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MULTIPLE COMPARISONS WITH A CONTROL FOR MULTIPLY-CLASSIFIED VARIANCES OF NORMAL POPULATIONS

by

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MULTIPLE COMPARISONS WITH A CONTROL FOR MULTIPLY-CLASSIFIED VARIANCES OF NORMAL POPULATIONS*

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Abstract

Dunnett has prepared tables for making multiple comparisons with a control for single-factor (or multi-factor) experiments involving means of normal populations. In the present paper the author uses the multiplicative model for variances which he introduced in an earlier paper, as the basis for making multiple comparisons with a control for multi-factor experiments involving variances of normal populations. A limited set of tables for use with this procedure is included.

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1. Introduction

In [1] it was shown how to design experiments for ranking multiply-classified variances of normal populations. For such experiments there will, of course, be situations in which ranking of the "effects" is not the objective of the experimenter, but rather he would like to estimate these effects. The purpose of the present note is to show how the theory developed in [1] provides a direct method of making certain joint confidence interval estimates of the effects. We shall use the notation of [1], and in particular that of Section 2.

2. Percentage points for joint interval estimation

For fixed $a \geq 2$, $b \geq 1$, $n \geq 1$, and for $i = 2,3,\ldots,a$ we define the $a-1$ chance variables

$$(1) \qquad G_n^{(i),b} = \prod_{j=1}^{b} \chi_n^2(i,j) \,/\, \chi_n^2(1,j) \; ,$$

where the $\chi_n^2(i,j)$ $(1 \leq i \leq a;\ 1 \leq j \leq b)$ are $a \cdot b$ independent chi-square variates, each based on n d.f. (Thus for fixed i, $G_n^{(i),b}$ is the product of b independent central F-statistics, each based on (n,n) d.f.; for $i_1 \neq i_2$, $G_n^{(i_1),b}$ and $G_n^{(i_2),b}$ are correlated.) For $0 < \lambda < 1$ let $G_n^{a,b}(1-\lambda)$ be defined by the equation

$$(2) \qquad P\{G_n^{a,b}(1-\lambda) < G_n^{(i),b} \qquad (i = 2,3,\ldots,a)\} = 1-\lambda \; .$$

We now show how the $G_n^{a,b}(1-\lambda)$ can be used to make certain types of inferences concerning the effects α_i defined by (2.1) of [1] when (2.2) holds. As in Section 3.3 of [1], we let $\dot{s}_{ij}^2$ denote the sample variance based on n d.f. associated with the i^{th} "level" of Factor A and the j^{th} "level" of Factor B.

3. Confidence statements

3.1 Multiple comparisons with a control

Suppose that the first level of Factor A is a "control" level, and that the remaining are "test" levels, i.e., that α_1 is the effect associated with the control level, and that $\alpha_2, \alpha_3, \ldots, \alpha_a$ are the effects associated with the test levels. Then it may be desired to make a joint confidence statement concerning the effect-ratios α_i/α_1 $(i=2,3,\ldots,a)$. Clearly the following statement holds with confidence coefficient $1-\lambda$:

$$(3) \qquad \left\{ \frac{\alpha_i}{\alpha_1} < \left[\frac{1}{G_n^{a,b}(1-\lambda)} \prod_{j=1}^{b} \frac{\hat{s}_{ij}^2}{\hat{s}_{1j}^2} \right]^{1/b} \quad (i=2,3,\ldots,a) \right\} .$$

This is sometimes referred to as a (one-sided) multiple comparison with a control, and is the analogue for multiply-classified variances of [2] (wherein $b=1$) for means. (See also [3] (wherein $b=1$) for the corresponding two-sided comparisons for means.)

3.2 Two-sided estimates when a=2.

When $a=2$ and $(0 \leq \lambda_1, \lambda_2;\ \lambda_1 + \lambda_2 < 1)$, the following statement holds with confidence coefficient $1 - \lambda_1 - \lambda_2$:

$$(4) \qquad \left\{ \left[\frac{1}{G_n^{2,b}(\lambda_2)} \prod_{j=1}^{b} \frac{\hat{s}_{2j}^2}{\hat{s}_{1j}^2} \right]^{1/b} < \frac{\alpha_2}{\alpha_1} < \left[\frac{1}{G_n^{2,b}(1-\lambda_1)} \prod_{j=1}^{b} \frac{\hat{s}_{2j}^2}{\hat{s}_{1j}^2} \right]^{1/b} \right\} .$$

This is a two-sided interval estimate of α_2/α_1.

4. Evaluation of the constants $G_n^{a,b}(\lambda)$

Comparison of (2) with (4.1) and (6.2) of [1] shows that

(5a) $$P_A(n|a,b;\ \theta_A^*) = 1-\lambda$$

(5b) $$(1/\theta_A^*)^b = G_n^{a,b}(1-\lambda).$$

Thus, for fixed $a \geq 2$, $b \geq 1$, $n \geq 1$ and specified λ, one can determine θ_A^* uniquely from (5a); this value of θ_A^* substituted in (5b) then yields $G_n^{a,b}(1-\lambda)$. Clearly, for $a=2$ we have $1/G_n^{2,b}(1-\lambda) = G_n^{2,b}(\lambda) = (\theta_A^*)^b$.

We have used these relationships and the formulae of Section 4.1 of [1] to determine exact values of $G_n^{2,b}(\lambda)$ for λ = 0.25, 0.15, 0.10, 0.05, 0.025, 0.01, 0.005, 0.001, 0.0005; Table I gives the G-values for b=2 and n=1(1)6,8 while Table II gives the G-values for b=3 and n=2,4. These percentage points are correct to the number of significant figures which are tabulated. The tables can be regarded as extensions of the tables [4]

Table I

Table of exact upper percentage points, $G_n^{2,2}(\lambda)$, of the product of <u>two</u> independent central F-statistics, each based on (n,n) degrees of freedom

$$P\{G_n^{2,2}(\lambda) < G_n^{(2),2}\} = \lambda$$

n \ λ	0.25	0.15	0.10	0.05	0.025	0.01	0.005	0.001	0.0005
1	15.081	72.626	226.91	1405.4	7917.0	71,272	35,941x10	$13,983\times10^3$	$65,707\times10^3$
2	5.1157	12.746	24.259	66.115	166.94	529.35	1224.2	7987.8	17,547
3	3.5038	6.9963	11.321	23.697	46.344	105.42	189.98	696.74	1195.2
4	2.8633	5.0940	7.5856	13.905	23.969	46.361	74.034	205.43	312.57
5	2.5166	4.1627	5.8860	9.9441	15.878	27.873	41.433	97.585	138.41
6	2.2970	3.6105	4.9246	7.8653	11.921	19.605	27.781	58.709	79.514
8	2.0310	2.9817	3.8773	5.7520	8.1480	12.322	16.433	30.312	38.767

Table II

Table of exact upper percentage points, $G_n^{2,3}(\lambda)$, of the product of <u>three</u> independent central F-statistics, each based on (n,n) degrees of freedom

$$P\{G_n^{2,3}(\lambda) < G_n^{(2),2}\} = \lambda$$

n \ λ	0.25	0.15	0.10	0.05	0.025	0.01	0.005	0.001	0.0005
2	7.6601	23.570	51.557	171.46	510.39	1939.4	5039.3	41,118	97,971
4	3.6712	7.4496	12.111	25.224	48.422	105.71	182.76	592.34	954.27

when the d.f. (ν_1, ν_2) of those tables are such that $\nu_1 = \nu_2 = n$. (Thus the diagonal entries in the tables [4] are the percentage points $G_n^{2,1}(\lambda)$.)

Additional exact values of $G_n^{a,b}(\lambda)$ for $a > 1, b, n$ small, and λ close to zero must await the evaluation of the integral (4.4) of [1] to obtain formulae like those of Section 4.1 of [1].

For $a=2$, Tables I and II can also be used for size and power calculations associated with the test $H_0: \alpha_1 = \alpha_2$ vs. a one-sided alternative $H_1: \alpha_1 > \alpha_2$ (say).

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[2] Dunnett, C.W.: "A multiple comparison procedure for comparing several treatments with a control," Journal of the American Statistical Association, Vol. 50 (1955), pp. 1096-1121.

[3] Dunnett, C.W.: "New tables for multiple comparisons with a control," Biometrics, Vol. 20 (1964), pp. 482-491.

[4] Merrington, M. and Thompson, C.M.: "Tables of percentage points of the inverted beta (F) distribution," Biometrika, Vol. 33 (1943), pp. 73-88.

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Dunnett has prepared tables for making multiple comparisons with a control for single-factor (or multi-factor) experiments involving means of normal populations. In the present paper the author uses the multiplicative model for variances which he introduced in an earlier paper, as the basis for making multiple comparisons with a control for multi-factor experiments involving variances of normal populations. A limited set of tables for use with this procedure is included.

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