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ON CONFIDENCE LIMITS FOR THE PERFORMANCE OF A SYSTEM WHEN FEW FAILURES ARE ENCOUNTERED

by

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Summary

In some situations encountered today the components are so reliable that no failures are observed within the time available for testing. This can pose a problem in data analysis and interpretation. We consider here the problem of determining lower confidence bounds on the reliability of a complex system, such as the Saturn 1-C, when each component is assumed to have an exponential life and different components have different multiplicities within the system. We discuss and compare the confidence limits obtainable from various interpretations of the data and several theories such as asymptotic likelihood ratio, asymptotic maximum likelihood, the Bayesian method and the new method presented here utilizing the conditional probabilities of malfunction of the components given that a malfunction in the system has occurred.

Introduction

The problem of determining the probability of successful operation of a large complex system when one has data only on the reliability of the components, has over the past decade, been the subject of many investigations. A large portion of these reports were proprietary.

However much of this vast literature either consists of amalgams of engineering judgement and untutored statistical intuition, or the study is based on an asymptotic theory for which the precision of the approximation is unknown or the analysis is based on subjective prior assumptions utilizing Bayesian methods.

The economic importance of correct assessment of the system reliability before full scale testing can hardly be underestimated in nearly any of the current aerospace programs. The desire to find an acceptable solution can be seen from the inclusiveness of the proprietary report of Dalton [2] which ostensibly compares many of these techniques. We also mention the comparison made in [11] and the many references given in [2].

The fact is estimating mean time until failure or the probability of failure (under many models), requires that at least one failure be observed. The plea which is often made to the preliminary testing program for more data is sometimes nothing but a covert wish for more failures. Thus the statistician is put in the uncomfortable position of having less and less confidence in his interval estimates of reliability when fewer and fewer failures are observed due to the fact the reliability is becoming higher and higher. Ultimately, when the system becomes perfect

and no failures are observed the statistician has no confidence if his statistical procedures are necessarily based on failure analysis. In this unsatisfactory situation it is an understandable reaction of program managers to form a distrust of statistical inference and its "numerologists" as well. And as indication of this disenchantment see [3].

In this note we address the problem of deriving appropriate statistical techniques which will be valid when there is a paucity of observed failures in the components which have been tested.

1. The Basic Model

Consider a large complex system which has been designed to perform in a specified manner when all of its components are operating. At question is the length of time during which it can perform adequately, or equivalently, the confidence we can have that it will perform adequately for a specified time.

Suppose we postulate that each component of the system, as well as the system itself, is either performing adequately or is in a failed state. Hence at any time t for the i^{th} component

$$X_i(t) = \begin{cases} 1 & \text{indicates performance} \\ 0 & \text{indicates failure.} \end{cases}$$

The stochastic process $X_i(t)$ is called the *performance process* of the i^{th} component. Let $\underset{\sim}{X}(t) = (X_1(t),\ldots,X_m(t))$ indicate the state of all the components at any time $t > 0$. The state of the system formed from these components is given by the system structure function ϕ, which is a non-decreasing function on the vertices of the hypercube of dimension m and such that

$$\phi(1,\ldots,1) = 1 \quad \text{and} \quad \phi(0,\ldots,0) = 0.$$

This class of functions is called *coherent of order* m and has been studied previously in [4] and [6]. Thus the stochastic process

$$\Phi(t) = \phi\underset{\sim}{X}(t),$$

where juxtaposition indicates composition, defines the performance process

of the system. We are interested in the probability of perfect performance of the system for a mission of length t, namely

$$P[\Phi(s) = 1, \quad 0 < s < t]. \tag{1.1}$$

The problem is that one has only a limited amount of data about the reliability of each of the components so that at best one can obtain a lower confidence bound on (1.1), for a fixed mission of given length. We shall make the specific assumption

1° The random variables

$$U_i = \inf\{t : X_i(t) = 0\} \qquad i = 1, \ldots, m$$

are called the *times until failure* for each component and are independently distributed by

$$P[U_i > t] = e^{-Q_i(t)} \quad \text{for} \quad t > 0$$

where Q_i is the *hazard* function of U_i.

In this context a *minimal cut* has been defined, in [6], as a minimal set of components such that if all are simultaneously failed the system must be in a failed state. The performance process of the system can then be related to the performance process of the components in the usual fashion through the minimal cuts $C_1, \ldots, C_k$ by

$$\phi\underline{X}(t) = \prod_{j=1}^{m} \eta_j\underline{X}(t) \tag{1.1.1}$$

where

$$\eta_j\underline{X}(t) = 1 - \prod_{i \in C_j} (1 - X_i(t)) \tag{1.1.2}$$

for all $t > 0$. This representation is well known, see [6].

We can now prove

Theorem 1. *Under assumption* 1°, *there exists for any coherent structure* ϕ *with minimal cut sets* $C_1,\ldots,C_k$ *a set of integers* $\nu_1,\ldots,\nu_m$, *with* ν_i *the number of the* C_j *which contain* i *as a member, and*

$$P[\Phi(t) = 1] \geq \exp[-\sum_{1}^{m} \nu_j Q_j(t)] \quad \textit{for} \quad t > 0.$$

Proof: By a fundamental result of Esary, Proschan and Walkup, theorem 3.3 [5], we have

$$P[\phi\underset{\sim}{X}(s) = 1, \ 0 < s < t] \geq \prod_{j=1}^{k} P[\eta_j\underset{\sim}{X}(s) = 1, \ 0 < s < t]$$

where $\eta_j\underset{\sim}{X}(s)$ was defined in (1.1.2). But notice that as events

$$[\eta_j\underset{\sim}{X}(s) = 1, \ 0 < s < t] = [\Pi_{i\in C_j}(1-X_i(s)) = 0, \ 0 < s < t] \supset \bigcap_{i\in C_j} [X_i(t) = 1]$$

and it is the probability of this last event which is easily computed. Hence by assumption 1°

$$P[\eta_j\underset{\sim}{X}(s) = 1, \quad 0 < s < t] \geq \prod_{i\in C_j} P[X_i(t) = 1] = \prod_{i\in C_j} e^{-Q_i(t)}$$

$$\ln P[\Phi(t) = 1] \geq -\sum_{j=1}^{k} \sum_{i\in C_j} Q_i(t)$$

and reversing the summation yields the result claimed. ||

This result provides a lower bound on the reliability of any structure in terms of a series system with the multiplicities ν_i of the components suitably chosen. Moreover if component reliabilities are very high the system itself begins to behave like its weakest links, namely its minimal cuts, and its reliability becomes closely approximated by the bound given. In view of the theoretical simplicity and the approximate equality of the bound given in the many practical cases when hazard rates are small, we shall henceforth assume that we seek a lower confidence bound for such a series system reliability. This we take as

$$e^{-\sum_1^m \nu_i Q_i(t)} \qquad \text{for} \qquad t > 0, \tag{1.2}$$

where ν_i are known constants determined from the monotone functional representation ϕ of the system and Q_i are unknown hazard functions of the components.

Keeping in mind the missile systems which are the archtype of this model, it is often necessary to assume that a different representation of the form (1.2) is assigned for each separate phase of the missions. Henceforth we assume that such a phase is given and fixed.

2. Beta Factors

We now make a distinction between malfunction and failure of a component. In actuality a malfunction in the component may cause only degraded performance of the component instead of a complete failure. This, by definition, will result in a system failure (or vehicle loss in the archtype missile system) since we have reduced our assumption to that of a series system.

It becomes expedient to extend the model and allow the concept of *malfunction* which if occurring at a given time in the mission may or may not result in component failure. A failure thus is a special case of malfunction.

We now make two very specific assumptions

2^{o}: A malfunction in the i^{th} component during the time interval $(t,t+h)$ of a mission will result in a component failure causing vehicle loss with a certain probability $\beta_i^{o}(t)\cdot h$.

3^{o}: The time until malfunction of the i^{th} component is exponential with hazard rate λ_i.

It follows that

$$\lim_{h\to 0} \frac{1}{h} P[X_i(t+h) = 0 | X_i(t) > 0] = \lambda_i \beta_i^{o}(t).$$

From the definition of U_i we have $Q_i(t) = \int_0^t \lambda_i \beta_i^{o}(x)dx$

$$P[U_i > t+h | U_i > t] = \exp\{-\int_t^{t+h} \lambda_i \beta_i^{o}(x)dx\}.$$

For a mission of fixed length t_0 we define the *beta factors* β_i, which are constants, by the equation

$$\beta_i t_0 = \nu_i \int_0^{t_0} \beta_i^{o}(x)dx.$$

Thus the probability of vehicle safety over the time interval $(0,t_0)$, i.e. the reliability of the system, is from (1.2)

$$\exp\{-\sum_1^m \lambda_i \beta_i t_0\} \tag{2.1}$$

Without loss of generality we shall henceforth assume that all time intervals are expressed in fractions of this fixed mission length t_0, to wit, assume $t_0 = 1$.

For the i^{th} component we will refer to both the (average) beta factor β_i and β_i^o, the conditional probability of failure given a malfunction in that component, as β-factors. The usage will make clear which is meant.

Although we define β_i as an integral, in practice it is ordinarily a sum. Usually the mission phase is divided into periods and the β-factor is taken as constant during each period. Suppose the β-factor is β_{ij}^o for the i^{th} component during the j^{th} period of the mission phase. Hence the time intervals $(t^{(j-1)}, t^{(j)})$ $j=1,\ldots,r$, where $t^{(0)} = 0$, $t^{(r)} = 1$ by our convention, would constitute the r phases. We would then have

$$\beta_i = \sum_{j=1}^{r} \beta_{ij}^o [t^{(j)} - t^{(j-1)}]. \tag{2.2}$$

Notice that incorporating β-factors into the model to evaluate the probability of vehicle loss, as a consequence of assumptions 2° and 3° changes the entire nature of (1.2) since (2.1) involves only a known linear combination of the unknown failure rates λ_i.

3. The Data

In many cases the first type of data one has concerning the reliability of the components comes from environmental tests. This test data must be transformed by engineering evaluation into the equivalent operational time during the given phase of the mission. Those factors, which transform environmental test time into the equivalent phase time called the environmental factors or E-factors, are used in industrial practice, see [1].

An E-factor is a number used to modify one unit of test time so that it may be expressed in the appropriate units of equivalent mission phase time. For example, if one unit of test time is equivalent in severity to

one-half of a unit of time during a certain phase condition then the E-factor is .5. In other words two units of test time at this specified condition of environmental severity to one unit of mission phase time.

Specifically, during the first phase of a mission a component may experience several types of vibration, several temperature and humidity changes. Consequently, testing the component in these separate environments must yield results requiring a transformation into the appropriate mission phase equivalent time.

We do not discuss further the derivation of these transformations but we merely point out that in such instances the data on the operational behavior of the components are not god-given, but rather are the construct of prior engineering knowledge and judgement. We remark also that E-factors are not the same as β-factors. Beta factors are determined from the systems vulnerability to a malfunction of a given type during a certain phase of mission. On the other hand E-factors are used to accelerate the testing and reduce its expense.

Keeping this point in mind, the statistician is ultimately provided with data on all the components in the form

$$x_i = (t_i, n_i) \qquad i = 1,\ldots,m \tag{3.1}$$

where t_i is the total time (in fractions of the mission length) the i^{th} component nas been operated and n_i is the total number of malfunctions of the i^{th} component during time t_i.

We now make some interpretations of these data.

Interpretation I. Suppose it was decided to run the i^{th} component for a fixed period of time t_i and count the number of malfunctions. From assumption 2° it follows that the time t_i is a known parameter while n_i is an observed value of a Poisson random variable N_i with density

$$\frac{e^{-\lambda_i t_i}(\lambda_i t_i)^n}{n!} \qquad n = 0,1,... \tag{3.2}$$

Interpretation II. We now regard the time until the occurrence of the preassigned n_i^{th} malfunction $(n_i \geq 1)$ for the i^{th} component as the observed value of a random variable. Let

$$S_i = T_i(1) + \ldots + T_i(n_i) \tag{3.3}$$

denote the time of the n_i^{th} malfunction of the i^{th} component where $n_i \geq 1$ is given and where $T_i(k)$ is the exponentially distributed time between the $(k-1)$st and the k^{th} malfunction of the i^{th} component.

Now the data (t_i, n_i) means having observed the event

$$[S_i \leq t_i] \cap [S_i + T_i(n_i+1) > t_i]. \tag{3.4}$$

In words; the n_i^{th} malfunction took place on, or before time t_i and the (n_i+1)st malfunction had not occurred at that time.

If $n_i \geq 1$, then we must have observed the event

$$[S_i = s] \cap [T_i(n_i+1) > t_i - s] \tag{3.5}$$

for some $0 < s < t_i$. Or on the other hand if $n_i = 0$, we must have observed the event

$$[T_i(1) > t_i] \tag{3.6}$$

Now the density of S_i is the gamma density, (p. 74, ref. [7]) given by

$$\frac{\lambda_i^{n_i} s^{n_i-1} e^{-\lambda_i s}}{(n_i-1)!} \quad \text{for} \quad s > 0 \tag{3.7}$$

which one may use to evaluate the probability of the events (3.4), (3.5) and (3.6). We defer this until later.

4. The Asymptotic Maximum Likelihood Method

In case interpretation I holds, one can verify easily from the density given in (3.2) that the maximum likelihood estimate of λ_i is $\hat{\lambda}_i = n_i/t_i$. But also under interpretation II the likelihood of the event (3.5) using the density in (3.7) is, except for some constant not depending upon λ_i

$$n_i \ln\lambda_i + (n_i-1)\ln s - \lambda_i s - \lambda_i(t_i-s) \tag{4.0.1}$$

and the maximum likelihood estimate of λ_i is again $\hat{\lambda}_i$ as given above. Moreover the probability of the event (3.6) is $e^{-\lambda_i t_i}$, and the value of λ_i which maximize this is $\hat{\lambda}_i = 0$.

Thus under either interpretation of the data we obtain the maximum likelihood estimate of λ_i is

$$\hat{\lambda}_i = \frac{n_i}{t_i} \tag{4.1}$$

and we note that $\hat{\lambda}_i = 0$ if $n_i = 0$.

Let $\lambda = (\lambda_1, \ldots, \lambda_m)$ denote the vector of component rates and $\hat{\lambda}$ the corresponding vector of maximum likelihood estimates.

We now state the appropriate

Theorem 2. *If* $\hat{\lambda}$ *is the maximum likelihood estimate of the true component hazard rate vector* λ, *then* $g(\hat{\lambda})$ *is asymptotically normal with mean* $g(\lambda)$ *and variance*

$$\sigma^2 = \sum_{j=1}^{m} [\partial_j g(\lambda)]^2 \operatorname{var}(\hat{\lambda}_j)$$

whenever the partial derivatives of g *exist.*

The proof is well known and is given in many texts, e.g.[12]. To utilize this theorem we take $g(\lambda) = \sum_1^m \beta_i \lambda_i$ and first make Interpretation I. It follows from $\hat{\lambda}_i = N_i/t_i$ where N_i is a Poisson random variable with mean $(\lambda_i t_i)$, that

$$\operatorname{var}(\hat{\lambda}_i) = \frac{1}{t_i^2} \operatorname{var}(N_i) = \frac{\lambda_i}{t_i}$$

$$\sigma^2 = \sum_1^m \beta_i^2 \frac{\lambda_i}{t_i} .$$

Now we make the notational convention to be used subsequently that

$$\tau_i = \frac{t_i}{\beta_i} \qquad i = 1,\ldots,m. \tag{4.2}$$

Hence we can estimate the standard deviation by

$$\hat{\sigma} = \sqrt{\sum_1^m N_i \tau_i^{-2}} .$$

Thus by theorem 2 and the consistency of $\hat{\sigma}$ we know that $[g(\hat{\lambda}) - g(\lambda)]/\hat{\sigma}$ is asymptotically a normal zero, one random variable. (Strictly speaking the Theorem 2 is not needed with this interpretation since the sum of Poisson random variables is known to be asymptotically normal.)

Suppose we now make interpretation II, restricting our attention to the case when the event $[S_i = t_i]$ is observed. Since S_i has the density given in (3.7) one checks that for $b = 1,2$, given that $n_i \geq 3$

$$ES_i^{-b} = \lambda_i^b / \prod_{j=1}^{b} (n_i - j) .$$

From this one shows that

$$\text{var}(\hat{\lambda}_i) = \frac{n_i \lambda_i}{n_i - 1}^2 \left[\frac{1}{n_i - 2}\right] .$$

If we substitute $\hat{\lambda}_i$ for λ_i in the expression for σ^2 we obtain the estimate

$$\hat{\sigma}^2 = \sum_{i=1}^{m} n_i / \tau_i^2 (1 - \frac{1}{n_i})(1 - \frac{2}{n_i})$$

which can be compared with the estimate for σ^2 obtained under interpretation I. One sees they are nearly the same only if n_i, which is required to exceed 2, is large.

We then have, under either interpretation,

$$(4.3) \qquad P\left[\frac{[g(\hat{\lambda}) - g(\lambda)]}{\hat{\sigma}} < - z_\varepsilon\right] \doteq 1 - \varepsilon,$$

where z_ε is the $100\varepsilon^{\text{th}}$ percentile of the standard normal. By algebra we find (4.3) is equivalent with

$$P\left[e^{-g(\lambda)} > e^{-W\varepsilon}\right] \doteq \varepsilon$$

where W_ε is a random variable given, say under interpretation I, by

$$W_\varepsilon = \sum_1^m \frac{N_i}{\tau_i} + z_\varepsilon \sqrt{\sum_1^m \frac{N_i}{\tau_i}} \quad . \tag{4.4}$$

Thus for any ε (for our application near one, say .95), e^{-W_ε} gives asymptotically a lower confidence bound of level ε on the true reliability. Notice that if no failures occur for the i^{th} component no account is taken of them in the equation (4.4) and indeed under interpretation II we must have at least three failures to take account of a component.

5. The Asymptotic Likelihood Ratio Method

Again we make interpretation I and examine the data. The total test time t_j, we consider as a parameter, and the number of malfunctions is a random variable N_j. As is well known, p. 10 ref. [7], the number of malfunctions in an exposure period of length t_j has a distribution given by (3.2). Using the notation of section 4, the likelihood function, given the parameter vector λ, is

$$L^*(\lambda) = \sum_{j=1}^m \{-\lambda_j t_j + n_j \ln(\lambda_j t_j) - \ln(n_j!)\}.$$

The reliability for the system with a given mission length and given β-factors is

$$h(\lambda) = \exp\{-\sum_{j=1}^m \lambda_j \beta_j\}.$$

We now state

Theorem 3. *If the hypothesis* $h(\lambda) = r$ *is true then the likelihood ratio,* $-2\ell nL(r)$, *where*

$$L(r) = \sup_{\{\lambda : h(\lambda) = r\}} L^*(\lambda) \qquad -L^*(\hat{\lambda}) \tag{5.1}$$

has asymptotically a Chi-square distribution with one degree of freedom.

A proof is found in many texts for example Wilks, p. 419 [15]. At this juncture we notice, were we to make interpretation II, that the likelihood function given in (4.0.1) of the event defined in (3.5) is the same as $L^*(\lambda)$, except for some constant not depending upon λ_i. Hence whether we make interpretation I or II the function $L(r)$ defined in (5.1) would be the same.

Since $L^*(\hat{\lambda})$ is the supremum of $L^*(\lambda)$ over all admissible values of λ it is easily found. The first term is more difficult since we seek the maximum of $L^*(\lambda)$ under the constraint $h(\lambda) = r$ with r fixed. We follow a procedure for binomial data first proposed by Madansky in [9] for series systems and then generalized to arbitrary systems in [10]. The procedure is to maximize L^* under the constraint by using a LaGrange multiplier δ. Setting

$$\frac{\partial}{\partial \lambda_j}[L^*(\lambda) - \delta \ell n h(\lambda)] = -t_j + \frac{n_j}{\lambda_j} + \delta \beta_j = 0,$$

and solving yields

$$\lambda_j(\delta) = \frac{n_j}{t_j - \delta \beta_j} \qquad j=1,\ldots,m.$$

Call $\lambda(\delta)$ the vector solution. Note that $\hat{\lambda} = \lambda(0)$. We now define

$$\Lambda(x) = L^*[\lambda(x)] - L^*(\hat{\lambda})$$

$$= \sum_{j=1}^{m} n_j \left[1 - \frac{\tau_j}{\tau_j - x} + \ell n\left(\frac{\tau_j}{\tau_j - x}\right)\right]$$

where we have used the definition (4.2). We want to solve for the value of $x > 0$ such that for a given level ϵ

$$\Lambda(x) = -\frac{1}{2}\chi^2_\epsilon(1) \tag{5.2}$$

where $\chi^2_\epsilon(1)$ is the $100\epsilon^{th}$ percentile of the Chi-square random variable with one degree of freedom. Call the solution of (5.2) the value x_ϵ. Then

$$h[\lambda(x_\epsilon)] = \exp\{-\sum_1^m \frac{n_j}{\tau_j - x_\epsilon}\} \tag{5.3}$$

is an asymptotic lower bound for $h(\lambda)$ of level $\frac{1+\epsilon}{2}$ rather than of level ϵ, since we obtain only a one sided confidence bound.

We now exhibit a method for the determination of x_ϵ. For a given ϵ, set $B_\epsilon = \sum_1^m n_i + \frac{1}{2}\chi^2_\epsilon(1)$, then for each $B_\epsilon > 0$ there exists an $x > 0$ such that $f(x) = 0$, where $f(x) = -\Lambda(x) - \frac{1}{2}\chi^2_\epsilon(1)$.

We can write

$$f(x) = \sum_{i=1}^m n_i\left[\frac{\tau_i}{\tau_i - x} - \ell n\left(\frac{\tau_i}{\tau_i - x}\right)\right] - B_\epsilon .$$

The solution of $f(x) = 0$ must be unique since

$$f'(x) = x\sum_{i=1}^m \frac{n_i}{(\tau_i - x)^2} > 0$$

whenever $0 < x < \tau_{(1)} = \min(\tau_1, \ldots, \tau_m)$.

But note that

$$f''(x) = \sum_{i=1}^m n_i \frac{(\tau_i + x)}{(\tau_i - x)^3} .$$

Thus we see that both f', f'' are continuous and do not vanish for $0 < x < \tau_{(1)}$, which is a sufficient condition that the Newton iteration procedure, namely

$$x_n = x_{n-1} - \frac{f(x_{n-1})}{f'(x_{n-1})} \qquad n=1,2,\ldots$$

will converge to the value x_ε.

6. Bayesian Confidence Intervals

Let us now outline the Bayesian method of obtaining confidence limits. Suppose that $\pi(\lambda)$ is a given prior density of the parameter λ, here taken as one-dimensional for illustrative purposes. If λ is known then a sample x of independent identically distributed observations from a population with that parametric value is $p(x|\lambda)$. Then, following Lindley [8], pp. 1 and 2, the posterior density of λ based on the evidence x is

$$f(\lambda|x) \propto p(x|\lambda)\pi(\lambda).$$

Now if $I_\varepsilon(x)$ is any interval of λ depending on x and ε, $0 < \varepsilon < 1$, such that

$$\int_{I_\varepsilon(x)} f(\lambda|x)d\lambda = \varepsilon, \tag{6.1}$$

then $I_\varepsilon(x)$ (p. 15, Lindley, loc. cit.,) is called a *100ε% Bayesian confidence interval* of λ for given x.

If we desire a Bayesian confidence interval on $\lambda = \lambda_1 + \lambda_2$ and we have posterior densities of λ_1, λ_2, call them f_1 and f_2 then from an assumption of independence of these posterior densities one can obtain,

via the calculus of probabilities, a posterior density of $\lambda_1 + \lambda_2$ as the convolution

$$f_1 * f_2(\lambda) = \int_0^{\lambda} f_1(\lambda - t) f_2(t)\, dt$$

and use $f_1 * f_2$ to calculate an upper confidence bound on λ from (6.1). This is essentially the method that has been recently utilized to obtain confidence bounds on the reliability of series systems and is presently the subject of much interest. (See [14], [16] and the references given there.)

If the data $x = (x_1, \ldots, x_m)$ where $x_i = (t_i, n_i)$ is given we may make either interpretation I or II. We decide whether one regards the time until the n_i^{th} malfunction as an observation from a random variable with parameter $n_i (\neq 0)$ fixed or the number of malfunctions n_i as an observed value of a random variable when the time of testing t_i is given. In each interpretation as we now show we are lead to the following: the probability of observing the event x_i given a value of λ_i is

$$p(x_i | \lambda_i) = \frac{e^{-\lambda_i t_i} (\lambda_i t_i)^{n_i}}{n_i!} . \tag{6.2}$$

Under interpretation I we regard x_i as the event $[N_i = n_i]$ where N_i is the Poisson random variable with density (2.2). Hence we have (6.2) directly.

Under interpretation II, n_i is fixed and the time until the (n_i+1)st malfunction is the random variable $S_{i+1} = S_i + T_i(n_i+1)$. Now x_i becomes the event (3.4), and we want to evaluate

$$p(x_i | \lambda_i) = P[S_i \leq t_i, S_{i+1} > t_i].$$

But the right hand side is the difference of the convolutions $G^{n_i*}(t_i) - G^{(n_i+1)*}(t_i)$, where G is the exponential distribution of the time between failures with hazard rate λ_i. This difference, p. 11 [7], is equal to (6.2).

From Bayesian Principles, using either interpretation I or II, one obtains the same joint posterior density of $(\lambda_1,\ldots,\lambda_m)$

$$f(\lambda_1,\ldots,\lambda_m|x_1,\ldots,x_m) \propto \prod_{i=1}^{m} p(x_i|\lambda_i)\pi(\lambda_1,\ldots,\lambda_m)$$

where π is the *joint prior density* of $(\lambda_1,\ldots,\lambda_m)$.

7. The Principle of Insufficient Reason

I now mention an approach which has been called a major breakthrough in the analysis of confidence levels for system reliability. This is because it leads to an answer which is theoretically exact and does not depend upon an approximation to an asymptotic distribution.

This approach is the Bayesian method with the special prior density

$$\pi(\lambda) \equiv 1 \quad \text{for all} \quad \lambda_i > 0. \tag{7.1}$$

This assumption is justified by the so called *principle of insufficient reason*; since we know nothing specific about π we have insufficient reason to take $\pi(\lambda)$ anything but uniform. Strictly speaking π as defined in (7.1) is a non-probabilistic prior. But of course one could consider it an approximation to a prior density. There have been several attempts made to use this general approach for confidence intervals, in particular see [1].

However from (7.1) we have

$$f(\lambda|x) = \prod_{i=1}^{m} t_i \frac{(\lambda_i t_i)^{n_i}}{n_i!} e^{-\lambda_i t_i} \quad \text{for } \lambda_i > 0.$$

The mathematical problem becomes that of finding the distribution of

$$V = \sum_{1}^{m} \beta_i \lambda_i \tag{7.2}$$

where β_i are known constants and λ_i are fiducial* gamma variates with known scale and shape parameters. To wit, each λ_j is $\Gamma(t_j, n_j+1)$ where $\Gamma(t,\nu)$ denotes the law with density , give $\nu > 0$

$$\frac{1}{\Gamma(\nu)} t^{\nu} x^{\nu-1} e^{-tx} \quad \text{for } x > 0. \tag{7.3}$$

We also quote two related results: Feller II loc. cit.

If λ_j is $\Gamma(t_j, \nu_j)$ then $t_j\lambda_j$ is $\Gamma(1, \nu_j)$

If $\nu_j > 1$ then $\lambda_j = \lambda_j' + \lambda_j''$ where λ_j' is $\Gamma(t_j, 1)$ independent of λ_j'' which is $\Gamma(t_j, \nu_j - 1)$.

Thus by the first remark we see that in distribution

$$V = \sum_{1}^{m} b_j \lambda_j$$

where

$$b_j = \frac{\beta_j}{t_j} = \frac{1}{\tau_j} \qquad j=1,\ldots,m$$

and each λ_j is now $\Gamma(1, n_j+1)$. By the second remark, for the data, given in Table I from the S1-C, (where we have at most two failures) we see that in distribution

---

* We shall use the word "fiducial" to apply to variates which are the construct of prior knowledge and hence represent, to a degree, personal belief.

$$V = \sum_{1}^{m} b_i Z_i + \sum_{1}^{r} b_j Y_j + \sum_{1}^{s} b_k X_k \tag{7.4}$$

where s is the number of components with two failures during testing

$r-s$ is the number of components with one failure during testing

$m-r-s$ is the number of components with no failures during testing

and X_k, V_j, Z_i are all independent $\Gamma(1,1)$ i.e. exponential with unit mean, variates.

We now quote a well known result proved, for example, in [13] as a Lemma 1: If $Z_1,\ldots,Z_k$ are independent exponential random variables with unit mean then for $b_i > 0$, all distinct, we have

$$P[\sum_{1}^{k} b_i Z_i > u] = \sum_{j=1}^{k} B_j^{(k)} e^{-u/b_j} \tag{7.5}$$

where $B_1^{(1)} = 1$ and for $k \geq 2$

$$B_j^{(k)} = \prod_{\substack{i=1 \\ i \neq j}}^{k} \frac{b_j}{b_j - b_i} \quad \text{for } j=1,\ldots,k. \tag{7.5.1}$$

Also these recursion relations hold

$$B_j^{(k)} = B_j^{(k-1)} b_i/(b_j - b_k) \quad j=1,\ldots,k-1 \quad \text{and} \quad B_k^{(k)} = 1 - \sum_{j=1}^{k-1} B_j^{(k)}.$$

Also we have

Lemma 2: The distribution of $\sum_{1}^{m} b_i Z_i + \sum_{1}^{r} b_i Y_j$ is

$$\sum_{i=1}^{m} \sum_{j=1}^{r} B_i^{(m)} B_j^{(r)} \{1 - e^{-c_j t} - c_j \psi(i,j,t)\} \quad \text{for } t > 0,$$

where we set $c_j = 1/b_j$ for $j=1,\ldots,\max(m,r)$

$$\psi(i,j,t) = \int_0^t \exp\{-c_i(t-y) - c_j y\}dy$$

$$= \begin{cases} te^{-c_i t} & \text{if } i = j \\ \dfrac{e^{-c_i t} - e^{-c_j t}}{c_j - c_i} & \text{if } i \neq j \,. \end{cases}$$

The proof is accomplished by the convolution of two distributions each of the form given in lemma 1. Consider the more general definition

$$V_k = \sum_{j=1}^{k} \sum_{i=1}^{m_j} b_i X_{ij}$$

where the X_{jj} are all independent exponential variates with unit mean. Let V_k have distribution F_k then

$$V_k = V_{k-1} + \sum_{i=1}^{m_k} b_i X_{ik} \,.$$

Defining $\bar{F} = 1 - F$ with any affixes, and taking c_i as given in lemma 2,

$$\bar{F}_k(u) = \int_0^\infty P\left[\sum_{i=1}^{m_k} b_i X_{ik} > u-v\right] dF_{k-1}(v)$$

$$= \bar{F}_{k-1}(u) + \sum_{i=1}^{m_k} B_i^{(m_k)} \left\{ \int_0^u e^{-(u-v)c_i} \, dF_{k-1}(v) \right\} .$$

But the quantity in braces in the equation above becomes

$$\{\ldots\} = \bar{F}_{k-1}(u) - e^{-uc_i} - c_i \int_0^u \bar{F}_{k-1}(v) e^{-(u-v)c_i} dv \,.$$

Hence we have shown the following

Lemma 3: The survival probability of V_k as defined in equation (7.6) is given in terms of the survival probability of V_{k-1} as

$$\bar{F}_k(u) = \sum_{i=1}^{m_k} B_i^{(m_k)} \left[e^{-uc_i} + \tau_i \int_0^u \bar{F}_{k-1}(v) e^{-(u-v)c_i} dv \right] . \tag{7.7}$$

Note that (7.7) can be used to prove lemma 2 and used recursively, by computing machine, to find the distribution of V_k easily for small k.

8. Malfunction Modes

A subsystem containing several components can fail by having different components fail but moreover each component itself may fail in different ways. For example a gimbal may shatter or crack, or a hose coupling can either rupture or leak. These separate ways of failure are called failure modes. Of course we talk about failure modes in an intuitive sense when in fact we mean the ways a component can malfunction and we do not mean to imply that a malfunction need necessarily cause a system failure. We shall without deference to established convention use the nomenclature *malfunction modes*.

We assume

4^0 The time until malfunction in each mode has constant hazard rate for each component and all are independent.

Suppose we separate the possible modes of malfunction for the j^{th} component into k classes which are functionally independent, then label the time until malfunction T_{ij} for the i^{th} mode of the j^{th} component. The time until malfunction of the component by any mode is

$$T_j = \min(T_{1j}, \ldots, T_{kj})$$

and the hazard rate of T_j is $\lambda_j = \sum_{i=1}^{k} \lambda_{ij}$ with the obvious interpretation of λ_{ij} as the hazard rate of T_{ij}. Consequently we can substitute either component hazard rates or hazard rates by separate failure modes interchangeably within the fundamental series system equation and use the same mathematical treatment. The introduction of this classification by failure mode may be an extension of the model for classification by components or it might be a refinement.

Without any real loss of clarity to the fundamental ideas let us fix our att ntion on a component with two separate modes of malfunction with fixed hazard rates λ_1 and λ_2, say.

Suppose this component was operated for a time t and no failure of either type were observed. Hence by the first Bayesian method the posterior fiducial distribution of the component hazard rate $\lambda(=\lambda_1+\lambda_2)$ considering the component as a unit, is

(8.1) $$1 - e^{-at} \quad \text{for} \quad a > 0.$$

Note that for a mission of length t

$$F[\lambda < a] = P[e^{-\lambda t} > e^{-at}]$$

so that e^{-at} is a lower bound on the mission reliability and the confidence level is $P[\lambda < a]$.

But on the other hand by considering the posterior fiducial distribution of $\lambda_1 + \lambda_2$ using data $x_1 = x_2 = (t,0)$ and the first Bayesian method (and the principle of insufficient reason) we have

(8.2) $$1 - e^{-at} - at\, e^{-at} \quad \text{for} \quad a > 0$$

as the posterior fiducial distribution of the hazard rate of λ of the component. But notice that (8.2) is much less than (8.1) which was the fiducial distribution from the same data for the same component. If by changing our interpretation of what constitutes a component, we arrive at different answers when using the same data, then something must be wrong. In reliability theory one man's component is another man's system and the fiducial distribution from the same data should be the same regardless of the labeling of the components.

To continue this point further, let us suppose that we have a series system with separate malfunction modes with hazard rates $\lambda_1,\ldots,\lambda_k$ each of which has acquired the same operational experience namely $x_i = (t,0)$ for $i=1,\ldots,k$ i.e. no failures during operation for a length of time t. Again by using the principle of insufficient reason and the non-probabilistic prior density we have by the first Bayesian method the fiducial distribution of $\lambda = \lambda_1+\ldots+\lambda_k$ as

$$1 - \sum_{j=1}^{k} \frac{e^{-at}(at)^j}{j!}$$

which approaches zero as k approaches infinity regardless of the value of $ta > 0$.

This effect is why reliability managers say "the system is all right, it is the statistics that is killing us," see [3]. I maintain that a proper model in these cases should yield exactly the same fiducial confidence under any interpretation as to what set of components constitutes a subsystem. It is to this point that we continue to write.

9. Alpha Factors

We do not regard the λ_i as independent fiducial random variables having a distribution which is to be constructed from prior knowledge but essentially as constants which are unknown. If the λ_i were unknown positive real numbers then there would exist a constant of proportionality between any two λ_i's which would be fixed even though it was unknown.

Thus we make the assumption

5° There exists a constant of proportionality say α_{ij} between any two λ_i and λ_j.

If we have m different modes of malfunction we define α_i for $i=1,\ldots,m$ as the *probability of malfunction in the i^{th} mode given that a malfunction in the system has occurred.* One sees that

$$\alpha_i = P[\, T_i < t \mid \sum_{j=1}^{m} [T_j < t]],$$

where we made the convention that the summation of events denotes the disjoint union. It follows that

$$\alpha_i = \frac{\lambda_i}{\sum_{j=1}^{m} \lambda_j} \qquad i=1,\ldots,m.$$

Thus 5° is equivalent to taking the prior distribution, say $\Pi(\lambda)$, to be singular with all measure concentrated along a ray out from the origin with the direction of the ray determined by the constants of proportionality. Specifically we assume

$$(9.1) \qquad d\Pi(\lambda_1,\ldots,\lambda_m) \begin{cases} > 0 & \text{if some } \lambda_i = \dfrac{\alpha_i \lambda_j}{\alpha_j} \quad \text{for } i \neq j \\ = 0 & \text{otherwise.} \end{cases}$$

In the case $m = 2$, $\Pi(\lambda_1,\lambda_2)$ is zero everywhere but along the ray $\lambda_2 = \frac{\alpha_2}{\alpha_1}\lambda_1$ out from the origin in the (λ_1,λ_2) plane.

Following the general Bayes procedure

$$f(\lambda|x) \propto \prod_{i=1}^{m} (t_i\lambda_i)^{n_i} e^{-t_i\lambda_i}\, d\Pi(\lambda_1,\ldots,\lambda_m).$$

We wish to find the posterior density of $\sum_1^m \beta_i\lambda_i$. We make the change of variables $\rho_i = \beta_i\lambda_i$ and by (3.2) and $\tau_i = t_i/\beta_i$ we have

$$f(\rho|x) \propto \sum_{i=1}^{m} (\tau_i\rho_i)^{n_i} e^{-\tau_i\rho_i}\, d\Pi^*(\rho_1,\ldots,\rho_m)$$

where

$$\Pi^*(\rho_1,\ldots,\rho_m) = \Pi(\frac{\rho_1}{\beta_1},\ldots,\frac{\rho_m}{\rho_m}).$$

Thus

$$(9.2)\qquad d\Pi^*(\rho_1,\ldots,\rho_m) \begin{cases} > 0 & \text{if some } \rho_i = \dfrac{\alpha_i\beta_i\rho_j}{\alpha_j\beta_j} \quad \text{for } i \neq j \\ = 0 & \text{otherwise.} \end{cases}$$

The density we seek is proportional to

$$(9.3)\qquad \int\cdots\int_{\{\rho:\ \Sigma\rho_i=\alpha\}} \prod_{i=1}^{m} (\tau_i\rho_i)^{n_i} e^{-\tau_i\rho_i}\, d\Pi^*(\rho_1,\ldots,\rho_m)$$

Consider the line in m-space

$$\ell(\rho_1) = (\rho_1, \frac{\alpha_2\beta_2}{\alpha_1\beta_1}\rho_1,\ldots,\frac{\alpha_m\beta_m}{\alpha_1\beta_1}\rho_1).$$

By equation (9.2) all the mass of Π^* is concentrated along the ray $\ell(\rho_1)$ for $\rho_1 > 0$. In effect the only quantity that has a distribution is ρ_1 and we shall later see it makes no difference what this distribution is

as long as it has support on $(0,\infty)$. This line intersects the plane $\sum_1^m \rho_i = a$ at a single point, namely ρ_1 such that

$$\rho_1 + \sum_{i=2}^{m} \frac{\alpha_i \beta_i}{\alpha_1 \beta_1} \rho_1 = a$$

and solving for ρ_1 we find $\rho_1 = \gamma_1$ where we define

$$\gamma_i = \frac{\alpha_i \beta_i}{\sum_{j=1}^{m} \alpha_j \beta_j} \qquad i=1,\ldots,m. \tag{9.3.1}$$

Then the value of ρ_i at the point of intersection of the line ℓ with the plane $\Sigma\rho_i = a$ is $\rho_i = a\gamma_i$ for $i=1,\ldots,m$. Since all the measure of Π^* is concentrated along the line ℓ, the integration over the plane in (9.3) yields a single value at the singularity of the measure Π^*. It follows that the density we seek is proportional to the value of the integrand at that point, namely

$$\prod_{i=1}^{m} (\tau_i a\gamma_i)^{n_i} \exp\{-\sum_{i=1}^{m} \tau_1 a\gamma_i\}.$$

If we define

$$\theta = \sum_{i=1}^{m} \tau_i \gamma_i = \frac{\sum t_i \alpha_i}{\sum \beta_i \alpha_i} \quad \text{and} \quad k = \sum_1^m n_i, \tag{9.4}$$

which are a weighted mean of the t_i, and

the total number of failures, respectively, we can write the posterior fiducial density of $\sum_1^m \rho_i$ as

$$\frac{\theta(\theta a)^k e^{-a\theta}}{k!} \quad \text{for} \quad a > 0. \tag{9.5}$$

The distribution then is

$$P[\Sigma\rho_i < u] = \int_0^{\theta u} \frac{s^k e^{-s}}{k!}\, ds = 1 - \sum_{j=0}^{k} \frac{e^{-\theta u}(\theta u)^j}{j!}. \tag{9.6}$$

But we recognize this as a Chi-square distribution and if we set

$$u = \frac{1}{2\theta}\chi_\varepsilon^2(2k+2),$$

where we have defined this notation in (5.2), we have

$$P[e^{-\Sigma\rho_i} > e^{-u}] = \varepsilon.$$

And this provides a lower confidence bound of level ε.

We now make some observations about this result.

1) Notice the confidence bound is the same regardless of how the components are apportioned to subsystems within the system. In particular if $t_1 = \cdots = t_m$ we obtain the same fiducial density of $\Sigma\beta_i\lambda_i$ as we would by considering the system as a single unit.

2) The addition of components to the system none of which have failed, i.e. data of the type $(t_i,0)$, do not necessarily cause the confidence to go rapidly to zero, (of course the confidence does depend upon t_i through θ). It is clear from (9.5) that it is not the number of components but the number of failures which force the confidence to zero.

3) A knowledge of α-factors is required to utilize this posterior density. However this does not mean that the values of λ_i need be known but merely

that engineering experience be used to classify all the failure rates as multiples of a fixed one, say the lowest one. Usually that can be done at least in a conservative manner. But we assert that if one believes 5° to be true then engineering knowledge should be expended to obtain information about the α_j's rather than elsewhere.

4) It is apparent from (9.4) that increasing the length of testing for a component for which α_i is high will yield the greatest improvement in confidence, provided no failures are observed.

5) In the special case when $\beta_i^o(t) \equiv \beta_i$ for all $t > 0$ we can make an intuitive interpretation of γ_i as the conditional probability of failure of the i^{th} component given that a component has failed.*

---

* Label the events F_i, the i^{th} component fails and M_i, the i^{th} component malfunctions. Now by definition

$$\beta_i = P[F_i|M_i] \qquad \alpha_i = P[M_i|\Sigma M_j]$$

and from the calculus of probabilities, since $F_i \subset M_i$

$$\alpha_i\beta_i = \frac{P(F_i)}{P[\Sigma M_j]}$$

and hence from (9.3.1) follows $\gamma_i = P[F_i|\Sigma F_i]$.

10. Numerical Comparisons and Commentary

In this section we perform the actual computations for the 95% lower confidence limits on the reliability of the Saturn 1-C using some data which was available at a particular time during its preflight program. Since this will be an illustrative example we simplify the actual situation by taking $\beta_i^o \equiv 1$ so that $\beta_i = \nu_i$. The data is given in Table I.

Method (1) - Asymptotic Maximum Likelihood

Using interpretation II we can take account only of data for which $n_i > 2$ of which there is none. Using interpretation I, we can take account of only that data for which $n_i > 0$. These are

$$\begin{aligned} \tau_1 &= 32.5 & n_1 &= 2 \\ \tau_2 &= 7.5 & n_2 &= 2 \\ \tau_3 &= 51.8 & n_3 &= 1 \\ \tau_4 &= 17.7 & n_4 &= 1 \\ \tau_5 &= 8.0 & n_5 &= 1 \\ \tau_6 &= 7.5 & n_6 &= 1 \end{aligned} \tag{10.1}$$

Now set $\epsilon = .95$, $z_\epsilon = 1.645$ and by equation (4.4)

$$w_\epsilon = \sum_1^6 \frac{n_i}{\tau_i} + 1.645\sqrt{\sum_1^6 n_i \tau_i^{-2}}$$

$$= .663 + 1.645\sqrt{.0746} = .898$$

Hence based on the validity of the assumptions made, $e^{-w_\epsilon} = .406$ is the approximate 95% lower confidence bound on the true reliability of the system.

Method (2) - Asymptotic Likelihood Ratio

We can now make either interpretation however, as in Method 1, we can utilize only the data with $n_i > 0$ given in equation (10.1) above. Applying the Newton procedure as mentioned we obtain for this data the solution of equation (5.2), for $\epsilon = .90$, as $x_\epsilon = 3.84$. From equation (5.3) we find

$$\sum_{1}^{6} \frac{n_i}{\tau_i - x_\epsilon} = 1.23$$

Thus based on the validity of the assumptions for this case we find $e^{-1.23} = .292$ is an approximate 95% lower confidence limit for the system reliability.

It would seem that one could have no confidence, in the intuitive sense, in the confidence (in the precise sense) level which is obtained by either of these two methods. This is because no account is taken of the larger part of the data namely, those components for which no failure was observed during testing. Thus whether there were sixty or sixty thousand components in the system which experienced no failures during the test program the confidence limits would be the same.

Second, the conditions are just not known which are necessary to enable one to assume that the distribution of the statistics in question, for the finite sample sizes which are obtained, can be adequately approximated by their asymptotic distribution.

Method (3) - Bayesian Method with Uniform Prior

Using the computational methods set forth in section 7, a machine program was written for the IBM-360, using double precision for the computation

of the $B_j^{(k)}$, which tabulates the distribution of the random variable V defined in (7.4). This distribution is graphed in figure 1 from the machine run using the data presented in Table 1.

The fiducial random variable $V = \sum_1^m \beta_i \lambda_i$ has distribution F which we calculate in the region of interest. Since

$$P[\exp\{-\Sigma\beta_i\lambda_i\} \geq e^{-v}] = F(v)$$

we find that if $v = 12.8$, $F(v) = .95$ and then based on the assumptions necessary for this method the 95% lower Bayesian confidence limit for the system reliability is $e^{-12.8}$. This result needs no comment. Every person responsible for the reliability program had at least 95% confidence, in the intuitive sense, that the true reliability exceeded this number before any statistical analysis was initiated! Making confidence statements, such as the above, to managers of reliability programs tends to put statistics in a bad light. (Recall the statements just preceding section 7.)

A word about the computational difficulties of this method. It is clear from Table 1 that the differences of the τ_j are small compared with their magnitude. A glance at the formula for the $B_j^{(k)}$ in equation (7.5.1) shows that in absolute value they can become very large. (In fact for such data as we have for 70 components, values as high as 10^{20} are not impossible). Since all $B_j^{(k)}$ summed over j must add to unity, some must be positive and some negative. Hence at $v = 0$, by definition we should have $F(v) = 0$.

However because of the nature of the machine decimal arithmetic, the summands will be rounded off and the machine value will not be unity. For example referring to figure 1, had we continued to plot the values computed at $v = 3.2$, the machine computation was $F(3.2) = .699 \times 10^{-3}$ but at $v = 2.4$, $F(2.4) = -.123 \times 10^{-1}$ with wider fluctuations for smaller values of v. Fortunately we are interested in those values of the argument for which $F(v)$ is near one and the values of v necessarily become large enough to eliminate the errors due to this circumstance.

Method (4) - Alpha Factor Method

The computation necessary for this method is trivial: we need compute only the two quantities k and θ and then from a table of the Chi-square distribution calculate $u = \chi^2_\varepsilon(2k+2)/2\theta$ and e^{-u} becomes the lower confidence limit at level ε.

Example 1:

Let us suppose that $v_i = \frac{1}{m}$ for $i=1,\ldots,m$. We recall that under certain conditions this would mean the event any one particular component had failed, knowing that exactly one component was in a failed state, was equally likely with the event any other component had failed.

From Table I we find $k = 8$ and compute from (9.4), $\theta = \Sigma\tau_i/m = 25.5$ and hence for $\varepsilon = .95$, using the Chi-square value for 16 degreed of freedom, we have $u = (28.87)/51 = .565$. Thus $e^{-u} = .57$ is a lower 95% confidence limit for the systems reliability, based on the validity of the assumptions.

Example 2:

Let us suppose $\alpha_i = \frac{1}{m}$ for $i=1,\ldots,m$ and from Table I, we again use (9.4) to compute $\theta = (\Sigma t_i)/(\Sigma \beta_i) = 16.27$. For $k = 8$, $\varepsilon = .95$ we find $u = (28.87)/(32.54) = .887$ and $e^{-u} = .412$ is the lower 95% confidence limit for the reliability of the system.

Example 3:

Assume the failure rate λ_i is proportional to the complexity of the component. Thus α_i is proportional to the (fictitious) weights given in the last column in Table I, to wit, we take $w_i = K\alpha_i$. We then use (9.4) to compute $\theta = (\Sigma t_i \alpha_i)/(\Sigma \beta_i \alpha_i) = 32.11$. For $k = 8$, $\varepsilon = .95$, $u = (28.87)/(64.22) = .45$ and $e^{-u} = .64$ is the lower 95% confidence limit for the reliability of the system.

One notices that for ease of computation method (4) is to be preferred since it does not require more than a desk calculator. Moreover the answers seem to be uniformly higher than those obtained by the other methods considered. Lastly, the three examples show there is a robustness concerning the value of v_i that are obtained. This helps to allay the fears of limitation of this method, namely, the necessity of exact determination of the α_i. Nonetheless, knowledge of the α_i are what is really needed to determine the confidence limit, thus it would seem that engineering and/or statistical effort should be expended in an attempt to determine them (or approximate bounds) rather than being expended in the determination of the parameters of prior distribution of the components failure rates.

Concerning the alpha factor method the author's sentiments are best described by a remark of L.J. Savage. "If Bayesian techniques are applied to situations in which they give ridiculous answers I want no part of them, however if they can be utilized to give wise answers in situations which cannot be treated by other methods then I favor their use."

TABLE 1

Summary of test data for Saturn I-C

t_i = test time in mission lengths, n_i = number of failures observed

$\beta_i = \nu_i$ = component multiplicity, $\tau_i = t_i/\beta_i$

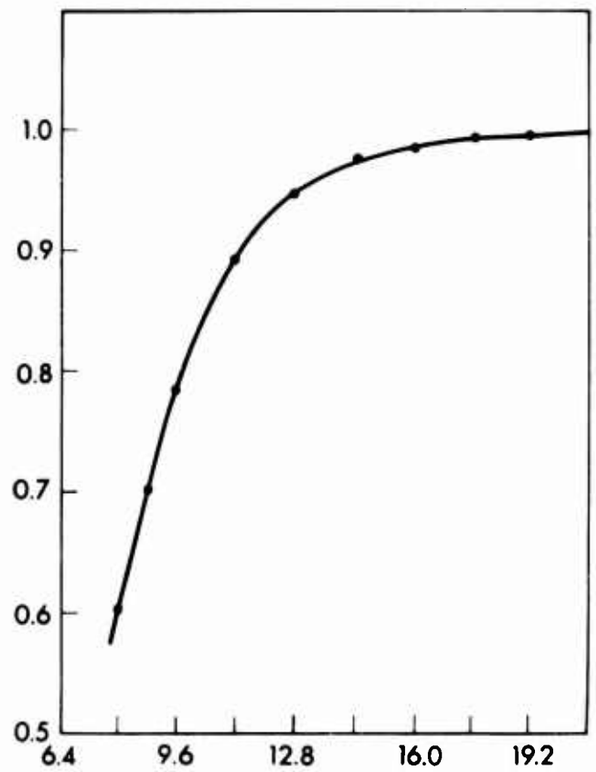
w_i = weights of relative complexity

t_i	n_i	β_i	τ_i	w_i	t_i	n_i	β_i	τ_i	w_i
318.5	0	16	19.9	5	48.7	0	1	48.7	100
138.8	0	16	8.7	5	33.9	0	1	33.9	100
69.4	0	16	4.3	5	30.2	0	1	30.2	100
159.2	0	4	39.8	100	45.7	0	1	45.7	100
187.9	0	8	23.5	100	36.5	0	1	36.5	100
144.9	0	4	36.2	100	50.6	0	2	28.3	100
69.7	0	4	17.4	10	45.2	0	2	22.6	100
148.7	0	4	37.2	100	22.6	0	2	11.3	10
146.8	0	4	36.7	100	37.7	0	2	18.8	10
15.1	2	2	7.5	5	49.9	0	1	49.9	100
7.5	0	12	.63	1	34.3	0	1	34.3	100
120.7	0	4	30.2	100	37.7	0	1	37.7	100
113.1	0	3	37.7	100	11.3	0	1	11.3	100
98.1	0	1	98.1	100	15.1	0	2	7.5	1
92.8	0	1	92.8	100	226.1	0	30	7.5	1
9.0	0	1	9.0	5	30.3	0	10	3.0	10
97.9	0	13	7.6	5	32.0	0	2	16	10
87.3	0	4	21.8	5	179.0	0	8	22.4	25
26.4	0	1	26.4	25	32.5	2	1	32.5	5
83.7	0	4	20.9	25	75.4	0	10	7.5	5
14.1	0	1	14.1	5	191.9	0	8	24	5
41.5	0	1	41.5	5	34.6	0	2	17.3	25
7.5	0	2	3.8	5	17.7	1	1	17.7	25
15.1	0	3	5.0	5	73.6	0	4	18.4	25
11.3	0	2	5.7	5	88.6	0	4	22.2	25
29.3	0	5	5.9	5	8.0	1	1	8.0	5
82.9	0	1	82.9	1	1.6	0	3	.5	1
7.5	1	1	7.5	1	2.4	0	6	.4	1
20.8	0	2	10.4	1	18.9	0	1	18.9	25
52.8	0	1	52.8	1	11.1	0	1	11.1	25
51.8	1	1	51.8	100	14.9	0	1	14.9	25
65.3	0	1	65.3	100	13	0	1	13	25
66.2	0	1	66.2	100	7.3	0	1	7.3	5
65.2	0	2	32.6	100					

Figure 1

Graph of the distribution F of the fiducial variable

$V = \sum \beta_i \lambda_i$ in the region near unity



If v is the abscissa value, the value F(v) of the ordinate is the confidence the system reliability exceeds e^{-v}.

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