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SIMILARITY OF A PROBLEM IN CONCENTRATION-DEPENDENT DIFFUSION AND FLOW IN A FREE BOUNDARY LAYER

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NOTICES

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FOREWORD

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ABSTRACT

The mathematical similarity of a problem in one-dimensional diffusion in a semi-infinite medium when the diffusion coefficient varies linearly with concentration to the problem of the fully developed boundary layer between two fluid streams has been demonstrated. By applying the von Mises transformation in reverse, the one-dimensional diffusion equation was transformed to the Prandtl boundary-layer equations, which were subsequently transformed to a single non-linear ordinary differential equation. The transformed boundary and initial conditions of the diffusion problem were shown to correspond to the boundary conditions for the mixing of two uniform fluid streams. Numerical results for the diffusion problem were obtained from existing solutions to the fluid mechanics problem.

CONTENTS

ILLUSTRATIONS

NOMENCLATURE

C	Concentration (temperature in the case of heat diffusion)
D	Diffusion coefficient (thermal diffusivity in the case of heat diffusion)
$f(\eta)$	Nondimensional concentration variable or nondimensional stream function
u	Concentration variable or velocity component
v	Velocity component
x	Position coordinate or time
y	Position coordinate
z	Position coordinate or stream function
α	Constant in the linear diffusion coefficient equation
η	Transformation variable

ν	Kinematic viscosity
τ	Time
ψ	Stream function

SUBSCRIPTS

o	Constant reference conditions
1	For diffusion problem, refers to conditions at face of slab for $\tau > 0$, and for fluid mechanics problem refers to conditions in free stream
2	For diffusion problem, refers to initial conditions of slab for $0 < z < \infty$, and for fluid mechanics problem, refers to conditions along the dividing streamline

SECTION I
INTRODUCTION

The purpose of this report is to demonstrate the mathematical similarity of a problem in one-dimensional diffusion in a semi-infinite medium when the diffusion coefficient depends linearly on concentration to the problem of the boundary layer between two fluid streams having different velocities.[1] The problem of diffusion when the diffusion coefficient varies linearly with concentration has been treated by Wagner (Ref. 1) and Stokes (Ref. 2). Both of these solutions, along with many others of interest, are discussed by Crank (Ref. 3) in his treatise on diffusion.

There are many practical problems where the diffusion coefficient varies with the concentration, and over a limited concentration range, the diffusion coefficient can often be reasonably approximated by a linear equation in concentration. For example, the diffusion coefficient for the exchange of lead and silver ions in solid silver chloride as solvent is expected to be essentially proportional to the concentration of lead chloride (Ref. 1). Moreover, there are many practical situations where the thermal conductivity may be taken to vary linearly with temperature (Ref. 4).

The similarity between the boundary-layer equations and the one-dimensional unsteady diffusion equation is well known. Korst and Chapman (Ref. 5) linearized the boundary-layer equations and obtained solutions for the velocity distribution of a laminar or turbulent free-jet boundary problem including an initial boundary layer by making use of known techniques for solving the heat diffusion equation. The transformation of von Mises (Ref. 6), published in 1927, allows the Prandtl boundary-layer equations to be transformed into the unsteady one-dimensional diffusion equation. In this case, however, the resulting equation is non-linear and corresponds to diffusion in a medium whose diffusion coefficient depends on the concentration. In particular, the diffusion coefficient corresponding to the constant pressure boundary layer must vary linearly with concentration.

It follows that one can apply the transformation of von Mises in reverse and, at least for the case in which the diffusion coefficient varies linearly

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[1] Although this problem is discussed from the standpoint of the diffusion of matter, it has a counterpart in the diffusion of heat.

with concentration, transform the one-dimensional diffusion equation into the Prandtl boundary-layer equations, which can, in turn, be transformed to a single ordinary differential equation. The resulting ordinary differential equation has been solved and tabulated for a number of different boundary conditions; thus, it is possible to make use of existing solutions in fluid mechanics in solving a class of problems in concentration-dependent diffusion.

SECTION II
THEORETICAL DEVELOPMENT

The equation for one-dimensional diffusion when the diffusion coefficient, D, is a function of the concentration, C, is

$$\frac{\partial C}{\partial \tau} = \frac{\partial}{\partial z}\left(D \frac{\partial C}{\partial z}\right) \tag{1}$$

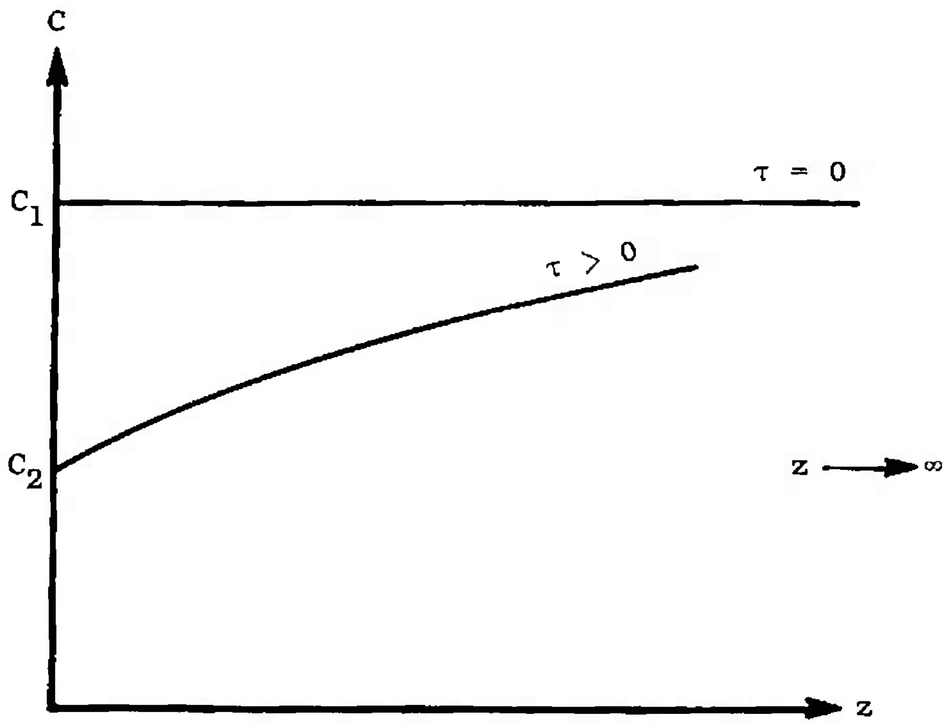

Fig. 1 Coordinate System for Diffusion Problem

Figure 1 illustrates the coordinate system. For the case of $D = D_0 C/C_0$, Eq. (1) can be recognized immediately as having the same form as the boundary-layer equations in stream function coordinates (the von Mises form of the boundary-layer equation). However, for the more general case, $D = D_o(1 + \frac{\alpha}{C_o} C)$, it is necessary to make a simple transformation for Eq. (1) to take the form of the boundary-layer equations. If $u = 1 + \frac{\alpha}{C_o} C$, Eq. (1) becomes

$$\frac{\partial u}{\partial \tau} = D_o \frac{\partial}{\partial z}\left(u \frac{\partial u}{\partial z}\right) \tag{2}$$

which now has the form of the boundary-layer equations in stream function coordinates.

The boundary and initial conditions to be considered are as follows:

at $z = 0$	$C = C_2$	$\tau > 0$	(3)
as $z = \infty$	$C \to C_1$	$\tau > 0$	(4)
at $\tau = 0$	$C = C_1$	$0 < z < \infty$	(5)

These conditions correspond to diffusion in a semi-infinite medium initially at a uniform concentration C_1 and having the concentration of its face suddenly changed to C_2. In terms of the new variable (u), the above conditions are:

at $z = 0$	$u = 1 + \alpha \frac{C_2}{C_o} = u_2$	$\tau > 0$	(6)
as $z \to \infty$	$u \to 1 + \alpha \frac{C_1}{C_o} = u_1$	$\tau > 0$	(7)
at $\tau = 0$	$u = 1 + \alpha \frac{C_1}{C_o} = u_1$	$0 < \tau < \infty$	(8)

If Eq. (2) is considered to be the boundary-layer equation, the boundary and initial condition (Eqs. (6), (7), and (8)) correspond to the boundary conditions for the mixing of two laminar streams of fluid where τ and u are the longitudinal position and velocity, respectively, D_o is the kinematic viscosity, and z is the stream function. Here u_1 corresponds to the free-stream velocity of one stream, and u_2 corresponds to the velocity along the streamline which divides the two streams.

In order to use a notation that is more consistent with the notation of boundary-layer theory, the following are defined:

$$\nu = D_o$$
$$x = \tau$$
$$\psi = z$$

Thus, Eq. (2) becomes

$$\frac{\partial u}{\partial x} = \nu \frac{\partial}{\partial \psi}\left(u \frac{\partial u}{\partial \psi}\right) \tag{9}$$

with corresponding boundary conditions,

at $x = 0$	$u = u_1$	$0 \leq \psi < \infty$
at $\psi = 0$	$u = u_2$	$0 < x < \infty$

If new variables y and v are defined as

$$y = \int_0^{\psi} \frac{1}{u}\, d\psi \tag{10}$$

and

$$v = -\left(\frac{\partial \psi}{\partial x}\right)_y \tag{11}$$

it follows that Eq. (9) can be transformed to

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = \nu \frac{\partial^2 u}{\partial y^2} \tag{12}$$

with

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{13}$$

The boundary conditions become

at $x = 0$	$u = u_1$	$0 \leq y < \infty$	(14)
at $y = 0$	$u = u_2$	$0 < x < \infty$	(15)
	$v = 0$		

Now Eqs. (12) and (13) may be recognized as the Prandtl boundary-layer equations. Although it may not be obvious, the boundary conditions (14) and (15) are the boundary conditions for the mixing of two uniform streams of fluid having different velocities when u_1 is the initial velocity of the upper stream and u_2 the velocity along the streamline which divides the two streams. Figure 2 illustrates the coordinate system and flow field for this problem.

If one defines

$$f(\eta) = \psi / \sqrt{\nu u_1 x} \tag{16}$$

where

$$\eta = y \sqrt{\frac{u_1}{\nu x}} \tag{17}$$

then substitution into Eq. (12) yields

$$2f''' + ff'' = 0 \tag{18}$$

where primes denote differentiation with respect to η.

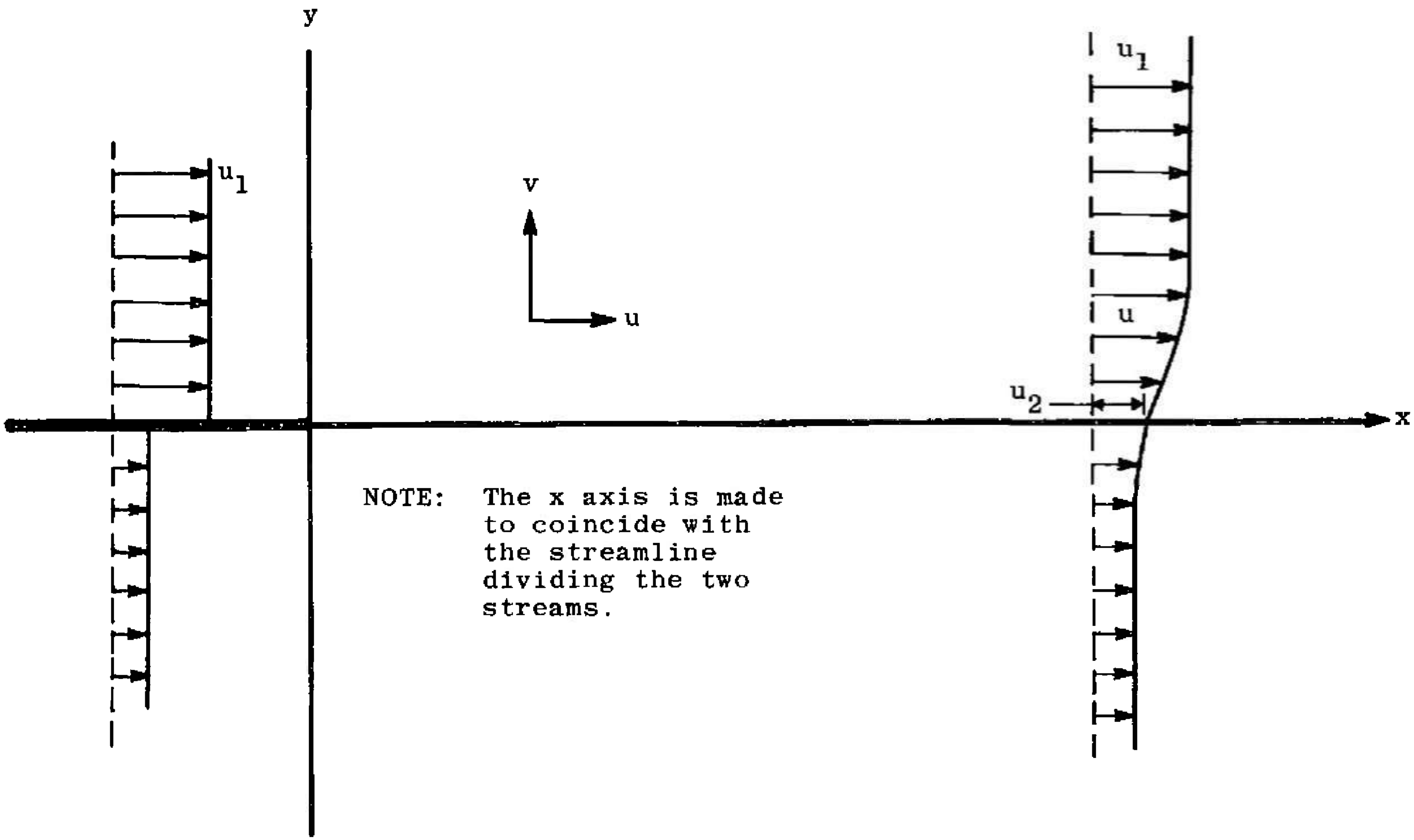

Fig. 2 Coordinate System for Fluid Mechanics Problem

The corresponding boundary conditions are

$$f'(0) = u_2/u_1 \tag{19}$$
$$f(0) = 0 \tag{20}$$
$$f'(\infty) = 1.0 \tag{21}$$

Blasius (Ref. 7) discussed the solution to Eq. (18) for the case $\frac{u_2}{u_1} = 0$ in connection with flow over a flat plate. Goertler (Ref. 8) obtained a power-series solution of Eq. (18) in terms of a parameter (λ), which is related to $\frac{u_2}{u_1}$, in connection with his study of the turbulent free-jet boundary. Other solutions have been given by Howarth, Lock, and Crane (Refs. 9, 10, and 11).

It can be shown that

$$u = u_1 f'(\eta) \tag{22}$$

and

$$\psi = f(\eta)\sqrt{\nu u_1 x} \tag{23}$$

Thus ψ and u are related parametrically through η. In the original notation,

$$\frac{C + C_o/\alpha}{C_1 + C_o/\alpha} = f'(\eta) \tag{24}$$

and

$$z\Big/\sqrt{D_o\left(1 + \frac{\alpha C_1}{C_o}\right)\tau} = f(\eta) \tag{25}$$

SECTION III
DISCUSSION OF RESULTS

From Lock's investigation of the velocity distribution in the laminar boundary layer between two incompressible streams having different densities and viscosities, solutions to Eq. (18) subject to boundary conditions (Eqs. (19) through (21)) for several values of u_2/u_1 can be deduced. It should be noted that Lock's f_1 and f_1' correspond to f and f' of this work, respectively. Figure 3 is a plot of $(C + C_o/\alpha)/(C_1 + C_o/\alpha)$ versus $z\Big/\sqrt{D_o(1 + \alpha C_1/C_o)\tau}$ for several values of $(C_2 + C_o/\alpha)/(C_1 + C_o/\alpha)$ as obtained from the tabulated results of Lock (Ref. 10). Also shown in Fig. 3 is the solution of Wagner (Ref. 1) for the case of $(C_2 + C_o/\alpha)/(C_1 + C_o/\alpha) = 0.585$, and it is seen that Wagner's solution does not agree with the solution obtained from Lock's results. Although they are not given here, numerical results can also be obtained from the solution of Crane (Ref. 11) for other values of the parameter $[(C_2 + C_o/\alpha)/(C_1 + C_o/\alpha)]$ than given in Fig. 3.

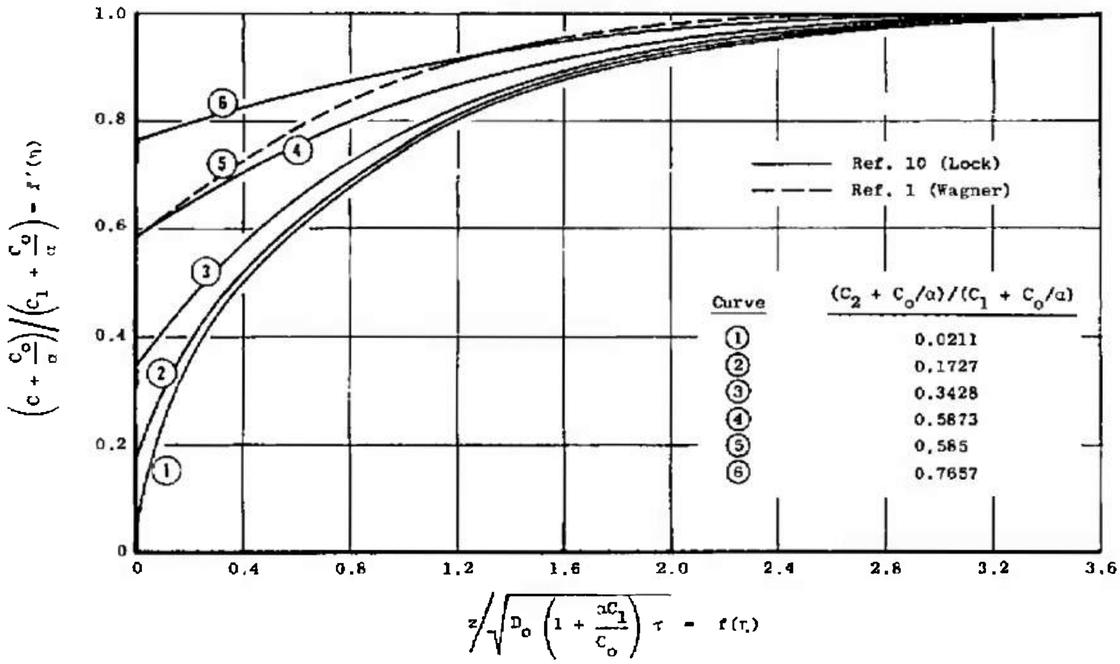

Fig. 3 Nondimensional Concentration versus the Nondimensional Position-Time Parameter

The mathematical similarity of a problem in one-dimensional diffusion in a semi-infinite medium when the diffusion coefficient varies linearly with concentration to the problem of the fully developed boundary layer between two fluid streams has been demonstrated. Numerical results were obtained from Lock's (Ref. 10) analysis of the laminar boundary layer between two fluid streams and are presented in graphical form.

Of particular interest is the disagreement between the solution of Wagner (Ref. 1) and the solution obtained here from the results of Lock. In order to resolve this discrepancy, Wagner's differential equation and boundary conditions were programmed for solution by the method of Runge-Kutta on a digital computer. It was found that the resulting numerical solution was in excellent agreement with the solution obtained from Lock's results, which suggests that Wagner's solution may be in error.

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