A NOTE ON LAGRANGE MULTIPLIERS

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A fundamental assumption common to all economic analyses is the maximization or minimization of some objective function (representing, say, utility, cost, welfare or whatnot) subject to certain constraints. Statements of the type: "A consumer with given income maximizes his total utility only if his marginal utilities for the various commodities are proportional to their prices," are almost commonplace in economic texts and are generally described as "equilibrium conditions" of the optimization process under consideration. Nevertheless, when these meaningful theorems are presented to even the more advanced students, the argument is usually shrouded with a complete or partial mystery around the so-called Lagrange multipliers. Very little explanation is given to these multipliers themselves except that they are the coefficients used to form a certain Lagrangian function, the extremization of which leads to the desired equilibrium

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conditions.* This note is devoted to an intrinsic (i.e., geometric) characterization of these multipliers and a natural reformulation of the equilibrium conditions that permits a better insight into the nature of constrained extremum problems in economics.

Let

$$y = f(x) \tag{1}$$

be a real-valued function of a single variable x. The function f may represent, for example, the short-run cost curve of a production process with only one variable factor. If f is sufficiently smooth (i.e., x is infinitesimally divisible), a necessary condition for a (relative) minimum of (1) is, as is well known,

$$\frac{dy}{dx} = f'(x) = 0, \tag{2}$$

and a sufficient condition for a (relative) minimum of (1) is (2) plus

$$\frac{d^2y}{dx^2} = f''(x) = \frac{d}{dx} f'(x) > 0. \tag{3}$$

Geometrically, (2) states that the tangent vector to the curve C defined by (1) must be horizontal; and (3) states that it is increasing in slope around any root x^o of (2). (Figure 1.)

A more easily generalizable geometric interpretation of (3) is

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*Cf. inter alia the following well known economic texts: R. G. D. Allen, Mathematical Analysis for Economists, London: Macmillan, 1949, pp. 366-367; idem, Mathematical Economics, London: Macmillan, 1956, pp. 610, 614; D. W. Bushaw and R. W. Clower, Introduction to Mathematical Economics, Homewood, Illinois: Irwin, 1957, p. 331; J. M. Henderson and R. E. Quandt, Microeconomic Theory, A Mathematical Approach, New York: McGraw-Hill, 1958, pp. 273-274; J. R. Hicks, Value and Capital, 2nd ed., London: Oxford, 1956, p. 305; P. A. Samuelson, Foundations of Economic Analysis, Cambridge, Massachusetts: Harvard, pp. 362-365; and Taro Yamane, Mathematics for Economists, Englewood, New Jersey: Prentice-Hall, 1962, pp. 116-123.

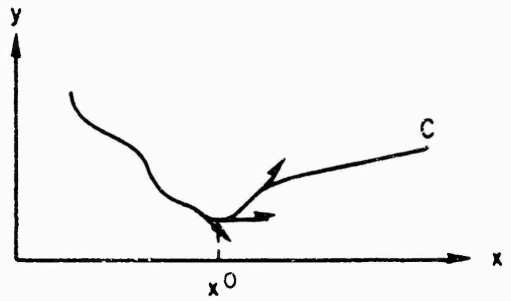

Figure 1

the following: Any function f possessing sufficient number of derivatives (i.e., sufficiently smooth) may be expanded into a Taylor's series:*

$$f(x) = f(x^0) + \frac{df}{dx}(x - x^0) + \frac{1}{2}\frac{d^2f}{dx^2}(x - x^0)^2 + \ldots + \frac{1}{n!}\frac{d^nf}{dx^n}(x - x^0)^n + \ldots \quad (4)$$

where all derivatives are to be evaluated at x^0. (4) states roughly that the value of f at x may be represented by its value at x^0 together with those of all derivatives of f at x^0. Consequently, if x^0 is to be a relative minimum, all sufficiently close neighboring x must not yield a smaller $y = f(x)$, i.e.,

$$f(x) - f(x^0) \geq 0 , \quad (5)$$

in terms of (4),

$$\frac{1}{2}\frac{d^2f}{dx^2}(x - x^0)^2 \geq 0 \quad (6)$$

*See, e.g., R. C. Buck, Advanced Calculus, New York: McGraw-Hill, 1956, pp. 75-77.

since, at x^o, $\frac{df}{dx} = 0$; and if x is sufficiently close to x^o, the term shown in (6) will dominate the combined effect of all succeeding terms because these remaining terms contain $x - x^o$ to a higher degree. That (6) is equivalent to (3) is obvious.

If f is now a function of two independent variables, (1) may be rewritten as:

$$y = f(x_1, x_2) \tag{7}$$

and a pair of necessary conditions corresponding to (2) are

$$\frac{\partial y}{\partial x_1} = f_{x_1} = 0 \quad , \qquad \frac{\partial y}{\partial x_2} = f_{x_2} = 0 \ . \tag{8}$$

These conditions state that the tangent vectors to the surface S defined by (7) in the directions of increasing x_1 and increasing x_2 must be horizontal, that is, the two tangent vectors must be parallel to the x_1x_2 - plane. (Figure 2.) If f is sufficiently smooth, its Taylor expansion around any root x^o of (8) is given by

$$\begin{aligned} f(x_1, x_2) = f(x_1^o, x_2^o) &+ \frac{\partial f}{\partial x_1}(x_1-x_1^o) + \frac{\partial f}{\partial x_2}(x_2-x_2^o) \\ &+ \tfrac{1}{2}\left\{ \frac{\partial^2 f}{\partial x_1^2}(x_1-x_1^o)^2 + 2\frac{\partial^2 f}{\partial x_1 \partial x_2}(x_1-x_1^o)(x_2-x_2^o) \right. \\ &\left. + \frac{\partial^2 f}{\partial x_2^2}(x_2-x_2^o)^2 \right\} + \ldots \end{aligned} \tag{9}$$

By an argument similar to that used to derive (6), a sufficient condition for x^o to be a (relative) minimum is (8) plus

$$\begin{aligned} \tfrac{1}{2}\left\{ \frac{\partial^2 f}{\partial x_1^2}(x_1-x_1^o)^2 + 2\frac{\partial^2 f}{\partial x_1 \partial x_2}(x_1-x_1^o)(x_2-x_2^o) \right. \\ \left. + \frac{\partial^2 f}{\partial x_2^2}(x_2-x_2^o)^2 \right\} \geq 0 \ . \end{aligned} \tag{10}$$

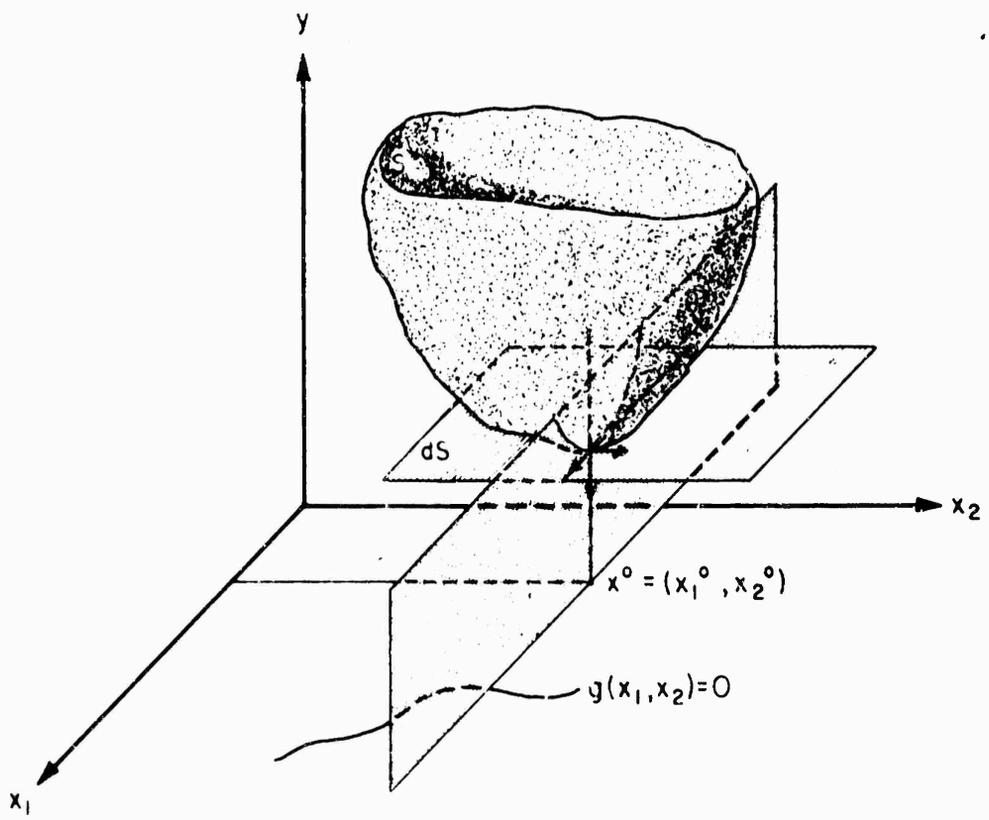

Fig. 2

The tangent vectors $(dx_1, 0) = (x_1 - x_1^o, 0)$ and $(0, dx_2) = (0, x_2 - x_2^o)$ to the surface (7) at x^o determine a 2-dimensional tangent plane $dS(x^o)$ to S. Since x^o is to be a relative minimum, all sufficiently close neighboring points must not yield a smaller y, points on the tangents $(dx_1, 0)$, $(0, dx_2)$ being only special cases. More generally, points on any tangent vector $d\bar{x} = (d\bar{x}_1, d\bar{x}_2)$ to S at x^o which is a linear combination of $(dx_1, 0)$, $(0, dx_2)$ must also not yield a smaller y. Since $(dx_1, 0)$, $(0, dx_2)$ span (i.e., form a basis of) $dS(x^o)$, $d\bar{x}$ may be represented as

$$(d\bar{x}_1, d\bar{x}_2) = \cos \alpha_1 \, (dx_1, 0) + \cos \alpha_2 \, (0, dx_2) \qquad (11)$$

where $\cos \alpha_1$, $\cos \alpha_2$ are the direction cosines of $d\bar{x}$ with respect to the local (orthogonal) coordinate system on $dS(x^o)$ with origin at x^o and directions $(dx_1, 0)$, $(0, dx_2)$. Consequently, a strengthened necessary condition for a relative minimum at x^o, which includes the two conditions in (8), is

$$\nabla_{d\bar{x}} f \equiv \frac{\partial f}{\partial x_1} \cos \alpha_1 + \frac{\partial f}{\partial x_2} \cos \alpha_2 = 0 \qquad (12)$$

where $\cos \alpha_1$, $\cos \alpha_2$ are the direction cosines of an arbitrary tangent vector $d\bar{x}$ to S at x^o, and $\partial f/\partial x_1$, $\partial f/\partial x_2$ are the components of the normal to dS (i.e., the gradient vector ∇f to S). $\nabla_{d\bar{x}} f$, defined to be the left side of (12), is called the directional derivative of f in the direction of $d\bar{x}$. For $\alpha_1 = 0$, $\alpha_2 = \pi/2$, $d\bar{x} = (dx_1, 0)$ and (12) yields the first condition in (8); for $\alpha_1 = \pi/2$, $\alpha_2 = 0$, $d\bar{x} = (0, dx_2)$ and the second condition in (8) obtains.

Moreover, a strengthened sufficient condition analogous to (3)

for a relative minimum at x^o is, by taking again the directional derivatives of f_{x_1}, f_{x_2} in the direction $d\bar{x}$,

$$\nabla^2_{d\bar{x}} f = \nabla_{d\bar{x}} (\nabla_{d\bar{x}} f) = \nabla_{d\bar{x}} (\frac{\partial f}{\partial x_1} \cos \alpha_1 + \frac{\partial f}{\partial x_2} \cos \alpha_2) =$$

$$(\nabla_{d\bar{x}} f_{x_1}) \cos \alpha_1 + (\nabla_{d\bar{x}} f_{x_2}) \cos \alpha_2 = (\frac{\partial}{\partial x_1} f_{x_1} \cos \alpha_1 + \qquad (13)$$

$$\frac{\partial}{\partial x_2} f_{x_1} \cos \alpha_2) \cos \alpha_1 + (\frac{\partial}{\partial x_1} f_{x_2} \cos \alpha_1 + \frac{\partial}{\partial x_2} f_{x_2} \cos \alpha_2) \cos \alpha_2 =$$

$$\frac{\partial^2 f}{\partial x_1^2} \cos^2 \alpha_1 + 2 \frac{\partial^2 f}{\partial x_1 \partial x_2} \cos \alpha_1 \cos \alpha_2 + \frac{\partial^2 f}{\partial x_2^2} \cos^2 \alpha_2 \geq 0 .$$

The tangent plane $dS(\hat{x})$ at any point $\hat{x}$ on S is defined by the linear terms in the expansion (9), i.e.,

$$y - f(\hat{x}_1, \hat{x}_2) = \frac{\partial f}{\partial x_1} (x_1 - \hat{x}_1) + \frac{\partial f}{\partial x_2} (x_2 - \hat{x}_2) \qquad (14)$$

where (x_1, x_2, y) is now a point in $dS(\hat{x})$, and the partial derivatives are to be evaluated at $\hat{x}$. To put the matter differently, if S itself is a plane, then the expansion (9) at any point on it must be exact up to and including the linear terms, i.e., all higher-order terms must vanish identically. The normal to the tangent plane $dS(\hat{x})$ has components proportional to $(\partial f/\partial x_1, \partial f/\partial x_2, -1)$.

At a relative minimum point x^o where S and $dS(x^o)$ are tangent to each other, $y = f(x_1^o, x_2^o)$ and the left side of (14) vanishes. Consequently, $dS(x^o)$ must be parallel to the x_1x_2-plane (called the base plane) but at distance $f(x_1^o, x_2^o)$ from it. For, in that case the right side of (14) can just as well be written as

$$\frac{\partial f}{\partial x_1} (x_1 - x_1^o) + \frac{\partial f}{\partial x_2} (x_2 - x_2^o) + (-1) \left[y - f(x_1^o, x_2^o) \right] = 0 \qquad (15)$$

where $y = f(x_1^o, x_2^o)$ identically in $dS(x^o)$, showing the orthogonality between $(\partial f/\partial x_1, \partial f/\partial x_2, -1)$ and $(x_1 - x_1^o, x_2 - x_2^o, 0)$, a vector in the base plane. However, the right side of (14) is the same as $\nabla_{d\bar{x}} f$ defined in (12) if we choose a point $(x_1, x_2, f(x_1^o, x_2^o))$ in $dS(x^o)$ such that

$$x_1 - x_1^o = \cos \alpha_1 \ , \ x_2 - x_2^o = \cos \alpha_2 \ , \ y - f(x_1^o, x_2^o) \equiv 0 \ . \qquad (16)$$

Since $(\cos \alpha_1, \cos \alpha_2)$ represents a unit vector with respect to the local coordinate system in $dS(x^o)$, $\nabla_{d\bar{x}} f$ is precisely the component (i.e., projection) of ∇f in the direction $d\bar{x}$. (12) states, therefore, that at a critical point x^o on S, the projection of the gradient vector ∇f in every direction $d\bar{x}$ in $dS(x^o)$ vanishes, where $dS(x^o)$ is parallel to the base plane. As α_1, α_2 in (16) can be arbitrarily chosen, the right side of (14) can vanish only if f_{x_1}, f_{x_2} themselves vanish, justifying (8). Moreover, at a noncritical point $\hat{x}$ of S, a point (x_1, x_2, y) in $dS(\hat{x})$ will generally have its last component y not identically equal to $f(\hat{x}_1, \hat{x}_2)$. In fact, these will usually be equal only for the tangency point between S and $dS(\hat{x})$; therefore, the right side of (14) does not now vanish always. It may vanish for some directions $d\bar{x}$ in $dS(\hat{x})$. These results apply, in general, to spaces of dimensions greater than 2 also.

On the basis of the above geometric argument, it is now possible to give an intrinsic characterization of Lagrange multipliers. Consider, for example, a constrained minimum problem of the following type: Minimize (7) subject to

$$g(x_1, x_2) = 0 \ . \qquad (17)$$

(17) defines a curve in the base plane, and minimum of f is to be sought over all points $x = (x_1, x_2)$ lying on this curve C. At any such (relative) minimum point x^o, the directional derivative ∇f of f along the tangent to C must vanish by (12), where $\cos \alpha_1$, $\cos \alpha_2$ denote the components of the unit tangent dx to C at x^o. However, (17) shows that

$$\nabla_{dx} g \equiv \frac{\partial g}{\partial x_1} \cos \alpha_1 + \frac{\partial g}{\partial x_2} \cos \alpha_2 = 0 \quad . \tag{18}$$

also at this point. Consequently, $\nabla_{dx} f$ and $\nabla_{dx} g$ must be collinear, i.e., for some scalar λ

$$\nabla_{dx} f = \lambda \nabla_{dx} g \ . \tag{19}$$

But, (19) is equivalent to

$$f_{x_1} = \lambda g_{x_1} \ , \quad f_{x_2} = \lambda g_{x_2} \tag{20}$$

which are the usual conditions derivable from differentiation of the Lagrangian function. A sufficient condition for a relative minimum at x^o is (13) with $\cos \alpha_1$, $\cos \alpha_2$ being again the components of dx (tangent to C at x^o) with respect to the local coordinate system at x^o.

Generalization of the above geometric characterization of Lagrange multipliers to spaces of higher dimensions is immediate. Let

$$y = f(x_1, x_2, \ldots, x_n) \tag{21}$$

again denote the objective function to be extremized, and

$$g_j(x_1, x_2, \ldots, x_n) = 0 \qquad (j = 1, \ldots, r < n) \tag{22}$$

denote a set of independent side constraints. Each g_j defines a

hypersurface S_j in the base plane [i.e., $x_1x_2\ldots x_n$-plane in $(n+1)$-dimensional space E^{n+1} with the last axis denoted by y]. The intersection

$$S_{12\ldots r} = \bigcap_{j=1}^{r} S_j \tag{23}$$

of these hypersurfaces in general yields an $(n-r)$-dimensional surface in the base plane. At a critical point $x^o = (x_1^o, \ldots, x_n^o)$ on $S_{12\ldots r}$ (which is C in the preceding example), a tangent space $dS_{12\ldots r}$ generally exists with basis vectors $(dx)^1$, ..., $(dx)^{n-r}$, and the directional derivative ∇f of f along each such basis vector or their linear combinations must vanish. This says that ∇f must be orthogonal to $dS_{12\ldots r}$, or ∇f lies in $dS^{\perp}_{12\ldots r}$, the orthogonal complement of $dS_{12\ldots r}$ at x^o. But (22) shows that

$$\sum_{i=1}^{n} \frac{\partial g_j}{\partial x_i} dx_i = 0 \qquad (j = 1, \ldots, r) \tag{24}$$

at x^o also. Hence, if $dx = (dx_1, \ldots, dx_n)$ is chosen to range over the basis vectors $(dx)^1$, ..., $(dx)^{n-r}$ of $dS_{12\ldots r}$, (24) merely shows that each ∇g_j (j=1, 2, ..., r) is also orthogonal to $dS_{12\ldots r}$. But if g_j (j=1, ..., r) are independent, ∇g_1, ..., ∇g_j (j=1, ..., r) would form a basis for $dS^{\perp}_{12\ldots r}$ since

$$\dim dS_{12\ldots r} + \dim dS^{\perp}_{12\ldots r} = n \tag{25}$$

at a regular point on $S_{12\ldots r}$. Therefore, for some scalars λ_1, ..., λ_r we must have

$$\nabla f = \sum_{j=1}^{r} \lambda_j \nabla g_j \tag{26}$$

expressing linear dependence of ∇f on ∇g_j (j=1, ..., r). (19) gives, in component form,

$$\frac{\partial f}{\partial x_i} = \sum_{j=1}^{r} \lambda_j \frac{\partial g_j}{\partial x_i} \qquad (i = 1, \ldots, n) , \tag{27}$$

the more familiar equilibrium conditions. In (27), there are n equations in n+r unknowns $x_1, \ldots, x_n; \lambda_1, \ldots, \lambda_r$. But since $(x_1, \ldots, x_n)$ must also satisfy (22), r additional equations must be included. Consequently, the Lagrange multipliers are merely coefficients used in expressing a certain necessary linear dependence relation of the gradient vector to the surface defined by f on those to surfaces defined by g_j (j=1, ..., r).

The sufficiency condition is also easily generalized. With respect to the basis vectors $(dx)^1, \ldots, (dx)^{n-r}$ of $dS_{12\ldots r}$, a typical unit tangent vector $d\bar{x}$ in $dS_{12\ldots r}$ has the form

$$d\bar{x} = \sum_{k=1}^{n-r} (dx)^k \cos \alpha_k \tag{28}$$

where $\cos \alpha_1, \ldots, \cos \alpha_{n-r}$ are the direction cosines of $d\bar{x}$ with respect to $(dx)^1, \ldots, (dx)^{n-r}$. Then

$$\nabla^2_{d\bar{x}} f = \sum_{i,j=1}^{n-r} \frac{\partial^2 f}{\partial x_i \partial x_j} \cos \alpha_i \cos \alpha_j \geq 0 \tag{29}$$

together with (27) yields a relative constrained minimum at x^0. Alternatively, if $z = (z_1, \ldots, z_n)$ is any vector in the base plane, a relative constrained minimum at a point x^0 is assured by (27) and

$$z'Hz = (z_1, \ldots, z_n) \begin{pmatrix} \frac{\partial^2 f}{\partial x_1^2} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ & \cdots & \\ \frac{\partial^2 f}{\partial x_n \partial x_1} & \cdots & \frac{\partial^2 f}{\partial x_n^2} \end{pmatrix} \begin{pmatrix} z_1 \\ \cdot \\ \cdot \\ \cdot \\ z_n \end{pmatrix} \tag{30}$$

$$= \sum_{i,j=1}^{n} \frac{\partial^2 f}{\partial x_i \partial x_j} z_i z_j \geq 0$$

for all z orthogonal to $g_1, \ldots, g_r$ -- that is, for all $z_1, \ldots, z_n$ satisfying

$$\sum_{i=1}^{n} \frac{\partial g_j}{\partial x_i} z_i = 0 \qquad (j = 1, \ldots, r) . \tag{31}$$

That (30) and (31) may be translated into appropriate properties of the bordered Hessian

$$\begin{vmatrix} 0 & \cdots & 0 & \frac{\partial g_1}{\partial x_1} & \cdots & \frac{\partial g_1}{\partial x_n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & \frac{\partial g_r}{\partial x_1} & \cdots & \frac{\partial g_r}{\partial x_n} \\ \frac{\partial g_1}{\partial x_1} & \cdots & \frac{\partial g_r}{\partial x_1} & \frac{\partial^2 f}{\partial x_1^2} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{\partial g_1}{\partial x_n} & \cdots & \frac{\partial g_r}{\partial x_n} & \frac{\partial^2 f}{\partial x_n \partial x_1} & \cdots & \frac{\partial^2 f}{\partial x_n^2} \end{vmatrix} \tag{32}$$

may also be readily established.