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ON SUBOPTIMIZATION: AN EXAMPLE

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ON SUBOPTIMIZATION: AN EXAMPLE

By J.M. Danskin

17 June 1966

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ABSTRACT

In certain allocation problems concerning groups of systems, it is possible to allocate by allocating within each system and then combining the results. This paper shows that while this method is correct for pure maximum problems and for cases in which the overall problem is a game, it is not true for Max-Min problems.

Suppose

$$H(x,\xi) = F(x) + G(\xi), \qquad (1)$$

where x and ξ are vectors satisfying $\Sigma x_i + \Sigma \xi_j = 1$, $x_i \geq 0$, $\xi_j \geq 0$. Suppose the pair x^0, ξ^0 maximizes (1). Let $\sum x_i^0 = \alpha$. It is a trivial fact that then x_i^0 maximizes $F(x)$ subject to $\Sigma x_i = \alpha$, $x_i \geq 0$. It follows from this in particular that the solution for (1) can be found by solving the problem for $F(x)$ for $\Sigma x_i = \alpha$ and the problem for $G(\xi)$ for $\Sigma \xi_j = 1 - \alpha$, and then turning through all values of α . This process is sometimes called "suboptimization". The idea is that if one solves the problem for the whole, one gets the solution for the parts, and vice-versa.

This "suboptimization principle" also holds when the maximum problem for $H(x,\xi)$ is replaced by a game problem

$$H(x,\xi,y,\eta) = F(x,y) + G(\xi,\eta), \qquad (2)$$

where the first player seeks to maximize subject to $\Sigma x_i + \Sigma \xi_j = 1$, $x_i \geq 0$, $\xi_j \geq 0$, and the second player seeks to minimize subject to $\Sigma y_i + \Sigma \eta_j = 1$, $y_i \geq 0$, $\eta_j \geq 0$. Suppose that there is a pair of pure strategy solutions (x^0,ξ^0) and (y^0,ξ^0) for the two players respectively and suppose that $\sum x_i^0 = \alpha$, $\sum y_i^0 = \beta$. Let the value of the game defined by H be v, and suppose that $G(\xi^0,\eta^0) = \gamma$. Then, for any x satisfying $\Sigma x_i = \alpha$, $x_i \geq 0$,

$$F(x,y^0) = H(x,\xi^0,y^0,\eta^0) - G(\xi^0,\eta^0) \leq v - \gamma ,$$

and for any y satisfying $\Sigma y_i = \beta$, $y_i \geq 0$,

$$F(x^0,y) = H(x^0,\xi^0,y,\eta^0) - G(\xi^0,\eta^0) \geq v - \gamma .$$

Thus the components x^0 and y^0 are optimal strategy solutions for F; and similarly ξ^0 and η^0 for G.

It follows that games can be solved piece-by-piece. One picks pairs α,β and solves the games separately. One then runs through all pairs with $0 \leq \alpha, \beta \leq 1$; this will surely lead to a solution of the overall game.

The object of this paper is to show that this suboptimization principle does not always hold for Max-Min problems which are not games.

Suppose that

$$F(x,y) = x_1 e^{-y_1} + .9x_2 e^{-y_2/x_2}$$

and

$$G(\xi,\eta) = \xi e^{-\eta/\xi} .$$

Here $x_1 + x_2 + \xi = 1$, $y_1 + y_2 + \eta = 1$, and $x_i,\xi,y_i,\eta \geq 0$. If we solve the problem $\underset{x,\xi}{\text{Max}} \ \underset{y,\eta}{\text{Min}} \ H(x,\xi,y,\eta)$ for

$$H(x,\xi,y,\eta) = x_1 e^{-y_1} + .9x_2 e^{-y_2/x_2} + \xi e^{-\eta/\xi}$$

subject to the above side conditions, we get (for the methods see Chapter V of [1])

$$x_1 = .465 \quad x_2 = 0 \quad \xi = .535$$

$$y_1 = .615 \quad y_2 = 0 \quad \eta = .385 .$$

Now consider the problem

$$\underset{x_1+x_2=.465}{\text{Max}} \ \underset{y_1+y_2=.615}{\text{Min}} \ x_1 e^{-y_1} + .9x_2 e^{-y_2/x_2} . \qquad (3)$$

The solution to this problem turns out to have both x_2 and y_2 positive. In fact, it is

$$x_1 = .434 \quad x_2 = .031$$

$$y_1 = .575 \quad y_2 = .040 .$$

The return to x at this point for the game (3) is .248. At $x_1 = .465$ the value is .226. The reader will easily verify that the y-solution is correct against $x_1 = .434$, $x_2 = .031$ by checking the derivatives with respect to y_1 and y_2, which are both approximately -.248. These calculations are carried out to slide rule accuracy.

It follows that for the above problem one would never have arrived at the solution to the overall problem by grouping the first and the second system together, optimizing, and then bringing in the third. The three variables must be considered simultaneously.

The meaning for operations research or economic analysis is clear; one cannot, in the presence of conflict, be assured of arriving at a solution for the whole by considering the parts one at a time.

[1] Danskin, J. M., The Theory of Max-Min, Springer-Verlag, Berlin - Heidelberg - New York, 1966.

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