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THE INFLUENCE OF DAMPING ON THE SNAPPING OF A SHALLOW ARCH UNDER A STEP PRESSURE LOAD[1]

by

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ABSTRACT

The snapping of a simply-supported shallow sinusoidal arch under a sinusoidally distributed step pressure load is considered. In the presence of velocity-dependent damping of any nonzero magnitude, it is shown that there exists no difference between static and dynamic snap loads for arch rises above a certain magnitude and for sufficiently small external disturbances. Below the foregoing value of arch rise snapping is governed entirely by symmetric snap-through. The results obtained herein, when compared to an existing analysis on the subject, imply that a jump in the critical snap-through load occurs at the boundary between the damped and undamped systems. Similar results concerning cylinders and rings are mentioned.

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1. Introduction

In several recent theoretical investigations concerning the dynamic stability of elastic shell-type structures subjected to transient loads, it has been found that stability boundaries corresponding to undamped and slightly damped mathematical structural models differ markedly. For example, Lock [1], who considered the snapping of a geometrically perfect sinusoidal arch under a time-step pressure load, found a dramatic dynamic influence on snap-through for the undamped arch and certain arch rises; on the other hand, his analysis of the same problem assuming a small amount of velocity-dependent damping indicates little or no difference between static and dynamic snap loads in some of the same arch-rise regions. Similar results were observed by Huang and Nachbar [2], who investigated the geometrically imperfect sinusoidal arch with and without material damping of the viscoelastic type. Further information is to be found in a paper by Bieniek, Fan, and Lackman [3], who considered the dynamic stability of a geometrically perfect cylindrical shell under a spatially uniform lateral pressure, applied as a step and ramp-step in time. There it was found that an undamped analysis predicted a significant dynamic effect; the authors' "remark", however, that a small amount of material damping resulted in the same stability boundary for both the dynamically and statically applied loads.

The foregoing investigations indicate the importance of including proper damping mechanisms in a stability analysis involving transiently applied loads. Further, they raise some serious doubts as to the validity (without proper interpretation) of a non-dissipative analysis under such loading conditions. It appears, in fact, that for a certain class of stability problems a jump condition exists in the stability boundary with respect to the damping coefficient, say γ, at $\gamma = 0$.

In this paper we discuss the role of damping in stability problems of the type [1 - 3]. The sinusoidal arch problem of [1] is employed as the vehicle for discussion. This, however, is a matter of convenience. Similar analyses and conclusions, as will be made here, can be made for other geometries and damping mechanisms (e.g., cylinders, spherical caps; viscoelasticity, etc.). It was thought, however, that a specific example would better serve the goal of this article than would a general and perhaps undiscernible analysis. The extension of the concepts put forth here to other systems is considered to be more or less obvious.

2. Formulation

We consider a shallow, simply supported sinusoidal arch (Fig. 1), which is governed by the following nonlinear differential equation of motion:

$$EI \frac{\partial^4}{\partial x^4}(w - w_o) - P\frac{\partial^2 w}{\partial x^2} + \beta\frac{\partial w}{\partial t} + \rho_s h \frac{\partial^2 w}{\partial t^2} = -p(x, t) \qquad (2.1)$$

where

$$P = \frac{EA}{2L}\int_0^L \left[\left(\frac{\partial w}{\partial x}\right)^2 - \left(\frac{\partial w_o}{\partial x}\right)^2\right]dx \quad .$$

Here $w_o(x)$ represents the initial unstressed position of the arch mid-surface and $w(x)$ is the displacement of said midsurface due to the load $p(x, t)$; both $w_o(x)$ and $w(x)$ are measured from the line $z = 0$ (Fig. 1). The quantities ρ_s, h, E, I, A, L denote arch density, thickness, Young's modulus, second moment of area of the arch cross section, cross-sectional area, and length respectively. The term multiplying the constant β is representative of velocity-dependent damping. The coordinate x is illustrated in Fig. 1; t represents time. The notation is that of [1].

In addition to (2.1) we have, for the simply supported sinusoidal arch, the following initial shape and boundary conditions, respectively:

$$w_o = \bar{w}_o \sin\frac{\pi x}{L} \qquad (\bar{w}_o = \text{const.}) \quad , \qquad (2.2)$$

$$w(0, t) = \frac{\partial^2 w}{\partial x^2}(0, t) = w(L, t) = \frac{\partial^2 w}{\partial x^2}(L, t) = 0 \quad . \qquad (2.3)$$

In the sequel the load, $p(x, t)$, is assumed to be of the form

$$p(x, t) = p_1 \left(\sin \frac{\pi x}{L} \right) H(t) \qquad (2.4)$$

where $H(t)$ denotes the Heaviside step function.

The following initial conditions are specified along with equations (2.1) to (2.4):

$$w(x, t_o) = G_1(x) \quad , \quad \frac{\partial w}{\partial t}(x, t_o) = G_2(x) \quad . \qquad (2.5)$$

Equations (2.5) can be physically envisioned as the result of a disturbance which terminates at the time $t = t_o$.

Consider now Figs. 2, 3. In [1] Lock, utilizing the foregoing equations of motion, boundary conditions, and loading function, and by virtue of a numerical integration of the governing equations (following their reduction to two ordinary differential equations by an approximate two-term modal analysis), found the locus of the encircled points of Fig. 2 to be the snap-through boundary[1] with a small value of the damping coefficient, γ, and a small disturbance (initial conditions, (2.5)) of a certain type. (In Fig. 2, e denotes a nondimensional arch rise; $\bar{q}_1$ is the nondimensional load amplitude at which snap-through occurs if p_1 is applied statically; q_1^* is the nondimensional load amplitude leading to snap-through when p_1 is applied as

[1] The concept of dynamic snap-through is defined mathematically in §4.

a step in time: thus $q_1^*/\bar{q}_1$ is the ratio of dynamic to static snap loads). In particular, note that little or no difference between statically and dynamically applied loads is apparent for sufficiently large arch rises.

In this paper we show that, subject to sufficiently small disturbances of a reasonably general class, the snap-through boundary for $\gamma > 0$ is a curve in the $q_1^*/\bar{q}_1$ vs. e-plane (Fig. 2) which is defined by <u>symmetric</u> snap-through up to the intersection with the line $q_1^*/\bar{q}_1 = 1$, and $q_1^*/\bar{q}_1 = 1$ thereafter. This curve is illustrated in Fig. 2 for $\gamma \to 0^+$, and also for γ corresponding to Lock's 1 percent critical damping. Upon comparing the $\gamma = 0^+$ case with the $\gamma = 0$ example of Lock (Fig. 3), one observes a jump in the snap-through boundary at $\gamma = 0$.

In contrast to [1], the analysis of this paper does not rely upon numerical methods, but upon a rigorous stability analysis incorporating the full nonlinear equations of motion. In this respect the work contained herein should complement [1].

3. Mathematical Preliminaries

In this section we collect a few elementary mathematical tools which will be of use in the ensuing analysis.

Norms

Let the norm, $\|\underset{\sim}{z}\|$, of an n-dimensional vector $\underset{\sim}{z}$ be defined as

$$\|\underset{\sim}{z}\| = \sum_{i=1}^{n} |z_i| \tag{3.1}$$

where z_i denote the elements of $\underset{\sim}{z}$. Further, let the norm, $\|\underset{\sim}{B}\|$, of any square $(n \times n)$ matrix $\underset{\sim}{B}$ be defined as

$$\|\underset{\sim}{B}\| = \sum_{\substack{i=1 \\ j=1}}^{n} |b_{ij}| \tag{3.2}$$

where b_{ij} are the elements of $\underset{\sim}{B}$.

Local Stability

Let $\underset{\sim}{z} \equiv 0$ be a solution of the following system of ordinary differential equations:

$$\frac{d\underset{\sim}{z}}{d\tau} = \underset{\sim}{g}(\underset{\sim}{z}, \tau) \quad . \tag{3.3}$$

Here $\underset{\sim}{z}$ is an n-dimensional vector and $\underset{\sim}{g}$ is an n-dimensional vector-valued function. We shall say that $\underset{\sim}{z} \equiv 0$ is (locally) asymptotically stable (in the sense of Liapunov) if 1) for each $\epsilon > 0$ there exists a δ, depending only on ϵ and a constant τ_o, such that $\|\underset{\sim}{z}(\tau)\| < \epsilon$ for all $\tau \geq \tau_o$ provided $\|z(\tau_o)\| < \delta$, and 2) $\|\underset{\sim}{z}(\tau)\| \to 0$ as $\tau \to \infty$.

Comparison Theorem

The following comparison theorem[a] will aid our discussion:

Theorem: Let $\underset{\sim}{z} \equiv 0$ be a solution of

$$\frac{d\underset{\sim}{z}}{d\tau} = \underset{\sim}{A}\,\underset{\sim}{z} + \underset{\sim}{B}(\tau)\underset{\sim}{z} + \underset{\sim}{f}(\underset{\sim}{z}, \tau) \tag{3.4}$$

where $\underset{\sim}{z}$ is an n-dimensional vector, $\underset{\sim}{A}$ is a constant $n \times n$ matrix whose elements depend continuously on τ, and $\underset{\sim}{f}$ denotes a continuous n-dimensional vector valued function of $\underset{\sim}{z}$ and τ. (It will be assumed that $\underset{\sim}{f}$ is a nonlinear function of $\underset{\sim}{z}$, any linear terms being included in $\underset{\sim}{A}$ or $\underset{\sim}{B}$.) Now, if 1) the trivial solution of

$$\frac{d\underset{\sim}{z}}{d\tau} = \underset{\sim}{A}\,\underset{\sim}{z} \tag{3.5}$$

is asymptotically stable, 2) $\underset{\sim}{B}(\tau)$ is impulsively small, i. e.,

$$\int_0^\infty \|\underset{\sim}{B}(\tau)\|\, d\tau < M \tag{3.6}$$

where M is a positive constant, and 3) the function $\underset{\sim}{f}(\underset{\sim}{z}, \tau)$ satisfies the nonlinearity condition

$$\lim_{\|\underset{\sim}{z}\| \to 0} \frac{\|\underset{\sim}{f}(\underset{\sim}{z}, \tau)\|}{\|\underset{\sim}{z}\|} = 0 \tag{3.7}$$

uniformly for $\tau \geq 0$, then the trivial solution of (3.4) is asymptotically stable.

---

[a] Theorems of this type, of which there exist many, are frequently referred to as Poincaré-Liapunov stability theorems. The reader is referred to Struble [4] and Bellman [5] for discussions.

Proof. Any solution of (3.4) satisfies the following integral equation

$$z(\tau) = Y(\tau)\, c + \int_0^{\tau} Y(\tau - s)\{B(s)\, z(s) + f(z(s), s)\}\, ds \qquad (3.8)$$

where $Y(\tau)$ denotes the principal matrix solution of (3.5) and c is an initial vector. The hypothesis 1) implies that

$$\|Y(\tau)\| \le c_1\, e^{-\alpha\tau} \qquad (3.9)$$

where c_1 and α are positive constants. From (3.9) and (3.8) we obtain

$$\|z(\tau)\| \le c_1 \|c\| e^{-\alpha\tau} + c_1 \int_0^{\tau} e^{-\alpha(\tau - s)}\{\|B(s)\|\ \|z(s)\| + \|f(z(s), s)\|\}\, ds$$

or equivalently,

$$\|z(\tau)\|\, e^{\alpha\tau} \le c_1 \|c\| + c_1 \int_0^{\tau} \Big[\|B(s)\|\ \|z(s)\| + \|f(z(s), s)\|\Big] e^{\alpha s}\, ds. \qquad (3.10)$$

Now, the nonlinearity condition (3.7) implies that there exists a positive number δ such that

$$\|f(z, \tau)\| \le \frac{\alpha}{2c_1} \|z\|$$

for $\|z\| < \delta$ and for all $\tau \ge 0$. Suppose $\|c\| < \delta/c_1$. Then, in view of the continuity of f, $\|z(s)\| < \delta$ for some interval $0 \le s \le \tau$, and from (3.10) we find

$$\|\underset{\sim}{z}(\tau)\| e^{\alpha\tau} \le c_1 \|\underset{\sim}{c}\| + c_1 \int_0^{\tau} \left[\|\underset{\sim}{B}(s)\| + \frac{\alpha}{2c_1} \right] \|\underset{\sim}{z}(s)\| e^{\alpha s}\, ds \tag{3.11}$$

on this interval. Application of Gronwall's lemma [4] to (3.11) yields

$$\|\underset{\sim}{z}(\tau)\| e^{\alpha\tau} \le c_1 \|\underset{\sim}{c}\| \exp\left[c_1 \int_0^{\tau} \|\underset{\sim}{B}(s)\| + \frac{\alpha}{2c_1} \right] d\tau \quad .$$

Utilising hypothesis 2) and multiplying both sides by $e^{\alpha\tau}$ we obtain

$$\|\underset{\sim}{z}(\tau)\| \le c_1 \|\underset{\sim}{c}\| e^{c_1 M} e^{-(\alpha/2)\tau} \quad . \tag{3.12}$$

Thus, if $\|\underset{\sim}{c}\|$ further satisfies $\|\underset{\sim}{c}\| < \delta/c_1 e^{c_1 M}$, then the equality (3.12) will hold for all $\tau \ge 0$ and the theorem is proved.

4. Analysis

Basic Equations.

It will be convenient, as in [1], to nondimensionalize equations (2.1) to (2.5) by introducing the new variables

$$\xi = \frac{\pi x}{L} \quad , \quad \tau = \omega_o t$$
$$\eta = \frac{w}{k} \quad , \quad \eta_o = \frac{\bar{w}_o}{k} \tag{4.1}$$

where

$$\omega_o = \frac{\pi^2}{L^2}\left(\frac{EI}{\rho_s h}\right)^{1/2}$$

and k is the cross-sectional radius of gyration. With use of (4.1), equations (2.1) to (2.5) reduce to

$$\frac{\partial^4}{\partial \xi^4}(\eta - \eta_o) - \frac{1}{2\pi}\frac{\partial^2 \eta}{\partial \xi^2}\int_0^\pi \left[\left(\frac{\partial \eta}{\partial \xi}\right)^2 - \left(\frac{\partial \eta_o}{\partial \xi}\right)^2\right] d\xi + \gamma \frac{\partial \eta}{\partial \tau} + \frac{\partial^2 \eta}{\partial \tau^2} = -q(\xi, \tau) \tag{4.2}$$

where

$$q(\xi, \tau) = q_1 \sin \xi \, H(\tau) \quad , \quad q_1 = \frac{p_1}{EIk}\left(\frac{L}{\pi}\right)^4 ,$$
$$\gamma = \frac{\beta}{\rho_s h \omega_o} . \tag{4.3}$$

In addition, we have

$$\eta_o = e \sin \xi \quad , \quad e = \bar{w}_o/k \quad , \tag{4.4}$$

$$\eta(0, \tau) = \frac{\partial^2 \eta}{\partial \xi^2}(0, \tau) = \eta(\pi, \tau) = \frac{\partial^2 \eta}{\partial \xi^2}(\pi, \tau) = 0 \quad , \tag{4.5}$$

$$\eta(\xi, 0) = g_1(\xi) \quad , \quad \frac{\partial \eta}{\partial \tau}(\xi, 0) = g_2(\xi) \quad , \tag{4.6}$$

where $g_i = G_i/k$, $i = 1, 2$.

An exact solution of equation (4.2), satisfying the boundary conditions (4.5) termwise, is

$$\eta(\xi, \tau) = e \sin \xi + \sum_{n=1}^{N} a_n(\tau) \sin n\xi \tag{4.7}$$

where the coefficients $a_n(\tau)$ are solutions of the following system of ordinary differential equations:

$$\frac{d^2 a_1}{d\tau^2} + \gamma \frac{da_1}{d\tau} + a_1 + \frac{1}{4}\left(2ea_1 + \sum_{n=1}^{N} n^2 a_n^2\right)(a_1 + e) = -q_1 H(\tau) ,$$

$$\frac{d^2 a_m}{d\tau^2} + \gamma \frac{da_m}{d\tau} + m^4 a_m + \frac{1}{4}\left(2ea_1 + \sum_{n=1}^{N} n^2 a_n^2\right) m^2 a_m = 0 , \tag{4.8}$$

$$m = 2, 3, \ldots, N$$

To avoid the question of convergence, we shall assume that the number N in (4.7) and (4.8) is finite. This, of course, restricts the class of disturbances that the arch may experience. In particular, (4.7) constitutes an exact solution only if g_1 and g_2 can be expressed in the form

$$g_1(\xi) = \sum_{n=1}^{N} a_n(\tau_0) \sin n\xi \quad , \quad g_2(\xi) = \sum_{n=1}^{N} \frac{da_n}{d\tau}(\tau_0) \sin n\xi . \tag{4.9}$$

The initial conditions to (4.8) are the Fourier coefficients

$$a_n(\tau_0) = \frac{2}{\pi}\int_0^{\pi} g_1(\xi) \sin n\xi \, d\xi \quad , \quad \frac{da_n(\tau_0)}{d\tau} = \frac{2}{\pi}\int_0^{\pi} g_2(\xi) \sin n\xi \, d\xi . \tag{4.10}$$

From a physical point of view, the above finite N restriction is not considered to be overly important.

Symmetric Motion and Symmetric Snap-through.

Under the assumption that $g_i(\xi) \equiv 0$, $i = 1, 2$, the response of the arch is purely symmetric, i.e., $a_n \equiv 0$, $n = 2, 3, \ldots, N$ and a_1 is governed by

$$\frac{d^2 a_1}{d\tau^2} + \gamma \frac{d a_1}{d\tau} + a_1 + \frac{1}{4}(2ea_1 + a_1^2)(a_1 + e) = -q_1 H(\tau) \quad ,$$
$$a_1(0) = \frac{d a_1(0)}{d\tau} = 0 \quad . \tag{4.11}$$

Now, let a_e denote the equilibrium position(s) the arch assumes if q_1 is applied statically. Then, a_e satisfies

$$a_e + \frac{1}{4}(2e a_e + a_1^2)(a_e + e) + q_1 = 0 \quad . \tag{4.12}$$

For $q_1 < (\bar{q}_1)_s$, and for $e > 2$, there exist three real roots of the cubic (4.12) which, in turn, define three equilibrium positions of the arch. For $q_1 > (\bar{q}_1)_s$, there exists only one real root and hence only one equilibrium position (the inside-out position of the arch). The number $(\bar{q}_1)_s$ is the classical static, symmetric, snap-through-load.

Let us assume that $q_1 < (\bar{q}_1)_s$, and let us introduce the transformation

$$a_1(\tau) = \bar{a}_1 - a_1^{\circ}(\tau) \tag{4.13}$$

where $\bar{a}_1 = \max a_e$, i.e., the unsnapped, symmetric equilibrium position.

By virtue of (4.11), $a_1^{(0)}(\tau)$ satisfies

$$\frac{d^2 a_1^{(0)}}{d\tau^2} + \gamma \frac{d a_1^{(0)}}{d\tau} + \mathcal{J}(a_1^{(0)}; \bar{a}_1) = 0 \quad , \tag{4.14a}$$

$$a_1^{(0)}(0) = \bar{a}_1 \quad , \quad \frac{d a_1^{(0)}}{d\tau}(0) = 0 \tag{4.14b}$$

where

$$\mathcal{J} = a_1^{(0)} + \frac{1}{4}\left\{ a_1^{(0)}\left[6e\bar{a}_1 + 3\bar{a}_1^2 + 2e^2\right] - a_1^{(0)^2}\left[3(\bar{a}_1 + e)\right] + a_1^{(0)^3}\right\} . \tag{4.14c}$$

In terms of the new variable $a_1^{(0)}$, the equilibrium positions a_e are the critical points of (4.14), i.e., $da_1^{(0)}/d\tau = 0$, $a_1^{(0)} = (a_1^{(0)})_i$, $i = 1, 2, 3$, where the $(a_1^{(0)})_i$ are defined by $\mathcal{J}(a_1^{(0)}; \bar{a}_1) = 0$ and ordered such that $(a_1^{(0)})_1 < (a_1^{(0)})_2 < (a_1^{(0)})_3$. The point $(a_1^{(0)})_1$ is the origin $a_1^{(0)} = 0$ and corresponds to the unsnapped configuration of the arch; the point $(a_1^{(0)})_3$ is the inverted arch position.

Consider next the phase plane: $da^{(0)}/d\tau$ vs. $a_1^{(0)}$. In the phase plane it can be shown that $(a_1^{(0)})_1$ is a center point, $(a_1^{(0)})_2$ is a saddle point, and $(a_1^{(0)})_3$ is another center point. If q_1 (i.e., the initial data (4.14b)) is such that the trajectories in the phase plane encompass the saddle point $(a_1^{(0)})_2$, we shall say (see [8]) that dynamic, symmetric, snap-through has occurred. On the other hand, if all trajectories are closed about the origin and do not extend to or beyond the saddle point, we shall say that dynamic, symmetric, snap-through has not occurred. The value of q_1 leading to a trajectory which intersects the saddle point represents the transition between snapped and unsnapped arches; we shall denote this value as $(q_1{}^*)_s$.

In addition to the assumption that $q_1 < (q_1)_s$, let us assume q_1 is less than that required to cause dynamic, symmetric, snap-through, i.e., $q_1 < (q^*)_s$. Then, for $\gamma > 0$, the trajectories in the phase plane spiral toward the origin. Thus, for a given q_1 and γ, and for any $\epsilon > 0$, there exists a number $T_o(\epsilon ; q_1)$ such that $|a_1{}^{(0)}(T_o)| + |da_1{}^{(0)}/d\tau (T_o)| < \epsilon$ for all $\tau \geq T_o$. Further, since the nonlinear portion, $n\mathfrak{Z}$, of the function $\mathfrak{Z}$ satisfies the nonlinearity condition

$$\lim_{|a_1{}^{(0)}| \to 0} \frac{|n\mathfrak{Z}|}{|a_1{}^{(0)}|} = 0 \quad ,$$

a well-known comparison theorem [4] indicates that

$$|a_1{}^{(0)}(\tau)| \leq \beta e^{-(\gamma/2)\tau} \quad , \quad \beta = \text{const.} \quad , \tag{4.15}$$

for sufficiently small ϵ, or for sufficiently large T_o. From (4.15), we therefore conclude that $a_1{}^{(0)}(\tau)$ is impulsively small as $\tau \to \infty$, i.e.,

$$\int_0^\infty |a_1{}^{(0)}(\tau)| \, d\tau < M_1 = \text{const.} \tag{4.16}$$

We shall make use of this fact later.

Definition of Dynamic Snap-through

The concept of dynamic snap-through has been mentioned in connection with the purely symmetric motion. In order to avoid a vague discussion, we must attempt to pin down this illusive butterfly for asymmetric as well as symmetric motions. The following definition will suffice for purposes of the present analysis:

Definition.[3] Let us perturb the symmetric motion according to

$$
\begin{aligned}
a_1(\tau) &= \bar{a}_1 - a_1^{(0)}(\tau) + a_1^*(\tau) \quad , \\
a_m(\tau) &= a_m^*(\tau) \quad , \quad m = 2, 3, \ldots, N.
\end{aligned}
\tag{4.17}
$$

Next, let us assume that q_1 is given. We shall say that snap-through, under load q_1, has not occurred if there exists a $\delta > 0$ such that

$$\sum_{n=1}^{N} \left\{ |a_n^*(\tau_o)| + |da_n^*/d\tau(\tau_o)| \right\} < \delta \text{ implies } \max_{0<\tau<\infty} [\bar{a}_1 - a_1(\tau)] < (a_1^{(0)})_3$$

for any constant τ_o (recall that $(a_1^{(0)})_3$ is the inside-out equilibrium position). Conversely, we shall say that snap-through has occurred if there exists at least one set of initial data such that

$\max_{0<\tau<\infty} [\bar{a}_1 - a_1(\tau)] \geq (a_1^{(0)})_3$ for arbitrary small δ.

Local Stability of Symmetric Motion.

Let us now investigate the local stability of the symmetric motion

[3] This definition is compatible with the snap-through concepts employed in both [1] and [2].

$a_1^{(0)}(\tau)$ with respect to all possible symmetric and asymmetric motions resulting from the disturbance (4.10). To accomplish this we perturb $a_1^{(0)}(\tau)$ according to (4.17). Upon substitution of (4.17) into (4.8) one finds that the perturbations a_m^* satisfy the following system of equations:

$$
\begin{aligned}
&\frac{d^2 a_1^*}{d\tau^2} + \gamma \frac{da_1^*}{d\tau} + a_1^*\left[1 + \frac{1}{4}\left(6e\bar{a}_1 + 3\bar{a}_1^2 + 2e^2\right)\right] \\
&\quad + \frac{3a_1^* a_1^{(0)}(\tau)}{4}\left[a_1^{(0)}(\tau) - 2(\bar{a}_1 + e)\right] + \frac{3a_1^{*2}}{4}(\bar{a}_1 + e) - \frac{3a_1^{*2}}{4} a_1^{(0)}(\tau) \\
&\quad + \frac{a_1^{*3}}{4} + \frac{(\bar{a}_1 + e)}{4}\sum_{n=2}^{N} n^2 a_n^{*2} - \frac{a_1^{(0)}(\tau)}{4}\sum_{n=2}^{N} n^2 a_n^{*2} \qquad (4.18a) \\
&\quad + \frac{a_1^*}{4}\sum_{n=2}^{N} n^2 a_n^{*2} = 0 \quad ,
\end{aligned}
$$

$$
\begin{aligned}
&\frac{d^2 a_m^*}{d\tau^2} + \gamma \frac{da_m^*}{d\tau} + a_m^*\left[m^4 + \frac{\bar{a}_1 m^2}{4}(2e + \bar{a}_1)\right] + \frac{a_m a_1^{(0)}(\tau) m^2}{4}\left[a_1^{(0)}(\tau) - 2(\bar{a}_1 + e)\right] \\
&\quad + \frac{m^2}{2} a_m^* a_1^*(\bar{a}_1 + e) - \frac{m^2}{2} a_m^* a_1^{(0)}(\tau) + \frac{m^2}{4} a_m^* a_1^{2} \qquad (4.18b) \\
&\quad + \frac{m^2}{4} a_m^* \sum_{n=2}^{N} n^2 a_m^{*2} = 0 \quad , \qquad m = 2, \ldots, N.
\end{aligned}
$$

With the substitutions

$$z_i = \frac{d a^*_{(i+1)/2}}{d\tau} \quad , \quad i = 1, 3, 5, \ldots, 2N-1$$
$$z_i = a^*_{(i/2)} \quad , \quad i = 2, 4, 6, \ldots, 2N \tag{4.19}$$

equations (4.18) can be reduced to the following 2N system of first order nonlinear differential equations:

$$\frac{d\underset{\sim}{z}}{d\tau} = \underset{\sim}{A}\,\underset{\sim}{z} + \underset{\sim}{B}(\tau)\underset{\sim}{z} + \underset{\sim}{f}(\underset{\sim}{z}, \tau) \quad . \tag{4.20}$$

Here $\underset{\sim}{z}$ denotes a 2N-dimensional vector whose elements, z_i, are defined by (4.19); $\underset{\sim}{A}$ is a $2N \times 2N$ constant matrix, the form of which is not pertinent to our discussion; $\underset{\sim}{B}(\tau)$ is a $2N \times 2N$ matrix having the form

$$\underset{\sim}{B}(\tau) = a_1^{(0)}(\tau)\left[a_1^{(0)}(\tau) - 2(\bar{a}_1 - e)\right]\underset{\sim}{C} \tag{4.21}$$

where $\underset{\sim}{C}$ is a $2N \times 2N$ constant matrix. The 2N-dimensional vector $\underset{\sim}{f}$ in (4.20) consists of autonomous (say $\underset{\sim}{f}^{(1)}$) and nonautonomous (say $\underset{\sim}{f}^{(2)}$) parts. Writing $\underset{\sim}{f} = \underset{\sim}{f}^{(1)} + \underset{\sim}{f}^{(2)}$ we have:

$$-f_1^{(1)} = \frac{3z_2^2}{4}(\bar{a}_1 + e) + \frac{z_2^3}{4} + \frac{(\bar{a}_1 + e + z_2)}{4} \sum_{n=2}^{N} n^2 z_{2n}^2$$

$$-f_i^{(1)} = 0\,, \quad i = 2, 4, 6, \ldots, 2N\,, \tag{4.22a}$$

$$-f_i^{(1)} = \frac{m^2}{2} z_2 z_{2m} (\bar{a}_1 + e) + \frac{m^2}{4} z_2^2 z_{2m} + \frac{m^2}{4} z_{2m} \sum_{n=2}^{N} n^2 z_{2m}^2\,,$$

$$i = 3, 5, 7, \ldots, 2N-1; \quad m = (i+1)/2 \quad .$$

$$f_1^{(2)} = \frac{a_1^{(0)}(\tau)}{4}\left[3z_2^2 - \sum_{n=2}^{N} n^2 z_{2n}^2\right]$$

$$f_i^{(2)} = 0 \quad , \quad i = \text{even} \tag{4.22b}$$

$$f_i^{(2)} = \frac{m^2}{2} z_{2m} z_2 a_1^{(0)}(\tau) \quad , \quad i = 3, 5, 7, \ldots, 2N-1 \quad ; \quad m = (i+1)/2 \quad .$$

Let us now assume that $q_1 < (\bar{q}_1)_s$, $q_1 < (q_1^*)_s$. Then, by virtue of (4.16), we conclude that

$$\int_0^\infty \|\underset{\sim}{B}(\tau)\|\, d\tau < M_2 = \text{const.} \tag{4.23}$$

Further, noting that $a_1^{(0)}(\tau)$ is a bounded function of τ, and (from (4.22)) that the elements of $\underset{\sim}{f}$ are polynomials in z_i with no linear terms, it is evident that

$$\lim_{\|\underset{\sim}{z}\| \to 0} \|\underset{\sim}{f}\| / \|\underset{\sim}{z}\| = 0 \quad . \tag{4.24}$$

Applying now the comparison theorem of Section 3, we conclude that

$\underset{\sim}{z} \equiv 0$ (the symmetric motion) is asymptotically stable provided the origin of

$$\frac{d\underset{\sim}{z}}{d\tau} = A\underset{\sim}{z} \tag{4.25a}$$

is asymptotically stable.

In terms of the original system of N second-order equations, (4.25a) can be written

$$\begin{gathered}
\frac{d^2 a_1^*}{d\tau^2} + \gamma \frac{d a_1^*}{d\tau} + a_1^*\left[1 + \frac{1}{4}\left(6e\bar{a}_1 + 3\bar{a}_1^2 + 2e^2\right)\right] = 0 \quad , \\
\frac{d^2 a_m^*}{d\tau^2} + \gamma \frac{d a_m^*}{d\tau} + m^2 a_m^*\left[m^2 + \frac{\bar{a}_1}{4}(2e + \bar{a}_1)\right] = 0 \quad , \\
m = 2, 3, \ldots, N.
\end{gathered} \tag{4.25b}$$

One recognizes (4.25b) as the classical perturbation equations for the statically loaded arch [1, 6, 7]. Traditionally, the critical static snap-through load, $\bar{q}_1$, is based on such equations. In particular, if $\gamma \equiv 0$, $\bar{q}_1$ corresponds to the transition between bounded and unbounded solutions of (4.25). If $\gamma > 0$, and $q_1 < \bar{q}_1$, all solutions of (4.25) decay exponentially to zero. Thus, by definition of $\bar{q}_1$, if $q_1 < \bar{q}_1$ and $\gamma > 0$, the origin of (4.25) is asymptotically stable. We thus conclude that the origin of (4.20) is also asymptotically stable if $q_1 < \bar{q}_1$, $q_1 < (q_1)_s$, $q_1 < (q_1^*)_s$.

Local Stability and Snap-through.

Consider now the foregoing result in terms of snap-through. Local asymptotic stability implies that, under sufficiently small disturbances, all motions of the arch can be maintained within an arbitrarily small neighborhood of the symmetric motion. To be more precise, we have the following: if $\gamma > 0$ and q_1 satisfies i) $q_1 < (\bar{q}_1)_s$, ii) $q_1 < (q_1^*)_s$, iii) $q_1 < \bar{q}_1$, then for any $\epsilon > 0$, there exists a $\delta > 0$ (depending in general on τ_0, γ, q_1 and the arch geometry) such that $\|z(t)\| < \epsilon$ for all $\tau \geq \tau_0$ provided $\|z(\tau_0)\| < \delta$. Thus, under such conditions asymmetric snap-through, as defined previously, is not possible. One therefore concludes that, if $\gamma > 0$ and q_1 lies in the region defined by i), ii), iii) above, then for sufficiently small disturbances snap-through will not occur. On the other hand, if $q_1 > (q_1^*)_s$, it is evident that symmetric snap-through will take place. Further, the analysis of Huang and Nachbar [2] implies there exists a set of initial conditions (resulting in an excitation of only two modes, i.e., for $\tau = 0$, $a_m^{(0)} = da_m^{(0)}/d\tau = 0$ for $m > 2$) such that snap-through takes place for an arbitrarily small initial disturbance if $q_1 > \bar{q}_1$ and $\gamma > 0$. Therefore the stability boundary for infinitesimal disturbances must be a curve consisting of dynamic, symmetric snap-through up to its intersection with $q_1^*/q_1 = 1.0$ and $q_1 = 1.0$ beyond. Finally, it is evident that this curve also constitutes an upper bound on stability for disturbances of finite amplitude.

The foregoing curve is illustrated in Fig. 2 as $\gamma \to 0^+$, and for γ corresponding to 1% critical damping in the symmetric mode (Lock's case). The slight disagreement with Lock's numerical results is attributed to the size of the initial conditions employed in that analysis, i.e., presumably the smaller the initial data, the better the agreement would be. The symmetric snap-through portion of the curves exhibited in Fig. 2 was obtained from data supplied by Lock for $\gamma \neq 0$ (numerical integration of the differential equation (4.11) is necessary if $\gamma \neq 0$.) An exact relation for $(q_1^*)_s$ vs. e can be obtained for $\gamma = 0^+$, e.g., from [2]. In the latter case a combination of Eqs. (32), (35) and (48) of [2] yield the following relations for the snap-through boundary (dotted lines of Fig. 2) when $\gamma = 0^+$:

(i) $e \le \sqrt{22} = 4.68$:

$$\frac{q_1^*}{\bar{q}_1} = \frac{(1/27)\left[e\left(9 + \frac{e^2}{2}\right) + \left(\frac{e^2}{2} - 3\right)(e^2 - 6)^{1/2}\right]}{\frac{e}{2} + \frac{2}{9}\sqrt{3}\left(\frac{e^2}{4} - 1\right)^{3/2}}, \tag{4.26a}$$

(ii) $4.68 \le e \le 6.38$:

$$\frac{q_1^*}{\bar{q}_1} = \frac{(1/27)\left[e\left(9 + \frac{e^2}{2}\right) + \left(\frac{e^2}{2} - 3\right)(e^2 - 6)^{1/2}\right]}{\frac{e}{2} + 3\left(\frac{e^2}{4} - 4\right)^{1/2}}, \tag{4.26b}$$

(iii) $e \ge 6.38$:

$$q_1^*/\bar{q}_1 = 1 \ . \tag{4.26c}$$

5. Concluding Remarks

As an example of a structure exhibiting a particular sensitivity to damping, the snapping of a simply-supported, geometrically perfect sinusoidal arch, subjected to a sinusoidal spatial pressure distribution applied as a step in time, was considered. It was found that, in the presence of velocity-dependent damping of any magnitude (> 0), and under sufficiently small disturbances (taking the form of initial arch displacements and velocities), there exists no difference between dynamic and static snap-through pressures for sufficiently large arch rises. For infinitesimal disturbances the snap-through boundary is a curve in a load vs. arch rise plane defined by symmetric snap up to its intersection with the static snap-load, and the static value thereafter. For finite disturbances this represents an upper bound on this load. Said curve is illustrated in Fig. 2 for vanishingly small damping. Upon comparison with the purely elastic case of Lock [1], (Fig. 3), one observes a jump in the snap-boundary with respect to damping.

A similar analysis as has been conducted here can be applied to other structures. In particular, an analysis of geometrically perfect rings and cylinders under spatially uniform loading applied as a step in time will yield the result that, under sufficiently small disturbances, there exists no difference between statically and dynamically applied buckling loads if damping is present.

Finally, we close on an interesting note. Suppose the foregoing load were applied as an impulse rather than a step; then $\bar{a}_1 = 0$. In this case (4.20) is asymptotically stable for all q_1. Therefore, one concludes that, in the presence of damping, and under sufficiently small disturbances, only symmetric snapping is possible. In the case of the ring and cylinder mentioned previously, no dynamic buckling whatsoever would result. The physical explanation of this is as follows: in the presence of damping a sufficiently small disturbance is damped out before it can initiate a "mode conversion" (by virtue of parametric resonance) which results in a buckling phenomena. The foregoing examples indicate that the magnitude of both disturbances and damping are important for near perfect structures under certain transient loadings.

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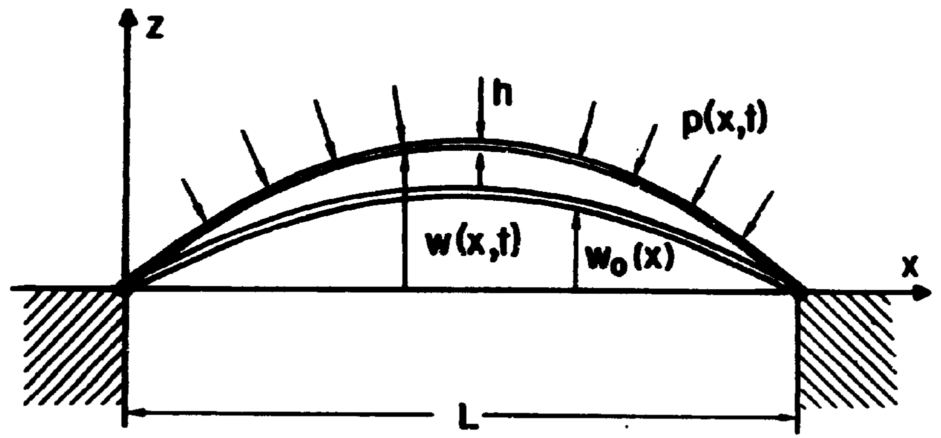

Fig. 1: Arch Geometry.

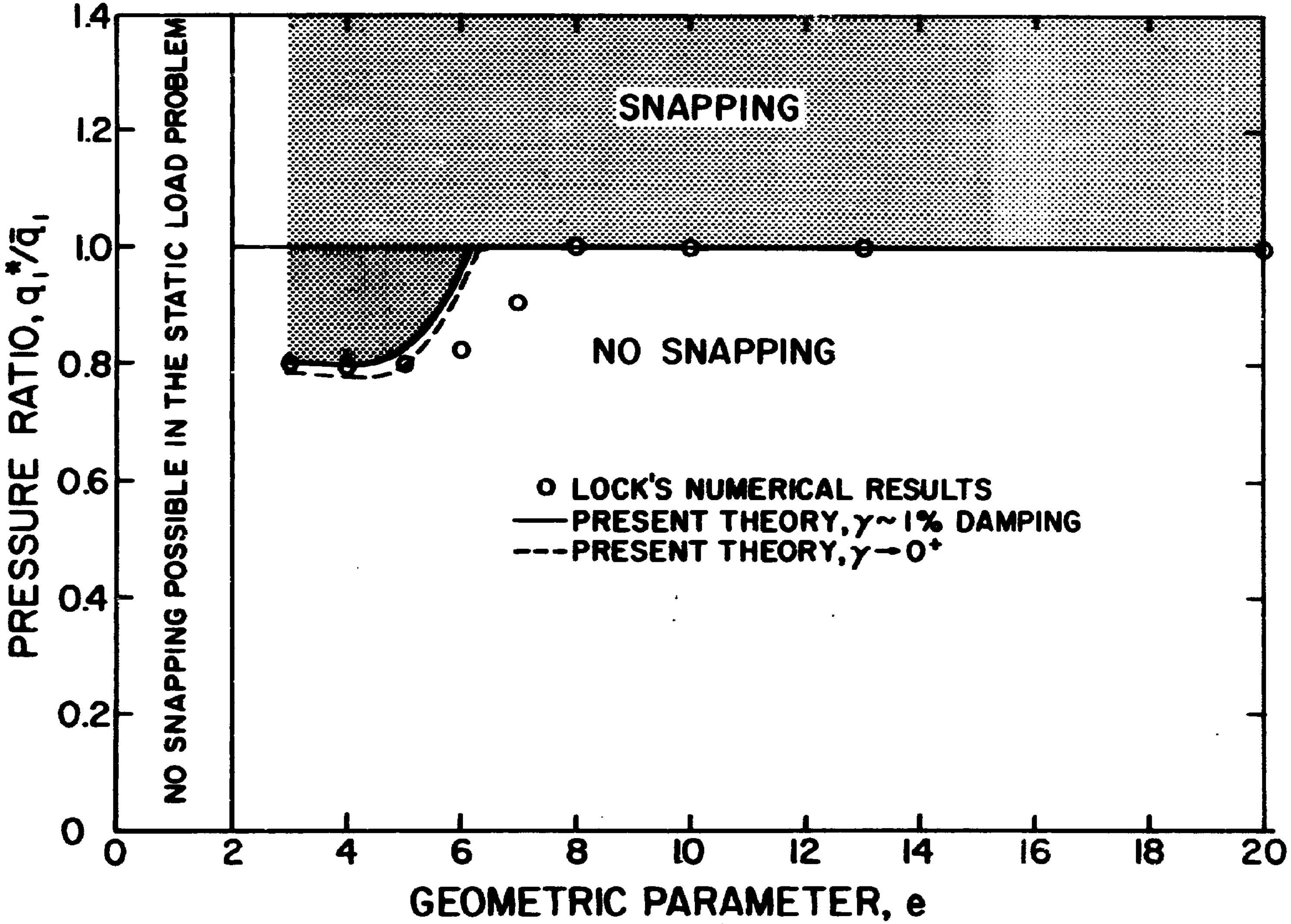

Fig. 2: Critical Pressure Ratio vs. Arch Rise.

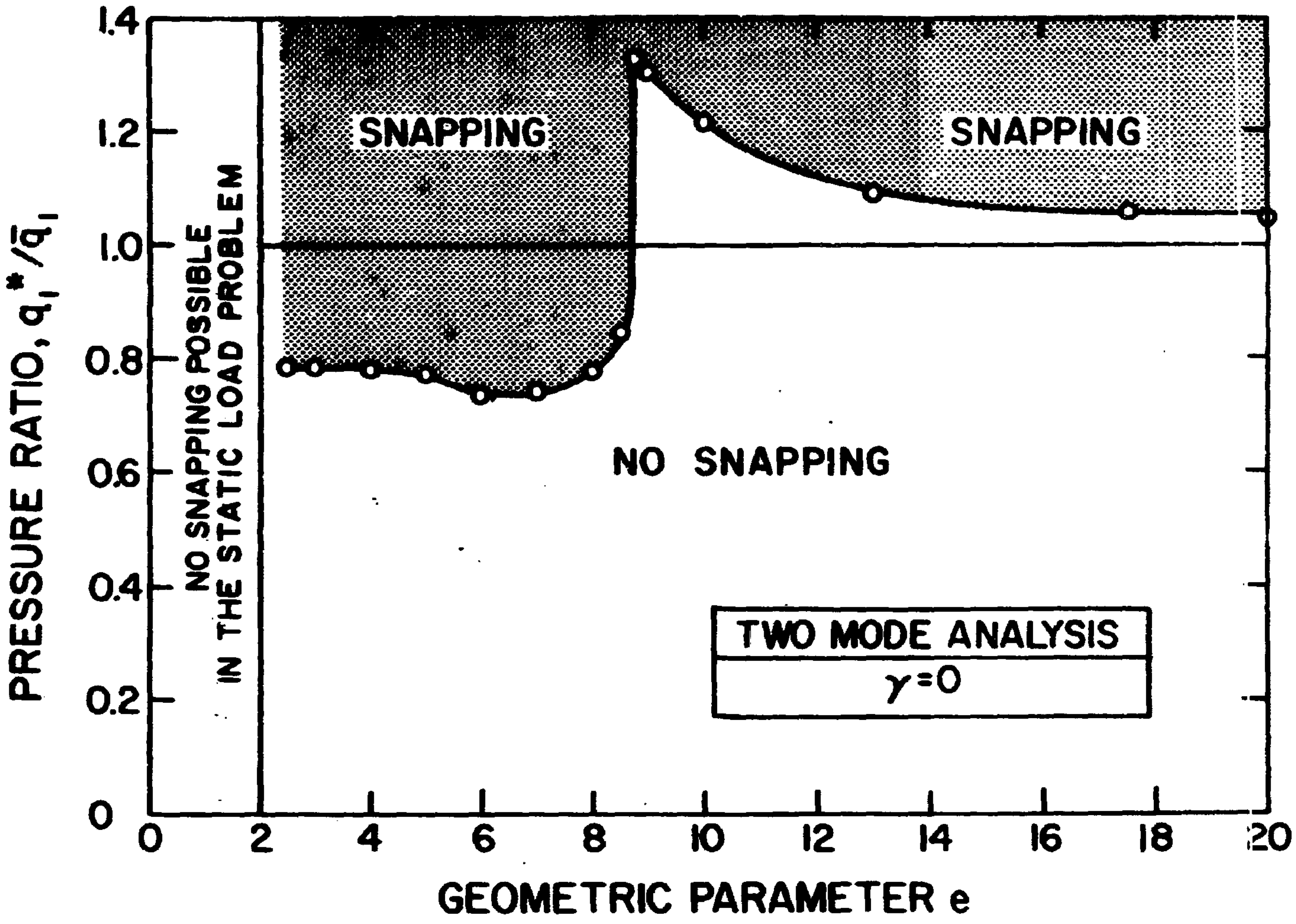

Fig. 3: Critical Pressure Ratio vs. Arch Rise for Zero Damping as obtained from Lock [1].

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13. ABSTRACT

The snapping of a simply-supported shallow sinusoidal arch under a sinusoidally distributed step pressure load is considered. In the presence of velocity-dependent damping of any nonzero magnitude, it is shown that there exists no difference between static and dynamic snap loads for arch rises above a certain magnitude and for sufficiently small external disturbances. Below the foregoing value of arch rise snapping is governed entirely by symmetric snap-through. The results obtained herein, when compared to an existing analysis on the subject, imply that a jump in the critical snap-through load occurs at the boundary between the damped and undamped systems. Similar results concerning cylinders and rings are mentioned.

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