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RM-5288-1-PR
OCTOBER 1967

DIGITAL COMPUTER SIMULATION: THE ALLOCATION OF COMPUTER TIME IN COMPARING SIMULATION EXPERIMENTS

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PREFACE

This Memorandum is part of RAND's continuing research into the statistical analysis of computer simulation experiments. Preceding work on this subject has been described in G. S. Fishman and P. J. Kiviat, Spectral Analysis of Time Series Generated by Simulation Models, The RAND Corporation, RM-4393-PR, February 1965, and G. S. Fishman, Problems in the Statistical Analysis of Simulation Experiments: The Comparison of Means and the Length of Sample Records, The RAND Corporation, RM-4880-PR, February 1966. The purpose of this research is to find methods for efficiently extracting useful information from time series generated by these experiments. This study compares two simulation experiments and investigates how to efficiently allocate computer time between them. In addition, it provides a two-step procedure for implementing the derived allocation rule.

This Memorandum replaces RM-5288-PR. The method described here leads to improved population estimators by reducing the error due to sampling fluctuations and correcting for a suboptimization performed in the earlier work.

SUMMARY

This Memorandum investigates the problem of efficiently allocating computer time between two simulation experiments when the objective is to make a statistical comparison of means. For a given level of accuracy our results show that significantly less computer time is required when the sample sizes are determined according to a certain rule than when the sample sizes are equal.

A graphical analysis suggests that small errors in estimating the population parameters of the allocation rule do not significantly affect the efficient allocation of time. The influence that the degree of autocorrelation has on the time allocation is also investigated; results show that small differences in the autocorrelation functions are important when each process is highly autocorrelated.

Positively correlated samples for the two experiments are examined and incorporated into the efficient allocation rule. It is shown that their use leads to a saving in computer time.

A two-stage procedure is described wherein initial estimates of the population parameters are computed which permit the experimenter to estimate how many more observations to collect on each experiment. The procedure is simple and straightforward to implement and should be of practical value.

When the computer time requirements turn out to be prohibitive, we suggest using negatively correlated replications on each experiment. This may be accomplished by using antithetic variates. The two-stage procedure also applies in this case.

CONTENTS

THE ALLOCATION OF COMPUTER TIME IN COMPARING SIMULATION EXPERIMENTS

1. INTRODUCTION

In performing computer simulation experiments, it is desirable to minimize the computer time required to obtain a statistical result with a given level of accuracy. This minimization is also desirable when comparing two experiments, but minimizing the time needed to obtain a given accuracy for each experimental result is not necessarily the same as minimizing the time needed to obtain this same accuracy for comparing the two results. It is therefore instructive to investigate the problem: When comparing two experiments, how many observations should be collected from each one to obtain a given level of statistical accuracy with a minimum expenditure of computer time?

This problem is investigated here with regard to comparing the means of two computer simulation experiments. A number of considerations enter into the problem's solution. The processes being observed are generally autocorrelated, thereby precluding the use of the statistical methodology applicable to independent observations. The variances of the sample means as well as the computation times required to collect one observation in each experiment strongly influence the choice of efficient sample sizes. Variance-reduction techniques also play a role. Inducing positive correlation in the samples of the two experiments and negative correlation between replications of the same experiment reduces the time required to obtain a given accuracy.

In this Memorandum a rule for choosing sample sizes is described. The rule incorporates the above-mentioned considerations and determines the sample sizes necessary to meet a specified reliability with

a minimum expenditure of computer time. The variance of the difference of the sample means measures reliability.

A two-stage procedure is also presented wherein initial samples collected on each experiment are used to estimate the relevant population parameters. These estimates are then used to determine how many more observations should be collected to efficiently meet the specified reliability.

2. DEFINITIONS

Consider a covariance stationary stochastic sequence $\{X_t;\ t = 0, \pm 1, \pm 2, \ldots, \pm\infty\}$ with mean

(1a) $$\mu = E(X_t),$$

variance

(1b) $$\sigma^2 = E(X_t-\mu)^2,$$

autocorrelation function

(1c) $$\rho_s = \sigma^{-2}E[(X_t-\mu)(X_{t+s}-\mu)],$$

(1d) $$\lim_{s\to\infty} \rho_s = 0,$$

and spectrum

(1e) $$g(\lambda) = \pi^{-1}\sigma^2\sum_{s=-\infty}^{\infty}\rho_s e^{-i\lambda s} \quad 0 \le \lambda \le \pi.$$

Given a time series of N observations, we compute the sample mean as

(2a) $$\bar{X} = N^{-1}\sum_{t=1}^{N} X_t,$$

with variance

(2b) $$\mathrm{var}(\bar{X}) = N^{-1}\sigma^2 \sum_{s=-N+1}^{N-1} (1-|s|/N)\rho_s.$$

As N becomes large, we have

(2c) $$\lim_{N\to\infty} N \,\mathrm{var}\,(\bar{X}) = m < \infty$$

(2d) $$m = \pi g(0) = \sigma^2 \sum_{s=-\infty}^{\infty} \rho_s < \infty$$

so that the large sample variance of $\bar{X}$ is approximately m/N. In addition the statistic $(\bar{X}-\mu)/(m/N)^{\frac{1}{2}}$ asymptotically has the normal distribution with zero mean and unit variance. We hereafter assume N is large enough so that these limiting properties hold reasonably well.

Condition (1d) assumes the absence of perfectly periodic components in $\{X_t\}$.* If any are present, they should be removed prior to performing the analysis presented here. Presumably the mechanism generating $\{X_t\}$ will be well enough understood to permit removal. An analysis of the sample spectrum of $\{X_t\}$ as described in [4] will reveal periodicities if they are present.

Extremely narrow peaks in the spectrum indicate the presence of fairly regular periodic components in $\{X_t\}$. The magnitudes of these

*See [1], p. 254.

peaks compared to that of the remaining spectrum determines the relative contributions of these components to the mean-square variation in $\{X_t\}$. In general a spectrum analysis can only be performed after the experiment is run. It nevertheless serves as a convenient check on assumption (1d) in addition to providing other useful information.*

The reader should not construe the term "time series" to apply only to a set of observations collected at equal intervals along a time axis. The subscript t is simply an index and any series of events indexed on t can generate a time series. For example, in a single-server queueing problem, t may denote the t^{th} job to receive service and X_t may be the waiting time of this job.**

Our purpose is to collect observations on a particular process in two simulation experiments and to compare the sample means of the two time series. Let the subscript i denote experiment i. Then $X_{i,t}$ with mean μ_i, variance σ_i^2 and autocorrelation function ρ_i denotes the t^{th} observation on experiment i.

Suppose that $\{X_{1,t}\}$ and $\{X_{2,t}\}$ are correlated with covariance function

$$R_{12,\tau} = E[(X_{1,t}-\mu_1)(X_{2,t+\tau}-\mu_2)], \tag{3a}$$

$$\lim_{\tau\to\infty} R_{12,\tau} = 0. \tag{3b}$$

Given N_1 and N_2 observations on experiments 1 and 2, respectively, we have

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*See [4].

**For a further discussion of the statistical analysis of series of events, see [2].

$$\text{(3c)} \qquad \operatorname{cov}(\bar{X}_1,\bar{X}_2) = (N_1N_2)^{-1}\sum_{s=1}^{N_1}\sum_{\tau=1}^{N_2} R_{12,s-\tau}$$

If $N_1 > N_2$ we may write

$$\text{(3d)} \qquad \operatorname{cov}(\bar{X}_1,\bar{X}_2) = (N_1N_2)^{-1}\left[\sum_{\tau=1}^{N_2}\tau R_{12,\tau-N_1} + N_2\sum_{\tau=N_2-N_1+1}^{0} R_{12,\tau} + \sum_{\tau=1}^{N_2-1}(N_2-\tau)R_{12,\tau}\right].$$

$$\text{(3e)} \qquad \lim_{N_1\to\infty} N_1 \operatorname{cov}(\bar{X}_1,\bar{X}_2) = \sum_{\tau=-\infty}^{0} R_{12,\tau} + \sum_{\tau=1}^{N_2-1}(1-\tau/N_2)R_{12,\tau},$$

$$\text{(3f)} \qquad \lim_{N_2\to\infty}\lim_{N_1\to\infty} N_1 \operatorname{cov}(\bar{X}_1,\bar{X}_2) = m_3$$

$$m_3 = \sum_{\tau=-\infty}^{\infty} R_{12,\tau} < \infty .$$

Similarly for $N_2 > N_1$ we have

$$\text{(3g)} \qquad \lim_{N_1\to\infty}\lim_{N_2\to\infty} N_2 \operatorname{cov}(\bar{X}_1,\bar{X}_2) = m_3.$$

For large N_1 and N_2 we therefore may write

$$\text{(21)} \qquad \operatorname{cov}(\bar{X}_1,\bar{X}_2) \sim m_3/\max(N_1,N_2).$$

The variance of interest may now be written

$$\text{(4a)} \qquad \operatorname{var}(\bar{X}_1-\bar{X}_2) \sim m_1/N_1 + m_2/N_2 - 2m_3/\max(N_1,N_2),$$

where m_1 and m_2 are obtained from (2d) for $\{X_1,t\}$ and $\{X_2,t\}$, respectively. For convenience of exposition we first assume $\{X_1,t\}$ to be independent so that

$$\text{(4b)} \qquad \operatorname{var}(\bar{X}_1 - \bar{X}_2) \sim m_1/N_1 + m_2/N_2.$$

3. THE EFFICIENT ALLOCATION OF TIME

We denote the computer time necessary to collect N_1 and N_2 observations on experiment 1 and 2, respectively, as

(5a) $$T = c_1 N_1 + c_2 N_2$$

where c_i is the computer time required to collect one observation on experiment i. Suppose we require that

(5b) $$\text{var}(\bar{X}_1 - \bar{X}_2) \sim V = m_1/N_1 + m_2/N_2,$$

V being specified by the experimenter. We choose N_1 and N_2 so as to minimize T for fixed V by considering

(5c) $$L = c_1 N_1 + c_2 N_2 + \lambda(V - m_1/N_1 - m_2/N_2),$$

λ being a Lagrange multiplier. Minimization yields:

(6a) $$N_1^* = m_1[1 + (br)^{\frac{1}{2}}]/V,$$

(6b) $$N_2^* = N_1^*(r/b)^{\frac{1}{2}}$$

(6c) $$T_0 = [(c_1 m_1)^{\frac{1}{2}} + (c_2 m_2)^{\frac{1}{2}}]^2/V,$$

(6d) $$b = c_2/c_1,$$

(6e) $$r = m_2/m_1.$$

The converse problem of minimizing V for a given T has the same solution.

It is convenient to first assay the special but common case

$$c_1 = c_2 = c, \qquad b = 1.$$

Table 1 shows the relationship between r and T_2^*/T_0 for this case. For r equal to 8, roughly 75 percent of the computer time is spent on experiment 2. When r lies in the interval 64 to 128, approximately nine observations are collected on experiment 2 for every observation collected on experiment 1.

Table 1

RELATIONSHIP BETWEEN r AND $T_2^*/T_0 = c_2N_2^*/T_0$ FOR b = 1

r	T_2^*/T_0
1	.5000
2	.5858
4	.6667
8	.7387
16	.8000
32	.8456
64	.8889
128	.9188

In many simulation experiments it is customary to set

(7a) $$N_1 = N_2 = T/(2c)$$

so that the computer time is equally shared. Then for a given V, we have

(7b) $$T = 2cm_1(1+r)/V,$$

whereas the derived rule yields

(7c) $$T_0 = cm_1(1+r^{\frac{1}{2}})^2/V.$$

To measure efficiency we use the ratio

$$T/T_0 = 2(1+r)/(1+r^{\frac{1}{2}})^2. \tag{8a}$$

For arbitrary N_1, N_2 and b we measure efficiency by

$$T/T_0 = [(1+ab)(1+r/a)]/[1+(br)^{\frac{1}{2}}]^2, \tag{8b}$$

$$a = N_2/N_1 \qquad a > 0. \tag{8c}$$

Notice that:

$$\lim_{r\to 0} T/T_0 = 1 + ab, \tag{9a}$$

$$\lim_{r\to \infty} T/T_0 = 1 + 1/(ab), \tag{9b}$$

$$\lim_{b\to 0} T/T_0 = 1 + r/a, \tag{9c}$$

$$\lim_{b\to \infty} T/T_0 = 1 + a/r. \tag{9d}$$

Expression (8b) suffices for comparison purposes.

4. THE SENSITIVITY OF EFFICIENCY TO THE CHOICE OF a

Estimating c_1 and c_2, and hence b, can be accomplished by straightforward observation. The true value of r is more elusive and it is therefore instructive to examine how sensitive the efficiency measure T/T_0 is to different values of a. Figure 1 shows T/T_0 for a equal to 1, 2 and 3, and b equal to unity. The values of a correspond to devoting 1/2, 2/3 and 3/4 of the computer time, respectively, to experiment 2.

If $6 \le r \le 20$, choosing a equal to 3 yields an allocation close to the efficient one. For $r < 6$ the allocation a equal to 2 appears

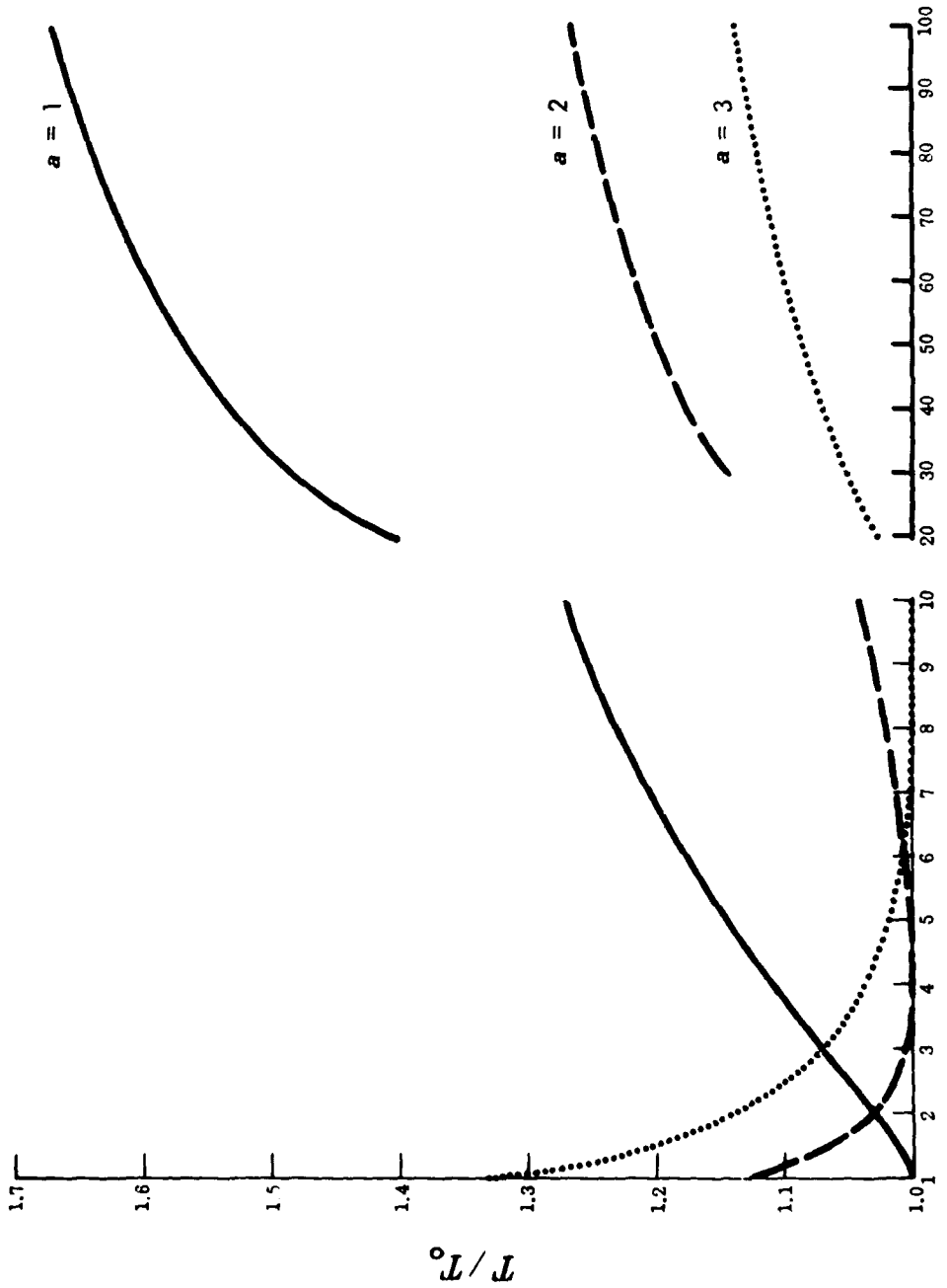

Fig. 1 -- T/T_0 versus r, $b = 1$

reasonably efficient. Now if r exceeds 100, the time allocation a equal to 3 requires 45 percent less time to meet the specified accuracy than an equal time allocation does. For r of this magnitude, an a equal to 9 would be even more reasonable since

$$\lim_{r\to\infty} T/T_0 = 10/9, \quad a = 9.$$

Figure 1 shows that, while the true value of r may be unknown, a rough estimate permits the experimenter to choose an a that brings the time allocation close to the efficient one.

Table 2 shows several interesting comparisons. It is evident that as b increases, the a equal to 3 allocation becomes more desirable for larger r.

Table 2

T/T_0 FOR a = 1, 3 AND b = .25, .5, 2 AND 4

	b = .25		b = .5		b = 2		b = 4	
r	a = 1	a = 3	a = 1	a = 3	a = 1	a = 3	a = 1	a = 3
1	1.1111	1.0370	1.0294	1.1438	1.0294	1.6013	1.1111	1.9259
2	1.2868	1.0008	1.1250	1.0417	1.0000	1.2963	1.0234	1.4783
3	1.4359	1.0052	1.2122	1.0102	1.0085	1.1766	1.0036	1.3047
4	1.5625	1.0208	1.2868	1.0008	1.0234	1.1144	1.0000	1.2133
5	1.6718	1.0403	1.3509	1.0007	1.0390	1.0775	1.0019	1.1577
6	1.7679	1.0607	1.4067	1.0048	1.0538	1.0538	1.0058	1.1208
7	1.8533	1.0811	1.4560	1.0111	1.0675	1.0378	1.0105	1.0947
8	1.9302	1.1009	1.5000	1.0185	1.0800	1.0267	1.0155	1.0757
9	2.0000	1.1200	1 5396	1.0264	1.0915	1.0187	1.0204	1.0612
10	2.0639	1.1382	1.5756	1.0345	1.1020	1.0130	1.0252	1.0500
20	2.5067	1.2812	1.8182	1.1063	1.1743	1.0003	1.0618	1.0079
30	2.7724	1.3772	1.9582	1.1581	1.2158	1.0066	1.0846	1.0006
40	2.9582	1.4478	2.0538	1.1967	1.2438	1.0146	1.1004	1.0002
50	3.0990	1.5029	2.1250	1.2269	1.2645	1.0220	1.1122	1.0017
60	3.2111	1.5476	2.1809	1.2514	1.2805	1.0286	1.1214	1.0037
70	3.3034	1.5850	2.2265	1.2718	1.2935	1.0344	1.1289	1.0059
80	3.3813	1.6169	2.2647	1.2892	1.3044	1.0396	1.1352	1.0081
90	3.4483	1.6446	2.2973	1.3044	1.3136	1.0441	1.1405	1.0102
100	3.5069	1.6690	2.3257	1.3176	1.3215	1.0482	1.1451	1.0121

5. THE EFFECTS OF VARIANCE AND AUTOCORRELATION

It is instructive to examine the effect of population variance and autocorrelation on the efficient time allocation. Assuming the two samples are independent, we consider two cases of special interest:

$$\text{Case 1:} \quad \sigma_1^2 \neq \sigma_2^2 \qquad \rho_{1,s} = \rho_{2,s},$$

$$\text{Case 2:} \quad \sigma_1^2 = \sigma_2^2 \qquad \rho_{1,s} \neq \rho_{2,s}.$$

For Case 1 we note that

$$r = \sigma_2^2/\sigma_1^2, \tag{10}$$

which is easily estimated. Figure 1 can then be used to choose an a. One might therefore conclude that if the nature of autocorrelation in each sample is roughly alike, an efficient choice of a can be based on the ratio of the population variances. This conjecture may be contrary to fact as we now show using Case 2.

For convenience of exposition we assume

$$\rho_{i,t} = \alpha_i^{|t|} \qquad 0 \leq \alpha_i < 1 \qquad i = 1, 2. \tag{11a}$$

This autocorrelation function corresponds to the Markov process,

$$X_{i,t} = \alpha_i X_{i,t-1} + \epsilon_{i,t}, \tag{11b}$$

where $\{\epsilon_{i,t}\}$ is a sequence of mutually independent, identically distributed random variables. Now we have

$$r = [1 - \alpha_1\alpha_2 + (\alpha_2 - \alpha_1)]/[1 - \alpha_1\alpha_2 - (\alpha_2 - \alpha_1)]. \tag{11c}$$

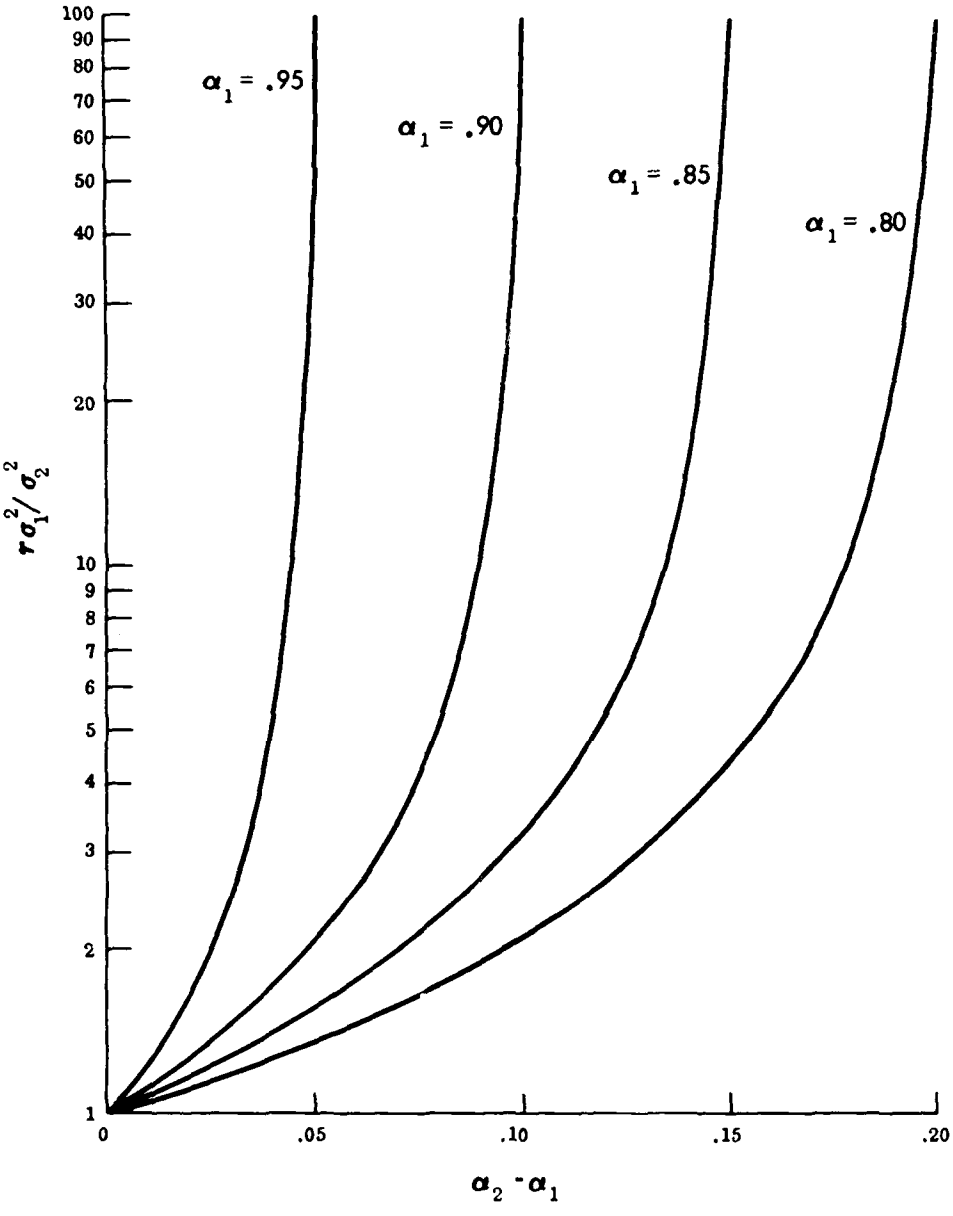

Fig. 2 -- $r\sigma_1^2/\sigma_2^2$ versus $\alpha_2 - \alpha_1$ for α_1 = .80, .85, .90 and .95

Figure 2 shows r as a function of $(\alpha_2-\alpha_1)$ for several values of α_1. Notice that for a given difference, $(\alpha_2-\alpha_1)$, r increases rapidly as α_1 and therefore α_2 increase. This result implies that small differences in the difference $(\alpha_2-\alpha_1)$ are important when each sequence is highly autocorrelated. Hence, while it may appear that two sequences are quite similar because their autocorrelation functions are, these minor differences may strongly affect the efficient allocation of time between the experiments.

6. CORRELATED SAMPLES

It is often suggested that one may reduce the sample sizes needed to obtain a given accuracy V by working with positively correlated samples on the two experiments. This is clearly true for equal sample sizes. Here the effects of correlated sampling on the efficient choice of N_1 and N_2 are investigated.

The expression to be minimized, in analogy to (5c), is

$$L = c_1N_1 + c_2N_2 + \lambda[V-m_1/N_1-m_2/N_2+2m_3/\max(N_1,N_2)]. \tag{12}$$

This yields

$$N_1^* = \{m_1 + [bm_2(m_1-2m_3)]^{\frac{1}{2}} - 2m_3\}/V \tag{13a}$$

$$N_2^* = \{m_2 + [m_2(m_1-2m_3)/b]^{\frac{1}{2}}\}/V \tag{13b}$$

for

$$m_2 \leq b(m_1-2m_3) \tag{13c}$$

and

(14a) $$N_1^* = \{m_1 + [bm_1(m_2-2m_3)]^{\frac{1}{2}}\}/V$$

(14b) $$N_2^* = \{[m_1(m_2-2m_3)/b]^{\frac{1}{2}} + m_2 - 2m_3\}/V$$

for

(14c) $$m_2 \geq bm_1 + 2m_3.$$

It is easily shown that correlated samples reduce the required computer time when either (13c) or (14c) holds.

It may occur that m_2 satisfies neither constraint so that

(15) $$b(m_1-2m_3) < m_2 < bm_1 + 2m_3.$$

Suppose we then choose our sample sizes as

(16a) $$N_1' = N_2' = (m_1+m_2-2m_3)/V.$$

The corresponding computer time is

(16b) $$T' = (c_1+c_2)(m_1+m_2-2m_3)/V.$$

As an alternative, in the case where (15) holds, we may work with independent samples and choose N_1 and N_2 as in (6a) and (6b). The required computer time is then

(17) $$T'' = [(c_1m_1)^{\frac{1}{2}} + (c_2m_2)^{\frac{1}{2}}]^2/V.$$

For independent sampling to be desirable, we require

(18) $$T' - T'' > 0.$$

Expression (18), however, implies that

(19a) $$m_2 > bm_1 + 2(1+b)m_3 + 2[2b(1+b)m_1m_3]^{\frac{1}{2}},$$

(19b) $$bm_1 > m_2 + 2(1+b)m_3 + 2[2(1+b)m_2m_3]^{\frac{1}{2}}$$

which clearly violate (15). When (15) holds it is therefore desirable to work with correlated samples with equal sample sizes determined by (16a).

In practice the quantities m_1, m_2 and m_3 are unknown prior to experimentation. To estimate them an initial sample of say Q observations on each experiment is needed. The estimation procedure is described in Sec. 7. For now we assume m_1, m_2 and m_3 are known and we concern ourselves with several contingencies that may arise.

Suppose that (13c) or (14c) holds. If

(20) $$N_1^* \geq Q, \qquad N_2^* \geq Q,$$

no difficulty occurs. If, however,

(21a) $$N_1^* > Q, \qquad N_2^* < Q,$$

then we recompute N_1^* and N_2^* as

(21b) $$N_1^* = Q(m_1 - 2m_3)/(QV - m_2)$$

(21c) $$N_2^* = Q.$$

Similarly when

(22a) $$N_1^* < Q, \qquad N_2^* \geq Q,$$

then we set

(22b) $$N_1^* = Q$$

(22c) $$N_2^* = Q(m_2 - 2m_3)/(QV - m_1).$$

If both required sample sizes are less than Q, we terminate the experiments.

7. ESTIMATION PROCEDURE

The quantities m_1, m_2 and m_3 seldom are known a priori and therefore a method of estimating them is required. The presence of autocorrelation in $\{X_{1,t}\}$ and $\{X_{2,t}\}$ makes estimation somewhat difficult. Ideally we desire a method that can easily be incorporated into the performance of the experiments. Then sample sizes can be automatically determined within the simulations.

The method suggested here accounts for autocorrelation but unfortunately cannot be readily incorporated into the experiments. It nevertheless provides valuable assistance to the experimenter especially when large amounts of computer time are involved. The method has been described in [3]. Here we briefly describe and extend it to the estimation of m_3. Several clarifying remarks on [3] should facilitate its application.

Suppose we initially collect Q observations on experiment 1 and wish to estimate m_1. Since the same procedure applies to m_2 in experiment 2, we momentarily drop the subscript i. The sample autocovariance function is

$$\hat{R}_\tau = Q^{-1}\sum_{t=1}^{Q-\tau}[(X_t-\bar{X})(X_{t+\tau}-\bar{X})] \tag{23a}$$

$$\tau = 0,1,\ldots,M < Q$$

$$\bar{X} = Q^{-1}\sum_{t=1}^{Q} X_t, \tag{23b}$$

which for large Q has expectation

$$E(\hat{R}_\tau) \sim \sigma^2 \rho_\tau - m/Q. \tag{23c}$$

An initial estimate of m is

$$\tilde{m} = \hat{R}_0 + 2\sum_{\tau=1}^{M} (1-\tau/M)\hat{R}_\tau \tag{24a}$$

which for appropriately chosen M has expectation

$$E(\tilde{m}) \sim m(1-M/Q). \tag{24b}$$

Dividing (24a) by (1-M/Q) compensates for the bias so that our estimator of m is

$$\hat{m} = [\hat{R}_0 + 2\sum_{\tau=1}^{M} (1-\tau/M)\hat{R}_\tau]/(1-M/Q). \tag{25a}$$

If $\{X_t\}$ is a normal process then

$$\text{var}(\hat{m}) \sim \Psi m^2 \tag{25b}$$

$$\Psi = 4M/(3Q). \tag{25c}$$

The experimenter must choose an appropriate value for the design parameter M. Our clarifying remarks concern this choice. Notice that as M becomes large

$$\lim_{M\to\infty} \sum_{\tau=-M}^{M} (1-|\tau|/M)R_\tau = m \qquad R_\tau = \sigma^2\rho_\tau, \tag{26}$$

the desired mathematical limit. In (24a) we use $\hat{R}_\tau$ since R_τ is unknown. As M increases here the variance Ψm^2 also increases so that the estimate becomes less statistically reliable. This dilemma between good resolution (mathematical convergence) and good reliability (convergence

in probability) is fundamental to spectrum analysis. Clearly a compromise must be made which permits both acceptable resolution and reliability.

The compromise depends on the choice of M. To guide us in our choice we use the ratio

$$p = [\mathrm{var}(\hat{m})]^{\frac{1}{2}}/E(\hat{m}) \sim [4M/(3Q)]^{\frac{1}{2}}. \tag{27a}$$

If

$$M = 3Q/4 \tag{27b}$$

the ratio p is unity which implies that sampling fluctuations in $\hat{m}$ are roughly of the same order as its mean m. This is clearly undesirable. If

$$M = Q/4 \tag{27c}$$

then the ratio is roughly .57. It is suggested that M not exceed Q/4 and in general be kept much smaller. One may easily compute $\hat{m}$ for several values of M, say Q/32, Q/16, Q/8, 3Q/16 and Q/4. Doing so permits the experimenter to decide, albeit subjectively, when $\hat{m}$ is well resolved.

There is the possibility that Q may be too small to permit good resolution. If the estimate $\hat{m}$ fails to become well resolved as Q increases to M/4, then more observations should be collected and used. Failure to obtain a well-resolved estimate can lead to a serious underestimate of m.

The estimation of m_3 is accomplished in the same manner. Suppose we have Q observations on each experiment. We form the sample covariance function

$$\hat{R}_{12,\tau} = Q^{-1}\sum_{t=1}^{Q-|\tau|}(X_{1,t}-\bar{X}_1)(X_{2,t+\tau}-\bar{X}_2) \qquad \tau = 0,\pm 1,\ldots,\pm M, \tag{28a}$$

which for large Q has expectation

$$E(\hat{R}_{12,\tau}) \sim R_{12,\tau} - m_3/Q. \tag{28b}$$

Our estimator is then

$$\hat{m}_3 = \left[\sum_{\tau=-M}^{M}(1-|\tau|/M)\hat{R}_{12,\tau}\right]/(1-M/Q) \tag{28c}$$

with expectation

$$E(\hat{m}_3) \sim m_3 \tag{28d}$$

and variance, when both processes are normal [6],

$$\mathrm{var}(\hat{m}_3) \sim \Psi(m_1m_2+m_3^2)/2. \tag{28e}$$

In spectrum analysis the quantity m_3/π is called the co-spectrum of $\{X_{1,t}\}$ and $\{X_{2,t}\}$ at frequency zero. The same procedure concerning Q applies here as in the estimation of m_1 and m_2.

Our main purpose is to estimate N_1^* and N_2^*. When (13c) holds, we have initial estimates

$$\tilde{N}_1^* = \{\hat{m}_1 + [b\hat{m}_2(\hat{m}_1-2\hat{m}_3)]^{\frac{1}{2}} - 2\hat{m}_3\}/V \tag{29a}$$

$$\tilde{N}_2^* = \{\hat{m}_2 + [\hat{m}_2(\hat{m}_1-2\hat{m}_3)/b]^{\frac{1}{2}}\}/V. \tag{29b}$$

When (14c) holds, the initial estimates are

$$\tilde{N}_1^* = \{\hat{m}_1 + [b\hat{m}_1(\hat{m}_2-2\hat{m}_3)]^{\frac{1}{2}}\}/V \tag{30a}$$

(30b) $$\tilde{N}_2^* = \{\hat{m}_2 + [\hat{m}_1(\hat{m}_2-2\hat{m}_3)/b]^{\frac{1}{2}} - 2\hat{m}_3\}/V.$$

When (15) holds, the sample sizes are equal so that

(31) $$\tilde{N}_1^* = \tilde{N}_2^* = (\hat{m}_1+\hat{m}_2-2\hat{m}_3)/V.$$

If $\{X_{1,t}\}$ and $\{X_{2,t}\}$ are normal processes we may use the results in [6] to derive the sampling properties of $\tilde{N}_1^*$ and $\tilde{N}_2^*$ to order Q^{-1}. Defining

(32a) $$\hat{\phi}_i \equiv \hat{m}_i - 2\hat{m}_3$$

(32b) $$\theta_i \equiv (m_3^2 - m_1 m_2)(m_1+m_2-2m_3)/(4V^2 m_{3-i}\phi_i^3)^{\frac{1}{2}} \qquad i = 1,2$$

(32c) $$\delta_i \equiv \begin{cases} b & i=1 \\ 1/b & i=2 \end{cases}$$

and noting that

(32d) $$E(\hat{\phi}_i) \sim \phi_i \equiv m_i - 2m_3,$$

we may show that

(32e) $$\mathrm{cov}(\hat{m}_i, \hat{m}_3) \sim \tfrac{1}{4} m_i m_3$$

(32f) $$\mathrm{var}(\hat{\phi}_i) \sim \tfrac{1}{4}[m_i^2 - 4m_i m_3 + 2(m_1 m_2 + m_3^2)]$$

(32g) $$\mathrm{cov}(\hat{\phi}_i, \hat{m}_j) \sim \tfrac{1}{4}(m_3^2 - 2m_j m_3) \qquad i,j = 1,2;\ i \neq j.$$

When (13c) holds we have

(33a) $$E(\tilde{N}_i^* - N_i^*) \sim \tfrac{1}{4}\theta_1 \delta_i$$

(33b) $V^2\mathrm{var}(\hat{N}_1^*) \sim [1 + (\delta_1 m_2/\phi_1)^{\frac{1}{2}}]^2 \mathrm{var}(\hat{\phi}_1) + 2[(\delta_1\phi_1/m_2)^{\frac{1}{2}} + \delta_1] \mathrm{cov}(\hat{\phi}_1,\hat{m}_2)$
$+ (\delta_1\phi_1/m_2) \mathrm{var}(\hat{m}_2)$

(33c) $V^2\mathrm{var}(\hat{N}_2^*) \sim (\delta_2 m_2/\phi_1) \mathrm{var}(\hat{\phi}_1) + 2[(\delta_2 m_2/\phi_1)^{\frac{1}{2}} + \delta_2] \mathrm{cov}(\hat{\phi}_1,\hat{m}_2)$
$+ [1 + (\delta_2\phi_1/m_2)^{\frac{1}{2}}]^2 \mathrm{var}(\hat{m}_2)$.

When (14c) holds we have

(34a) $$E(\hat{N}_i^* - N_i^*) \sim \Psi\theta_2\delta_i$$

(34b) $V^2\mathrm{var}(\hat{N}_1^*) \sim (\delta_1 m_1/\phi_2) \mathrm{var}(\hat{\phi}_2) + 2[(\delta_1 m_1/\phi_2)^{\frac{1}{2}} + \delta_1] \mathrm{cov}(\hat{\phi}_2,\hat{m}_1)$
$+ [1 + (\delta_1\phi_2/m_1)^{\frac{1}{2}}]^2 \mathrm{var}(\hat{m}_1)$

(34c) $V^2\mathrm{var}(\hat{N}_2^*) \sim [1 + (\delta_2 m_1/\phi_2)^{\frac{1}{2}}]^2 \mathrm{var}(\hat{\phi}_2) + 2[(\delta_2\phi_2/m_1)^{\frac{1}{2}} + \delta_2] \mathrm{cov}(\hat{\phi}_2,\hat{m}_1)$
$+ (\delta_2\phi_2/m_1) \mathrm{var}(\hat{m}_1)$.

Since

(35a) $$m_3^2 < m_1 m_2$$

(35b) $$m_1 + m_2 > 2m_3,$$

the bias is negative. To compensate for this bias we estimate N_i^* by

(35c) $$\hat{N}_i^* = \hat{N}_i^* - \Psi\hat{\theta}_1\delta_i$$

when (13c) holds, and by

(35d) $$\hat{N}_i^* = \hat{N}_i^* - \Psi\hat{\theta}_2\delta_i$$

when (14c) holds. Here we define

(35e) $$\hat{\theta}_i \equiv (\hat{m}_3^2 - \hat{m}_1\hat{m}_2)(\hat{m}_1 + \hat{m}_2 - 2\hat{m}_3)/(4V^2\hat{m}_{3-i}\hat{\phi}_i^3)^{\frac{1}{2}} \qquad i = 1,2.$$

To order Q^{-1} the variances remain unchanged.

When (15) holds equal sample sizes are used and we have straightforwardly

$$E(\hat{N}_i^*) \sim N_i^* \tag{36a}$$

$$\mathrm{var}(\hat{N}_i^*) \sim \gamma[(m_1+m_2)^2 + 4m_3(m_3-m_1-m_2)]. \tag{36b}$$

No bias adjustment is necessary here.

For normal processes the distributions of $\hat{m}_1/m_1$ and $\hat{m}_2/m_2$ are usually approximated by that of chi-square. This suggests that we also approximate the distribution of $\hat{N}_i^*/N_i^*$ by that of chi-square with degrees of freedom

$$f_i = 2[E(\hat{N}_i^*)]^2/\mathrm{var}(\hat{N}_i^*). \tag{37}$$

Estimating f_i by replacing the terms in (37) by their corresponding sample values will assist the experimenter in determining how reliable his estimate of $\hat{N}_i^*$ is. This is especially important when $c_i\hat{N}_i^*$ turns out to be large.

8. ANTITHETIC VARIATES

It may occur that the sample sizes determined by the foregoing analysis require a prohibitively large amount of computer time. In this case positively correlated samples have not sufficed to bring the cost of experimentation within reasonable bounds. One may improve this situation by working with negatively correlated replications of each experiment. The suggested technique is called the method of antithetic variates [5]. If ξ_i is the i^{th} uniformly distributed psuedorandom number

in the first replication of the experiment, then using $(1 - \xi_i)$ as the uniformly distributed pseudorandom number in the corresponding event of the antithetic replication induces a negative correlation between the observations in the two replications.

Let $X_{ij,t}$ be an observation on the j^{th} replication of the i^{th} experiment at time t. Suppose we collect N_i observations on each replication of experiment i. Then we have

$$\bar{X}_{ij} = N_i^{-1} \sum_{t=1}^{N_i} X_{ij,t} \qquad j = 1, 2, \tag{38a}$$

$$\bar{X}_i = (\bar{X}_{i1} + \bar{X}_{i2})/2. \tag{38b}$$

If the second replication uses the antithetic pseudorandom number sequence of the first, then

$$\mathrm{cov}(\bar{X}_{i1}, \bar{X}_{i2}) = m_{i,4}/N_i \tag{38c}$$

$$m_{i,4} \leq 0$$

so that

$$\mathrm{var}(\bar{X}_i) = (m_i - m_{i,4})/(2N_i). \tag{38d}$$

If $m_{i,4} = m_i/2$ then using two antithetic replications of length N_i is equivalent to collecting $4N_i$ observations on 1 replication. If $m_{i,4} = 3m_i/4$ then two antithetic replications each of length N_i is equivalent to $8N_i$ observations on one replication.

To incorporate antithetic variates into the preceding results, we need only form the observations defined by (38b) and compute m_1,

m_2 and m_3 as described. The resulting N_1^* means that $N_1^*/2$ observations should be collected on each of two antithetic replications of experiment 1. A similar result applies to N_2^*.

9. CONCLUSIONS

Our results indicate that for a given level of accuracy, significantly less computer time is required when the sample sizes are determined according to the derived rule than when the sample sizes are equal. This is true when one uses either independent or positively correlated samples on the two experiments.

An interesting conclusion also emerges about the significance of the degree of autocorrelation in the observed processes. Our results imply that small differences in the autocorrelation functions are important for the time allocation when each process is highly autocorrelated. Failure to take these differences into account can significantly affect the efficient allocation of time, thereby requiring unusually long running times to obtain a given level of precision.

The suggested two-stage procedure provides initial estimates for determining sample sizes and final estimates for testing hypotheses. The graphical analysis suggests that the efficient allocation is not very sensitive to small errors in the estimates and, therefore, the use of estimates for the population parameters is not of major concern. It is important, however, to get good estimates of m_1, m_2 and m_3. This depends on an appropriate choice of M using the suggested guidelines.

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10. ABSTRACT

An improved step-by-step procedure for minimizing the computer time needed to obtain a specified statistical precision in the comparison of simulation experiments. The simulations are considered as covariance stationary stochastic processes, as explained in RM-4493-PR. Well-known time-saving methods for reducing variance by inducing positive correlations between the experiments and high negative correlations between replications are included in the procedure. Results show that for a given level of accuracy, significantly less computer time is required when sample sizes are determined by the method suggested in this study than when they are equal. Also, small differences in the autocorrelation functions are important when each process is highly correlated. The suggested two-stage procedure provides initial estimates for determining sample sizes and final estimates for testing hypotheses. Graphical analysis suggests that the efficient allocation is not very sensitive to small errors in the estimates of population parameters.

11. KEY WORDS

Computer simulation
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