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PROPER EFFICIENCY AND THE THEORY OF VECTOR MAXIMIZATION

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This working paper should be regarded as preliminary in nature, and subject to change before publication in the open literature. It should not be quoted without prior consent of the author. Comments are cordially invited.

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SUMMARY

The concept of efficiency in problems with multiple criterion functions--sometimes under an alias such as "admissibility" or "Pareto optimality"--has long played an important role in economics, game theory, statistical decision theory, and in all optimal decision problems with noncomparable criteria. Here we propose a slightly restricted definition of efficiency that eliminates efficient points of a certain anomalous nature. This new definition, which we call proper efficiency, is related in spirit to the notion of "proper" efficiency introduced by Kuhn and Tucker in their celebrated paper of 1950; but the present definition avoids certain drawbacks inherent in the earlier one. A comprehensive theory of vector maximization is constructed using the new definition, with and without various constraint qualification, convexity, and differentiability assumptions. The theory includes as a special case the standard theory of nonlinear programming.

I. INTRODUCTION

Given a vector-valued criterion function $f(x) = (f_1(x), \ldots, f_p(x))$ and a set $X \subseteq R^n$ of feasible points, the Vector Maximum Problem

(VMP) V-MAX $f(x)$ subject to $x \in X$

is the problem of finding all points that are efficient: x^o is said to be efficient if $x^o \in X$ and there exists no other feasible point x such that $f(x) \geqq f(x^o)$ but $f(x) \neq f(x^o)$. The concept of efficiency--sometimes under an alias such as "admissibility," "maximality," noninferiority," or "Pareto optimality"--has long played an important role in economics, game theory, statistical decision theory, and in all optimal decision problems with noncomparable criteria.

In this study we propose a slightly restricted definition of efficiency that (a) eliminates efficient points of a certain anomalous type; and (b) lends itself to more satisfactory characterization (see Theorem 2 below, and Sec. II). We shall call this new definition proper efficiency, although Kuhn and Tucker [7] have previously used the same term. Their intent appears to have been much the same as ours but, as we shall see, the present definition is of greater generality and seems to be somewhat more natural.

PROPER EFFICIENCY

Definition: x^o is said to be a proper efficient solution of (VMP) if it is efficient and if there exists a scalar $M > 0$ such that, for each i, $f_i(x) > f_i(x^o)$ and $x \in X$ implies

$$\frac{f_i(x) - f_i(x^o)}{f_j(x^o) - f_j(x)} \leq M$$

for some j such that $f_j(x) < f_j(x^o)$.

An efficient point that is not properly efficient is said to be <u>improperly</u> efficient. Thus for x^o to be improperly efficient means that to every scalar $M > 0$ (no matter how large) there is a point $x \in X$ and an i such that $f_i(x) > f_i(x^o)$ and

$$\frac{f_i(x) - f_i(x^o)}{f_j(x^o) - f_j(x)} > M$$

for all j such that $f_j(x) < f_j(x^o)$. If we take a sequence $\langle M^\nu \rangle \to \infty$ and remember that there is but a finite number of criteria, we see that for some criterion i_o, the marginal gain in f_{i_o} can be made arbitrarily large relative to each of the marginal losses incurred by other criteria. Assuming that the decisionmaker's desire for f_{i_o} is not satiated, x^o certainly seems undesirable. An example of improper efficiency is given in Sec. III.

CHARACTERIZATION

A matter of great interest, both computationally and theoretically, is the relation of the Vector Maximum Problem to the following scalar maximum problem:

$$(P_\lambda) \qquad \text{MAX} \sum_{i=1}^{p} \lambda_i f_i(x) \text{ subject to } x \in X,$$

where the λ_i are nonnegative parameters often normalized according to $\sum_{i=1}^{p} \lambda_i = 1$. The fundamental results characterizing proper vector maxima in terms of the solutions of (P_λ) are given in Theorems 1 and 2.

> **Theorem 1.** Let $\lambda_i > 0$ $(i = 1, \ldots, p)$ be fixed. If x^o is optimal in (P_λ), then x^o is properly efficient in (VMP)

Proof: It is easy to show by contradiction that x^o is efficient. Let $\bar{\lambda} = \operatorname{Max}_{i,j} \{\lambda_j/\lambda_i\}$. We shall show that x^o is properly efficient in (VMP) with $M = p\bar{\lambda}$; that is, we shall show that for each i, $f_i(x) > f_i(x^o)$ and $x \in X$ implies $f_i(x) - f_i(x^o) \le p\bar{\lambda}\,(f_j(x^o) - f_j(x))$ for some $j \in J_x$, where $J_x \triangleq \{1 \le j \le p : f_j(x) < f_j(x^o)\}$. Suppose to the contrary that, for some i and $x \in X$, we have $f_i(x) > f_i(x^o)$ and

$$f_i(x) - f_i(x^o) > p\bar{\lambda}\,(f_j(x^o) - f_j(x)),\ \forall j \in J_x.$$

Summing the latter inequalities and dividing the result by the number q of elements of J_x, we obtain

$$(f_i(x) - f_i(x^o)) > \frac{p}{q}\sum_{j\in J_x} \bar{\lambda}(f_j(x^o) - f_j(x)).$$

Using $p/q > 1$ and the definitions of $\bar{\lambda}$ and J_x, in that order, it follows that

$$(f_i(x) - f_i(x^o)) > \sum_{j\ne i}(\lambda_j/\lambda_i)(f_j(x^o) - f_j(x))$$

Multiplying through by λ_i and rearranging, we see that this contradicts the optimality of x^o in (P_λ). Hence x^o must be properly efficient.

> Theorem 2. Let X be a convex set, and let the f_i be concave on X. Then x^o is properly efficient in (VMP) if and only if x^o is optimal in (P_λ) for some λ with strictly positive components.

Proof: The "if" part of the theorem is provided by Theorem 1. If x^o is properly efficient, then there exists a scalar $M > 0$ such that for each i $(i = 1, \ldots, p)$ the system

$$f_i(x) > f_i(x^o)$$

$$f_i(x) + M\, f_j(x) > f_i(x^o) + M\, f_j(x^o),\ \text{all } j \ne i$$

admits no solution in X. By a fundamental property of concave functions [2, p. 62], for the i^{th} system there exist $\lambda^i_j \geq 0$ ($j = 1, \ldots, p$) with $\sum_{j=1}^{p} \lambda^i_j = 1$ such that

$$\lambda^i_i f_i(x) + \sum_{j \neq i} \lambda^i_j (f_i(x) + M f_j(x)) \leq \lambda^i_i f_i(x^o) + \sum_{j \neq i} \lambda^i_j (f_i(x^o) + M f_j(x^o))$$

for all $x \in X$. Summing over i yields, after some simplification and rearrangement,

$$\sum_{j=1}^{p} (1 + M \sum_{i \neq j} \lambda^i_j) f_j(x) \leq \sum_{j=1}^{p} (1 + M \sum_{i \neq j} \lambda^i_j) f_j(x^o)$$

for all $x \in X$. This completes the proof.

Thus from a computational viewpoint, finding proper efficient solutions is reduced to a parametric programming problem; (P_λ) yields only properly efficient solutions as λ varies over

$$\Lambda^+ \triangleq \{\lambda \in R^p : \text{all } \lambda_i > 0 \text{ and } \sum_{i=1}^{p} \lambda_i = 1\},$$

and if concavity holds then this approach yields <u>all</u> properly efficient points.*

A more complete characterization theory for the Proper Vector Maximum Problem is developed in the next section. It provides, for example, necessary conditions for a proper vector maximum in the absence of concavity.

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*In this regard see, for example, Charnes and Cooper [3, Ch. 9], Markowitz [8], and Geoffrion [4].

II. THEORY

We shall give the theory of the Proper Vector Maximum Problem in terms of the relationships between the following six problems. In problems 3, 4, and 5, X is taken to be of the form $X = \{x : g(x) \geqq 0\}$, where $g(x) = (g_1(x), \ldots, g_m(x))$. In problems 3 and 4, the differentiability of all functions is presumed.

Problem 1 - Find a point $\bar{x}$ that is a proper efficient solution of (VMP).

Problem 2 - Find a point $\bar{x}$ that is a locally* proper efficient solution of (VMP).

Problem 3 - Find a feasible point $\bar{x}$ such that none of the p systems** $(i = 1, \ldots, p)$

$$\nabla_x f_i(\bar{x}) \cdot u > 0$$

$$\nabla_x f_j(\bar{x}) \cdot u \geqq 0, \text{ all } j \neq i$$

$$\nabla_x g_j(\bar{x}) \cdot u \geqq 0, \text{ all } j \ni g_j(\bar{x}) = 0$$

has a solution u in R^n.

Problem 4 - Find a feasible point $\bar{x}$, a point $\bar{y} \geqq 0$ in R^m, and a point $\bar{\lambda} \in \Lambda^+$ such that $\bar{y}\, g(\bar{x}) = 0$ and

$$\nabla_x [\bar{\lambda} \cdot f(\bar{x}) + \bar{y}\, g(\bar{x})] = 0$$

Problem 5 - Find a feasible point $\bar{x}$, a point $\bar{y} \geqq 0$ in R^m, and a point $\bar{\lambda} \in \Lambda^+$ such that $\bar{y} \cdot g(\bar{x}) = 0$ and $\bar{x}$ achieves the unconstrained maximum of $\bar{\lambda} \cdot f(x) + \bar{y} \cdot g(x)$.

Problem 6 - Find a point $\bar{x}$ and a point $\bar{\lambda} \in \Lambda^+$ such that $\bar{x}$ is optimal in $(P_{\bar{\lambda}})$.

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*$\bar{x}$ is said to be a locally proper efficient solution of (VMP) if it is properly efficient in $N_{\bar{x}} \cap X$, where $N_{\bar{x}}$ is some (open convex) neighborhood of $\bar{x}$.

**$\nabla_{\bar{x}} \varphi(\bar{x})$ represents the gradient vector of the function φ evaluated at $x = \bar{x}$.

Problem 1 is the central problem of interest. Problem 2 is its "local" equivalent, and problem 3 is the local problem in terms of directional derivatives. Problem 4 represents the generalized Lagrange multiplier or Kuhn-Tucker conditions in differential form associated with problem 1. Problem 5 is precisely equivalent to the following saddle-point problem:

> Find a point $\bar{x}$, a point $\bar{y} \geqq 0$ in R^m, and a point $\bar{\lambda} \in \Lambda^+$ such that the pair $(\bar{x},\bar{y})$ is a saddle-point subject to $y \geqq 0$ of the function $F(x,y) = \bar{\lambda}\cdot f(x) + y\cdot g(x)$; i.e., such that $F(\bar{x},y) \leq F(\bar{x},\bar{y}) \leq F(x,\bar{y})$ for all $x \in R^n$ and $y \geqq 0$ in R^m

Problem 5 is also of interest for its own sake. Problem 6 is just (P_λ)

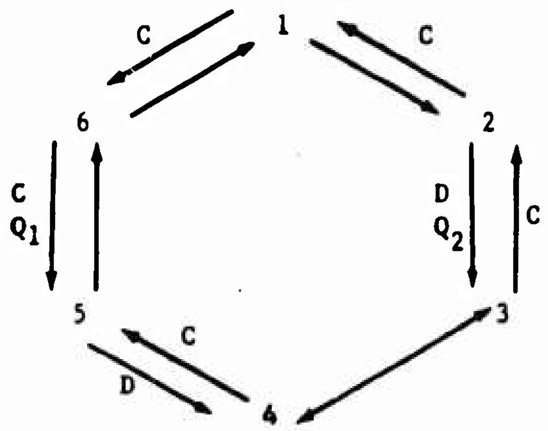
In stating the relations between these problems, we shall use the notation $j \xrightarrow{A_1,\ \ldots} k$, which is to be understood as follows. Let (u,v) be the unknowns of problem j and (u,w) the unknowns of problem k. Then this notation is to be read. "If $(\bar{u},\bar{v})$ solves problem j, and if assumptions A_1, ... hold, then there exists $\bar{w}$ such that $(\bar{u},\bar{w})$ solves problem k." Or, somewhat more loosely, "Under assumptions A_1, ..., every solution of problem j is also a solution of problem k."

The assumptions which will be used at one time or another are:

> Assumption C: All functions are concave on E^n.
>
> Assumption D: All functions are continuously differentiable on E^n.
>
> Assumption Q_1: The following constraint qualification holds: there exists a feasible point $\hat{x}$ such that $g_j(\hat{x}) > 0$ for g_j nonlinear.
>
> Assumption Q_2: The Kuhn-Tucker constraint qualification holds [7, p. 483].

We are now in a position to state the relationships between the six problems.

Theorem (Comprehensive)



For example, the Comprehensive Theorem asserts $(1 \rightarrow 2)$ that every proper efficient solution of (VMP) is a locally proper efficient solution of (VMP), and $(1 \xleftarrow{C} 2)$ that the converse is true under Assumption C. It also asserts $(5 \rightarrow 6)$ that if $(\bar{x},\bar{y},\bar{\lambda})$ solves problem 5, then $(\bar{x},\bar{\lambda})$ solves problem 6; and $(5 \xleftarrow{C,Q_1} 6)$ that if $(\bar{x},\bar{\lambda})$ solves problem 6, then there exists a point $\bar{y} \in R^m$ such that $(\bar{x},\bar{y},\bar{\lambda})$ solves problem 5.

Because of its length, we give the proof in Appendix A.

The Comprehensive Theorem is actually many theorems in one. Its significance is that it gives, under various assumptions, necessary and/or sufficient conditions for proper efficiency. In order to be explicit, we state the most important of these conditions as three simple corollaries of the Comprehensive Theorem. Corollary 1 asserts that under Assumptions D and Q_2, the conditions of problem 4 are necessary first order conditions for proper efficiency. Corollary 2 characterizes problem 1 as being equivalent (in the appropriate sense) to problems 2, 5, and 6 under Assumptions C and Q_1. Corollary 3 asserts that all six problems are equivalent under C, D, and either Q_1 or Q_2.

> <u>COR 1</u> - If Assumptions D and Q_2 hold, then problem 2 $\leftrightarrow$ problem 4.
>
> <u>COR 2</u> - If Assumptions C and Q_1 hold, then problem 1 $\leftrightarrow$ problem 2 $\leftrightarrow$ problem 5 $\leftrightarrow$ problem 6.
>
> <u>COR 3</u> - If Assumptions C, D, and either Q_1 or Q_2 hold, then problem 1 $\leftrightarrow$ problem j for $j = 2, \ldots, 6$.

The Comprehensive Theorem subsumes, of course, the cases in which there are no constraints or only equality constraints. Again for the sake of explicitness, we shall state the main results for these cases in Appendix B.

It is of interest to note that in the special case all of the f_i are identical or $p = 1$, the notion of proper efficiency coincides with the notion of a constrained maximum, so that the results of the Comprehensive Theorem reduce to well-known counterparts in the theory of nonlinear programming.

III. DISCUSSION

We turn now to further discussion of the notion of proper efficiency.

Just how slight a restriction proper efficiency is over efficiency can perhaps be better appreciated in the light of the following. Denote the set of all efficient (properly efficient) points by $X^{\geq}$ ($X^{\geq}_{pr.}$), and the image in R^p of $X^{\geq}$ under f by $f[X^{\geq}]$. If the f_i are continuous and concave on the closed convex set X, then $f[X^{\geq}_{pr.}] \subseteq f[X^{\geq}] \subseteq \overline{f[X^{\geq}_{pr.}]}$, where the bar denotes closure. This result is a consequence of Theorem 2 and a result* due to Arrow, Barankin and Blackwell [1]. Thus under the given conditions, which are almost always satisfied in concave programming, the outcome of any improperly efficient point is always the limit of the outcomes of some sequence of properly efficient points.

COMPARISON WITH THE DEFINITION OF KUHN AND TUCKER

The notion of "proper" efficiency introduced by Kuhn and Tucker applies only when assumptions D and Q_2 hold. Under these assumptions, x^o is said to be "properly" efficient if it is efficient and if it solves problem 3. Let us denote the problem of finding such a "properly" efficient point as $(X^{\geq}, 3)$. Then the results obtained by Kuhn and Tucker are** (in the presence of D and Q_2):

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*If S is a closed convex set in R^n, then the set of efficient points of S contains the subset of points of S for which there is a supporting hyperplane whose normal has all positive components, and is contained in the closure of the last mentioned set.

**Each of these assertions can be obtained as an immediate corollary of the Comprehensive Theorem.

(i) $(\alpha^{\geq}, 3) \longrightarrow 4$ [7, Theorem 4]

(ii) $4 \xrightarrow{C} (\alpha^{\geq}, 3)$ [7, Theorem 5]

(iii) $(\alpha^{\geq}, 3) \xleftrightarrow{C} 5$ [7, Theorem 6].

To justify excluding efficient solutions that are not "proper," Kuhn and Tucker give an example with $p = 2$ in which such a solution admits a first-order gain in one criterion at the expense of but a second-order loss in the other. Indeed, every "improperly" efficient solution poses an equally objectionable anomaly.* The converse, however, is not true--not every anomalous efficient point is "improper" in the sense of Kuhn and Tucker, as the following example shows. Put $n = 1$, $m = 1$, $p = 2$, $g(x) = x$, $f_1(x) = x^2$, $f_2(x) = -x^3$, $x^o = 0$. Assumptions D and Q_2 hold, and x^o is "properly" efficient, but for x positive and sufficiently small the gain in f_1 can be made arbitrarily large with respect to the loss in f_2 (the gain-to-loss ratio is $1/x$ for $x > 0$).

Since $1 \xrightarrow{D, Q_2} 3$ (see Comprehensive Theorem), the set of points "properly" efficient in the sense of Kuhn and Tucker contains all those properly efficient in the present sense. The above example (in which x^o is improperly efficient in the sense of Sec. I) shows that the containment can be strict.

To summarize, the advantages of the present definition of proper efficiency over that of Kuhn and Tucker seem to be that it excludes all of a precise class of anomalies, and that it applies even in the absence of Assumption D or Q_2.

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*For an explicit proof see Klinger [6]; his proof seems to require the locus of $\hat{x}(t)$ in the definition of Q_2 to be linear, but this restriction can be removed (cf. the proof of $2 \xrightarrow{D, Q_2} 3$ in Appendix A).

CONCLUSION

We began with the premise that, in optimization problems with multiple criteria, it is natural to restrict attention to efficient decisions that are properly so--in the sense that at least one potential marginal gain-to-loss ratio must be bounded. We then obtained, in Theorems 1 and 2, basic characterization results for proper efficiency in terms of the scalar parametric problem (P_λ). These results were extended in the Comprehensive Theorem to include the relationships with four other intimately related problem formulations, with and without various constraint qualifications, differentiability and convexity assumptions. The result is a coherent theory of the Proper Vector Maximum Problem which generalizes the well-known Kuhn-Tucker theory for nonlinear programming. This theory seems more satisfactory than that possible using either the usual definition of efficiency or the closely related definition of "proper" efficiency proposed by Kuhn and Tucker.

Appendix A

PROOF OF THE COMPREHENSIVE THEOREM

A. $1 \overset{C}{\rightleftarrows} 6$. This is a restatement of Theorems 1 and 2 (with λ normalized).

B. $6 \overset{C,Q_1}{\rightleftarrows} 5 \underset{C}{\overset{D}{\rightleftarrows}} 4$. These assertions are all known results from the theory of nonlinear programming applied to (P_λ).

$6 \xrightarrow{C,Q_1} 5$ is a consequence of a slightly more general form of the Farkas-Minkowski Theorem [2, p. 67].

$5 \longrightarrow 6$ is easily verified directly.

$5 \xrightarrow{D} 4$ occurs because the gradient of a continuously differentiable function must vanish at an unconstrained extremum.

$4 \xrightarrow{C} 5$ occurs because a concave function (which $\bar{\lambda}\cdot f(x) + \bar{y}\cdot g(x)$ must be, since $\bar{\lambda} \geqq 0$ and $\bar{y} \geqq 0$) achieves an unconstrained supremum at any point for which its gradient vanishes.

C. $1 \overset{C}{\leftrightarrows} 2$. $1 \longrightarrow 2$ is trivial.

Let $\bar{x}$ be a locally proper efficient solution in the neighborhood $N_{\bar{x}}$. Under Assumption C, Theorem 2 tells us that $\bar{x}$ maximizes $\bar{\lambda}\cdot f(x)$ on $N_{\bar{x}} \cap X$ for some $\bar{\lambda} \in \Lambda^+$. Again from Assumption C, $\bar{x}$ must maximize $\bar{\lambda}\cdot f(x)$ over X, and so by Theorem 1 $\bar{x}$ must be properly efficient. Thus $2 \xrightarrow{C} 1$.

D. $3 \longleftrightarrow 4$. This result can essentially be found in [7, Theorems 4 and 5].

$3 \longleftarrow 4$ can be shown as follows. If $(\bar{x}, \bar{y}, \bar{\lambda})$ is a solution to problem 4 then $\sum_{i=1}^{p} \bar{\lambda}_i \nabla_x f_i(\bar{x}) + \sum_{j\in J} \bar{y}_j \nabla_x g_j(\bar{x}) = 0$, where $J =$ $\{j : g_j(\bar{x}) = 0\}$, for the complementary slackness condition $\bar{y}\cdot g(\bar{x}) = 0$ implies $\bar{y}_j = 0$ when $g_j(\bar{x}) \neq 0$. Upon postmultiplication of the vector equation by u, we readily see by contradiction that $\bar{x}$ must be a solution of Problem 3.

To see $3 \longrightarrow 4$, let $\bar{x}$ be a solution of Problem 3 and apply the Farkas-Minkowski Theorem in turn to each of the p systems. As a result, there must exist numbers $w_j^i \geq 0$ and $z_j^i \geq 0$, such that, for $i = 1, \ldots, p$,

$$\nabla_x f_i(\bar{x}) + \sum_{j \neq i} w_j^i \nabla_x f_j(\bar{x}) + \sum_{j \in J} z_j^i \nabla_x g_j(\bar{x}) = 0.$$

Summing over i yields

$$\sum_{j=1}^{p} (1 + \sum_{i \neq j} w_j^i) \nabla_x f_j(\bar{x}) + \sum_{j \in J} (\sum_{i=1}^{p} z_j^i) \nabla_x g_j(\bar{x}) = 0.$$

Put $\hat{\lambda}_i = (1 + \sum_{i \neq j} w_j^i)$, $\hat{y}_j = (\sum_{i=1}^{p} z_j^i)$ for $j \in J$

and $\hat{y}_j = 0$ for $j \notin J$. Clearly $\bar{x}$, $\bar{\lambda}_i = \hat{\lambda}_i / (\sum_{i=1}^{p} \hat{\lambda}_i)$,

and $\bar{y}_j = \hat{y}_j \; / \; (\sum_{i=1}^{p} \hat{\lambda}_i)$ solves Problem 4.

E. $2 \underset{C}{\overset{D,\ Q2}{\rightleftarrows}} 3$. We have previously shown $3 \longrightarrow 4 \xrightarrow{C} 5 \xrightarrow{C} 6 \longrightarrow 1 \longrightarrow 2$; hence $3 \xrightarrow{C} 2$. To complete the proof of the Comprehensive Theorem it remains only to show $2 \xrightarrow{D,\ Q2} 3$.

Let $\bar{x}$ be a locally proper efficient solution of (VMP), and let Assumptions D and Q_2 hold. Suppose, contrary to what we desire to show, that $\bar{x}$ is not a solution of Problem 3. Then one of the p systems, say the first, has a solution: there exists $\bar{u}$ such that

$$\nabla_x f_1(\bar{x}) \cdot \bar{u} > 0$$

$$\nabla_x f_j(\bar{x}) \cdot \bar{u} \geq 0, \; j = 2, \ldots, p$$

$$\nabla_x g_j(\bar{x}) \cdot \bar{u} \geq 0, \text{ all } j \ni g_j(\bar{x}) = 0.$$

By Assumption Q_2 there exists a continuously differentiable arc $\hat{x}(t)$, $0 \leq t \leq 1$, contained in the feasible region, with $\hat{x}(0) = \bar{x}$ and some positive scalar α such that $\left(\frac{d\,\hat{x}_1(0)}{dt}, \ldots, \frac{d\,\hat{x}_n(0)}{dt}\right) = \alpha\,\bar{u}$. Consider the functions $f_i(\hat{x}(t))$. From Taylor's Theorem we have

$$f_i(\hat{x}(t)) = f_i(\hat{x}(0)) + t\,\frac{d\,f_i(\hat{x}(t_i))}{dt}$$

$$= f_i(\bar{x}) + t \sum_{j=1}^{n} \left.\frac{\partial\, f_i(x)}{\partial\, x_j}\right|_{\hat{x}(t_i)} \left.\frac{d\,\hat{x}_j(t)}{dt}\right|_{t_i},$$

where t_i is some scalar between 0 and t. Denote the summation in the last term by $s_i(t)$, so that $f_i(\hat{x}(t)) = f_i(\bar{x}) + t\, s_i(t)$. Evidently $s_i(0) = \alpha \nabla_x f_i(\bar{x}) \cdot \bar{u}$ and $s_i(t)$ is continuous (from the right) at $t = 0$. Now for t sufficiently near 0, $\hat{x}(t)$ is in the neighborhood within which $\bar{x}$ is properly efficient. Consider a sequence $< t^{\nu} > \to 0$, where $t^{\nu} > 0$. By taking a subsequence, if necessary, we may assume that the set $\{j : f_j(\hat{x}(t^{\nu})) < f_j(\bar{x})\}$ is constant for all ν -- call it J^-. We know that $< s_j(t^{\nu})> \to \alpha \nabla_x f_j(\bar{x}) \cdot \bar{u} \geq 0$, all $j \in J^-$. But $s_j(t^{\nu}) < 0$ by definition for all ν and $j \in J^-$, and so $< s_j(t^{\nu}) > \to 0$ for all $j \in J^-$. Furthermore, $< s_1(t^{\nu}) > \to \alpha \nabla_x f_1(\bar{x}) \cdot u > 0$. Therefore the sequences

$$\left\langle \frac{f_i\,(\hat{x}(t^\nu)) - f_i\,(\bar{x})}{f_j\,(\bar{x}) - f_j\,(\hat{x}(t^\nu))} \right\rangle \quad , j \in J^-,$$

which can be written

$$\left\langle \frac{t^\nu\; s_i(t^\nu)}{-t^\nu\; s_j(t^\nu)} \right\rangle, \; j \in J^-,$$

all diverge to $+\infty$. But this contradicts the local proper efficiency of $\bar{x}$, and so $\bar{x}$ must indeed be a solution of Problem 3.

Appendix B

NO CONSTRAINTS AND EQUALITY CONSTRAINTS

NO CONSTRAINTS

Here we consider the case in which X is an open set in R^n (perhaps the whole of R^n). Corollary 4 gives necessary, and Cor. 5 sufficient conditions for a locally proper efficient (l.p.e.) solution.

COR 4 - Let the f_i be continuously differentiable on X. If x^o is l.p.e., then $\nabla_x[\lambda \cdot f(x^o)] = 0$ for some $\lambda \in \Lambda^+$.

Proof: With $m = 0$ and $x^o \in X$, Q_2 becomes superfluous, and the Comprehensive Theorem yields $2 \xrightarrow{D} 3 \longrightarrow 4$.

COR 5 - Let the f_i be twice continuously differentiable on an open set $X \subseteq R^n$. If $x^o \in X$ satisfies $\nabla_x[\lambda \cdot f(x^{\cdot})] = 0$ for some $\lambda \in \Lambda^+$, and the Hessian $\nabla_x^2 [\lambda \cdot f(x^{\cdot})]$ is negative definite, then x^o is l.p.e.

Proof: The assumptions imply that $\lambda \cdot f(x)$ is strictly concave on some (convex) open neighborhood N_{x^o} of x^o. Hence x^o maximizes this function on N_{x^o}, and so by Theorem 1 x^o must be l.p.e.

It is clear from the proof of Cor. 5 that the hypothesis "f_i twice continuously differentiable and $\nabla_x^2[\lambda \cdot f(x^{\cdot})]$ negative definite" can be weakened to "f_i continuously differentiable and $\lambda \cdot f(x)$ concave on some neighborhood of x^o."

EQUALITY CONSTRAINTS

Here we consider the case $X = \{x : g_j(x) = 0,\ j = 1, \ldots, m\}$. The Comprehensive Theorem subsumes this case if we write X as $\{x : g_j(x) \geq 0,\ j = 1, \ldots, m \text{ and } -\sum_{j=1}^{m} g_j(x) \geq 0\}$. Assumption Q_1 is satisfied if and only if all constraints are linear; and the directions u of concern in Q_2 are those for which $\nabla_x g_j(\bar{x})\cdot u = 0$, $j = 1, \ldots, m$.

Corollary 6 is a Lagrange Multiplier Theorem, and Cor. 7 examines the linear constraints case.

COR 6 - Let the f_i and g_j be continuously differentiable on some neighborhood of x^o, and let Q_2 hold at x^o. If x^o is l.p.e., then

$$\nabla_x[\lambda \cdot f(x^o) + \mu \cdot g(x^o)] = 0$$

for some $\lambda \in \Lambda^+$ and $\mu \in R^m$.

Proof: The Comprehensive Theorem asserts $\vdash\xrightarrow{D,Q_2} 4$. Put $\mu_j = \bar{y}_j - \bar{y}_{m+1}$.

COR 7 - Let the g_j be linear, and the f_i concave. Then each of the following conditions is necessary and sufficient for x^o to be properly efficient:

(i) x^o maximizes $\lambda \cdot f(x)$ subject to $g(x) = 0$ for some $\lambda \in \Lambda^+$;

(ii) x^o is feasible, and maximizes $\lambda \cdot f(x) + \mu \cdot g(x)$ over all x for some $\lambda \in \Lambda^+$ and $\mu \in R^m$;

(iii) there exists $\mu^o \in R^m$ such that (x^o, μ^o) is a saddlepoint of the function $F(x, \mu) = \lambda^o \cdot f(x) + \mu \cdot g(x)$ for some $\lambda^o \in \Lambda^+$; i.e., $F(x^o, \mu) \leq F(x^o, \mu^o) \leq F(x, \mu^o)$ for all $x \in R^n$ and $\mu \in R^m$.

If, in addition, the f_i are continuously differentiable, then a fourth equivalent condition is:

(iv) x^o satisfies

$$\nabla_x[\lambda \cdot f(x^o) + \mu \cdot g(x^o)] = 0$$

$$g(x^o) = 0,$$

for some $\lambda \in \Lambda^+$ and $\mu \in R^m$.

<u>Proof:</u> Directly from the Comprehensive Theorem.

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The concept of efficiency in problems with multiple criterion functions--sometimes under an alias such as "admissibility" or "Pareto optimality"--has long played an important role in economics, game theory, statistical decision theory, and in all optimal decision problems with noncomparable criteria. Here we propose a slightly restricted definition of efficiency that eliminates efficient points of a certain anomalous nature. This new definition, which we call proper efficiency, is related in spirit to the notion of "proper" efficiency introduced by Kuhn and Tucker in their celebrated paper of 1950; but the present definition avoids certain drawbacks inherent in the earlier one. A comprehensive theory of vector maximization is constructed using the new definition, with and without various constraint qualification, convexity, and differentiability assumptions. The theory includes as a special case the standard theory of nonlinear programming.

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