_K \tag{17}$$

and which define the locally affine transformations

$$dX_o^L = p^L{}_i dx^i \quad , \quad dX^\beta = q^\beta{}_i dx^i \quad , \quad dX_o^L = {}^o q^L{}_\alpha dX^\alpha \quad . \tag{18}$$

These inverse deformation tensors are obtained by solving (17) by Cramer's rule; for example,

$$p^L{}_i = \mathrm{cof}(a^i{}_L)/\det(a^j{}_K) \quad . \tag{19}$$

Also, observation of the fact that the metric tensors do not depend explicitly on time and have vanishing covariant derivatives allows us to treat the metric tensors as constants when performing material differentiation. The vanishing of the covariant derivative of the metric tensor can be proven by direct substitution or by observing its truth in the underlying Cartesian coordinate system and invoking the invariance property of tensor equations.

The tensor D_{ij} is called the stretching tensor and is defined by

$$\dot{\overline{ds^2}} = 2D_{ij} dx^i dx^j \quad , \tag{20}$$

where the dot denotes material derivative. Substitution of (10) into (20) and performance of the indicated differentiation leads to

$$\begin{aligned} D_{ij} &= \frac{1}{2} \dot{C}_{KL} p^K{}_i p^L{}_j \\ &= \mathrm{sym}(g_{ik} \dot{a}^k{}_K p^K{}_j) \end{aligned} \tag{21}$$

The stretching is thus linear in the material derivative of the deformation tensor which implies that it is linear in the gradient of the velocity and, therefore, linear in the velocity itself. We cannot, however, expect to obtain a split of the stretching into a sum of elastic and plastic parts by writing the total particle velocity as a sum of the two particle velocities (which do indeed exist) corresponding to the elastic and plastic parts of the deformation, because these two velocities cannot be written as differentiable fields. The assumption, tacit in the statement following (21), of the existence of the usual kinematic relation stating that the time derivative of the deformation tensor equals the gradient of the velocity is invalid for each of the elastic and plastic deformations separately.

The total stretching tensor can, however, be separated into the sum of an elastic stretching tensor and a plastic stretching tensor. Substitution of (9) into (21) and using relations between metrics, such as (6), leads to the result

$$
\begin{aligned}
D_{ij} &= \operatorname{sym}(g_{ik}\dot{b}^{k}{}_{\alpha}q^{\alpha}{}_{j}) + \operatorname{sym}(g_{ik}b^{k}{}_{\alpha}\,{}^{o}\dot{b}^{\alpha}{}_{L}\,{}^{o}g^{L}{}_{\beta}q^{\beta}{}_{j}) \\
&= \operatorname{sym}(g_{ik}\dot{b}^{k}{}_{\alpha}q^{\alpha}{}_{j}) + g^{\mu}{}_{i}g^{\nu}{}_{j}\operatorname{sym}(G_{\alpha\nu}g^{\alpha}{}_{k}b^{k}{}_{\beta}\,{}^{o}\dot{b}^{\beta}{}_{L}\,{}^{o}q^{L}{}_{\gamma}q^{\gamma}{}_{m}g^{m}{}_{\mu}) \ . \qquad (22)
\end{aligned}
$$

It is clear that the first term on the right side of (22) defines the instantaneous rate of deformation of the current configuration with respect to the reference configuration, while the second term defines the instantaneous rate of deformation of the current configuration due to the changing reference configuration. The former term is therefore the elastic stretching and the latter is the plastic stretching. The plastic stretching takes on a much more complicated form than the elastic because it is actually the part of the instantaneous motion of the current configuration in terms of the instantaneous motion of the reference configuration. The interior term ${}^{o}\dot{b}^{\beta}{}_{L}\,{}^{o}q^{L}{}_{\gamma}$ characterizes the motion of the reference configuration. This term is then contracted with $b^{k}{}_{\beta}$ and $q^{\gamma}{}_{m}$ which deforms the characterization into the current configuration, where the actual measurement is made. Representing the elastic and plastic stretchings by D^{e}_{ij} and D^{p}_{ij}, (22) is written as

$$
D_{ij} = D^{e}_{ij} + D^{p}_{ij} = D^{e}_{ij} + g^{\mu}{}_{i}g^{\nu}{}_{j}D^{p}_{\mu\nu} \quad . \qquad (23)
$$

One additional quantity of interest is the stretching of the reference configuration (i.e., the measure of plastic strain rate). This rate $D^s_{\alpha\beta}$ is defined by

$$\dot{\overline{dS^2}} = 2D^s_{\alpha\beta} dX^\alpha dX^\beta \quad , \tag{24}$$

and is easily determined to be

$$D^s_{\alpha\beta} = \mathrm{sym}(G_{\alpha\gamma} {}^{\circ}\dot{b}^{\gamma}{}_{L} {}^{\circ}q^{L}{}_{\beta}) \quad . \tag{25}$$

Following Truesdell and Toupin, this rate is called the slippage tensor and is different from the plastic stretching above. (The terminology probably stems from the fact that plastic flow is the macroscopic manifestation of progressive microscopic slip, or dislocation motion, on discrete planes in the crystal lattice.)

In summary, expressions for the following rates have been derived:

- D^e_{ij} - measure of instantaneous motion of the current configuration due to elastic deformation
- D^p_{ij} - measure of instantaneous motion of the current configuration due to plastic deformation
- $D^p_{\alpha\beta}$ - measure of instantaneous motion of the current configuration due to plastic deformation, referred to the intermediate configuration
- $D^s_{\alpha\beta}$ - measure of instantaneous motion of the reference configuration.

Furthermore, although only stretching of lineal elements has been considered above, shearing (i.e., instantaneous shear rate) can also be determined from these tensors. For example, let ψ be the angle between elements dx^i_1 and dx^i_2 in the coordinate directions at x^i. For any two elements

$$ds_1 ds_2 \cos\psi = g_{ij} dx^i_1 \, dx^j_2 \quad .$$

Then at any instant $\psi = \pi/2$ and

$$\dot{\psi} = -2D_{ij} \frac{dx_1^i}{ds_1} \frac{dx_2^j}{ds_2} \tag{26}$$

where the element lengths are measured as in (4).

RESTRICTIONS IMPOSED BY PHYSICAL ASSUMPTIONS

In the preceding section the geometric and kinematic features of the deformation of a body of elastic-plastic material, undergoing finite displacements, were considered, without regard for actual deformation processes of such material. We now consider some of the simplifications introduced by applying assumptions resulting from experimental observation.

The most frequent assumption made concerning elastic-plastic deformation is that the volume change per unit volume is recoverable, that is, the volume change due to plastic deformation is zero. To determine the mathematical form of this assumption, consider a small volume element of material dV in the reference configuration which occupied the element dV_o in the initial configuration. In general, these volume elements are related by

$$dV = J_o \, dV_o \tag{27a}$$

where

$$J_o = \left\{\frac{\det (G_{\alpha\beta})}{\det ({}_oG_{KL})}\right\}^{1/2} \det ({}^ob^\gamma{}_M) \quad . \tag{27b}$$

To satisfy the condition of no plastic volume change, set $J_o = 1$. It can also be shown that

$$J_o^2 \equiv III_o \equiv \det ({}_oG^{KL} \, {}_oB_{LM}) \quad ,$$

so that an equivalent condition is $III_o = 1$. Yet another equivalent condition is $tr(G^{\alpha\gamma}D^s_{\gamma\beta}) = 0$.

A second assumption, concerning the elastic or recoverable part of the deformation, might be made. As stated above, the material volume change during elastic-plastic deformation is completely elastic, and we can conceive of deformation processes during which this dilatancy is large. This is not the case, however, for elastic shears. During elastic-plastic deformation of most metals yielding begins long before the elastic shears increase to the point where they must be considered large. It seems that in many applications the assumption that the elastic shears remain small, even though the plastic shears become large, is a reasonable one.

Before proceeding, some sort of meaning should be attached to the description "small." Here we say that a deformation tensor $h^i_{\ \alpha}$ is small if

$$h = \max_{i,\alpha}\left|h^i_{\ \alpha}\right| \quad \text{and} \quad h \ll 1 \ .$$

The above assumption is given a mathematical form by stating that the elastic deformation tensor is a small deviation from a uniform dilation. This is written in terms of physical components of tensors (denoted by hat ^) because $\beta\delta^i_{\ \alpha}$ are the physical components of a deformation tensor for uniform dilation, where β is a scalar function, $0 < \beta < \infty$. The assumption takes the form

$$\hat{b}^i_{\ \alpha} = \beta(\delta^i_{\ \alpha} + \hat{h}^i_{\ \alpha}) \quad , \tag{28}$$

where $h^i_{\ \alpha}$ is small. Without loss of generality, we can take $tr(\hat{h}^i_{\ \alpha}) = 0$.[3]

[3]Note that the corresponding quantity $tr(h^i_{\ \alpha})$ cannot be formed.

[4]Physical components of a deformation tensor measure change in length per unit length, while tensor components measure change in generalized coordinate per unit generalized coordinate [10].

For assume this is not true. Setting

$$\hat{h}^i{}_\alpha = \beta^{-1}[\beta + \frac{1}{3}\,\mathrm{tr}(\hat{h}^i{}_\alpha)]\,{}^*h^i{}_\alpha + \frac{1}{3}\,\delta^i{}_\alpha\,\mathrm{tr}(\hat{h}^i{}_\alpha) \quad ,$$

$$\beta^* = \beta + \frac{1}{3}\,\mathrm{tr}(\hat{h}^i{}_\alpha) \quad ,$$

relation (28) reduces to

$$\hat{b}^i{}_\alpha = \beta^*(\delta^i{}_\alpha + {}^*\hat{h}^i{}_\alpha)$$

where ${}^*h^i{}_\alpha$ is small. Redefining the starred quantities to be their unstarred counterparts, we again arrive at the form (28), with the trace of $\hat{h}^i{}_\alpha$ vanishing. The nine components of $\hat{b}^i{}_\alpha$ are thus replaced by β and the eight independent components of $\hat{h}^i{}_\alpha$.

Reverting to tensor components, (28) becomes

$$b^i{}_\alpha = \beta(g^i{}_\alpha + h^i{}_\alpha) \quad . \tag{29}$$

To determine the volume change of the deformation, suppose that the material volume element dv in the current configuration occupied the element dV_o in the initial configuration; then

$$dv = J\,dV_o \quad , \quad J = \left\{\frac{\det(g_{ij})}{\det({}_oG_{KL})}\right\}^{1/2} \det(a^k{}_M) \quad . \tag{30}$$

Substituting (9) into (30) and assuming zero plastic volume change, J reduces to

$$J = \left\{\frac{\det(g_{ij})}{\det(G_{\alpha\beta})}\right\}^{1/2} \det(b^k{}_\gamma) \quad .$$

Furthermore, substituting (29) into this relation and expanding in a nine-dimensional Taylor series about $h = 0$, we obtain

$$\det(b^i{}_\alpha) = \beta^3\left\{\det(g^i{}_\alpha) + \sum_{i,\alpha=1}^{3} h^i{}_\alpha \operatorname{cof}(g^i{}_\alpha)\right\} + o(h) \quad . \tag{31}$$

A similar expansion about $h = 0$ yields

$$\operatorname{cof}(b^i{}_\alpha) = \beta^2\left\{\operatorname{cof}(g^i{}_\alpha) + \varepsilon_{ijk}\varepsilon^{\alpha\beta\gamma} h^j{}_\beta h^k{}_\gamma\right\} + o(h) \quad . \tag{32}$$

This formula is of use in solving (17) for $g^\beta{}_j$ by Cramer's rule.

SPECIAL DEFORMATIONS

We now list some results for specific coordinate systems and special deformations. In particular, plane strain in Cartesian coordinates and axially symmetric strain in circular cylindrical coordinates are considered.

In Cartesian coordinates the material coordinates of an arbitrary particle of the body are $X^K_o = (X^1_o, X^2_o, X^3_o)$ before deformation begins. At time t this particle is at place x^i, and the fact that the deformation is plane strain (assumed independent of the 3 coordinate) yields the functional relations

$$x^1 = x^1(X^1_o, X^2_o, t) \quad , \quad x^2 = x^2(X^1_o, X^2_o, t) \quad , \quad x^3 = X^3_o \quad .$$

The deformation tensor has the form

$$[a^i{}_K] = \begin{bmatrix} a^1{}_1 & a^1{}_2 & 0 \\ a^2{}_1 & a^2{}_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad . \tag{33}$$

The elastic and plastic deformation tensors b^i_α and ${}^ob^\alpha_K$ have the same form as (33), except for the 33 components. These are not each unity but are related by $b^3_3 \, {}^ob^3_3 = 1$.

The metric tensor and Euclidean shifters reduce to unit tensors for Cartesian coordinate systems.

The condition that the plastic volume change is zero can be stated, from (27b), for this case as

$$ {}^ob^3_3 ({}^ob^1_1 \, {}^ob^2_2 - {}^ob^1_2 \, {}^ob^2_1) = 1 \quad . \tag{35} $$

Relation (35) may be used to eliminate ${}^ob^3_3$ from all other relations. The condition that elastic shears are small becomes

$$ b^i_\alpha = \beta(\delta^i_\alpha + h^i_\alpha) \tag{36a} $$

where

$$ h^1_3 = h^3_1 = h^2_3 = h^3_2 = 0 \quad . \tag{36b} $$

When the motion is prescribed in terms of material coordinates in a Cartesian system the material time derivative reduces to the usual partial derivative with respect to time, and deformation rates may be easily calculated. Also, in Cartesian coordinates, the covariant derivative reduces to the usual partial derivative. For the case of one-dimensional strain the geometric and kinematic relations are given by Lee and Liu [7].

As a second example consider a deformation which is rotationally symmetric about an axis. The circular cylindrical coordinates of an arbitrary particle in the initial configuration are $X^K_o = (R_o, \Theta_o, Z_o)$, and this particle occupies the place $x^i = (r, \theta, z)$ at time t, with the Z_o-axis being the axis of rotational symmetry. The deformation has the form

$$r = r(R_o, Z_o, t) \quad , \quad \theta = \Theta_o \quad , \quad z = z(R_o, Z_o, t) \tag{37}$$

and

$$[a^i{}_K] = \begin{bmatrix} a^1{}_1 & 0 & a^1{}_3 \\ 0 & 1 & 0 \\ a^3{}_1 & 0 & a^3{}_3 \end{bmatrix} . \tag{38}$$

As before, $b^i{}_\alpha$ and ${}^ob^\alpha{}_K$ have the same form as (38) except that the 22 components are not each unity, but that $b^2{}_2 \, {}^ob^2{}_2 = 1$.

The metric tensor and the Euclidean shifters depend on position in this coordinate system, and may be determined as

$$[{}_oG_{KL}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & R_o^2 & 0 \\ 0 & 0 & 0 \end{bmatrix} , \quad g^K{}_i = \left\{\frac{g_{ii}}{{}_oG_{KK}}\right\}^{1/2} \tag{39a}$$

where i and K are not summed.

The condition that there be no volume change due to plastic deformation is determined from (27b) to be

$${}^ob^2{}_2({}^ob^1{}_1 \, {}^ob^3{}_3 - {}^ob^1{}_3 \, {}^ob^3{}_1) = R_o/R \tag{40}$$

where R is the radial coordinate in the intermediate configuration $X^\alpha = (R, \Theta, Z)$.

In this coordinate system, the covariant derivative is written in terms of the Christoffel symbols Γ^i_{jk}. Of the 27 components only three are nonzero, and these are

$$\Gamma^2_{12} = \Gamma^2_{21} = \frac{1}{r} , \quad \Gamma^1_{22} = -r . \tag{41}$$

The material derivative of $b^i_{\ \alpha}$ when considered as a function of X^K_o and t, is

$$\begin{aligned} \dot{b}^i_{\ \alpha} &= \frac{\partial b^i_{\ \alpha}}{\partial t} + \Gamma^i_{jk}\, b^k_{\ \alpha}\, \dot{x}^j - \Gamma^\gamma_{\alpha\beta}\, b^i_{\ \gamma}\, \dot{X}^\beta \\ &= \frac{\partial b^i_{\ \alpha}}{\partial t} + \delta^i_{\ 2}\delta^2_{\ \alpha} b^2_{\ 2}\, \overline{\ell n \frac{r}{R}}^{\bullet} , \end{aligned} \tag{42}$$

with a similar formula holding for ${}^o b^\alpha_{\ K}$. The elastic and plastic stretching tensors may be calculated from (22) for this case to be

$$D^e_{ij} = \text{sym}\{(\delta^k_{\ j} - h^k_{\ \beta} g^\beta_{\ j}) \frac{\partial}{\partial t} [g_{ik}\ \ell n \beta + g_{i\ell} h^\ell_{\ \alpha} g^\alpha_{\ k}] + \delta^2_{\ i}\delta^2_{\ j}\ r^2\ \overline{\ell n \frac{r}{R}}^{\bullet} \} \tag{43a}$$

$$\begin{aligned} D^p_{ij} = \text{sym}\{&\frac{r}{R_o} \text{cof}(g^k_{\ \beta}\, {}^o b^\beta_{\ K}) \frac{\partial}{\partial t} (g^\ell_{\ \alpha}\, {}^o b^\alpha_{\ K})[\delta^k_{\ j} g_{\ell i} + \delta^k_{\ j}(g_{mi} h^m_{\ \alpha} g^\alpha_{\ \ell}) \\ &- g_{\ell i}(h^k_{\ \gamma} g^\gamma_{\ j})] + \delta^2_{\ i}\delta^2_{\ j}\ r^2\ \overline{\ell n \frac{R}{R_o}}^{\bullet} \} \end{aligned} \tag{43b}$$

The conditions of zero plastic volume change and small elastic shear deformation have been used in obtaining (43).

CONCLUSION

We have considered the finite deformation of a body of elastic-plastic material by introducing a particular intermediate configuration, distinct from the initial and current configurations of the deforming body, which is the state of the body due to its plastic deformation alone. The main result is that the total rate of deformation tensor, in an orthogonal curvilinear coordinate system, can be written as the sum of rates of elastic and plastic deformation as shown in (22). The elastic stratching takes on an expected form, while the plastic stretching tensor is not the intuitive rate of deformation (which is called the slippage tensor above). That is, the plastic stretching is not merely a shift to the current configuration of the rate of deformation of the reference configuration. It might be called instead a deformation of this rate, which reduces to a simple shift only when $b^i_{\ \alpha} = g^i_{\ \alpha}$. Whereas a shift leaves the proper numbers and proper vectors of a tensor unaltered, the proper numbers and vectors of $D^p_{i\cdot j}$ are different from those of $D^s_{\alpha\beta}$ and, in general, $D^p_{i\cdot j} \neq D^s_{\alpha\beta} g^\alpha_{\ \cdot i} g^\beta_{\ \cdot j}$.

Having established the additivity of the rates of elastic and plastic deformation, the general scheme for development of proper forms for constitutive equations becomes clear. The particular rates of (22) must be written in terms of material properties, stress, and temperature, and then added to yield a stress-deformation-temperature relation for the material. In a recent paper, Perzyna [12] assumed an additivity resembling that of (22) and from there proceeded to discuss forms for constitutive equations for viscoplastic materials.

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