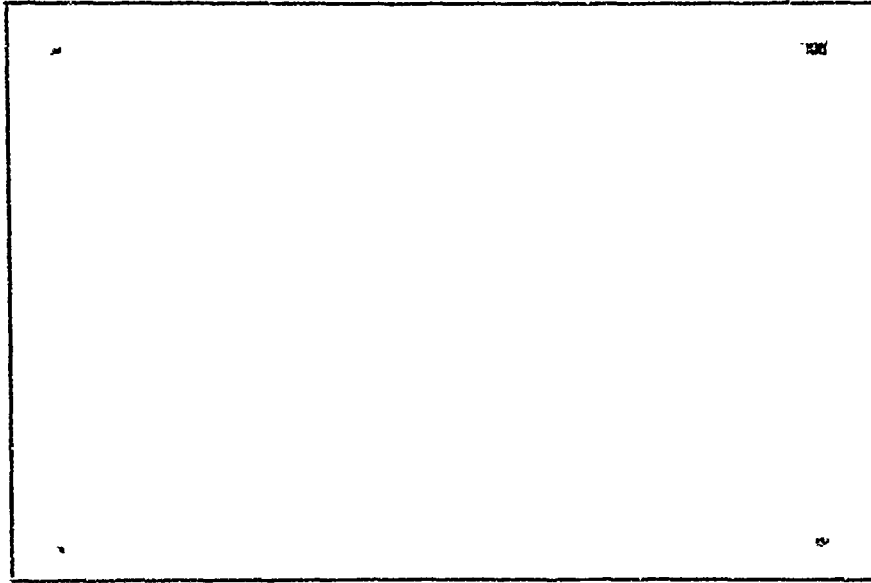


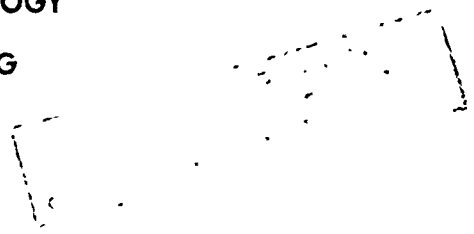
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62

SCIENTIFIC REPORT No. 2.

BUCKLING OF DISCRETELY RING STIFFENED CYLINDRICAL
SHELLS.

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SUMMARY

The buckling of ring-stiffened cylinders is studied by a "discrete" approach, in which the rings are considered as linear discontinuities represented by the Dirac delta function. The analysis is a linear Donnell type theory that takes account of the eccentricity of stiffeners. Buckling loads under hydrostatic pressure, lateral pressure and axial compression are compared with those obtained by "smeared-stiffener" theory for an extensive range of geometries. The discreteness effect depends very strongly on the geometry of the shell and the eccentricity of the rings. Significant discreteness effects are found for hydrostatic pressure loading.

LIST OF CONTENTS

	<u>Page</u>
SUMMARY	I
LIST OF TABLES	III
LIST OF FIGURES	IV
SYMBOLS	V - VII
1. INTRODUCTION	1 - 3
2. STABILITY EQUATIONS	3 - 6
3. SOLUTION	6 - 9
4. THE STABILITY DETERMINANT	10
5. LOCAL BUCKLING	10 - 11
6. NUMERICAL RESULTS AND DISCUSSION	11
6.1. Hydrostatic Pressure	11 - 15
6.2. Lateral Pressure with Non-Uniform Rings	15 - 16
6.3. Axial Compression	16 - 17
7. CONCLUSIONS	18
REFERENCES	19 - 22
ACKNOWLEDGEMENT	22
APPENDIX I - Reduction in Order and Simplification of Stability Determinants	23 - 27
APPENDIX II - Comparison of "Smearred" and "Discrete" Solutions	28 - 29
APPENDIX III - Convergence of the Solution	30
TABLES	31 - 43

LIST OF TABLES

- TABLE 1. Local Buckling of Typical Ring Stiffened Shells
- TABLE 2. Effect of Shell Geometry
- TABLE 3. Effect of Ring Parameters - Ring Cross-Sectional Area
- TABLE 4. Effect of Ring Parameters - Eccentricity of Rings
- TABLE 5. Influence of the Number of Rings on Discreteness Effect
for Different (L/R) and (R/h)
- TABLE 6. Influence of the Number of Rings on the Discreteness Effect
Between Cut-Off Lines for Rings of Varying Strength
- TABLE 7. Comparison Between Discrete and "Smeared" Non-Uniform 3-
Ring Stiffened Cylinders Under Lateral Pressure
- TABLE 8. Discreteness Effect in Ring-Stiffened Cylindrical Shells under
Axial Compression
- TABLE 9. Convergence of the Approximations for "Discrete" Buckling Loads

LIST OF FIGURES

- Fig. 1 Notation
- Fig. 2 The Influence of Shell Geometry on the Discreteness Effect.
 for Shells with One Ring Under Hydrostatic Pressure
- Fig. 3 Influence of Ring Stiffness on the Discreteness Effect
 (Shells with One Ring)
- Fig. 4 Effect of Eccentricity of Rings on the Discreteness Effect
- Fig. 5 Dependence of the Discreteness Effect on the Number of Rings
 (For Different R/h)
- Fig. 6 Dependence of the Discreteness Effect on the Number of Rings
 (For Different L/R)
- Fig. 7 Influence of the Number of Rings on the Discreteness Effect
 for Rings of Varying Strength
- Fig. 8 Convergence of Approximations.

SYMBOLS

A_n	coefficient of axial displacement
A_{1j}	cross sectional area of j^{th} stringer
A_{2i}	cross sectional area of i^{th} ring
a	distance between rings
B_n	coefficient of circumferential displacement
b	distance between stringers
b_{m1}, b_{m2}	expressions defined by Eqs. (9)
C_n	coefficient of radial displacement
c_{m1}, c_{m2}	expressions defined by Eqs. (9)
D	$[Eh^3/12(1-\nu^2)]$
E, E_1, E_2	moduli of elasticity of shell, stringers and rings respectively
e_1, e_2	distance between centroid of stiffener cross section and middle surface of shell, positive when stiffener is inside
G_1, G_2	shear moduli of stringers and rings respectively
h	thickness of shell
I_{11}, I_{22}	moment of inertia of stiffener cross section about its centroidal axis
I_{o1}, I_{o2}	moment of inertia of stiffener cross section about the middle surface of the shell
I_{t1}, I_{t2}	torsion constant of stiffener cross section
L	length of shell between bulkheads

$M_x, M_\phi, M_{x\phi}$	moment resultants acting on element
$N_x, N_\phi, N_{x\phi}$	membrane force resultants acting on element
$N_{x0}, N_{\phi0}, N_{x\phi0}$	prebuckling membrane force resultants
n	number of stringers
P	axial load
p	lateral or hydrostatic pressure
q	number of subshells for ring stiffened cylinder (number of rings is q-1)
R	radius of shell
t	number of circumferential waves
u	non-dimensional axial displacement ($=u^*/R$)
u^*	axial displacement
v	non-dimensional circumferential displacement ($=v^*/R$)
v^*	circumferential displacement
w	non-dimensional radial displacement ($=w^*/R$)
w^*	radial displacement
x	non-dimensional axial coordinate ($=x^*/R$)
x^*	axial coordinate
Z	$(1-\nu^2)^{1/2}(L/R)^2(R/h)$

β	$\pi R/L$
β_{1i}, β_{2i}	expressions defined by Eqs. (9)
$\delta_{imn}, \gamma_{1i}, \gamma_{2i}$	expressions defined by Eqs. (9)
ζ_{1j}	$E_{1j} A_{1j} e_{1j} / D$
ζ_{2i}	$E_{2i} A_{2i} e_{2i} / D$
n_{01j}	$E_{1j} i_{01j} / RD$
n_{02i}	$E_{2i} i_{02i} / RD$
n_{+1j}	$G_{1j} i_{+1j} / RD$
n_{+2i}	$G_{2i} i_{+2i} / RD$
λ	$(PR/\pi D)$
λ_p	$(R^3/D)p$
μ_{1j}	$(1-v^2)(E_{1j} A_{1j} / ERh)$
μ_{2i}	$(1-v^2)(E_{2i} A_{2i} / ERh)$
ν	Poisson's ratio
ϕ	circumferential coordinate
χ_{1j}	$[(1-v^2)E_{1j} A_{1j} e_{1j} / EhR^2]$
χ_{2i}	$[(1-v^2)(E_{2i} A_{2i} e_{2i} / EhR^2)]$

Subscripts following a comma indicate differentiation
 Subscripts i and j refer to i'th ring and j'th
 stringer respectively.

1. INTRODUCTION

Until a few years ago the customary approach to the stability analysis of a stiffened shell was to replace it by an equivalent orthotropic shell (see for example [1] - [3]). This approach, however, did not permit taking into account the eccentricity of stiffeners which was found to be of importance in heavily stiffened shells (for example [4] or [5]). A second approach also assumes the stiffeners to be "smeared" over the whole surface of the shell, but considers the "distributed" stiffness of the stiffeners separately, which permits inclusion of the effect of eccentricity. The eccentricity effect had been pointed out long ago, [6] and [7], but was studied in detail only in recent years (see for example [8], [9], [10], [11], [12] or [13]).

The "smearing" of stiffeners appears reasonable for shells with many closely spaced stiffeners, but is doubtful when their number is small. The study of discretely stiffened shells is therefore of interest.

Hence the buckling of stiffened cylindrical shells is analysed by a third approach in which, instead of "smearing" the stiffeners, they are considered as linear discontinuities represented by the Dirac delta function. This representation is satisfactory as long as the width of the stiffeners is not comparable to the distance between them. One may add that a "discrete" analysis includes also consideration of local instability between stiffeners, whereas the first two approaches dealt with general instability only.

This approach has been employed in [14] for buckling of cylindrical shells under torsion, in [15] for buckling under lateral pressure, and in [16] and [17] for the case of axial compression. In these analyses, however, the eccentricity of stiffeners was not considered. In [18] the equilibrium and stability equations for discretely stiffened shells are derived taking into account the eccentricity of stiffeners. The present study uses the formulation of [18] and some preliminary results for ring-stiffened cylindrical shells were given in [19].

It should be pointed out that the effect of discreteness of rings on the buckling of cylindrical shells under lateral pressure has been investigated in [20] by an analog method, and that parallel work on axially compressed shells by the approach employed here has recently been reported in [21]. These studies, however, treat only typical examples and do not extend to a wide range of geometries.

In this report ring-stiffened cylindrical shells under lateral or hydrostatic pressure and axial compression are studied in detail. The main purpose is to find the difference between the buckling loads predicted by discrete and "smeared" theories and the influence of the various geometrical parameters on this difference.

The rings considered are equally spaced, but not necessarily of constant cross-section. For the cases of varying ring cross-section, the variation is assumed symmetrical with respect to the mid-length of the shell, since there is no physical motivation for asymmetrical stiffness distribution.

The analysis is a linear Donnell type theory and for clarity the main assumptions are repeated here:

- (1) The shell is thin, $(R/h) \gg 1$, and hence high powers of (h/R) are neglected.
- (2) The number of circumferential waves is not small, or rather $n^2 \gg 1$.
- (3) The rings are concentrated in their planes, i.e. they have zero width in the axial direction.
- (4) The normal strains $\epsilon_x(z)$ and $\epsilon_\phi(z)$ vary linearly in the ring as well as in the sheet. The normal strains in the ring and in the sheet are equal at their point of contact.
- (5) The rings do not transmit shear out to their plane, hence the shear membrane force $N_{x\phi}$ is carried entirely by the sheet.
- (6) The rings carry torsional moments on account of their torsional rigidity.

2. STABILITY EQUATIONS

The stability equations and force and moment expressions for a cylindrical shell stiffened by rings and stringers derived in [18] are given here for convenience (it should be remembered that in [18] the stringers are assumed to be equal, whereas every ring may have a different cross section):

$$\begin{aligned}
 0 = \delta U = & \int_0^{L/R} \int_0^{2\pi} \{ [-N_{x,x} - N_{x\phi,\phi}] R \delta u \\
 & + [-N_{\phi,\phi} - N_{x\phi,x}] R \delta v \\
 & + [-M_{x,xx} - RN_{\phi} - M_{\phi,\phi\phi} + M_{x\phi,x\phi} - M_{\phi x,x\phi} \\
 & - R(N_{x\phi w,x})_{,x} - R(N_{\phi\phi w,\phi})_{,\phi} - R(N_{x\phi\phi w,\phi})_{,x} \\
 & - R(N_{x\phi\phi w,x})_{,\phi} \delta w \} R dx d\phi \\
 & + \int_0^{2\pi} [N_x R \delta u + N_{x\phi} R \delta v - M_x \delta(w_{,x}) \\
 & + (M_{x,x} - M_{x\phi,\phi} + M_{\phi x,\phi} + RN_{x\phi w,x} \\
 & + RN_{x\phi\phi w,\phi}) \delta w]_{x=0}^{x=L/R} R d\phi \quad (1)
 \end{aligned}$$

and

$$\begin{aligned}
 N_x = & \frac{Eh}{(1-\nu^2)} \{ [u_{,x} + \nu(u_{,\phi} - w)] + \sum_{j=1}^n \delta(\phi - \frac{2\pi j}{n}) (u_1 u_{j,x} \\
 & - x_1 w_{j,xx}) \}
 \end{aligned}$$

$$\begin{aligned}
 N_{\phi} = & \frac{Eh}{(1-\nu^2)} (v_{,\phi} - w + \nu u_{,x}) + \sum_{i=1}^{q-1} \delta(x - \frac{i}{q} \frac{L}{R}) [u_{2i} (v_{i,\phi} - w_i) \\
 & - x_{2i} w_{i,\phi\phi}]
 \end{aligned}$$

$$N_{x\phi} = N_{\phi x} = \frac{Eh}{2(1+\nu)} (u_{,\phi} + v_{,x}) \quad (2)$$

$$M_x = \left(-\frac{D}{R}\right) [w_{,xx} + v w_{,\phi\phi}] + \sum_{j=1}^n \delta\left(\phi - \frac{2\pi j}{n}\right) (\eta_{01} w_{j,xx} - \zeta_1 u_{j,x})]$$

$$M_\phi = \left(-\frac{D}{R}\right) \{ (w_{,\phi\phi} + v w_{,xx}) + \sum_{i=1}^{q-1} \delta\left(x - \frac{iL}{qR}\right) [\eta_{02i} w_{i,\phi\phi} - \zeta_{2i} (v_{i,\phi} - w_i)] \}$$

$$M_{x\phi} = \left(\frac{D}{R}\right) \left[(1-\nu) w_{,x\phi} + \sum_{j=1}^n \delta\left(\phi - \frac{j}{n} 2\pi\right) \eta_{1j} w_{j,x\phi} \right]$$

$$M_{\phi x} = \left(-\frac{D}{R}\right) \left[(1-\nu) w_{,x\phi} + \sum_{i=1}^{q-1} \delta\left(x - \frac{iL}{qR}\right) \eta_{2i} w_{i,x\phi} \right] \quad (3)$$

After substitution of Eqs. (2) and (3) into Eq. (1) and omission of the terms that relate to stringers, the stability equation for a ring-stiffened cylindrical shell becomes:

$$\begin{aligned} 0 = \delta U = \int_0^{L/R} \int_0^{2\pi} & \left\{ -\frac{Eh}{(1-\nu^2)} \left\{ u_{,xx} + \frac{1-\nu}{2} u_{,\phi\phi} + \frac{1+\nu}{2} v_{,x\phi} - v w_{,x} \right\} R \delta u \right. \\ & - \frac{Eh}{(1-\nu^2)} \left\{ \left(\frac{1+\nu}{2} u_{,x\phi} + v_{,\phi\phi} + \frac{1-\nu}{2} v_{,xx} - w_{,\phi} \right) \right. \\ & + \sum_{i=1}^{q-1} \delta\left(x - \frac{iL}{qR}\right) [\mu_{2i} (v_{i,\phi\phi} - w_{i,\phi}) - \chi_{2i} w_{i,\phi\phi\phi}] \} R \delta v \\ & + \frac{D}{R} \left[w_{,xxxx} + 2w_{,xx\phi\phi} + w_{,\phi\phi\phi\phi} + 12\left(\frac{R}{n}\right)^2 (w - v_{,\phi} - \nu u_{,x}) \right] \\ & + \sum_{i=1}^{q-1} \delta\left(x - \frac{iL}{qR}\right) \left[\zeta_{2i} (2w_{i,\phi\phi} - v_{i,\phi\phi\phi}) + \eta_{02i} w_{i,\phi\phi\phi\phi} \right. \\ & \left. + 12\left(\frac{R}{n}\right)^2 \mu_{2i} (w_i - v_{i,\phi}) \right] + \sum_{i=1}^{q-1} \delta_{,x} \left(x - \frac{iL}{qR}\right) \eta_{2i} w_{i,x\phi\phi} \end{aligned}$$

$$\begin{aligned}
 & + \frac{R^2}{D} [(N_{x\phi} w_{,x})_{,x} + (N_{\phi\phi} w_{,\phi})_{,\phi} + (N_{x\phi\phi} w_{,x})_{,\phi} + (N_{\phi\phi\phi} w_{,\phi})_{,x}] \delta w \left\{ R dx d\phi \right. \\
 & + \int_0^{2\pi} \left\{ \frac{Eh}{(1-\nu^2)} [u_{,x} + \nu(v_{,\phi} - w)] \right\} R \delta u \\
 & + \frac{Eh}{2(1+\nu)} (u_{,\phi} + v_{,x}) R \delta v + \frac{D}{R} \{w_{,xx} + \nu w_{,\phi\phi}\} \delta(w_{,x}) \\
 & - \frac{D}{R} \{[w_{,xxx} + (2-\nu)w_{,x\phi\phi}]\} - \frac{R^2}{D} (N_{x\phi} w_{,x} \\
 & + N_{\phi\phi} w_{,\phi}) \delta w \left. \right\}_{x=0}^{x=L/R} R d\phi \quad (4)
 \end{aligned}$$

3. SOLUTION

The displacement components are expanded into Fourier series in the axial direction

$$\begin{aligned}
 u &= \sin t\phi \sum_{n=1}^{\infty} A_n \cos n\beta x \\
 v &= \cos t\phi \sum_{n=1}^{\infty} B_n \sin n\beta x \\
 w &= \sin t\phi \sum_{n=1}^{\infty} C_n \sin n\beta x
 \end{aligned} \quad (5)$$

Each set of terms of series (5) satisfies the stability equations of a shell with smeared stiffeners (Eqs. 12 of [8]) and each term fulfills the classical simple support boundary conditions

$$\begin{aligned}
 w &= 0 \\
 M_x &= 0 \\
 v &= 0 \\
 N_x &= 0
 \end{aligned}
 \quad \text{at } x = 0, (L/R) \quad (6)$$

Substitution of the displacements (5) into Eq. (4) yields:

$$\begin{aligned}
 0 &= \int_0^{L/R} \int_0^{2\pi} \left\{ - \frac{Eh}{(1-\nu^2)} \sum_{n=1}^{\infty} \left\{ -n^2 \beta^2 A_n - \left(\frac{1-\nu}{2}\right) t^2 A_n \right. \right. \\
 &\quad \left. \left. - \left(\frac{1+\nu}{2}\right) t n \beta B_n - \nu n \beta C_n \right\} \cos n\beta x \sin t\phi \sum_{m=1}^{\infty} R \delta A_m \cos m\beta x \sin t\phi \right. \\
 &\quad \left. - \frac{Eh}{(1-\nu^2)} \sum_{n=1}^{\infty} \left\{ - \left(\frac{1+\nu}{2}\right) t n \beta A_n - t^2 B_n - \left(\frac{1-\nu}{2}\right) n^2 \beta^2 B_n \right. \right. \\
 &\quad \left. \left. - t C_n + \sum_{i=1}^{q-1} \delta\left(x - \frac{iL}{qR}\right) [-\mu_{2i} t^2 B_n - \mu_{2i} t C_n \right. \right. \\
 &\quad \left. \left. + \chi_{2i} t^3 C_n \right] \right\} R \sin n\beta x \cos t\phi \sum_{m=1}^{\infty} \sin m\beta x \cos t\phi \delta B_m \\
 &\quad + \frac{D}{R} \left\{ \sum_{n=1}^{\infty} C_n \left\{ (n^2 \beta^2 + t^2)^2 + 12 \left(\frac{R}{h}\right)^2 [C_n + t B_n + \nu n \beta A_n] \right. \right. \\
 &\quad \left. \left. + \sum_{i=1}^{q-1} \delta\left(x - \frac{iL}{qR}\right) [\zeta_{2i} (-2t^2 C_n - t^3 B_n) \right. \right. \\
 &\quad \left. \left. + \eta_{0?i} t^4 C_n + 12 \left(\frac{R}{h}\right)^2 \mu_{2i} (C_n + t B_n) \right] \right\}
 \end{aligned}$$

$$\begin{aligned}
 & + \frac{R^2}{D} [C_n (-n^2 \beta^2 N_{x_0} - t^2 N_{\phi_0} + 2t n \beta N_{x\phi_0})] \sin n\beta x \sin t\phi \\
 & + \sum_{i=1}^{q-1} \delta_{,x} (x - \frac{1L}{qR}) n \beta t^2 n_{+2i} C_n \sin t\phi \cos n\beta x \left\{ \sum_{m=1}^{\infty} \delta C_m \sin m\beta x \sin t\phi \right\} R dx d\phi
 \end{aligned} \tag{7}$$

It was tacitly assumed in Eq. (7) that the prebuckling stresses N_{x_0} , N_{ϕ_0} and $N_{x\phi_0}$ are constant all over the cylinder. In the case of hydrostatic or lateral pressure and axial compression this assumption is valid, if the membrane stresses represent the prebuckling stresses satisfactorily. Since each of the coefficients of δA_m , δB_m and δC_m ($m = 1 \dots \infty$) must be zero, Eq. (7) is equivalent to three infinite sets of algebraic equations.

The first set (that corresponding to δA_n) does not contain delta functions and can be solved directly by equating the integrand to zero.

$$-A_n [n^2 \beta^2 + (\frac{1-\nu}{2}) t^2] - B_n (\frac{1+\nu}{2}) t n \beta - C_n \nu n \beta = 0$$

or

$$A_n = \frac{-\nu n \beta C_n + t n \beta (\frac{1+\nu}{2}) B_n}{n^2 \beta^2 + (\frac{1-\nu}{2}) t^2} \tag{8}$$

Substitution of Eq. (8) into the non-vanishing parts of Eq. (7) yields after appropriate integrations two sets of equations

$$b_{m1} B_m + c_{m1} C_m + \sum_{n=1}^{\infty} \sum_{i=1}^{q-1} (\beta_{i1} B_n + \gamma_{i1} C_n) \sin \frac{mi\pi}{q} \sin \frac{ni\pi}{q} = 0$$

$$b_{m2} B_m + c_{m2} C_m + \sum_{n=1}^{\infty} \sum_{i=1}^{q-1} (\beta_{i2} B_n + \gamma_{i2} C_n) \sin \frac{mi\pi}{q} \sin \frac{ni\pi}{q} + \delta_{im} C_n \cos \frac{mi\pi}{q} \cos \frac{ni\pi}{q} = 0$$

$$m = 1 \dots \infty \quad (9)$$

where

$$b_{m1} = \left[\frac{(\frac{1+v}{2})^2 + m^2 \beta^2}{m^2 \beta^2 + (\frac{1-v}{2})^2} - t^2 - (\frac{1-v}{2}) m^2 \beta^2 \right] \frac{L}{2R}$$

$$c_{m1} = \left[\frac{v(\frac{1+v}{2}) m^2 \beta^2 t}{m^2 \beta^2 + (\frac{1-v}{2})^2} - t \right] \frac{L}{2R}$$

$$b_{m2} = -12 \left(\frac{R}{h}\right)^2 c_{m1}$$

$$c_{m2} = \left\{ (m^2 \beta^2 + t^2)^2 + 12 \left(\frac{R}{h}\right)^2 \left[1 - \frac{v^2 m^2 \beta^2}{m^2 \beta^2 + (\frac{1-v}{2})^2} \right] + \frac{R^2}{D} (-n^2 \beta^2 N_{x0} - t^2 N_{\phi 0} + 2tn \beta N_{x\phi 0}) \right\} \frac{L}{2R}$$

$$\beta_{i1} = -\mu_{2i}$$

$$\gamma_{i1} = \chi_{2i} t^3 - \mu_{2i} t$$

$$\beta_{i2} = -\zeta_{2i} t^3 + 12 \left(\frac{R}{h}\right)^2 \mu_{2i} t$$

$$\gamma_{2i} = -2\zeta_{2i} t^2 + \eta_{02i} t^4 + 12 \left(\frac{R}{h}\right)^2 \mu_{2i}$$

$$\delta_{im} = m \beta^2 t^2 \eta_{t2i}$$

4. THE STABILITY DETERMINANT.

The buckling load is found from the vanishing of the determinant of Eqs. (9).

For n terms (or n equations) the determinant is of the order $2n$. As shown in Appendix I, the equations separate into subgroups. Hence the stability determinant is reduced into subdeterminants. By choosing one equation (not necessarily the first one) one obtains a first approximation to the buckling load. It is also shown in Appendix II that in the case of uniform rings this approximation coincides with the "smeared" solution. In this case, the difference between the results of "discrete" theory and "smeared" theory is therefore the difference between the first order approximation and that of infinite order.

5. LOCAL BUCKLING

Obviously the effect of discreteness of rings is more pronounced for a small number of rings than for many rings. When there are only few rings, however, local buckling between rings usually occurs at a lower load than general instability.

When the torsional stiffness of the rings is neglected, the local buckling load is that of a simply supported cylinder of the same thickness and radius but of length L/q . Or, in other words, the local buckling load is equivalent to that of an unstiffened cylinder forced to buckle with $n = q$.

In Table I buckling loads are given for several wave numbers n . Comparison of this table with Table I of reference [11] (the complete report, TAE 42) shows that, in most cases of fairly strong rings, more than 3 rings are needed for general instability to be predominant (to yield the lower buckling loads). In some cases more than 20 rings are needed.

Hence the effect of discreteness of rings is studied in particular for shells with weak rings.

6. NUMERICAL RESULTS AND DISCUSSION

The critical pressure was computed for discretely stiffened cylinders covering a wide range of geometries. Most of the calculations relate to cylinders stiffened with one ring only, since the discreteness effect is most pronounced for one ring. Furthermore the results for one ring are more accurate, as discussed in Appendix III, in connection with convergence considerations.

As the local buckling load is very low for a cylinder stiffened with one ring, very weak rings are considered to ensure that the general instability loads will be lower.

6.1. Hydrostatic Pressure

Figure 2 shows the influence of the shell parameters L/R and R/h on the effect of discreteness. The effect is presented as the difference between the "discrete" and "smeared" loads (or rather the percentage reduction in

buckling load for "discrete" rings) versus the Batdorf shell geometry parameter $Z = (1-\nu^2)^{1/2}(R/h)(L/R)^2$, that combines L/R and R/h in the appropriate manner. Shells with one ring are considered for which the difference is most pronounced. It is seen that the dependence is monotonous and there is very small scatter. The scatter is mainly caused by the necessary integer values of the number of circumferential waves, that produce ripples in the separate "smeared" and "discrete" curves, the difference of which is presented in Fig. 2.

Rings with the following geometric parameters $e_2/h = 1$, $A_2/ah = 0.1$, $I_{22}/ah^3 = 0.01$ were chosen since they ensure for all shell geometries general instability at about 60%-80% of the local buckling load. This is high enough (see also Fig. 3) for a pronounced discreteness effect, but on the other hand not too close to the cut-off line (that represents the local buckling load).

Figure 3 shows the influence of the relative stiffness of the rings on the discreteness effect. When the rings are very weak, the buckling mode tends to remain in simple sinusoidal form, as for unstiffened shells, and the "discreteness" effect is therefore very small. For stronger rings, the shell seeks lower energy modes and hence the difference between the "smeared" and "discrete" theory becomes more pronounced. Again shells with one ring are considered, for which the ring spacing $a = L/2$. In the example of Fig. 2, the ring eccentricity is 1 and hence, for rings of rectangular cross-section, (A_2/ah) would be $(c/a) = (2c/L)$ where c is the width of the ring. For rings of rectangular cross-section (I_{22}/ah^3) would then be $0.0833(c/a) = 0.0833(2c/L)$

In Fig. 3 the ring cross-section differs slightly from the rectangular and is defined by $(A_2/ah) = (2c/L)$ and $(I_{22}/ah^3) = 0.1(2c/L)$. $f = (2c/L)$ appears as a parameter in the curves of Fig. 2.

Figure 3 also shows the effect is larger for external rings than for internal ones. The general behavior is however similar.

Figure 4 shows that effect of the eccentricity on the discreteness effect. Comparison of rings of equal A_2/ah and I_{22}/ah^3 would not be appropriate here, since the eccentricity noticeably affects the "smeared" buckling load, and the comparison would then actually be of rings of different strengths instead. The comparison in Fig. 4 is between rings of equal strength with cross-sections defined by $A_2/ah = 0.1f$, $I_{22}/ah^3 = 0.01f$ and where the parameter f was adjusted for each eccentricity to yield the same range of "smeared" buckling load.

As was pointed earlier, Figs. 2 to 4 show the influence of the discreteness effect for shells with one ring only. Figs. 5 and 6 show dependence of the effect on the number of rings. The general trend is, as expected, for the magnitude of the difference to go asymptotically to zero with increasing number of rings. It is however seen that for high L/R and R/h , when the one-ring difference is highest, the effect also decreases more slowly with increasing number of rings.

It should be noted that the results here are the 10th approximation and are not accurate for one or two rings for high L/R .

In Tables 2 to 6 some of the numerical results on which Figs. 2 to 7 are based are presented in tabular form. The values in the tables are in some cases given up to 5 or six significant figures, as obtained from the computer program, though usually only the first 4 figures are meaningful.

Figure 6 summarizes the importance of the discreteness of rings in the instability analysis of cylindrical shells under hydrostatic pressure. It shows the variation of the percentage reduction in buckling pressure (due to discreteness of the rings) with number of rings and increasing ring strength. As the "smeared" stiffness - represented in the figure by the critical pressure parameter of the stiffened shell - increases, the discreteness effect for 1 ring rises till the cutoff point is reached, when local buckling becomes dominant for 1 ring. For 2 rings the discreteness effect is at first smaller, but then rises beyond that for 1 ring, till the cutoff point for 2 rings is reached. This increase in discreteness effect with number of rings and ring strength continues till at 6 rings a maximum for this geometric configuration is reached. Then the effect starts to decrease with increase in number and strength of rings.

It should be noted that the largest discreteness effect does not occur with one ring, as one would intuitively assume, since local buckling becomes dominant before the ring can attain a high stiffness that would result in an appreciable discreteness effect. There are two opposing tendencies, one for the discreteness effect to decrease with number of rings for a given basic shell, and the other a postponement of the cut off point with number of rings. The

result is a certain maximum for every shell geometry as shown in Fig. 6.

6.2. Lateral Pressure with Non-Uniform Rings.

In order to investigate the effect of discreteness of stiffeners for non-uniform stiffeners, the buckling loads were computed for ring-stiffened cylinders under lateral pressure with rings of varying cross-section. Results of a "smeared" stiffener theory are available for sinusoidal height variation of rings with rectangular cross-section [22]. For the "smeared" case, the height of the rings d as a function of x can be written

$$d = d_0 + d_1 \sin \alpha x \quad (10)$$

where d_0 represents a constant part and d_1 the amplitude of the varying portion of the ring cross-section. For the corresponding discrete case the height of rings is d_i $i = 1 \dots (q-1)$ and the i^{th} ring is at $x_i = \frac{iL}{qR}$. The discretely varying portion of the height of the ring, d_i is chosen so that the mean height is the same as for the "smeared" case and also that

$$\frac{d_i - d_0}{d_j - d_0} = \frac{\sin \alpha x_i}{\sin \alpha x_j} \quad (11)$$

In Table 7 the critical lateral pressures are given for three non-uniformly stiffened shells with 3 rings. The pressure parameters for the non-uniform and uniform discrete cases are compared with the corresponding "smeared" theory values from [22]. It is seen that the difference between the uniform and non-uniform cases is considerable in the discrete analysis. This does not

invalidate the weight savings predicted by the "smeared" theory in [22]. For, even in the 3-ring case, the gain in critical pressure predicted by the "smeared" theory can be recovered, if a larger portion of the height is varied than in [22]. The results, however, indicate that for shells with few rings the optimal dimensions predicted by "smeared" theory may have to be revised in view of the discreteness effects. Discrete theory is therefore needed for a reliable analysis when one wishes to exploit the advantages of non-uniform stiffening for shells with few rings.

6.3. Axial Compression

The critical load of unstiffened cylinders under axial compression is a function of the wave parameter

$$\frac{(n^2 \beta^2 + t^2)^2}{n^2 \beta^2}$$

and not of t and n independently. (This applies strictly only to "classical" simple supports and to shells that buckle into many waves in the axial direction). Hence in many cases a ring-stiffened cylindrical shell under axial compression can buckle under the same load as a corresponding unstiffened cylinder, since for $n=q$ it buckles through the rings for some value of the wave parameter. This "ineffectiveness" of rings disappears only for large β or large n - for shells with small ring spacing. These shells, however, exhibit a very small discreteness effect (see shells Nos. 4 to 8 in Table 8). As small ring spacing means low effective Z , this is not surprising if one remembers that also for the case of

hydrostatic pressure only very small discreteness effects were found for shells of small Z .

In Table 8, eight typical ring-stiffened shells, chosen from the test data of [23] and [24], are studied for discreteness effects. Shells Nos. 1 to 3 show a very small, but noticeable discreteness effects. The local buckling load in these shells, for simple supports, would be slightly below the general instability, which would mean that the results presented are beyond the cut-off point and therefore meaningless; but as in reality there is some restraint at the boundaries of the sub-shells, the cut-off point is shifted and the results are of value. For shells Nos. 4 to 8, the local buckling load is larger than the general instability load, even for simple supports. The discreteness effect is, however, negligible as already pointed out above.

The computations of Table 8 neglect the rotational restraint of the rings ($n_{t2} = 0$ is assumed). Calculations with the proper n_{t2} values are in progress. It can, however, be expected that the conclusion - that the discreteness effect is very small for general instability of ring-stiffened cylinders under axial compression - will not be affected.

7. CONCLUSIONS

The results of the calculations for cylindrical shells under hydrostatic or lateral pressure show that the discreteness effect depends very strongly on the geometry of the shell and the eccentricity of the rings.

Though for most practical cases with many rings the effect is not important, one should be cautious when dealing with shells of high Z and rings of low eccentricity. The "discrete" load may then be about 20% lower than the "smeared" one even for ten rings or more.

In ring-stiffened shells under axial compression the discreteness effect is always very small.

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APPENDIX I.

REDUCTION IN ORDER AND SIMPLIFICATION OF STABILITY

DETERMINANTS.

Equations (9) consist of two infinite sets of equations. The n^{th} approximation yields $2n$ equations and the corresponding stability determinant consists of n^2 blocks of 2×2 . Denoting the blocks Δ_{ij} we have

$$\Delta_{mm} = \begin{vmatrix} b_{m1} + \sum_{i=1}^{q-1} \beta_{i1} \sin^2 \frac{m\pi i}{q} & c_{m1} + \sum_{i=1}^{q-1} \gamma_{i1} \sin^2 \frac{m\pi i}{q} \\ b_{m2} + \sum_{i=1}^{q-1} \beta_{i2} \sin^2 \frac{m\pi i}{q} & c_{m2} + \sum_{i=1}^{q-1} \gamma_{i2} \sin^2 \frac{m\pi i}{q} + \delta_{imm} \cos^2 \frac{m\pi i}{q} \end{vmatrix}$$

and

... 1 - 1

$$\Delta_{mn} = \begin{vmatrix} \sum_{i=1}^{q-1} \beta_{i1} \sin \frac{m\pi i}{q} \sin \frac{n\pi i}{q} & \sum_{i=1}^{q-1} \gamma_{i1} \sin \frac{m\pi i}{q} \sin \frac{n\pi i}{q} \\ \sum_{i=1}^{q-1} \beta_{i2} \sin \frac{m\pi i}{q} \sin \frac{n\pi i}{q} & \sum_{i=1}^{q-1} \gamma_{i2} \sin \frac{m\pi i}{q} \sin \frac{n\pi i}{q} \\ & + \delta_{imn} \cos \frac{m\pi i}{q} \cos \frac{n\pi i}{q} \end{vmatrix}$$

... 1 - 2

It is assumed that the rings are symmetrical with respect to the mid-length of the cylinders. Hence $\beta_{i1} = \beta_{(q-i)1}$, $\gamma_{i1} = \gamma_{(q-i)1}$ etc.

A sum of the form $\sum_{i=1}^{q-1} \alpha_i \sin \frac{m\pi i}{q} \sin \frac{n\pi i}{q}$ or $\sum_{i=1}^{q-1} \phi_i \cos \frac{m\pi i}{q} \cos \frac{n\pi i}{q}$

is zero when m is odd and n is even, or vice versa. This can be shown as follows:

$$\begin{aligned} & \alpha_{(q-i)} \sin \frac{m\pi(q-i)}{q} \sin \frac{n\pi(q-i)}{q} \\ = & \alpha_i \sin \left(m\pi - \frac{m\pi i}{q} \right) \sin \left(n\pi - \frac{n\pi i}{q} \right) \\ = & - \alpha_i \sin \frac{m\pi i}{q} \sin \frac{n\pi i}{q} \quad \dots 1 - 3 \end{aligned}$$

since for odd m

$$\sin \left(m\pi - \frac{m\pi i}{q} \right) = \sin \frac{m\pi i}{q}$$

and for even n

$$\sin \left(n\pi - \frac{n\pi i}{q} \right) = - \sin \frac{n\pi i}{q}$$

The i^{th} term in the sum therefore cancels the $(q-i)^{\text{th}}$ term. Hence all the blocks Δ_{mn} for which m is odd and n is even, or m is even and n is odd, vanish. The stability determinant reduces therefore to two subdeterminants, the first containing the odd n 's and second the even n 's.

This reduction of the stability determinant applies to the most general case. In some particular cases there are further possible subdivisions.

The first case is that of rings with zero torsional stiffness.

Rings of Zero Torsional Stiffness

Here $\delta_{imn} = 0$ and there are no sums of the type $\sum_{i=1}^{q-1} \phi_i \cos \frac{m\pi i}{q} \cos \frac{n\pi i}{q}$

The remaining sums, of the type $\sum_{i=1}^{q-1} \alpha_i \sin \frac{m\pi i}{q} \sin \frac{n\pi i}{q}$ are zero

when m or n are equal to a multiple of q . This means that the equations numbered $q, 2q, 3q, \dots$ separate and the corresponding buckling load can be calculated separately.

The physical reason is immediately apparent. These particular modes of buckling are the modes of local buckling, and when the rings have zero torsional stiffness local buckling is the same as buckling of the unstiffened shell (a problem that has a closed form solution). Other subdivisions are possible when all the rings are equal.

Equal Rings of Zero Torsional Stiffness

When all the rings are equal the sums of the type $\sum_{i=1}^{q-1} \alpha_i \sin \frac{m\pi i}{q} \sin \frac{n\pi i}{q}$

become $\alpha \sum_{i=1}^{q-1} \sin \frac{m\pi i}{q} \sin \frac{n\pi i}{q}$. The sum

$$S_{mnq} = \sum_{i=1}^{q-1} \sin \frac{m\pi i}{q} \sin \frac{n\pi i}{q} \quad \dots \quad (4)$$

vanishes if neither $m + n$ or $m - n$ are multiples of $2q$ or if both are.

This may be shown as follows:

$$\sum_{i=1}^{q-1} \sin \frac{m\pi i}{q} \sin \frac{n\pi i}{q} = \frac{1}{2} \sum_{i=1}^{q-1} \left[\cos \frac{(m-n)\pi i}{q} - \cos \frac{(m+n)\pi i}{q} \right] \quad \dots \quad (5)$$

Now if $(m - n)$ is a multiple of $2q$

$$\sum_{i=1}^{q-1} \cos \frac{(m-n)i\pi}{q} = q - 1 \quad \dots 1 - 6$$

and the first set of terms of the sum equals $\frac{q-1}{2}$.

If $(m + n)$ is a multiple of $2q$

$$\sum_{i=1}^{q-1} \cos \frac{(m+n)i\pi}{q} = q - 1 \quad \dots 1 - 7$$

and the second set of terms of the sum equals $-\frac{q-1}{2}$. Hence when both $(m-n)$ and $(m+n)$ are multiples of $2q$,

$$S_{mnq} = 0 \quad \dots 1 - 8$$

To consider the case when neither $m + n$ nor $m - n$ are multiples of $2q$, denote $(m-n) = r$ and $(m+n) = s$. Then

$$\sum_{i=1}^{q-1} \cos \frac{(m-n)i\pi}{q} = \sum_{i=1}^{q-1} \cos \frac{ri\pi}{q} = \frac{1}{2} \sum_{i=1}^{q-1} (e^{\frac{jri\pi}{q}} + e^{-\frac{jri\pi}{q}}) \quad \dots 1 - 9$$

where j denotes here $j = \sqrt{-1}$.

Since all the blocks Δ_{mn} for which m is odd and n is even, or m is even and n is odd, vanish in the general case, only even values of r and s need be considered here.

Now

$$\sum_{i=1}^{q-1} e^{\frac{jri\pi}{q}} = \frac{e^{\frac{jr\pi}{q}} (1 - e^{\frac{jr(q-1)\pi}{q}})}{1 - e^{\frac{jr\pi}{q}}} = -1 \text{ for even } r \quad \dots 1 - 10$$

Similarly

$$\sum_{i=1}^{q-1} e^{-\frac{jri\pi}{q}} = -1 \text{ for even } r \quad \dots 1 - 11$$

Hence, here,

$$\sum_{i=1}^{q-1} \cos \frac{(m-n)i\pi}{q} = -1 \quad \dots 1 - 12$$

In the same manner

$$\sum_{i=1}^{q-1} \cos \frac{(m+n)i\pi}{q} = \sum_{i=1}^{q-1} \cos \frac{si\pi}{q} = -1 \text{ for even } s \quad \dots 1 - 13$$

Therefore

$$S_{mnq} = \frac{1}{2} \sum_{i=1}^{q-1} \left[\cos \frac{(m-n)i\pi}{q} - \cos \frac{(m+n)i\pi}{q} \right] = 0 \quad \dots 1 - 14$$

and the subdeterminants are further simplified.

APPENDIX II

COMPARISON OF "SMEARED" AND "DISCRETE" SOLUTIONS.

To compare the "discrete" and "smeared" solutions one first has to clarify what are the corresponding cases.

The equivalence is defined here by taking the ring spacing as $a = \frac{L}{q}$.

The ring parameters $\mu_2, \chi_2, \zeta_2, \eta_0$ as defined for the "smeared" case contain $\frac{1}{a}$. In the "discrete" theory $\frac{1}{a}$ is replaced in the parameter by $\frac{1}{R}$. Hence

$$\mu_{2s} = \frac{\mu_{2d} \cdot qR}{L} \quad \dots \text{II} - 1$$

where the subscript s refers to parameters as defined in the "smeared" theory and the subscript d to those defined in the "discrete" theory.

The "smeared" solution is in a closed form whereas in the "discrete" theory the solution is the limit of a series.

It can be shown that the first term, the first approximation, of the "discrete" solution is equal to the "smeared" solution, for the case of equal rings. If one compares Eq. (12) of Ref. [3] with Eq. (4) of this report, and solves these equations by the Galerkin method (choosing u, v, w as in Eq.(5)) one sees that the terms corresponding to shell itself are equal.

For the rings, there are in the "smeared" case terms of the type

$$S(m,n) = \mu_{2s} \int_0^{L/R} \sin n\beta x \sin m\beta x dx = \mu_{2s} \left(\frac{1}{2R}\right) \delta_{mn} \quad \dots \text{II} - 2$$

where δ_{mn} is the Kronecker delta.

The corresponding term for the "discrete" solution is

$$D(m,n) = \int_0^{L/R} \left[\sum_{i=1}^{q-1} \mu_{2di} \delta(x-x_i) \sin n\beta x \sin m\beta x \right] dx$$

$$= \sum_{i=1}^{q-1} \mu_{2di} \sin n\beta x_i \sin m\beta x_i \quad \dots 11-3$$

Now for equal rings $\mu_{2di} = \mu_{2d}$ and therefore

$$D(m,m) = \mu_{2d} \sum_{i=1}^{q-1} \sin \frac{m\pi i}{q} \sin \frac{m\pi i}{q} = \frac{\mu_{2d} q}{2} \quad \dots 11-4$$

Hence, after substitution from Eq. (11-1)

$$D(m,m) = \frac{\mu_{2s} L}{2k} = S(m,m) \quad \dots 11-5$$

This means that the diagonal terms in the "discrete" stability determinant are equal to those of the "smeared" one. On account of the Kronecker delta in Eq. (11-2) one can conclude that the first approximation of the "discrete" case is equal to the "smeared" solution for any number of rings and any n .

APPENDIX III

CONVERGENCE OF THE SOLUTION

In the "smeared" case the solution is in a closed form, whereas in the "discrete" case there is a series which converges to the solution,

For the interesting case, when there is a considerable difference between the "smeared" and discrete solutions, the convergence is slow. This is to be expected since rapid convergence means that the shape of the deflection is close to a simple sine curve, or to a combination of the first harmonics. When the effect of discreteness is appreciable the deflection shape is usually different from a simple sine curve.

Fortunately, however, the series for one ring, though it converges slowly, can be extrapolated easily. As can be seen from Fig. 8, a semi-logarithmic plot of the difference between successive approximations approaches a straight line asymptotically. This means that the difference can be represented by a geometric series. Thus if we know three approximations, for example λ_i , λ_{i+1} , λ_{i+2} , we can estimate the limit

$$\lambda_{\infty} = \lambda_i - \frac{(\lambda_i - \lambda_{i+1})}{\frac{\lambda_{i+1} - \lambda_{i+2}}{\lambda_i - \lambda_{i+1}}}$$

When the shell has more than one ring, this is not true anymore and in the numerical work the 10th approximation was then usually considered adequate. This rather arbitrary decision was based on the results obtained for one ring where the difference between λ_{10} and λ_{∞} was usually less than 2-3%, except for large L/R values for which the difference was up to 5%.

TABLE 1.

LOCAL BUCKLING OF TYPICAL RING STIFFENED SHELLS

$\underline{A_2/ah = 0.5}$

$\underline{I_{22}/ah^3 = 5}$

$\underline{e_2 = 5}$

R/h	L/R	λ smeared	local buckling for one ring *	Number of rings required for general instability to pre- dominate
50	0.5	644.9	428.9	2
	1.0	644.9	173.2	5
	1.5	644.9	108.7	7
100	0.5	1303	533.8	3
	1.0	1288	229.9	7
	2.0	1281	107.4	14
250	0.5	3228	755.7	5
	1.0	3224	345.9	10
500	0.5	6398	1013	8
	1.0	6398	477.2	16
	2.0	4660	231.7	26
1000	0.5	12860	1381	11
	1.0	12180	661.1	22
	2.0	6881	323.8	29
2000	0.5	25590	1903	16
	1.0	18520	924.0	26
	2.0	9957	454.9	33

* Torsional stiffness of rings is neglected.

TABLE 2

INFLUENCE OF SHELL GEOMETRY ON DISCRETENESS EFFECT

$A_2/ah = 0.1$

$I_{22}/ah^3 = 0.01$

$e_2/h = 1$

R/h	L/R	Z	λ smeared	λ discrete*	†	$\Delta\lambda$ %
100	0.25	5.96	618.97	617.28	11	0.27
	0.5	23.85	324.44	315.76	10	2.68
	0.75	53.66	223.64	216.48	8	3.17
	1.0	95.39	172.76	163.24	7	4.92
	1.25	149.0	140.08	129.73	7	7.40
	1.5	214.6	117.20	108.80	6	7.16
	1.75	292.1	101.68	92.03	6	9.50
	2.0	381.6	91.90	81.10	6	11.75
	2.25	482.2	79.51	71.25	5	10.4
	2.50	596.3	71.78	62.52	5	12.9
500	0.25	29.8	1457.7	1417.5	21	2.76
	0.5	119.2	771.4	722.4	16	6.35
	0.75	268.3	525.3	481.7	13	8.30
	1.0	477.0	402.1	353.9	13	12.0
	1.25	745.2	323.4	278.1	11	13.9
	1.5	1073	272.5	228.3	10	16.2
	1.75	1461	231.8	189.8	9	18.2
	2.0	1908	204.4	163.4	9	20.0
	2.25	2415	181.3	141.2	8	22.1
	2.75	3607	148.6	114.7	7	22.8
	3.0	4293	136.8	102.5	7	25.1

TABLE 2. (CONT'D)

INFLUENCE OF SHELL GEOMETRY ON DISCRETENESS EFFECT

$A_2/ah = 0.1$

$I_{22}/ah^3 = 0.01$

$e_2/h = 1$

R/h	L/R	Z	λ smeared	λ discrete*	†	$\Delta\lambda\%$
1000	0.25	59.6	2125.2	2036.6	26	4.17
	0.5	238.5				
	0.75	536.6	753.60	664.6	16	11.8
	1.0	953.9				
	1.25	1490	461.48	373.7	13	19.05
	1.5	2146	385.87	303.8	12	21.3
	1.75	2921	332.16	261.5	11	21.2
	2.0	3816	288.95	218.8	10	24.3
	2.25	4829	259.24	193.3	10	25.4
	2.5	5963	231.90	170.0	9	26.5
	2.75	7214	214.73	155.6	9	27.5
	3.0	8585	193.66	144.6	8	25.3

* Extrapolation to infinite order approximation.

TABLE 3

INFLUENCE OF RING AREA ON DISCRETENESS EFFECT

L/R = 3

R/h = 250

e_2/h	A_2/ah	I_{22}/ah^3	\dagger	λ discrete *	λ smeared	$\Delta\lambda\%$
1	0.01	0.001	6	58.33	58.62	0.5
	0.03	0.003	6	64.20	67.68	5.1
	0.05	0.005	6	68.10	76.45	10.9
	0.06	0.006	6	69.60	80.72	13.7
	0.08	0.008	6	72.70	89.08	18.5
	0.10	0.010	6	76.00	97.17	21.8
	0.12	0.012	6	78.30	105.02	26.4
-1	0.01	0.001	6	57.17	57.79	0.7
	0.03	0.003	6	61.20	65.25	6.2
	0.05	0.005	6	64.05	72.47	11.6
	0.06	0.006	6	65.30	76.00	14.0
	0.08	0.008	6	67.25	82.89	18.9
	0.10	0.010	6	68.90	89.56	23.1
	0.12	0.012	6	70.20	96.05	26.8
	0.14	0.014	6	71.40	102.34	30.1

* Extrapolation to infinite approximation

TABLE 4

EFFECT OF RING PARAMETERS - ECCENTRICITY OF RINGS.

L/R = 1

R/h = 500

e_2/h	A_2/ah	I_{22}/ah^3	λ smeared	λ discrete *	t	$\Delta\lambda\%$
-1	0.01	0.001	252.45	251.56	12	0.35
	0.03	0.003	279.11	269.15	12	3.55
	0.05	0.005	307.71	284.50	12	7.50
	0.07	0.007	335.40	296.60	12	10.7
	0.085	0.0085	355.60	303.80	12	14.5
	0.10	0.010	373.70	310.10	12	17.0
	0.12	0.012	394.50	317.40	12	19.5
	0.14	0.014	414.70	324.20	12	21.5
	0.16	0.016	434.20	328.80	12	24.0
	0.18	0.018	453.30	334.00	12	26.2
-2	0.005	0.0005	263.96	261.60	12	0.89
	0.010	0.001	293.30	284.90	12	2.86
	0.015	0.0015	322.31	304.78	12	5.45
	0.020	0.002	351.10	322.10	12	8.29
	0.025	0.0025	378.76	337.27	12	10.9
	0.030	0.003	402.01	350.45	12	12.8
	0.035	0.0035	425.04	362.40	12	14.75
	0.040	0.004	447.87	373.00	12	16.7
	0.045	0.0045	470.50	381.90	12	18.8
-3	0.001	0.0001	247.63	247.25	13	0.15
	0.003	0.0003	274.87	271.60	12	1.05
	0.006	0.0006	314.32	303.70	12	3.40
	0.010	0.0010	367.12	340.05	12	7.36
	0.012	0.0012	390.74	355.85	12	8.94

TABLE 4 (Cont'd)

EFFECT OF RING PARAMETERS - ECCENTRICITY OF RINGS

e_2/h	<u>L/R = 1</u>		<u>R/h = 500</u>			
	Λ_2/ah	l_{22}/ah^3	λ smeared	λ discrete *	\dagger	$\Delta\lambda\%$
-3	0.015	0.0015	423.14	377.55	12	10.80
	0.018	0.0018	455.36	396.40	12	12.95
-4	0.001	0.0001	258.25	257.42	12	0.32
	0.003	0.0003	305.86	298.76	12	2.32
	0.005	0.0005	353.29	334.88	12	5.20
	0.007	0.0007	397.01	366.37	12	7.97
	0.009	0.0009	436.02	394.08	12	9.60
-10	0.0002	0.00002	264.52	263.65	12	0.33
	0.0004	0.00004	294.65	291.21	12	1.16
	0.0006	0.00006	324.77	317.18	12	2.32
	0.0008	0.00008	354.87	341.63	12	3.63
	0.0010	0.00010	384.66	364.42	12	5.27
	0.0012	0.00012	409.70	385.60	12	5.65
	0.0014	0.00014	434.74	407.20	12	6.35
	0.0016	0.00016	459.76	426.60	12	7.20

* Extrapolation to infinite approximation.

TABLE 5

INFLUENCE OF THE NUMBER OF RINGS ON DISCRETENESS EFFECT FOR DIFFERENT L/R

AND R/h

$e_2/h = -1$

$A_2/ah = 0.15$

$I_{22}/ah^3 = 0.015$

R/h	L/R	Z	Number of Rings	†	λ smeared	λ discrete *	Δλ%
500	0.5	119.2	1	16	842.92	692.22	17.9
			2	16		743.38	11.8
			3	16		781.16	7.34
			4	16		806.99	4.27
			5	15		821.45	2.55
			6	15		828.69	1.69
			7	15		832.97	1.18
			8	15		835.64	0.86
			9	15		837.38	0.66
			10	15		838.56	0.52
			11	15		839.41	0.42
			12	15		840.04	0.34
			13	15		840.56	0.30
			14	15		840.86	0.25
			15	15		841.15	0.21
500	1	477.0	1	12	424.5	330.8	22.1
			2	12		351.6	17.2
			3	12		367.6	13.4
			4	11		380.5	10.4
			5	11		389.2	8.34
			6	11		396.9	6.52
			7	11		403.2	5.03
			8	11		408.3	3.81
			9	11		412.1	2.93
			10	11		414.9	2.27
			11	11		417.0	1.77

TABLE 5 (Cont'd)

INFLUENCE OF THE NUMBER OF RINGS ON DISCRETENESS EFFECT FOR DIFFERENT L/R

AND R/h							
<u>$e_2/h = -1$</u>			<u>$A_2/ah = 0.15$</u>			<u>$I_{22}/ah^3 = 0.015$</u>	
R/h	L/R	Z	Number of Rings	†	λ smeared	λ discrete *	$\Delta\lambda\%$
500	1	477.0	12	11	424.5	418.52	1.42
			13	11		419.67	1.14
			14	11		420.53	0.95
			15	11		421.20	0.78
500	2	1908	1	9	213.17	165.73	22.2
			2	8		170.11	20.2
			3	8		174.37	18.2
			4	8		178.51	16.1
			5	8		182.37	14.4
			6	8		185.71	12.9
			7	8		188.61	11.5
			8	8		191.18	10.3
			9	8		193.50	9.23
			10	8		195.66	8.20
			11	8		197.67	7.28
			12	8		199.56	6.38
			13	8		201.29	5.57
			14	8		202.85	4.85
			15	8		204.25	4.18
100	1	95.39	1	8	188.41	156.80	15.85
			2	7		171.72	9.10
			3	7		178.17	5.45
			4	7		182.38	3.20
			5	7		184.67	1.99

TABLE 5 (Cont'd)

INFLUENCE OF THE NUMBER OF RINGS ON DISCRETENESS EFFECT FOR DIFFERENT L/R

		AND R/h									
		$a_2/h = -1$	$A_2/ah = 0.15$			$I_{22}/ah^3 = 0.015$					
R/h	L/R	Z	Number of Rings	†	λ smeared	λ discrete*	$\Delta\lambda\%$				
100	1	95.39	6	7	188.41	185.93	1.31				
			7	7		186.67	0.92				
			8	7		187.12	0.68				
			9	7		187.42	0.52				
			10								
			11	7		187.77	0.33				
			12	7		187.88	0.28				
			13	7		187.96	0.24				
			14	7		188.03	0.20				
			2000	1		1908	1	17	852.69	652.78	23.5
							2	17		670.08	21.4
							3	17		690.07	19.1
							4	17		709.45	16.8
							5	17		727.51	14.7
6	16	742.82			12.9						
7	16	754.46			11.5						
8	16	764.72			10.3						
9	16	774.01			9.20						
10	16	782.62			8.23						
11	16	790.70			7.28						
12	16	798.73			6.42						
13	16	805.15			5.49						
14	16	811.42			4.84						
15	16	816.98			4.18						

* Tenth order approximation

TABLE 6

INFLUENCE OF NUMBER OF RINGS ON THE DISCRETENESS EFFECT BETWEEN CUT-OFF LINES

FOR RINGS OF VARYING STRENGTH.

L/R = 1		R/h = 500			$e_2/h = -2$	
Number of Rings	A_2/ah	I_{22}/ah	\dagger	λ smeared	λ discrete*	$\Delta\lambda\%$
1	0.003	0.00225	12	255.13	254.22	0.36
	0.018	0.01350	12	357.4	331.1	7.36
	0.039	0.02925	12	475.5	400.5	15.7
2	0.042	0.03375	11	491.6	435.7	11.4
	0.060	0.04500	11	586.7	486.1	17.1
	0.096	0.07200	11	736.1	563.0	23.5
3	0.099	0.07425	11	748.0	611.1	18.7
	0.120	0.09000	11	830.2	655.1	21.1
	0.168	0.12600	10	1007	733.7	27.1
4	0.171	0.12825	10	1016	784.4	22.8
	0.210	0.15750	10	1123	843.0	25.0
	0.280	0.21000	10	1303	930.2	28.6
5	0.288	0.21600	10	1323	1010	23.6
	0.360	0.27000	10	1492	1097	26.4
	0.416	0.31200	10	1615	1155	28.5
6	0.432	0.32400	9	1650	1252	24.1
	0.512	0.38400	9	1806	1326	26.8
	0.608	0.45600	9	1943	1406	27.6
7	0.624	0.46800	9	1965	1525	22.4
	0.800	0.60000	9	2189	1671	23.7
	0.896	0.67200	9	2300	1741	24.3
8	0.912	0.68400	9	2318	1889	18.5

* tenth order approximation

TABLE 7
COMPARISON BETWEEN DISCRETE AND "SMEARED" NON-UNIFORM 3-RING
STIFFENED CYLINDERS UNDER LATERAL PRESSURE*

<u>R/h = 500</u>		<u>L/R = 1</u>		<u>e₂/h = 2</u>			
A_2/ah	I_{22}/ah^3	λ smeared	λ discrete	smeared non- uniform **	discrete non- uniform	smeared $\Delta\lambda\%$	discrete $\Delta\lambda\%$
0.08	0.06	758.1	727.7	967.1	767.4	27.6	5.41
0.10	0.0667	854.7	809.8	1109	858.1	29.8	5.96
0.12	0.08	948.4	896.2	1246	943.9	31.4	5.85

* The non-uniformity is a sinusoidal height variation of rings of rectangular cross-section.

** Half the cross-sectional area of the rings is constant and half varies sinusoidally.

TABLE 8

DISCRETENESS EFFECT IN RING-STIFFENED CYLINDRICAL SHELLS UNDER AXIAL COMPRESSION

Shell No.	R/h	L/R	e_2/h	A_2/ah	l_{22}/ah^3	Ring Spacing a/h	\dagger discrete	No. of rings λ	λ smeared	λ discrete *	$\Delta\lambda\%$
1	395	2.025	3.00	0.878	1.83	57.1	10	13	5515	5416	1.79
2	384	2.030	2.93	0.900	1.70	55.7	10	13	5354	5267	1.63
3	415	2.030	3.16	0.834	2.13	60.2	10	13	5805	5693	1.92
4	373	0.526	-0.977	0.347	0.0169	10.9	0	17	5689	5689	0.00
5	314	0.465	-0.690	0.175	0.00209	9.1	0	15	4453	4453	0.00
6	404	0.750	-0.936	0.447	0.00333	11.7	0	25	6387	6387	0.02
7	384	0.758	0.914	0.410	0.0227	10.8	15	26	5067	5067	0.00
8	667	0.515	1.39	0.878	0.2265	19.1	17	17	9044	9043	0.02

* eighth order: approximation.

TABLE 9

CONVERGENCE OF THE APPROXIMATIONS FOR "DISCRETE" BUCKLING LOADS

<u>$A_2/ah = 0.1$</u>		<u>$e_2/h = 1.$</u>		<u>$I_{22}/ah = 0.01$</u>	
L/R	R/h	No.	n	λ	$\Delta\lambda$
250	3	1	1	97.173	
					5.079
		2	1-3	92.094	2.413
		3	1-5	89.681	1.635
		4	1-7	87.856	1.527
		5	1-9	86.329	1.321
		6	1-11	85.008	1.158
		7	1-13	83.850	1.018
		8	1-15	82.832	0.889
		9	1-17	81.943	0.776
		10	1-19	81.167	
100	0.5	1	1	324.436	
					4.599
		2	1-3	319.837	1.493
		3	1-5	318.345	0.771
		4	1-7	317.574	0.471
		5	1-9	317.103	0.316
		6	1-11	316.757	0.226
		7	1-13	316.561	0.170
		8	1-15	316.391	0.133
		9	1-17	316.258	0.106
		10	1-19	316.152	

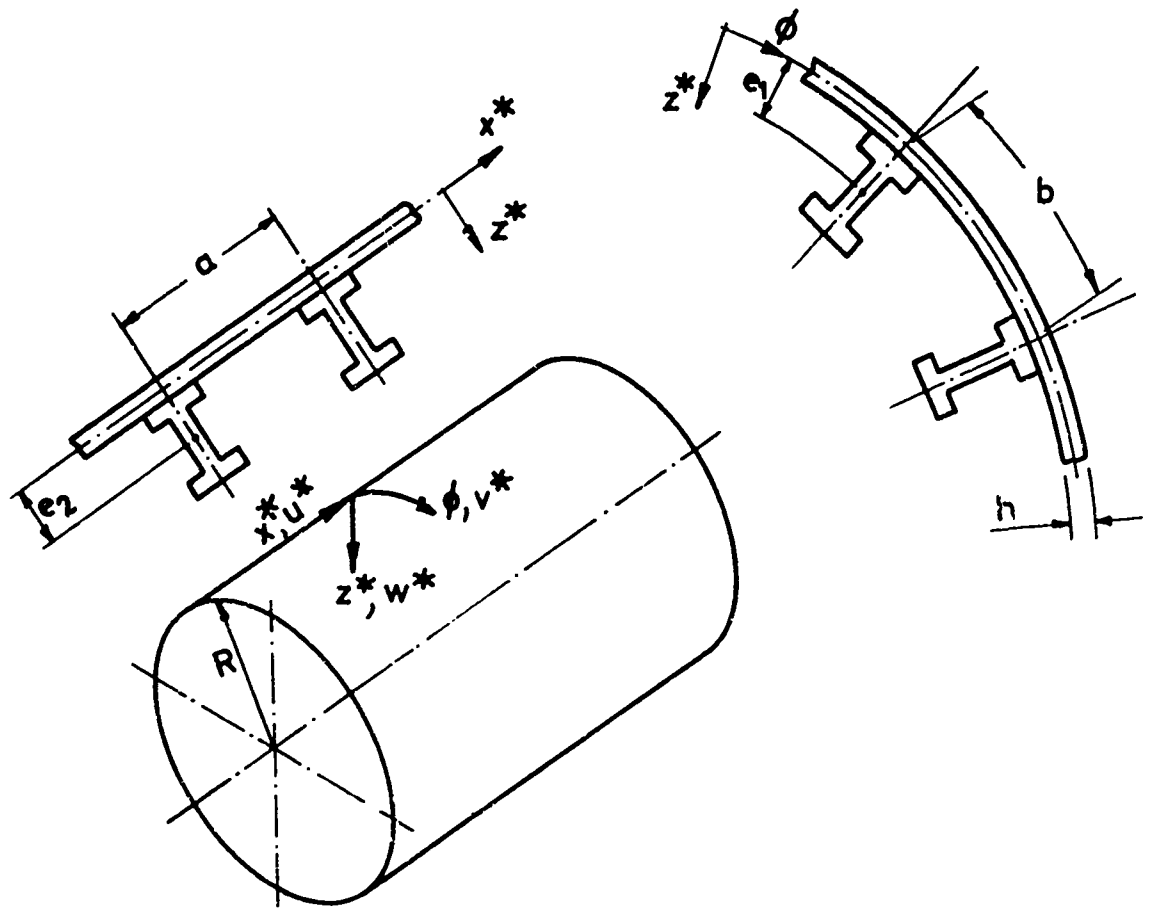


FIG. 1 NOTATION

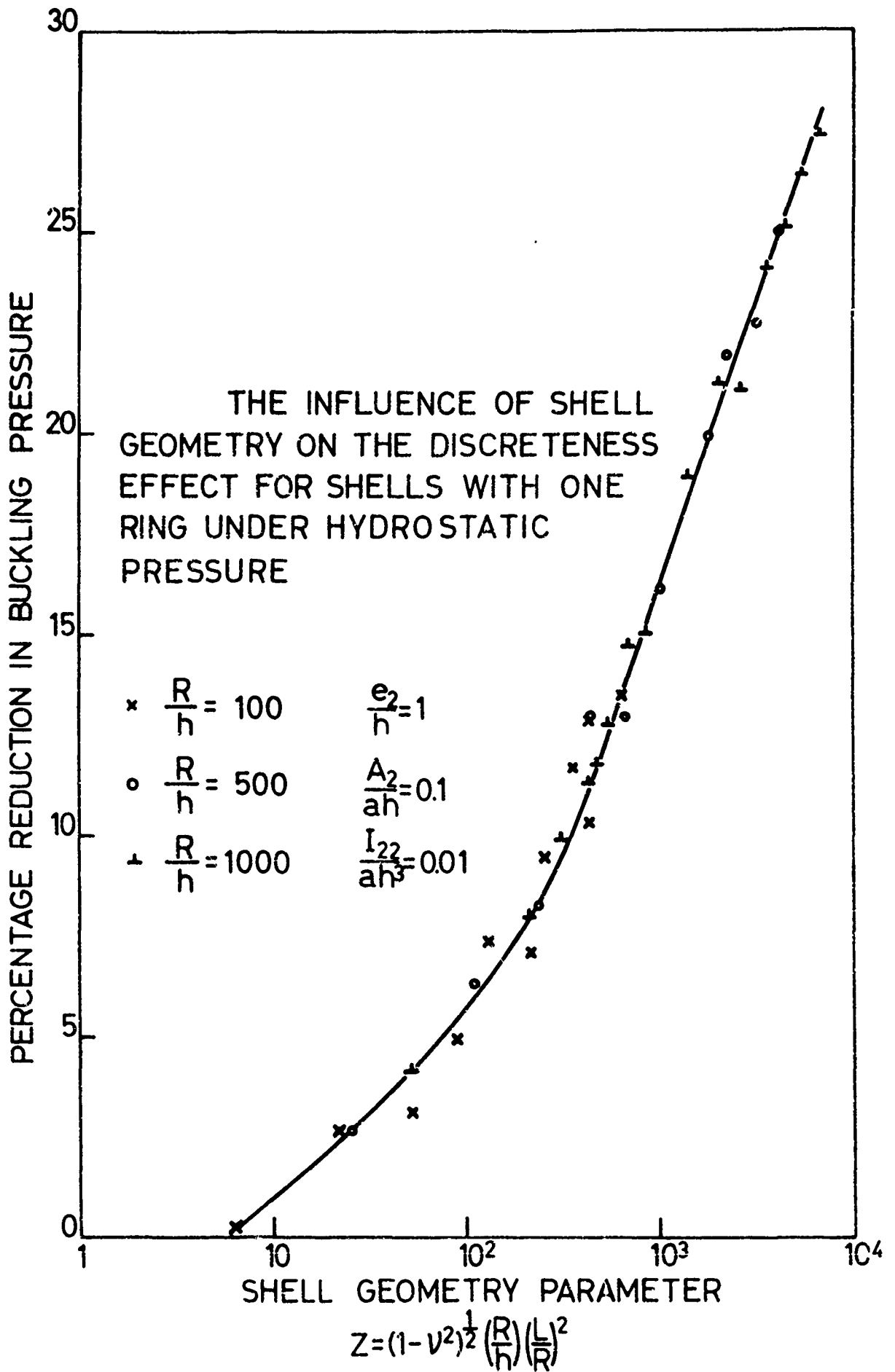
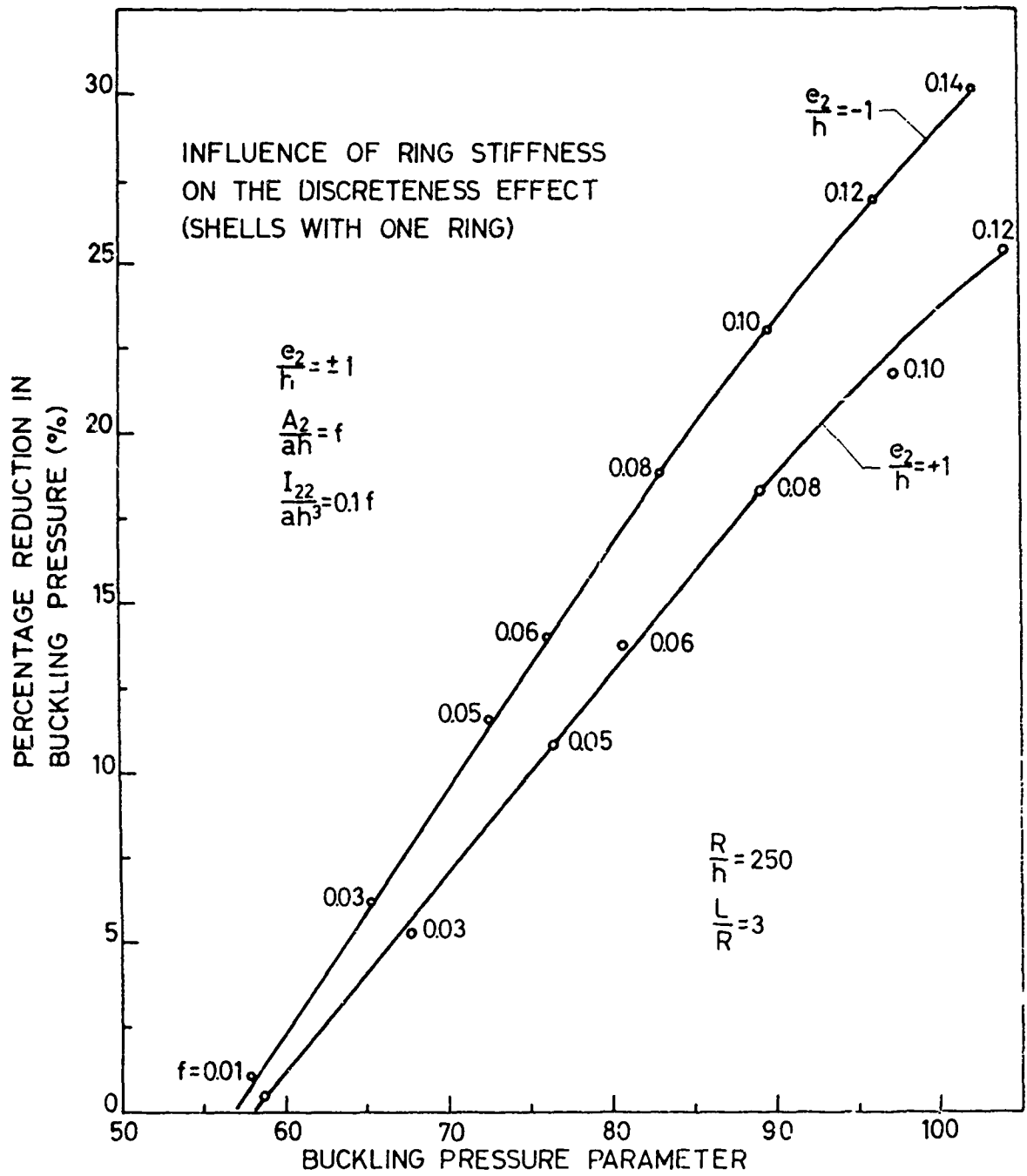
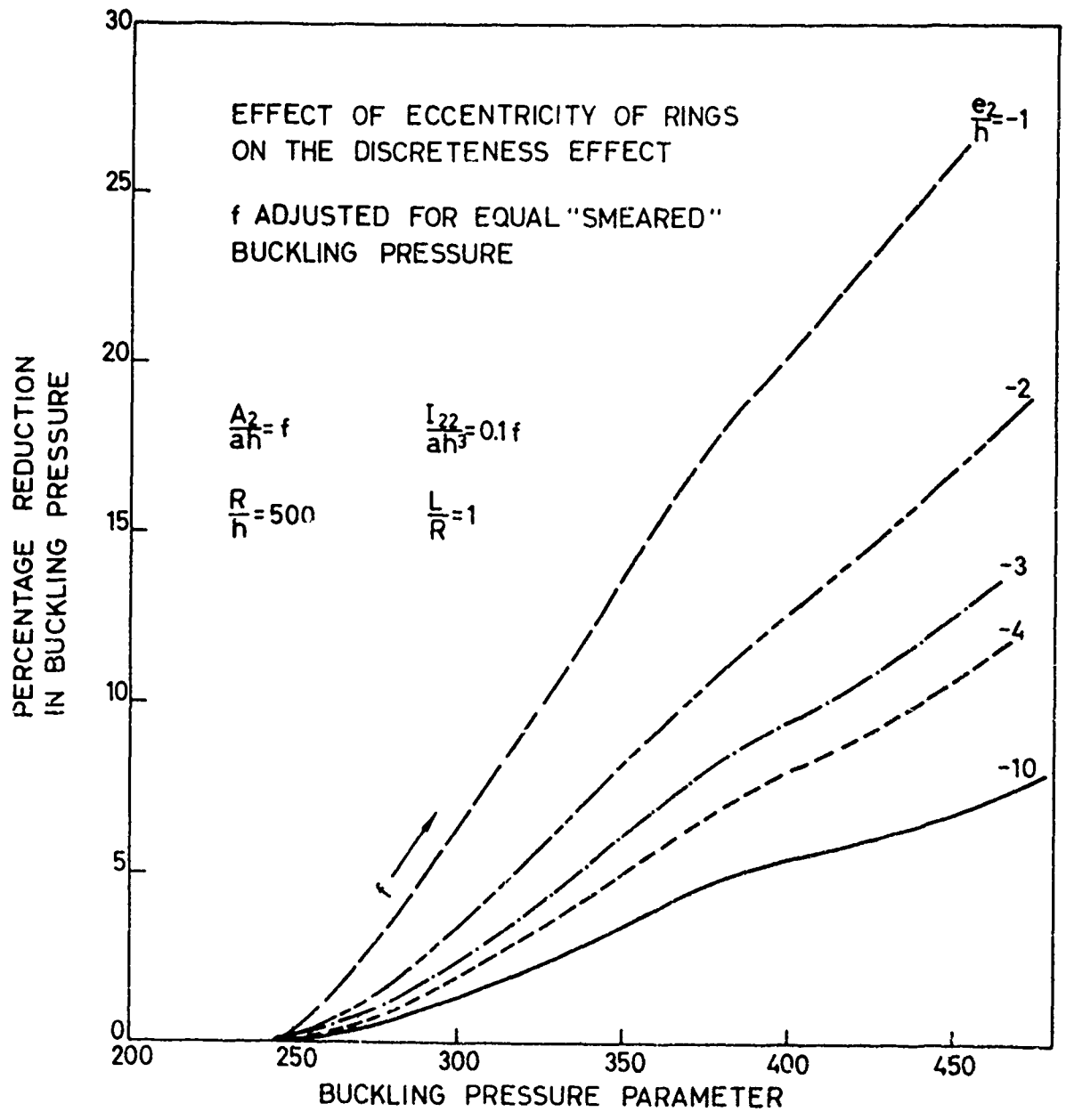


FIG. 2



$$\lambda = \left[\left(\frac{R}{h} \right)^3 \frac{12(1-\nu^2)}{E} \right] p$$

FIG 3



$$\lambda = \left[\frac{R}{h} \frac{12(1-\nu^2)}{E} \right] p$$

FIG 4

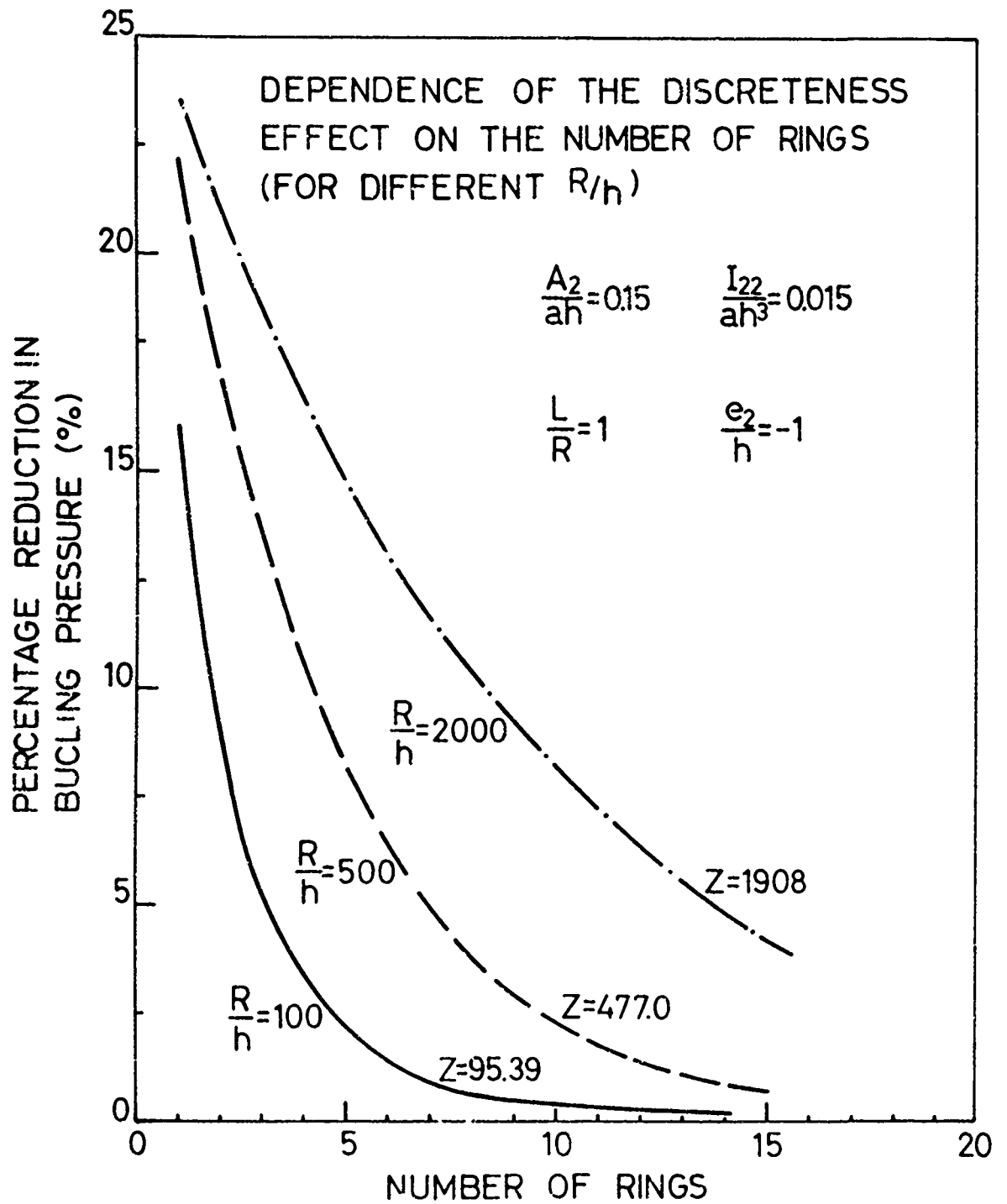


FIG. 5

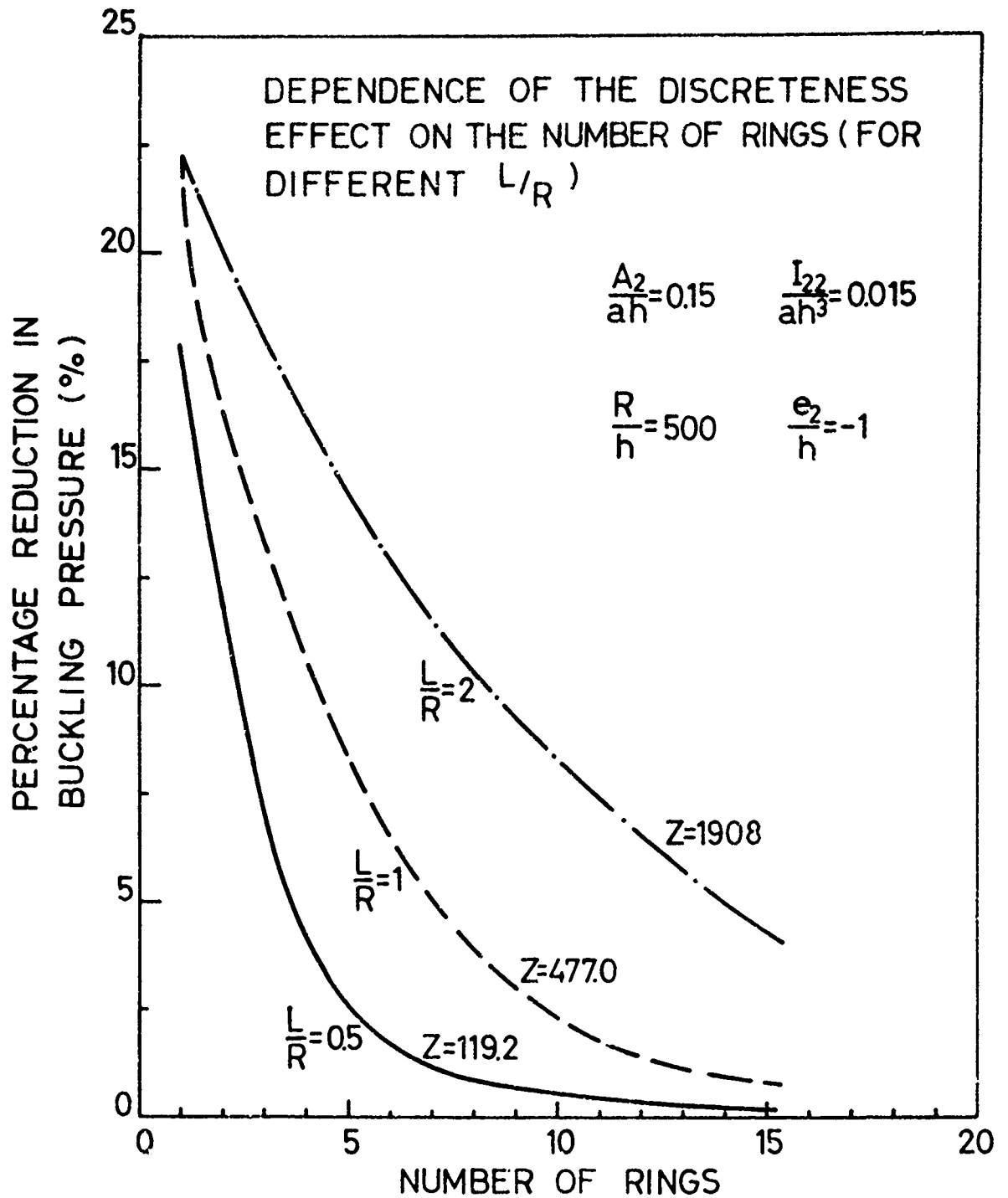
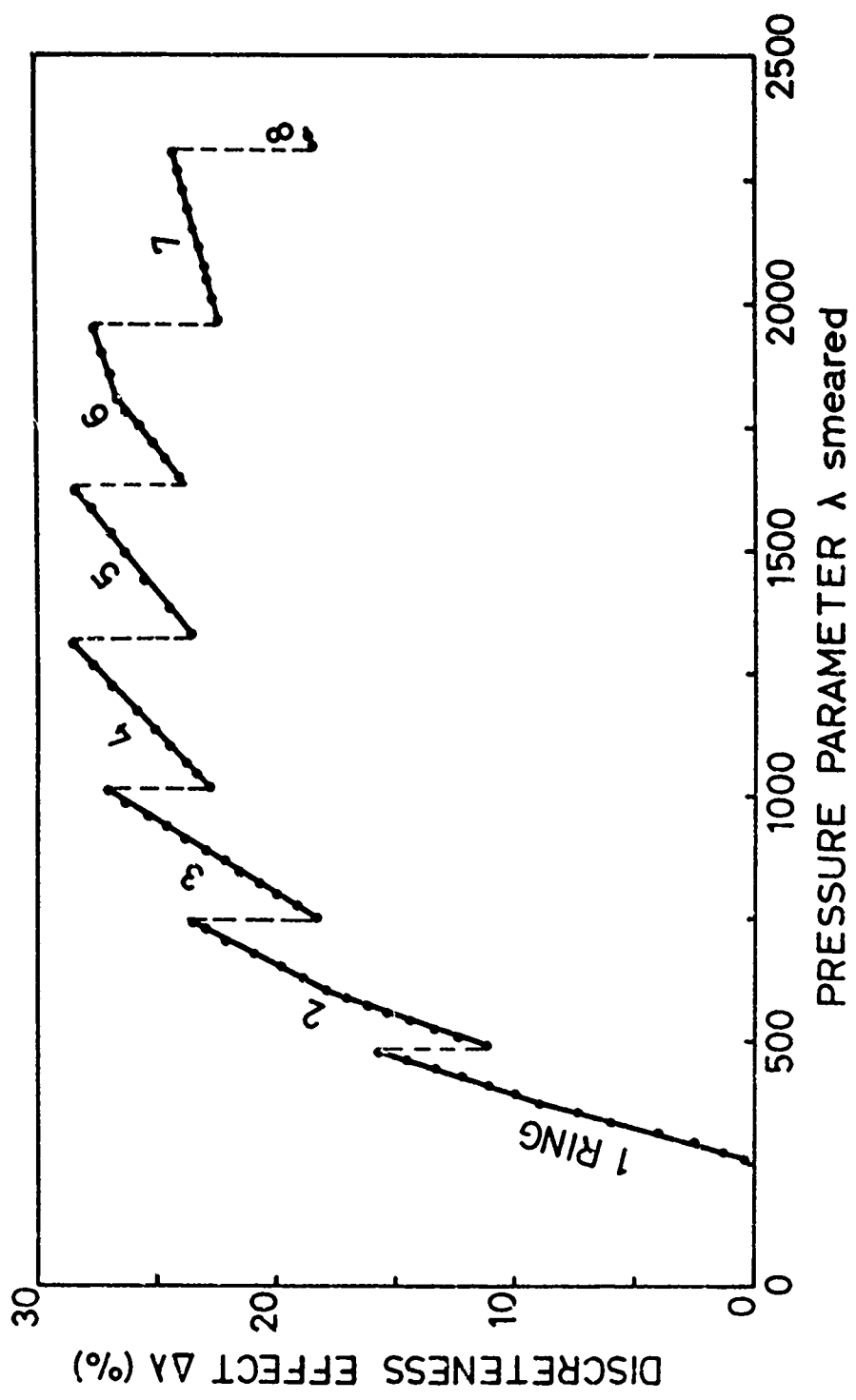


FIG. 6



INFLUENCE OF THE NUMBER OF RINGS ON THE DISCRETENESS EFFECT FOR RINGS OF VARYING STRENGTH

FIG. 7

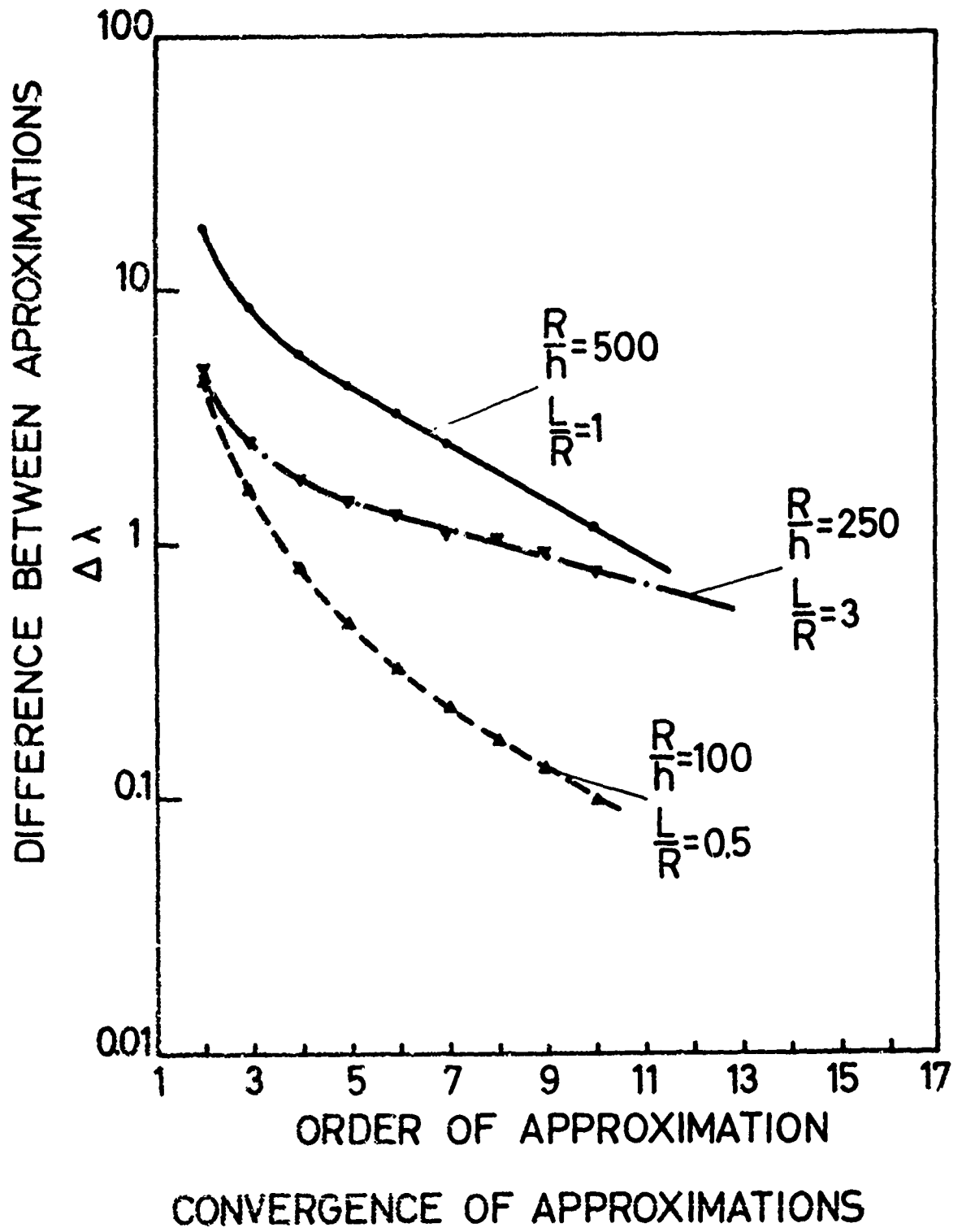


FIG. 8

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3 REPORT TITLE BUCKLING OF DISCRETE RING STIFFENED CYLINDRICAL SHELLS		
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11. SUPPLEMENTARY NOTES	12. SPONSORING MILITARY ACTIVITY AIR FORCE OFFICE OF SCIENTIFIC RESEARCH OAR, European Office of Aerospace Research	
13 ABSTRACT The buckling of ring-stiffened cylinders is studied by a "discrete" approach, in which the rings are considered as linear discontinuities represented by the Dirac delta function. The analysis is a linear Donnell type theory that takes account of the eccentricity of stiffeners. Buckling loads under hydrostatic pressure, lateral pressure and axial compression are compared with those obtained by "smeared-stiffener" theory for an extensive range of geometries. The discreteness effect depends very strongly on the geometry of the shell and the eccentricity of the rings. Significant discreteness effects are found for hydrostatic pressure loading.		

14 KEY WORDS	LINK A		LINK B		LINK C	
	ROLE	WT	ROLE	WT	ROLE	WT
1. Buckling Stiffened Cylindrical Shells						
2. Discrete Stiffeners						

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