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The Effects of Outage on Performance Statistics for Bipropellant Rockets

JULY 1967

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Prepared for SPACE AND MISSILE SYSTEMS ORGANIZATION
AIR FORCE SYSTEMS COMMAND
LOS ANGELES AIR FORCE STATION
Los Angeles, California

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FOREWORD

This report is published by the Aerospace Corporation, El Segundo, California under Air Force Contract No. F04965-67-C-0158. The report was prepared by Joseph W. Duroux of the Systems Design Subdivision, Applied Mechanics Division, in order to document research carried out from March 1966 to December 1966 under the above contract.

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Publication of this report does not constitute Air Force approval of the report's findings or conclusions. It is published only for the exchange and stimulation of ideas.

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ABSTRACT

The distribution of performance errors in bipropellant rockets is determined as a function of engine and propellant loading parameters and contributing errors. The distribution is found to be highly skewed, so that the probability of exceeding three sigma can be significantly greater than it would be if the distribution were normal. Thus, three sigma as a minimum performance limit is not justified. A detailed analysis is required in each case to determine the relationship between performance margin and probability of mission success. Also, a realistic method of optimizing the propellant loading bias is developed. The methods previously used have resulted in fuel biases that are usually much lower than the optimum. Computer programs have been developed to perform the analyses described in this report, both for the single-stage and the multistage case.

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CONTENTS

FIGURES

TABLES

SYMBOLS

Roman Uppercase Letters

A	the deviation of S from $\overline{S}$ in multiples of σ_s
C	the deviation of shutdown performance from mean performance
K	the error scaling factor for each stage
N	the number of bipropellant stages
S	$\lambda - 1$
W	the sum of non-outage performance errors in one stage
X	the total performance error caused by one stage
Z	the fractional outage equal to $\frac{W_z}{W_L}$

Roman Uppercase Letters With Superscripts

$F^*()$	the discrete probability that $() = \xi$

Roman Uppercase Letters With Subscripts

$F_z(Z)$	the discrete probability that the fractional outage has the value Z
P_α	the cumulative probability that $X > \overline{X} + \alpha\sigma_x$ or that $T > \overline{T} + \alpha\sigma_{tN}$
R_b	the oxidizer-to-fuel burning ratio
R_L	the loaded ratio of usable oxidizer to usable fuel equal to $\frac{W_o}{W_f}$

R_n	the nominal value of R_b
T_i	the total performance error caused by the first i stages
W_f	the weight of usable fuel
W_L	the total weight of usable propellant equal to $W_f + W_o$
W_o	the weight of usable oxidizer
W_z	the outage weight
Z_m	the maximum allowable outage

Roman Lowercase Letters With Superscripts

$f^*(\ )$	a probability density function truncated by command shutdown

Roman Lowercase Letters With Subscripts

$f_f(\ )$	the contribution to $f_z(\ )$ from fuel outage
$f_o(\ )$	the contribution to $f_z(\ )$ from oxidizer outage
$f_{[\ ]}(\ )$	the probability density function for [ ] unless otherwise defined

Greek Lowercase Letters

α	the deviation of X from $\overline{X}$ or of T from $\overline{T}$ in multiples of σ_x or σ_t
β	the propellant loading bias and the mean value of S
δ	the limiting magnitude of S for which zero outage occurs
λ	equal to $\frac{R_L}{R_b}$
υ	a dummy variable for integration
ξ	the total error corresponding to the command shutdown performance level equal to $\overline{T} + C\sigma_t$

Greek Lowercase Letters With Subscripts

β_o	the optimum propellant loading bias
$\sigma^2_{[\,]}$	the variance of [ ]

Superscripts and Subscripts

$(\ )_f$	refers to the situation in which fuel outage occurs unless otherwise defined
$(\ )_i$	(or numerical subscript) refers to the i^{th} or numbered stage
$(\ )_o$	refers to the situation in which oxidizer outage occurs unless otherwise defined

SECTION I

INTRODUCTION

This study primarily aims to determine the effects of outage on the distribution of errors in the performance of bipropellant-staged rockets, and to develop a method to optimize propellant loading bias.

In computing performance errors for rockets, three basic assumptons have been standard:

a. All contributing errors are normally distributed.

b. All contributing errors are independent of each other.

c. Except in one special case, all of the transfer functions which relate the contributing errors to the performance parameters are linear over the ranges of the errors. The special case is associated with the outage effect, characteristic of a liquid bipropellant system.

While investigation into the validity of all the preceding assumptions is desirable, this study deals only with the special case in Assumption c above. Therefore all the other assumptions are considered valid in this report. The special case combines three of the contributing errors to form a highly skewed distribution. This distribution often contributes more to the total performance error of a stage than all of the other contributing errors combined. Thus the distribution of all of the combined errors can be considerably skewed. Until now, this combined distribution has not been investigated adequately. The specific objective of this study is to determine certain distribution characteristics of combined performance errors. The important characteristics determine the probability of exceeding a certain maximum allowable adverse error or the adverse error which will not be exceeded with a particular allowable probability. Also for this purpose, the variances given for the contributing errors are assumed to be population variances estimated with sufficient confidence so that the probabilities computed from them are conservative.

SECTION II

THE OUTAGE FUNCTION

Outage is defined only in reference to a bipropellant. It is the usable weight of one propellant component (fuel or oxidizer) left in the tank when the usable portion of the other component is completely consumed. Outage degrades the stage performance by increasing the burnout weight and decreasing the consumed propellant. Errors in the amounts of fuel and oxidizer loaded, and in the predicted oxidizer/fuel burning ratio cause outage.

The basic outage function has been described in previous reports (e.g., Refs. 1, 2, 3, and 5). However, to derive the function here should aid in explanation. All quantities will be defined as the derivation progresses. Definitions are also given in the symbols section in the report front matter.

The loaded ratio R_L (or mixture ratio loaded) is the ratio of the weight of oxidizer W_o to weight of fuel W_f loaded on the stage and usable. The term "usable" excludes propellant trapped in the tanks or lines or otherwise wasted, even when no outage exists. A useful intermediate parameter in this analysis is λ, which is the ratio of R_L to R_b, where R_b is the oxidizer/fuel burning ratio averaged over the burning time. R_b is often called the mixture ratio burned.

$$\lambda = \frac{R_L}{R_b} = \frac{W_o}{R_b W_f} \qquad (1)$$

It is further useful to define a parameter S so that $S = \lambda - 1$. Then

$$\frac{W_o}{R_b} = W_f + W_f S$$
$$W_o = R_b W_f + R_b W_f S \qquad (2)$$

The actual amount of outage must be specified differently in two different regions. When the loaded ratio equals the burning ratio, no outage exists. At this point, $\lambda = 1$ and $S = 0$. When the loaded ratio is less than the burning ratio, an excess of fuel (a deficiency of oxidizer) is loaded and a fuel outage occurs. From Eq. (1) observe that $\lambda < 1$ and S is negative for this fuel outage situation. Similarly, when S is positive ($\lambda > 1$) an oxidizer outage occurs. Separate expressions are required for fuel outage and oxidizer outage. Fuel outage is equal to the usable fuel minus the fuel consumed by combining with the usable oxidizer. Similarly, oxidizer outage is equal to the usable oxidizer minus the oxidizer which combines with the usable fuel. Hence, if one uses Eq. (2)

$$W_{zf} = W_f - \frac{W_o}{R_b} = - W_f S \quad \text{when } S \le 0 \tag{3a}$$

$$W_{zo} = W_o - R_b W_f = R_b W_f S \quad \text{when } S \ge 0 \tag{3b}$$

where W_{zf} is the fuel outage and W_{zo} is the oxidizer outage.

The outage may be more conveniently expressed as a fraction of W_L, the total usable propellant. Z_f and Z_o are the fractional outages for the two cases.

$$W_L = W_f + W_o = W_f(1 + R_b\lambda) = W_f(1 + R_L) \tag{4}$$

$$Z_f = \frac{W_{zf}}{W_L} = - \frac{S}{1 + R_L} \quad \text{when } S \le 0 \tag{5a}$$

$$Z_o = \frac{W_{zo}}{W_L} = \frac{R_b S}{1 + R_L} \quad \text{when } S \ge 0 \tag{5b}$$

Although this outage function of S is nonlinear because of its discontinuous slope at S = 0, the two parts of it are assumed (Assumption c, Section I) to be individually linear within the ranges of the errors. The slopes at S = 0 are chosen as the slopes of the linear segments of the function. These segments are obtained by assuming the loaded ratio R_L and the burning ratio R_b are both constants in Eq. (5) and that they are equal. They are also assumed to be equal to the nominal value of the burning ratio, which is denoted by R_n. Actually, both R_L and R_b vary statistically, and their average values are functions of S. However, their variations from R_n are small and result in correspondingly small errors in outage. Thus the fractional outage function may be written

$$Z_f = -\frac{1}{1 + R_n} S \qquad \text{when } S \leq 0 \tag{6a}$$

$$Z_o = \frac{R_n}{1 + R_n} S \qquad \text{when } S \geq 0 \tag{6b}$$

A plot of a typical outage function is shown in Figure 1, represented by the solid lines.

In the exhaustion shutdown of a stage using pump-fed engines, an additional effect may be encountered as described in detail in Ref. 6. As one component of the propellant is exhausted, the other component is pumped to the engine at an increased rate, so that an extra quantity of the excess component is consumed. Thus the outage is reduced by an amount equal to this extra quantity. An outage that would be less than the extra quantity if the effect were not present, then becomes zero. A compensation is introduced into the analysis to allow for the burning ratio being so far off during the shutdown transient that the specific impulse is significantly reduced. An approximation to the over-all result is shown by the dashed lines of Figure 1. In Ref. 6, for a Titan III core stage 1, δ_f and δ_o were found to be equal, but in the present study they are treated as separate quantities to accommodate situations in which they are unequal.

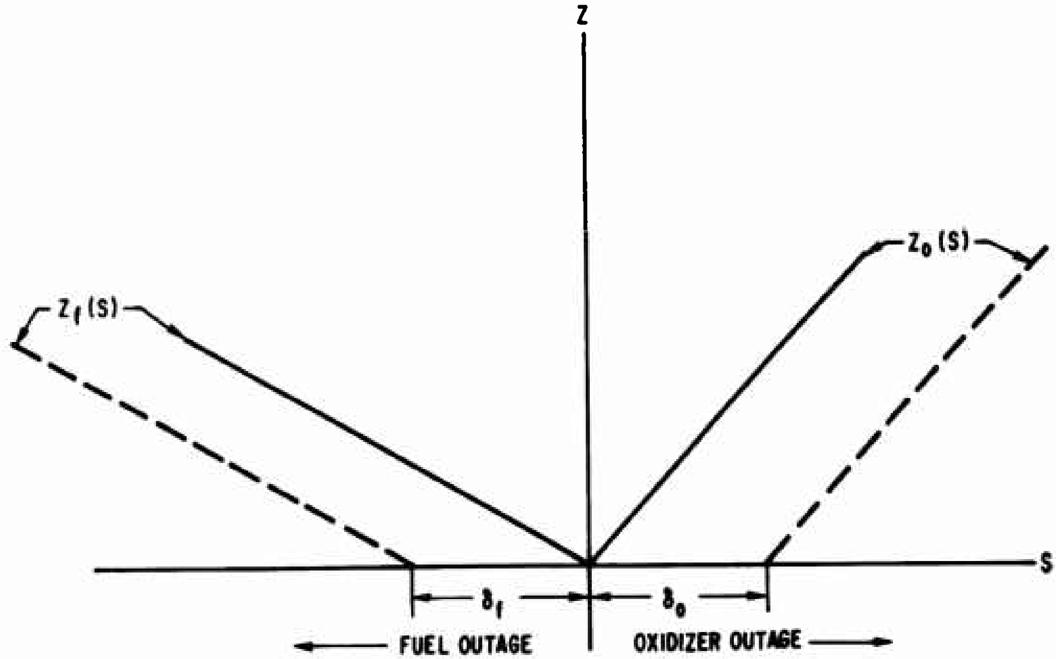

Figure 1. Outage Function

The analytical expression in exhaustion shutdown is a modification of Eq. (6) as

$$Z_f = -\frac{1}{1 + R_n}(S + \delta_f) \qquad \text{when } S \le -\delta_f \qquad (7a)$$

$$Z_o = \frac{R_n}{1+R_n}(S - \delta_o) \qquad \text{when } S \ge \delta_o \qquad (7b)$$

$$Z_f = Z_o = 0 \qquad \text{when } -\delta_f \le S \le \delta_o \qquad (7c)$$

For convenience, Eq. (7) will be used for all stages whether the exhaustion effect occurs or not. When the pump-fed exhaustion phenomenon does not occur, δ_f and δ_o will simply be set equal to zero, which will make the equation the same as Eq. (6).

SECTION III

THE OUTAGE DISTRIBUTION

If S is characterized by a probability distribution which has a density function $f_s(S)$, then the quantity Z which is a monotone function of S in a given interval will be characterized by a density function in that interval wherever the derivative exists according to (see Figure 2)

$$f_z(Z)\ |dZ| = f_s(S)\ |dS| \tag{8a}$$

$$f_z(Z) = \left|\frac{dS}{dZ}\right| f_s(S) \tag{8b}$$

Multiplication by the differentials simply compensates for the change in scale so that the integral of $f_z(Z)$ over any subinterval in Z will be equal to the integral of $f_s(S)$ over the corresponding subinterval in S. Thus, from Eqs. (7) and (8)

$$f_f(Z) = (1 + R_n)\ f_s(S) \qquad \text{when } S < -\delta_f \tag{9a}$$

$$f_o(Z) = \frac{1 + R_n}{R_n}\ f_s(S) \qquad \text{when } S > \delta_o \tag{9b}$$

where f_f and f_o represent the contributions to the density function from fuel outage and oxidizer outage. A density function does not exist for Z corresponding to the region between $-\delta_f$ and δ_o for S. Instead, there is a discrete probability that $Z = 0$. It is equal to the probability that S lies between $-\delta_f$ and δ_o. Solving Eq. (7) for S and substituting into Eq. (9) gives

$$f_f(Z) = (1 + R_n)\ f_s\ [-\delta_f - (1 + R_n)\ Z_f] \quad \text{when } Z_f > 0 \tag{10a}$$

$$f_o(Z) = \frac{1 + R_n}{R_n}\ f_s \left[\delta_o + \frac{1 + R_n}{R_n}\ Z_o\right] \quad \text{when } Z_o > 0 \tag{10b}$$

The restrictions on Z correspond to the restrictions on S in Eq. (9).

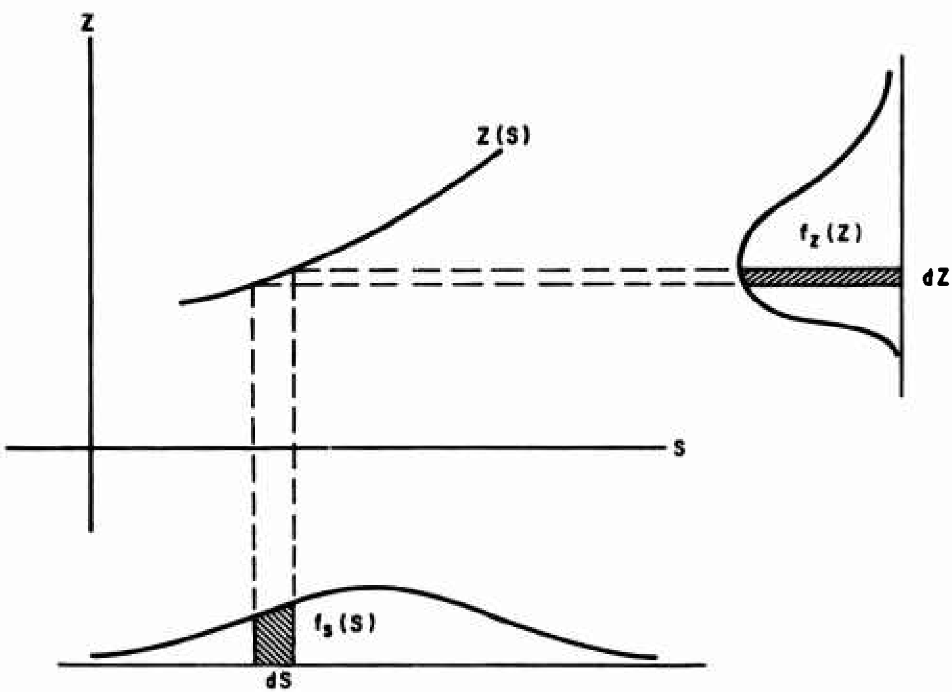

Figure 2. Transformation of Density Function

Now it is possible to write an analytical expression for the density function of Z in terms of the density function of S. It is a hybrid continuous and discrete density function.

$$f_z(Z) = f_f(Z) + f_o(Z) \qquad \text{when } Z > 0 \tag{11a}$$

$$F_z(Z) = \int_{-\delta_f}^{\delta_o} f_s(S)\, dS \qquad \text{when } Z = 0 \tag{11b}$$

$$f_z(Z) = 0 \qquad \text{when } Z < 0 \tag{11c}$$

where the functions f_f and f_o are given in Eq. (10). Since both these expressions cover the same region of Z, their probability densities are additive in that region. F_z is the discrete probability that $Z = 0$. Eq. (11c) is consistent with negative outage being meaningless.

The density function $f_s(S)$ is obtained by assuming that the errors in W_o, W_f, and R_b are all normally distributed. If the logarithms of both sides of Eq. (1) are differentiated at specific values of the variables, the result is

$$\frac{d\lambda}{\lambda} = \frac{dW_o}{W_o} - \frac{dR_b}{R_b} - \frac{dW_f}{W_f} \tag{12}$$

Since the fractional errors are thus additive, and since they are independent according to Assumption b, the means and variances of the fractional errors are also additive (Ref. 7). And since $\lambda = 1$ at zero error, the absolute error in λ is the same as the fractional error in λ. This means that S, which is equal to the absolute error in λ, is also equal to the fractional error in λ. Therefore, the variance of S is equal to the sum of the variances of the fractional contributing errors.

$$\sigma_S^2 = \sigma_{W_o}^2 + \sigma_{R_b}^2 + \sigma_{W_f}^2 \tag{13}$$

Also, the distributions of the contributing errors are normal according to Assumption a. Since the sum of independent normally distributed variables is normally distributed, S is normally distributed.

$$f_s(S) = \frac{1}{\sigma_s\sqrt{2\pi}} e^{-\frac{(S-\beta)^2}{2\sigma_s^2}} \qquad (14)$$

Above, β is an intentional bias on the quantity S to make one type of outage more likely than the other. Normally the weight of one component of the propellant is much less than the corresponding weight of the other component with which it combines. Thus an outage in one direction causes more penalty to the system than an outage in the other direction caused by the same magnitude of error. The error is intentionally biased to favor an outage of one of the components, usually the fuel, according to some criterion. If possible this criterion will minimize outage effects. Methods of arriving at a desirable bias are discussed in Section VI.

If the expression in Eq. (14) is substituted into Eqs. (10) and (11), an analytical expression is obtained for the distribution of outage. It is a highly skewed distribution which looks something like that shown in Figure 3. Its mean and variance may be computed by taking the first and second moments of the density function as

$$\overline{Z} = \int_0^\infty Z\, f_z(Z)\, dZ \qquad (15)$$

$$\sigma_z^2 = \overline{Z}^2\, F_z(0) + \int_0^\infty (Z - \overline{Z})^2\, f_z(Z)\, dZ \qquad (16)$$

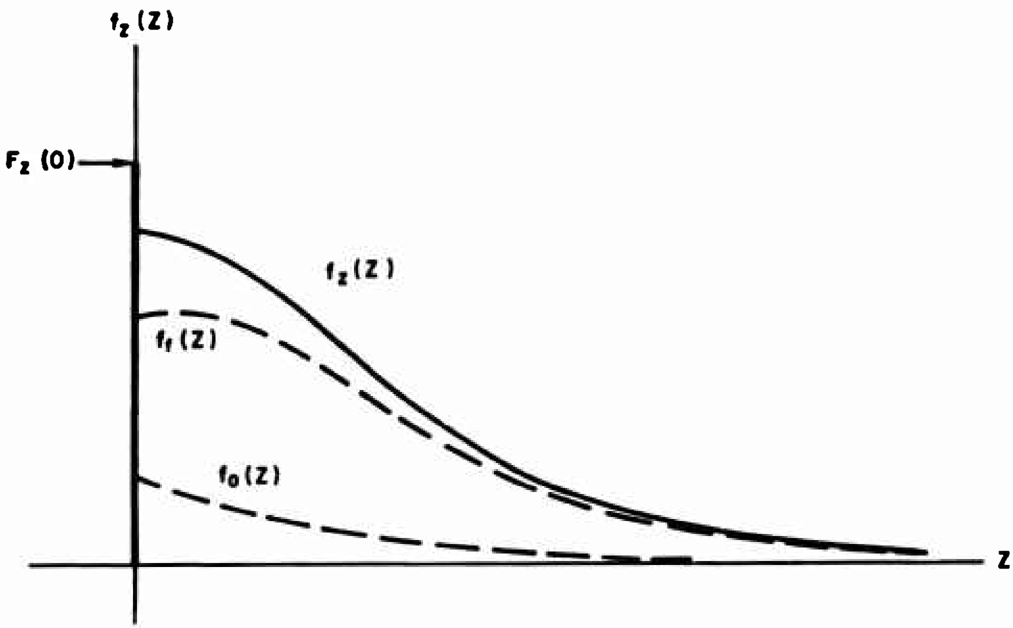

Figure 3. Outage Distribution

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SECTION IV

THE COMBINED DISTRIBUTION FOR ONE STAGE

The effects of contributing errors on rocket performance are determined as perturbations in a trajectory simulation. The results are a variation in some performance parameter caused by each contributing error. The performance parameters most commonly chosen are (1) propellant margin, which is the excess propellant at burnout of the last stage when the mission conditions have been met, and (2) velocity margin, which is the excess velocity over the mission conditions, which could be attained if all of the propellant were burned. Since the transfer functions used in the trajectory simulation are assumed to be linear within the ranges of the errors (Assumption c), the performance parameter variations have distributions that are the same, except in scale, as the distributions of the errors which caused them. If outage is treated as a contributing error in place of the three errors which contribute to it, this situation then holds for all contributing errors. Since the errors other than outage are independent and normally distributed, their combined effect on the performance parameter is also normally distributed with a variance that is the sum of the variances of the contributing effects on the performance parameter. The means of the contributing effects may be considered zero relative to nominal performance. If the effect on the performance parameter by the other errors is called W, then the frequency function of W is

$$f_w(W) = \frac{1}{\sigma_w\sqrt{2\pi}} e^{-\frac{W^2}{2\sigma_w^2}} \qquad (17)$$

If X is the total effect of all the errors in a single stage on the performance parameter of that stage, then the frequency function of X is

$$f_x(X) = \int_{-\infty}^{\infty} f_w(X-\nu)\, f_z(\nu)\, d\nu \tag{18a}$$

$$f_x(X) = F_z(0)\, f_w(X) + \int_{0}^{\infty} f_w(X-\nu)\, f_z(\nu)\, d\nu \tag{18b}$$

where ν is a dummy variable. The first form of this equation corresponds to Eq. (15.12.4) of Ref. 7. The second form breaks f_z into its three component parts. The first term on the right hand side accounts for part of the distribution of Z being discrete. It is simply the probability that Z = 0 multiplied by the distribution of X when Z = 0, which is the distribution of W. The lower limit of zero in the integral accounts for the f_z function being zero when the argument is negative.

Since the outage effect is independent of the other effects; and since the mean of the other effects is taken as zero

$$\overline{X} = \overline{Z} \tag{19}$$

$$\sigma_x^2 = \sigma_z^2 + \sigma_w^2 \tag{20}$$

The important characteristics of the distribution of X are those which determine the probability of not exceeding a certain adverse error or the adverse error which will not be exceeded with a certain probability. Only the regions of high probability, in which the errors are several times the standard deviation of the distribution, are of interest. A plot of the entire density curve would provide little useful information. Another function, P_α, is defined as

$$P_\alpha = \int_{\overline{X}+\alpha\sigma_x}^{\infty} f_x(X)\, dX \tag{21}$$

In other words, P_α shows the probability that the error differs from the mean error by more than α times the standard deviation. A computer program has been written to compute P_α as a function of R_n, σ_s, β, δ_f, δ_o, σ_w, and α. An example of a set of curves of P_α versus α is shown in Figure 5.

Although it is possible to integrate Eq. (18) analytically, as shown in Refs. 5 and 8, the integrand must be split into several parts and the limits become very complicated functions of the parameters. If hand computation is necessary, this method may be easier because the resulting functions are of the form of the normal distribution function and can be found in tables. However, for a machine, numerical integration is easier. Even Eq. (11b), which can be evaluated simply by looking up the limits in the tables of the normal distribution function, can be evaluated more easily by numerical integration when a machine is used. In the computer program, each value of the integrand in the numerical integration of Eq. (21) is evaluated by numerically integrating Eq. (18), and each value of the integrand in that numerical integration is evaluated by using Eqs. (17), (11) and (10).

The values of R_n and σ_s are available from the error analysis of the vehicle, such as in Ref. 9. In this case, the criterion for choosing β has already been selected, so that β is also available from the report. Thus far, values of δ_f and δ_o have been determined only for Titan III Stage 1 as given in Ref. 6. The information is not included in the official error analysis of any of the Titan III series vehicles, and so the figures given in Ref. 6 are just assumed to be representative of all Titan III first stages. For Titan III, the value of σ_w may be found as a multiple of σ_z by noting the relative effects of the two types of errors on the final performance of the vehicle. This information may be obtained from the error perturbations in a trajectory analysis such as those presented in Ref. 10, Table III. Thus, the value of the ratio of σ_w to σ_z will normally be entered into the computer program to determine the error distribution when trajectory analysis is available. Once the value of σ_w has been determined, the values of the other parameters may then be varied with σ_w held constant. The remaining parameter α is used as the independent variable for plotting P_α. Usually values of α from 1.5 to 5 will cover the useful range.

Notice that the above analysis for a single stage is useful only when a particular stage has a requirement to meet certain performance standards independently. Otherwise, the stages in a multistage vehicle must be considered together as shown in Section V.

SECTION V

THE MULTISTAGE CASE

When more than one bipropellant stage is used in a rocket, each such stage has its own outage distribution which affects the performance parameter. In addition for range safety, it is sometimes necessary to command a stage shutdown at a certain maximum level of performance. Therefore, the distribution of performance errors caused by each stage must be determined separately and then combined with those caused by the previous stages.

To combine the distributions of errors from the different stages, the error variables must be converted to a common scale in the final-stage performance parameter which they affect. Eq. (18) must be modified accordingly as

$$f_{xi}(X_i) = F_{zi}(0)\, K_i f_{wi}(K_i X_i) + K_i^2 \int_0^{\infty} f_{wi}\,[K_i(X_i - \nu)] f_{zi}(K_i \nu) d\nu \qquad (22)$$

where X_i is the contribution to the performance error by the i^{th} stage. This modification simply scales the f_{wi} and f_{zi} functions so that their forms remain the same except the scaling; their standard deviations become σ_{wi}/K_i and σ_{zi}/K_i; and the mean of the f_{zi} function becomes $\overline{Z}i/K_i$. The mean of the f_{wi} function remains zero. K_i is obtained by dividing σ_{zi} by the standard deviation of the final-stage performance error caused by outage in the i^{th} stage. This figure is available from the trajectory perturbation analysis. If any stage is preceded by stages not subject to outage, the performance errors caused by these preceding stages may simply be combined with those caused by non-outage effects in the bipropellant stage, for they are assumed to be normally distributed.

First, Eq. (22) determines the distribution of errors caused by the first bipropellant stage. If this stage is to have a command shutdown, the distribution of performance errors caused by this stage is truncated at a point which corresponds to the level of performance ξ_1 at which the stage is shut down. The density function beyond that point in performance is integrated to determine the probability that the shutdown performance or better could be achieved if shutdown did not occur. This number then represents the discrete probability of occurrence of ξ_1 when shutdown does occur. It is combined with the portion of the density function on the low performance side of ξ_1 to form a new hybrid continuous and discrete density function, $f^*_{x1}(X_1)$.

$$f^*_{x1}(X_1) = f_{x1}(X_1) \qquad \text{when } X_1 > \xi_1 \tag{23a}$$

$$F^*_{x1}(X_1) = \int_{-\infty}^{\xi_1} f_{x1}(X_1)\, dX_1 \qquad \text{when } X_1 = \xi_1 \tag{23b}$$

$$f^*_{x1}(X_1) = 0 \qquad \text{when } X_1 < \xi_1 \tag{23c}$$

where $f_{x1}(X_1)$ is given by Eq. (22) and F^*_{x1} is the discrete probability that $X_1 = \xi_1$. The shutdown performance may be conveniently expressed as a deviation C_1 from the mean of X_1 in multiples of the standard deviation of X_1.

$$\xi_1 = \overline{X}_1 + C_1 \sigma_{x1} \tag{24}$$

where $\overline{X}_1$ and σ_{x1} are found by modifying Eqs. (19) and (20) as

$$\overline{X}_1 = \frac{\overline{Z}_1}{K_1} \tag{25}$$

$$\sigma_{x1}^2 = \frac{\sigma_{z1}^2}{K_1^2} + \frac{\sigma_{w1}^2}{K_1^2} \tag{26}$$

The deviation of shutdown performance from mean performance is given as a multiple of the standard deviation for convenience if the information is available in that form and also to provide an indication of how much of the density curve is being cut off. Usually, the available information is in the form of the actual deviation, which is the full quantity $C_1\sigma_{x1}$.

The next function that must be defined is the distribution of combined performance errors caused by the first i stages. It is called $f_{ti}(T_i)$, where T_i is the total performance error caused by the first i stages. If the i^{th} stage is to have a command shutdown, the f_{ti} function is truncated as the f_{x1} function was truncated so that a new hybrid function f^*_{ti} is formed. Notice that

$$f^*_{t1}(T_1) \equiv f^*_{x1}(X_1) \qquad \text{for all } T_1 \equiv X_1 \tag{27}$$

Eq. (23) may now be written as a form that applies to the sum of the errors caused by the first i stages.

$$f^*_{ti}(T_i) = f_{ti}(T_i) \qquad \text{when } T_i > \xi_i \tag{28a}$$

$$F^*_{ti}(T_i) = \int_{-\infty}^{\xi_i} f_{ti}(T_i)\, dT_i \qquad \text{when } T_i = \xi_i \tag{28b}$$

$$f^*_{ti}(T_i) = 0 \qquad \text{when } T_i < \xi_i \tag{28c}$$

The expression $f_{ti}(T_i)$ is obtained by combining the error distribution for the i^{th} stage with the truncated distribution for the sum of the previous stages.

$$f_{ti}(T_i) = F^*_{t(i-1)}(\xi_{i-1})\, f_{xi}(T_i - \xi_{i-1}) + \int_{\xi_{i-1}}^{\infty} f_{xi}(T_i - \nu)\, f^*_{t(i-1)}(\nu)\, d\nu \tag{29}$$

This equation has the same form as Eq. (18), which was used to combine the two distributions of errors within a stage. As before, the lower limit of integration simply accounts for the f^*_{ti} function being zero when the argument is less than ξ_i. Also, since the distributions are independent

$$\overline{T}_i = \overline{T}^*_{i-1} + \overline{X}_i \tag{30}$$

$$\sigma^2_{ti} = \sigma^{*2}_{t(i-1)} + \sigma^2_{xi} \tag{31}$$

and, as in Eqs. (15) and (16)

$$\overline{T}^*_i = \xi_i F^*_{ti}(\xi_i) + \int_{\xi_i}^{\infty} T_i f_{ti}(T_i)\, dT_i \tag{32}$$

$$\sigma^{*2}_{ti} = (\overline{T}^*_i - \xi_i)^2 F^*_{ti}(\xi_i) + \int_{\xi_i}^{\infty} (T_i - \overline{T}^*_i)^2 f_{ti}(T_i)\, dT_i \tag{33}$$

If the rocket has N stages, $f_{tN}(T_N)$ is the distribution of all the combined errors for the entire rocket. As in single stages, P_α is defined as

$$P_\alpha = \int_{\overline{T}_N + \alpha\sigma_{tN}}^{\infty} f_{tN}(T_N)\, dT_N \tag{34}$$

The truncated form of the error distribution is used in the general case for every stage, and when a command shutdown is not used, ξ is simply set at $-\infty$ or the computer is told to ignore the truncation process. Note that the exhaustion shutdown effect described in the latter part of Section II still applies to the case of a command shutdown. Exhaustion occurs whenever shutdown performance is not achieved, and the performance obtained in this situation can be significantly affected by the exhaustion effect occurring in some engines. However, data on the exhaustion effect may sometimes not be available for engines shut down by command.

In the multistage case, the only feasible method of computation is numerical integration. The process is similar to that of the single stage except there are many more levels of numerical integration. For example, in evaluating the expression for $f_{ti}(T_i)$, each value of the integrand is evaluated by numerically integrating the f_{xi} function for the same stage and the f^*_{ti} function for the previous stage.

With two additions, the multistage parameters are the same for each stage as those used in the single stage. The scale factors K_i are included by entering the ratio of σ_{zi} to K_i into the program. This ratio is obtained from the trajectory perturbation analysis. The shutdown performance level is also included in the trajectory analysis, so that the difference between nominal performance and shutdown performance is easily determined. Now that the multistage computer program is available, it may be used for the single stage by setting $N = 1$.

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SECTION VI

THE CHOICE OF BIAS

Bias is an intentional deviation of the loaded ratio from the nominal value of the burning ratio, so that the mean value of λ is different from 1 and the mean value of S is different from zero. A desirable bias increases the expected outage of the lighter component of propellant and diminishes the expected outage of the heavier component.

The following criteria for optimizing bias have been suggested in previous studies, and methods for using them to optimize bias have been formulated:

a. Minimum mean outage
b. Minimum mean square outage (minimum variance about Z = 0)
c. Maximum probability that outage is less than a given value
d. Minimum value of outage which is not exceeded with a given probability
e. Maximum probability that performance is greater than a given value
f. Maximum value of performance which is exceeded with a given probability

Criterion a was used only for comparison with other criteria in Refs. 2 and 4. Criterion b has had the widest use thus far. It is described in Refs. 1, 2, and 6. Both of these criteria have the advantage that they carry over from the outage variable to the performance variable in a single stage. Eq. (19) shows that minimum mean outage is equivalent to minimum mean performance degradation or maximum mean performance. Also, the variance of X about zero is equal to the square of the mean plus the variance about the mean.

$$\overline{X^2} = \overline{X}^2 + \sigma_X^2 \qquad (35)$$

Thus, from Eqs. (19) and (20)

$$\overline{X^2} = Z^2 + \sigma_z^2 + \sigma_w^2 = \overline{Z^2} + \sigma_w^2 \quad (36)$$

Since the variance of W is independent of the bias, σ_w^2 is a constant in Eq. (36) when the bias is varied. Therefore, when $\overline{Z^2}$ is minimized, $\overline{X^2}$ is minimized, and the mean square of the performance variable is maximized. In spite of this advantage, Criterion b is not a desirable criterion because it does not necessarily maximize the probability of mission success.

Criteria c and d are related only to outage and do not consider performance to optimize bias. They may be used where the outage error alone must be held below a certain level. They are also useful as bounds in Criteria e and f as shown below.

Since the assurance of mission success is why errors and penalties are investigated in the first place, the only really meaningful criteria are e and f. Usually, and especially in a man-rated mission, an acceptable probability of success is chosen, and the payload and mission requirements are tailored to fit that probability. In this case, if we use the specified probability of success, Criterion f would optimize the bias. However, sometimes performance for a specific mission may be critical and the decision may be to maximize the probability of achieving the required performance. In this case, we employ Criterion e to optimize the bias and use the assigned performance level. Note that if the probability obtained by Criterion e is equal to that assigned for Criterion f, the two criteria are equivalent, and the optimum bias is the same. However, if these probabilities differ significantly, then the optimum biases may also considerably differ.

Criteria c and d are similar to e and f but concern themselves only with outage. However, they do have a useful application, and they have the advantage that the optimum bias is very easy to compute. Formulas are derived in Ref. 5, but the exhaustion effects (δ terms) are not included. For an alternate derivation, note in Figure 4 that when the outage is less than some specific value Z_m the value of S lies within a corresponding interval from S_1 to S_2. The probability that the outage is less than the indicated value is equal to the integral of $f_s(S)$ from S_1 to S_2. Since $f_s(S)$ is a normal density function, this integral is a maximum when the mean lies halfway between S_1 and S_2. This situation must hold for either Criterion c, when the value of Z_m is specified, or for Criterion d, when the value of the integral is specified. Thus, if one uses values of S_1 and S_2 obtained from Eq. (7)

$$\beta_o = \frac{1}{2}(S_1+S_2) = \frac{1}{2}\left[-\delta_f - (1+R_n)Z_m + \delta_o + \frac{1+R_n}{R_n}Z_m\right]$$

$$\beta_o = \frac{1}{2}\left(\frac{1}{R_n} - R_n\right)Z_m + \frac{1}{2}(\delta_o - \delta_f) \tag{37}$$

where β_o is the optimum bias. If Criterion c is used, so that Z_m is specified, the final expression of Eq. (37) is all that is needed to find the optimum bias. When δ_o and δ_f are equal, the last term of the expression is zero.

If Criterion d is used, so that the probability of success is specified, the length of the interval from S_1 to S_2 will be determined as a function of the probability of success. From the tables of the normal distribution, a multiple A of the standard deviation of $f_s(S)$ can be found such that the integral of the function from $-A\sigma_s$ to $+A\sigma_s$ will be equal to the required probability. Thus the probability is specified by specifying A. Then

$$2A\sigma_s = S_2 - S_1 = \delta_o + \frac{1+R_n}{R_n}Z_m + \delta_f + (1+R_n)Z_m \tag{38}$$

Solving for Z_m

$$Z_m = \frac{2A\sigma_s - \delta_o - \delta_f}{(1+R_n)^2}R_n \tag{39}$$

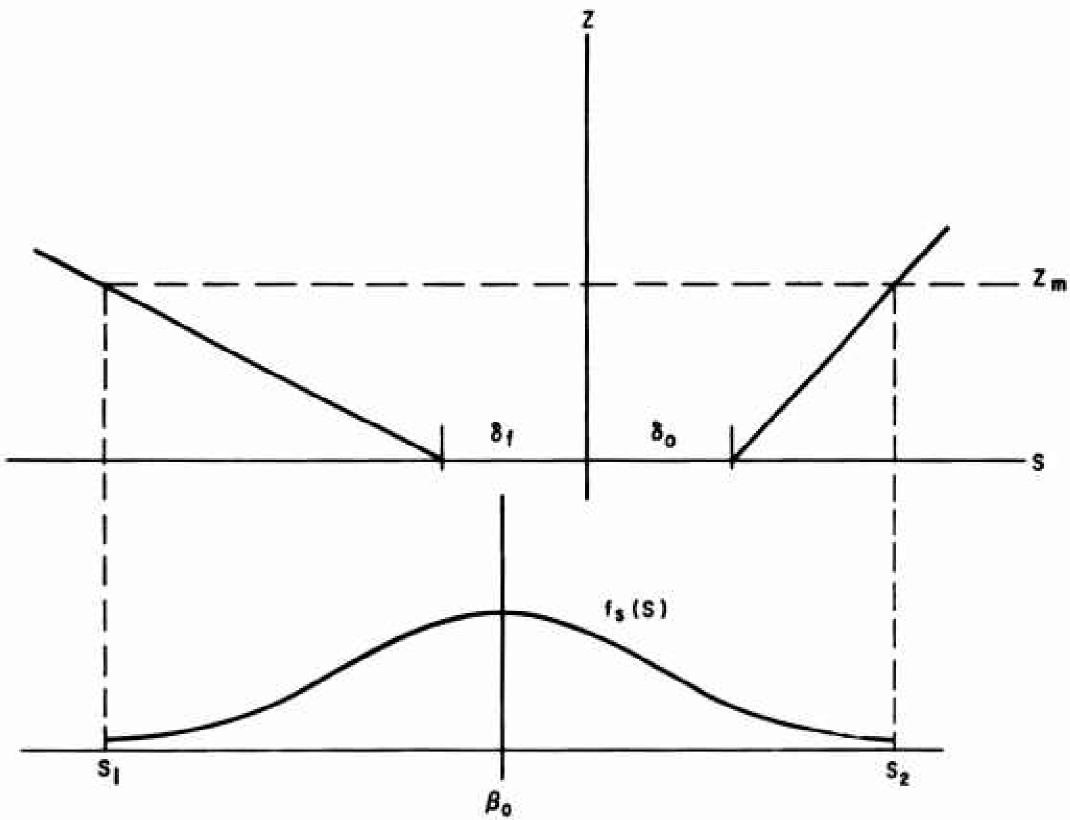


Figure 4. Distribution of S

and substituting into Eq. (37)

$$\beta_o = \frac{1 - R_n}{1 + R_n} (A\sigma_s - \delta) + \frac{1}{2} (\delta_o - \delta_f) \qquad \text{when } A\sigma_s \geq \delta$$

$$\beta_o = \frac{1}{2} (\delta_o - \delta_f) \qquad \text{when } A\sigma_s \leq \delta \tag{40}$$

where δ is the average of δ_o and δ_f. Once again, when the deltas are equal the last term is zero. When A is specified, this expression determines the optimum bias.

One useful application of Criteria c and d is the location of bounds for determining the bias by Criteria e and f. First we note the two extremes of the shape of the probability density function of the total error $f_x(X)$ [determined by Eq. (18)]. One extreme occurs when $\sigma_w = 0$, and the entire error is just the outage error. In this case, the density function is highly skewed, looking like the example in Figure 3, and Criteria c and d are the same as Criteria e and f for optimizing the bias. At the other extreme, the outage makes a negligible contribution toward the total error and the density function is characterized by a normal distribution with zero mean. As this limit is approached, the contribution of outage to the variance of the total error approaches the mean of the square of outage (from Eqs. (35) and (36) with $\overline{X}^2$ approaching zero). Thus minimizing the mean square outage (Criterion b) approaches equivalence with minimizing the variance (and thus the standard deviation) of a normally distributed total error. This situation is exactly equivalent to minimizing any given multiple of the standard deviation of this same total error and thus minimizing the allowable error which is not exceeded with any given probability (as long as the probability exceeds 0.5). Therefore, in this extreme, Criterion b approaches equivalence with Criterion f, and thus also with Criterion e, for any assigned probability or performance level.

Actually, the shape of the total error density function lies somewhere between the extremes mentioned above. For example, the P_α curves in Figure 5 show a gradual variation as σ_w is varied for a constant σ_z. Thus the optimum bias according to Criterion e or f lies somewhere between that obtained by Criterion b and that obtained by Criterion c or d. The probability used for Criterion d must of course be the same as that to be used for Criterion f. If Criterion e is to be used, an assigned value of outage for Criterion c is more difficult to choose. The value should be one which results in about the same probability of success as that expected from Criterion e. Since accuracy is not important for these bounds, a guess should be adequate. If the probability of success involved in these criteria is greater than about 0.7, Criterion b will provide the lower bound and Criterion c or d will furnish the upper bound for the optimum bias.

Criterion e has been discussed in Ref. 5 for a single stage, including the derivation of an implicit expression for determining the optimum bias. However, the expression is very complex and must be solved by trial and error for each assigned value of the total error. Also, it does not include exhaustion effects and is useful only in a single stage. The straightforward approach to bias optimization is to try different values of the bias and see which value best meets the desired criterion. The computer program which calculates P_α has been modified so that it can be used for this purpose. Only the multistage program has been modified because it is more convenient even for the single stage case (N = 1). In the form used for the multistage analysis, scale factors are introduced which make the total error $\overline{X} + \alpha\sigma_x$ equal to the actual decrement in the chosen measure of performance. The original single-stage program did not provide for these scale factors.

To optimize the bias in a single stage, a set of possible bias values spanning the region between the bounds determined as described above, is set into the program. Additional inputs are a set of levels of the performance decrement $\overline{X} + \alpha\sigma_x$, or a set of values of α which will generate a set of values of $\overline{X} + \alpha\sigma_x$ if the first value of β in the set is used. In either case, the performance decrement is held fixed at each value and used as the lower limit of integration in Eq. (34) to compute P_α for all values of β. Thus a set of curves of P_α versus β may be plotted for the different values of the performance decrement. An example is shown in Figure 7. On each curve, the minimum P_α represents the maximum probability of achieving the performance represented by that curve. If the optimum β for each curve is plotted against the performance decrement for that curve, as in Figure 8, the optimum bias for any assigned level of performance (Criterion e) can be found by simply reading it off the curve.

Criterion f minimizes the performance decrement that is exceeded with the probability P_α, for P_α is the probability of being beyond a certain level of adverse error. If one observes the curves of Figure 7, he can see for a given P_α the minimum performance decrement is the one whose curve is just tangent to the horizontal line representing that P_α. For a lesser performance decrement, the given P_α cannot be achieved. Since the point of tangency is the minimum point, the β at the minimum point minimizes the performance decrement for the P_α at the same minimum point. Therefore, the P_α at the minimum points are plotted against the β at the same minimum points as shown by the broken line in Figure 7. The optimum bias for any assigned probability (Criterion f) can be found by simply reading it off this curve. At the time of the analysis, if it is not known whether Criterion e or Criterion f is to be used, both curves (Figure 8 and the broken line in Figure 7) may be plotted.

In multistage rockets the problem is more complex. The optimum bias for each stage is not independent of the other stages. An optimum set of biases must be found by comparing how different sets of biases affect rocket performance. Two versions of this approach have been used in the two-stage Gemini Launch Vehicle. In the earlier method (Ref. 4), the means and variances of the nonnormal outage distributions are computed and then simply combined with those of the normal distribution of other errors. The combined distribution is assumed to be normal. The bias is optimized by minimizing the value of the mean plus three sigma with no knowledge of the actual probability of success. In the later method, effectively the same thing is done, but the actual probability is found by combining the distributions. The combination is done by a method similar to that described in this report, but the numerical integrations are done by hand, and the results are not very accurate. Also, the criterion thus far is not a valid one, for neither the probability nor the performance is set at a desired level. The result is still three sigma, but the probability of success is not that associated with three sigma for a normal distribution (0.99865). Minimum performance and the probability associated with it are both accepted at whatever values they turn out to be. The process would have to be repeated for different multiples of sigma to use Criterion e or f. Also, how exhaustion shutdown affects the outage distribution has not been considered in the Gemini Launch Vehicle. Since the first stage is essentially the same as that of Titan III series, the effects described in Ref. 6 should still be present.

In the solution of the multistage problem, the bounds computed for the individual stages will still be applicable if none of the stages uses a command shutdown. The bound generated by Criterion b still holds because it maximizes the mean square performance for the entire vehicle when applied to each stage, so that the effects of the individual biases are independent in this case. This fact can be seen from Eqs. (30) and (31), with the asterisks not being applicable because there is no command shutdown. As the number of stages increases, the combined distribution becomes more like a normal distribution according to the central limit theorem. Therefore, the optimum biases become closer to those obtained by Criterion b than they were for the single stages. This puts them even further inside the bounds generated by Criterion c or d than for the single stages. But a command shutdown distorts the combined distribution of all stages up to the shutdown. Because of this distortion, the generalizations from a single stage do not hold. However, the "bounds" computed for the single stages will still provide a rough indication of the regions within which the biases should be investigated.

The procedure for optimizing the multistage biases is simply an extension of that used in the single stage. A set of biases spanning the region of interest is inserted into the program for each stage, and P_α is computed for all combinations of the biases at each level of performance decrement. The results are plotted in whatever way is most convenient to find the over-all minimum P_α, and the combination of biases that resulted in the minimum P_α for each level of performance decrement. An example for a two-stage vehicle at a single level of performance decrement is shown in Figure 9. A curve of P_α versus β_1 is plotted for each β_2. Then the minimum values of P_α for these curves are plotted versus β_2 in Figure 10. The minimum P_α in this curve is the over-all minimum, and the corresponding abcissa is the optimum β_2. As shown, the optimum β_1 is apparent from Figure 9, for it is almost independent of β_2 within the region of values considered. Where the dependence is greater, a curve through the minimum points of the curves of Figure 9 could be drawn. Its minimum point would occur at an abcissa equal to the optimum β_1. As an aid in drawing this curve, notice that its minimum value is the same as that of the curve of Figure 10.

In a three-stage vehicle, this procedure would be carried out for each β_3. Thus a set of sets of curves like those in Figure 9 and a set of curves like the one in Figure 10 would be plotted. The minimum values of P_α for the Figure 10 curves would then be plotted versus β_3 on a separate page. The minimum point of this new plot would be at the optimum β_3. The optimum β_2 would be found from the Figure 10 curves, as the optimum β_1 was found from the Figure 9 curves in the two-stage case. If the optimum β_1 is sensitive to β_2 and β_3, a curve through the minimum points of each set of Figure 9 curves could be drawn, and the minimum points of the resulting curves could then be connected by a single curve. The abcissa of the minimum point of the single curve would be the optimum β_1. The method may similarly be extended to more than three stages.

If more than one level of performance decrement is considered in the bias optimization, as it frequently will be, the procedure described above is carried out for each of a set of levels covering the region of interest. The optimum bias for each stage may then be plotted versus the performance decrement as was done in the single stage. From these curves, if one uses Criterion e, all of the optimum biases may be found for any desired level of performance. Also, the over-all minimum value of P_α for each performance level, obtained most accurately from the single curve used to optimize the bias of the last stage, may be plotted against the optimum bias for each stage. From this set of curves, if one uses Criterion f, all of the optimum biases may be found for any desired probability of success.

SECTION VII

EXAMPLES OF RESULTS

A. SINGLE STAGE CASE

The error distribution in a single stage is investigated for a Titan III first stage. A nominal set of parameters is taken from Ref. 9, except the deltas are taken from Ref. 6. The bias in Ref. 9 was chosen on the basis of minimum mean square outage (Criterion b of Section VI). The effects of a more realistic bias will be shown in later examples of results. Plots of P_α versus α are shown in Figure 5 for a number of values of σ_w. The curve labeled $\sigma_w = \infty$ represents a pure normal distribution. The points on the curve are taken from the tables of the normal distribution function. The computer program described in Section IV was used to compute the points on the other curves. The curve labeled $\sigma_w = 0$ represents a pure outage distribution with no other contributing errors. The value of σ_z turns out to be 0.002127. The value of σ_w will be different for different versions of Titan III, but will generally be about the same as σ_z within a factor of two. Thus, the curve labeled $\sigma_w = 0.002$ is a typical example.

The curves of Figure 5 can be interpreted two ways. The error distributions may be compared with a normal distribution either by choosing a fixed multiple of the standard deviation and comparing the corresponding probabilities of failure or by choosing a fixed probability of failure and comparing the multiples of the standard deviation that correspond to that probability. Since the error analyses to date have almost always used $3\sigma(\alpha=3)$ as the maximum allowable error under the assumption that the distribution is normal, comparisons with this point on the normal distribution curve are taken as a useful example. Note that for the normal distribution, the probability P_α that the error will exceed the mean plus 3σ in the adverse direction is 0.00135 (0.135 percent chance). This value appears to be the probability counted on when a normal distribution is assumed. However, for a nonnormal

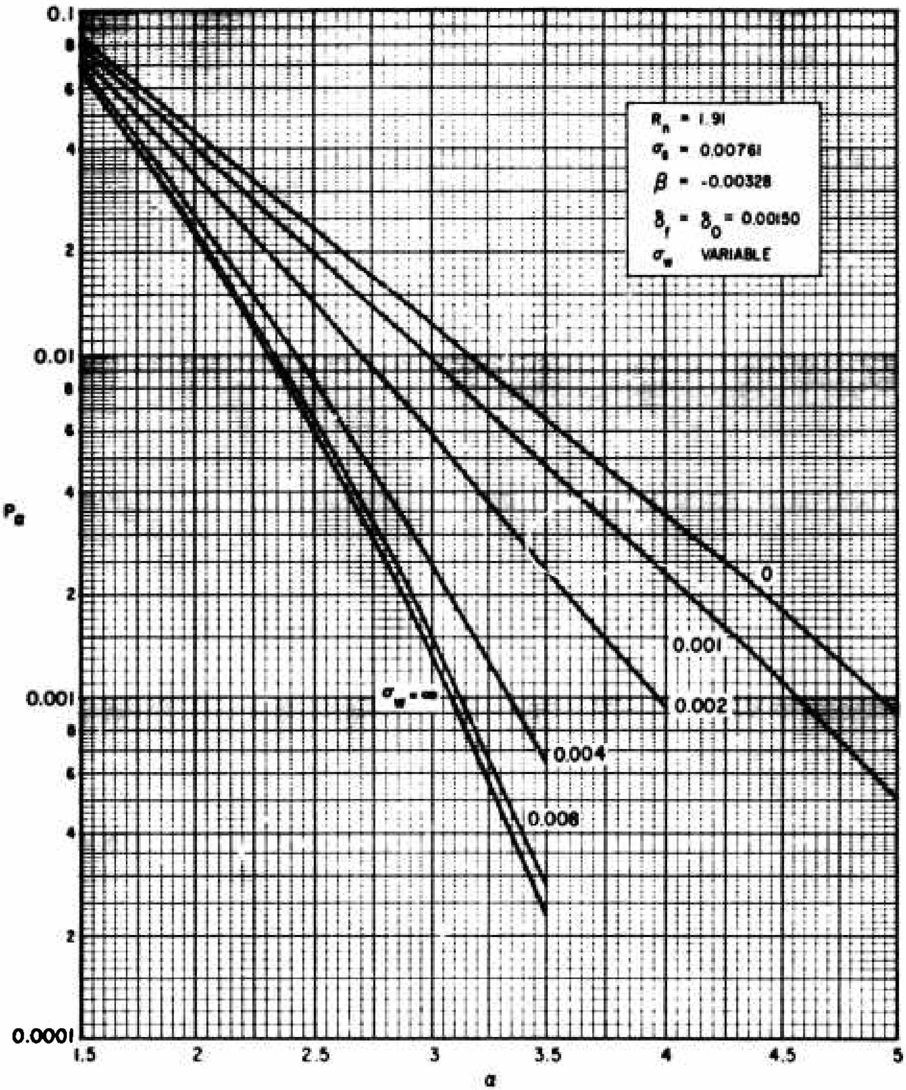


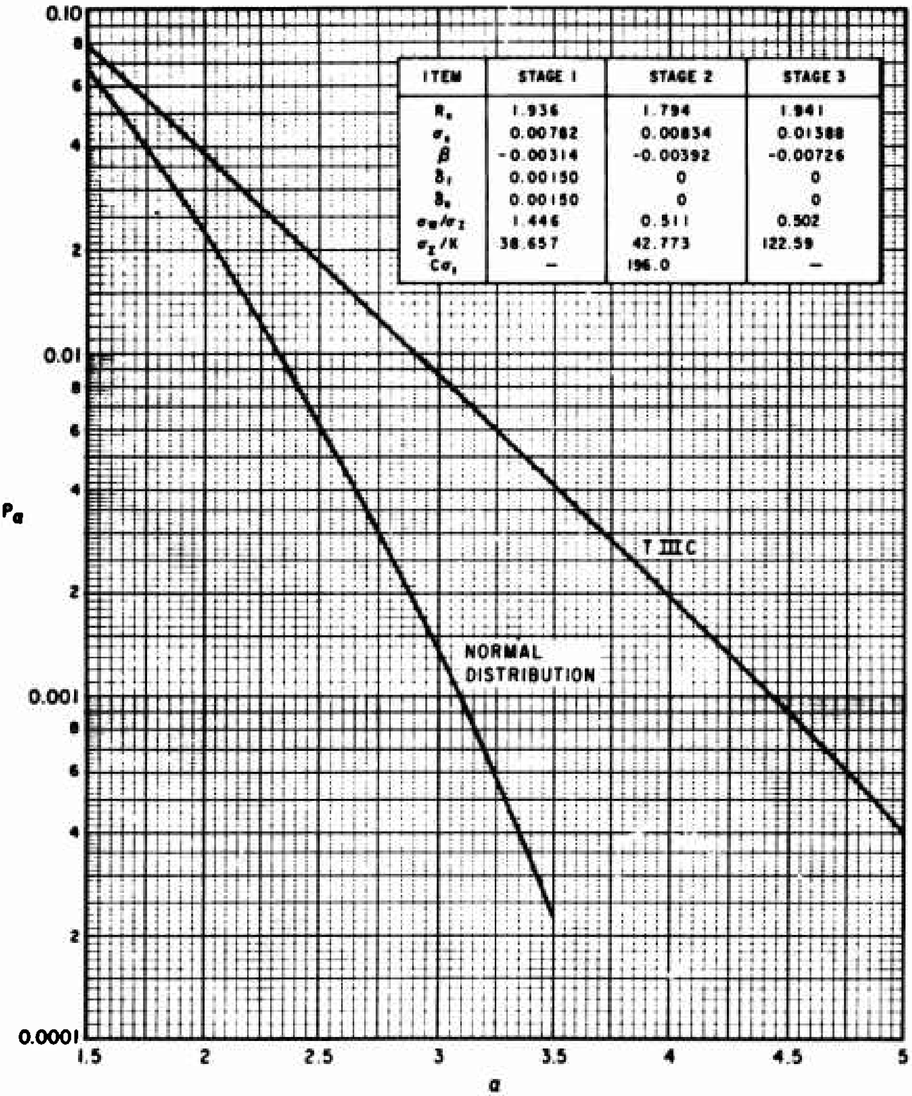
Figure 5. Probability (P_α) Versus Deviation From Mean Error; Titan III, Stage 1

distribution, the probability is different. For example, if σ_w were equal to 0.002, the probability of exceeding the 3σ level would actually be 0.0059 (0.59 percent chance). By our looking at it another way, we see that if the probability usually identified with a 3σ error is required, then the allowable error which corresponds to that probability is actually 3.8σ. The situation may be better or worse than this example depending on the actual value of σ_w. In any event, the use of 3σ as a specified minimum performance level is not justified. A detailed analysis is required in each case to determine the maximum allowable error as a function of probability of failure.

B. MULTISTAGE CASE

The multistage analysis has been used for Titan IIIC, Vehicle 11. Some of the parameters were obtained from the Titan III program office, and the others were taken from Ref. 10. The plot of P_α versus α is shown in Figure 6. Because the second stage is shut down by command at a level of performance far below the nominal level so that the probability of achieving the shutdown level of performance is very high, a single-stage computation for the third stage only was made for comparison with the multistage case. The resulting curve is indistinguishable from that representing the multistage case, which shows that the shape of the distribution of total performance error is the same. Also, the means and variances of the two distributions turn out to be equal within a little more than one-tenth of one percent. Therefore, the performance of the over-all vehicle may be assumed to be independent of the performance of the first two stages when the second stage is shut down at the chosen level of performance.

Since the errors in stage three can be considered independently, the bias for this stage may be optimized separately using the single-stage optimization procedure. Then a two-stage optimization may be used for the first two stages. Also, the criterion for bias optimization of the first two stages is not a matter of choice. One simply maximizes the probability of achieving the shutdown level of performance (Criterion e).



ITEM	STAGE 1	STAGE 2	STAGE 3
R_e	1.936	1.794	1.941
σ_e	0.00782	0.00834	0.01388
β	-0.00314	-0.00392	-0.00726
δ_1	0.00150	0	0
δ_e	0.00150	0	0
σ_M/σ_I	1.446	0.511	0.502
σ_I/K	38.657	42.773	122.59
$C\sigma_1$	—	196.0	—

Figure 6. Probability (P_α) Versus Deviation From Mean Error; Titan IIIC, Vehicle 11

For the third-stage bias optimization, the single-stage procedure outlined in Section VI is used. For the lower bound of the magnitude of β, -0.00726 is used. It is the value computed by the contractor using Criterion b. For the upper bound, if we assume a maximum allowable outage of 1.59 percent (taken from the figures of Ref. 10), Criterion c [Eq. (37)] yields a bias of about -0.0113. If we assume A = 3.2 in order to correspond approximately to the 0.00135 probability of being outside the interval of S, Criterion d [Eq. (40)] yields a bias of about -0.0137. On the basis of these estimates, the biases chosen for the optimization procedure are -0.00726, -0.009, -0.011, -0.012, -0.013, and -0.014. The performance decrements chosen are 400, 500, 600, and 700 fps, which correspond to values of α from less than 2 to almost 4. Curves of P_α versus β for the different performance decrements are shown in Figure 7. A curve of optimum bias versus performance decrement, plotted from the minimum points of Figure 7, is shown in Figure 8. From this curve, the maximum probability of achieving any specified level of performance (minimum P_α -- Criterion e) can be found. The broken line in Figure 7 shows a curve of minimum P_α versus bias, also plotted through the minimum points. From this curve, the maximum performance (minimum performance decrement) that can be achieved with any specified probability of success $(1 - P_\alpha)$ can be found.

The following examples of the uses of Criteria e and f with reference to Figures 7 and 8 are illustrative, though not necessarily realistic. Suppose first that a critical performance level 550 fps less than nominal must be achieved with maximum probability of success (Criterion e). From Figure 8, the optimum bias is seen for this situation to be about -0.01085. Now suppose instead that a certain probability of success is required and that the performance to be achieved with that probability is to be maximized (Criterion f). By our referring to the dashed line in Figure 7, we see if the required probability is to be 99.865 percent (P_α = 0.00135, corresponding to three sigma for a normal distribution), then the optimum bias is about -0.0132. On the other hand, if only 99 percent confidence is required (P_α = 0.01), the optimum bias is about -0.0107.

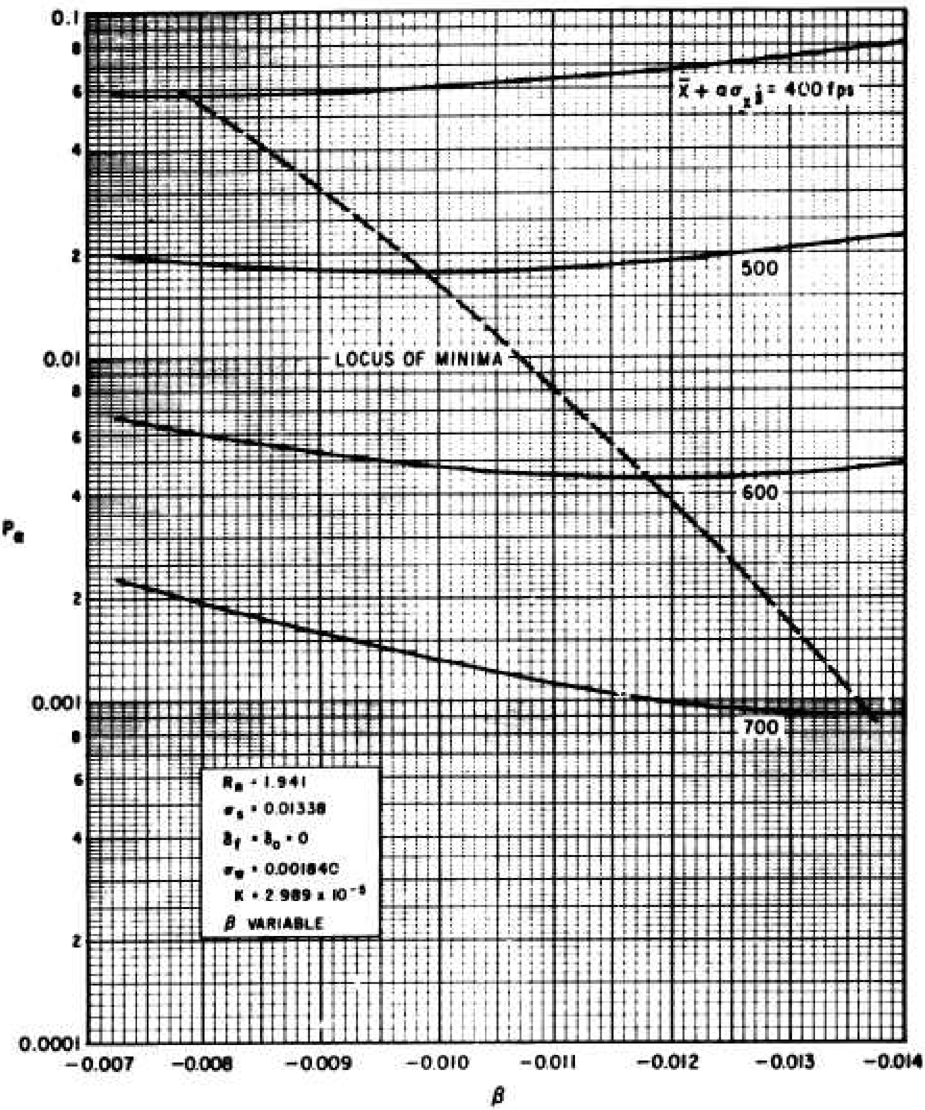


Figure 7. Probability (P_α) Versus Bias; Titan IIIC, Stage 3

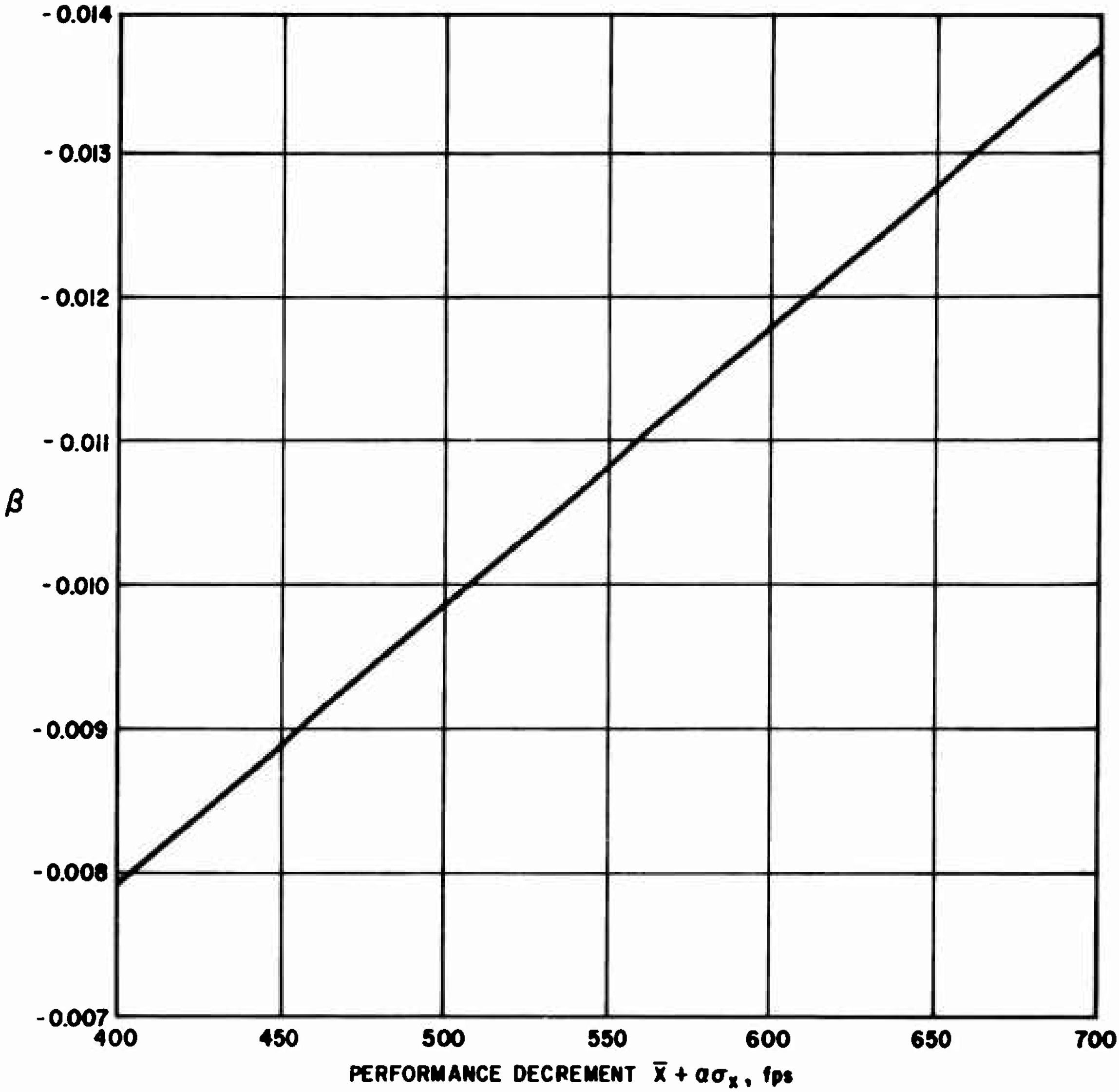
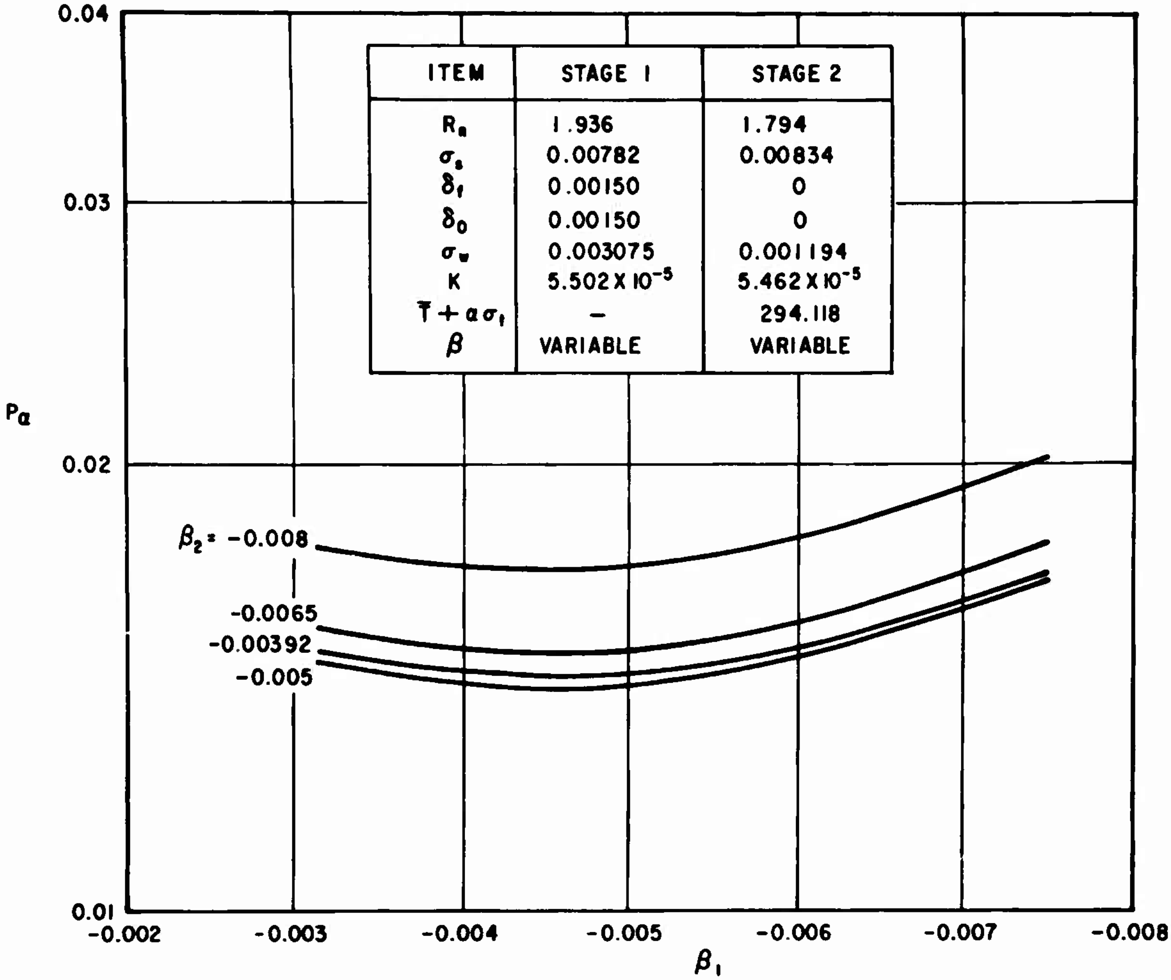


Figure 8. Optimum Bias Versus Performance Decrement; Titan IIIC, Stage 3

In general, as a higher probability of success is needed, a larger negative bias is required. However, the maximum performance achievable with the required probability is lower. Where hard requirements for either performance or probability are not present, a tradeoff may be conducted to determine the best compromise between performance and probability. Figures 7 and 8 provide all the information needed for such a tradeoff. Remember that none of these optima is critical in either performance or probability. Although the optima may be found to the fifth decimal place in this problem, they are not accurate beyond the fourth because of the graphical solution. Only the first three decimal places are important, as can be seen by the flatness of the curves in Figure 7 near the minima.

Optimization of the biases in the first two stages is less important than for the third stage because of the high probability of achieving shutdown velocity at the end of second-stage burning. However, the pro ability of achieving this shutdown velocity can be maximized by choosing the best combination of biases. The shutdown velocity represents a performance decrement of 294.118 fps with respect to nominal performance at stage-two burnout. This value is the only one used. The lower and upper bounds for the bias, based on the same criteria as in the third stage are the following: -0.00314 and -0.00750 for the first stage, and -0.00393 and -0.00759 for the second stage. From these figures, the values chosen for β_1 are -0.00314, -0.0045, -0.006, and -0.0075. Those chosen for β_2 are -0.00392, -0.005, -0.0065, and -0.008. Curves of P_α versus β_1 for the four values of β_2 are shown in Figure 9. The minimum points of these curves, which are essentially the values at β_1 = -0.0045, are plotted versus β_2 in Figure 10. The optimum value of β_2 corresponds to the minimum point of this curve, which is about -0.0049. The optimum value of β_1 is seen from Figure 9 to be almost independent of β_2 in the region of interest. Since all the minima occur at approximately -0.0045, it is the optimum value.



ITEM	STAGE 1	STAGE 2
R_a	1.936	1.794
σ_s	0.00782	0.00834
δ_f	0.00150	0
δ_0	0.00150	0
σ_w	0.003075	0.001194
K	5.502×10^{-5}	5.462×10^{-5}
$\bar{T} + a\sigma_t$	–	294.118
β	VARIABLE	VARIABLE

Figure 9. Probability (P_α) Versus First Stage Bias; Titan IIIC, Stages 1 and 2

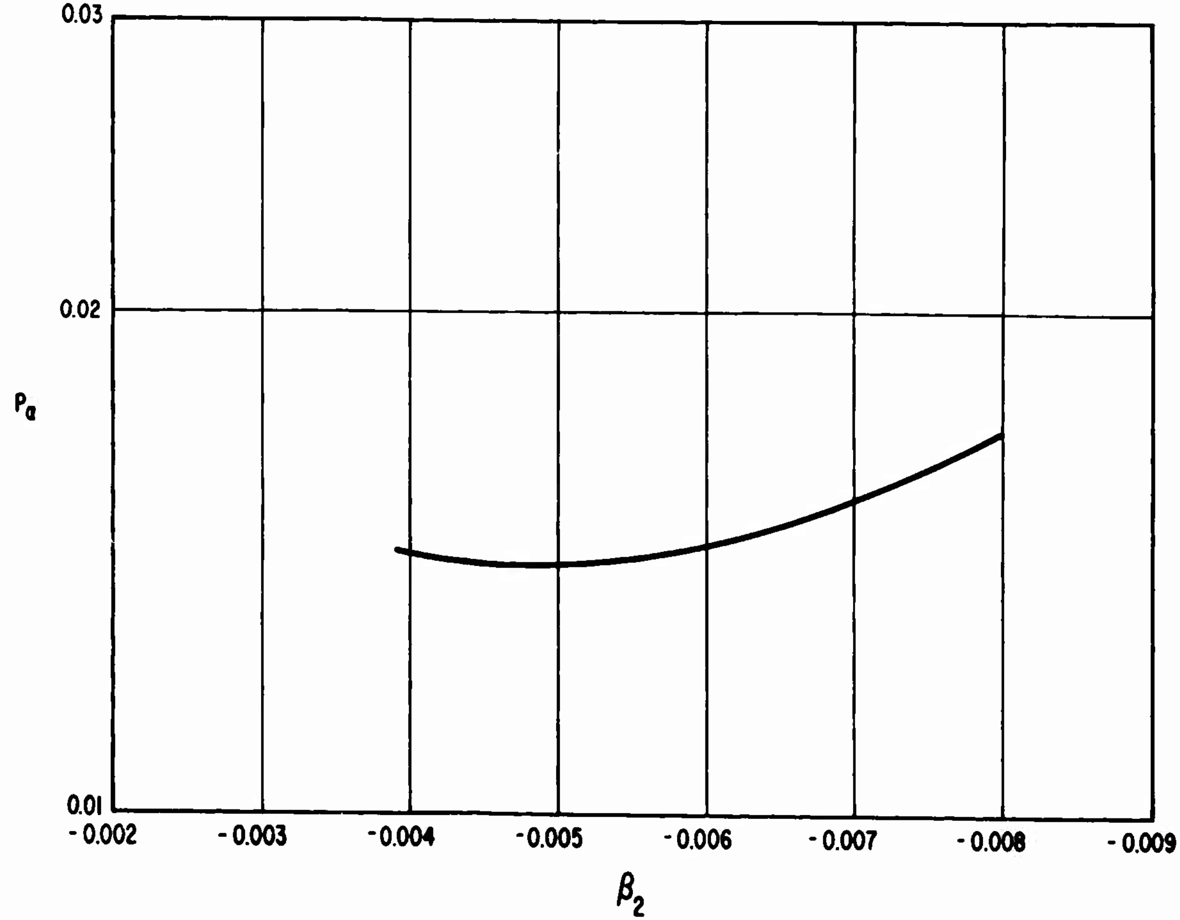


Figure 10. Minimum P_α Versus Second-Stage Bias; Titan IIIC, Stages 1 and 2

By our using the optimum values for the biases in the first two stages, a P_α about 0.0141, or a probability of success about 98.59 percent, can be achieved. If we use the biases chosen by the contractor according to Criterion b, P_α is 0.0150, or the probability of success is about 98.50 percent. Therefore, in this particular case, the more sophisticated bias optimization is not very helpful. However, where higher probabilities are involved, significant improvement over Criterion b can be achieved.

Notice how bias affects the form of the error distribution. As the magnitude of the bias increases, the distribution becomes more like a normal distribution because a greater portion of the $f_s(S)$ function lies completely to the left of the nonlinear portion of the outage function (see Figure 4). Figure 11 repeats the curve of Figure 6, which shows the error distribution for Titan IIIC, Vehicle 11, with a third-stage bias of -0.00726. This value used by the contractor is based on Criterion b. For comparison, two additional curves are plotted using different third-stage biases. As discussed previously in this section, the bias of -0.0107 represents an optimization based on a required probability of success of 99 percent and the bias of -0.0132 represents an optimization based on a required probability of 99.865 percent. This probability is associated with the three-sigma level of a normal distribution.

The most significant fact to be learned from Figure 11 is that when the bias is optimized for a high probability of success so that a large (negative) bias is used, the combined error distribution is closer to a normal distribution than when less bias is used. The distribution is still significantly nonnormal, having a probability of exceeding three sigma about 0.0045 as compared with 0.00135 for the normal distribution. But when the bias based on minimum mean square outage is used, the probability of exceeding three sigma is about 0.0086. If we look at it the other way, with the high-probability bias it takes about 3.55 sigma to give the same probability of failure that three sigma gives for a normal distribution; however, with the minimum mean square outage bias it takes about 4.23 sigma to give that probability.

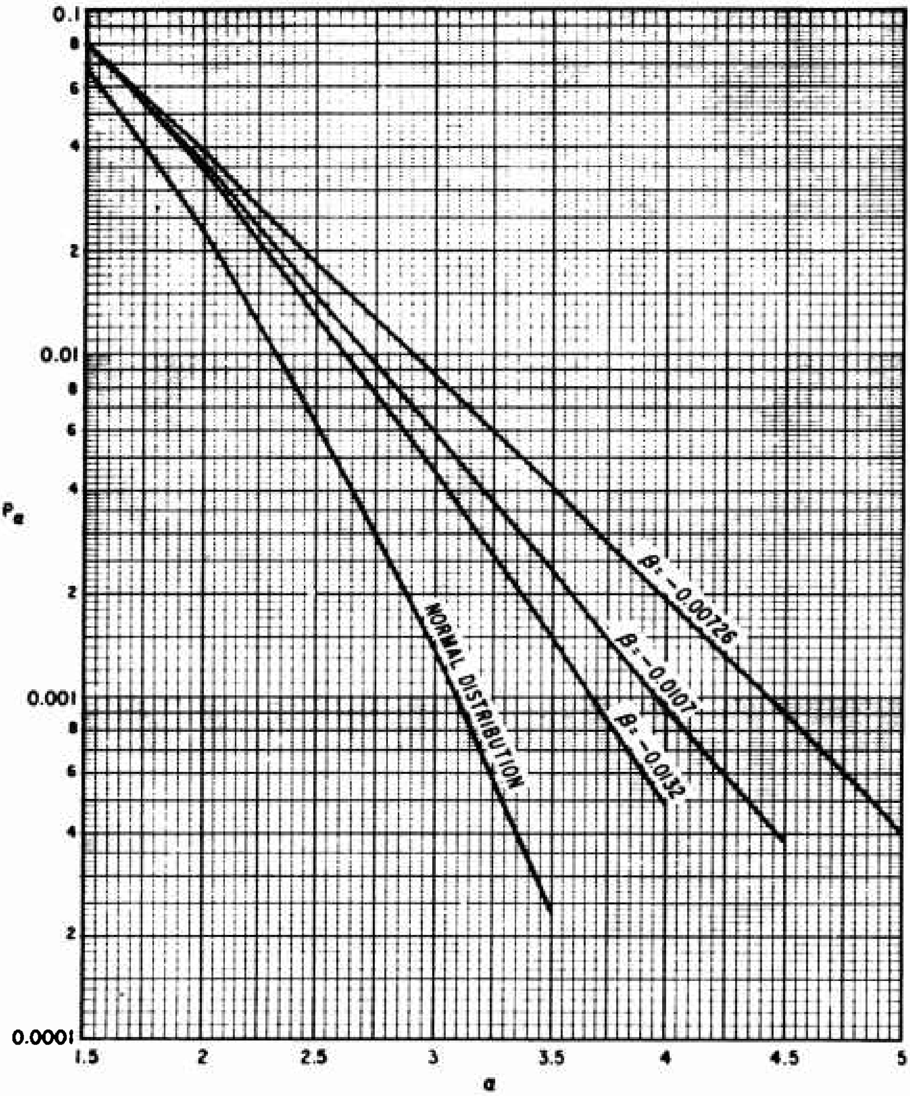


Figure 11. Probability (P_α) Versus Deviation From Mean Error; Titan IIIC, Vehicle 11

SECTION VIII

CONCLUSIONS AND RECOMMENDATIONS

A. CONCLUSIONS

The following three conclusions are reached from this investigation of the effects of outage on performance statistics for bipropellant rockets:

a. The over-all distribution of performance errors of a bipropellant rocket can be significantly nonnormal.

b. The probability associated with the achievement of a given performance for a bipropellant rocket can be significantly less than that obtained by assuming the distribution to be normal.

c. If a high probability of success is required, similar to that associated with three sigma for a normal distribution, a propellant loading bias significantly greater than that based on maximum mean square outage may be needed.

B. RECOMMENDATIONS

After consideration of the report text and conclusions, these three recommendations are suggested:

a. Performance requirements for bipropellant rockets should be specified in actual probability of success rather than as a multiple of sigma.

b. For each nominal vehicle, an analysis of the total error distribution should be made using the multistage computer program described in the Appendix or its equivalent.

c. Propellant loading bias should be chosen according to procedures similar to those described in this report to insure maximum performance for a required probability of success or maximum probability of success for a required performance.

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APPENDIX
MULTISTAGE COMPUTER PROGRAM

by

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A. GENERAL DESCRIPTION

The computer program carries out the numerical implementation of the analysis in the preceding sections. The main task is to perform accurately and efficiently the numerical integrations necessary to obtain P_α. To do this, the Legendre-Gauss numerical integration was determined to be the most appropriate method (see Ref. 11). To do an N-stage analysis an (N + 1) level multiple integration must be done (together with several integrals of lower level). The time required increases an order of magnitude for each level of integration. It was therefore thought best to vary the number of points in the integration technique at each level of integration. This enables the user to select the order of approximation that gives the accuracy needed commensurate with the number of stages and the amount of computer time available.

B. NUMERICAL INTEGRATION

Since the Legendre-Gauss numerical integration method requires that the limits of integration be finite and that the upper limit be +1 and the lower limit be -1, all the infinite integrals involved had to be truncated at some finite value, and all the integrals transformed to the interval (-1, 1). Because the integrands involved were probability frequency functions they could be truncated at some positive or negative multiple of σ from the mean, where the area outside that point could be assumed to be negligible. The multiple was determined quickly by trial and error to be about nine for the functions that have been tried.

At each level in the numerical integration process a different order of approximation can be used. The order here is the number of integrand values used in the numerical integration. The choices of the order are 2, 4, 8, 16, 20, 24, 32, 40, and 48 and the values of the abscissas and weights for all these orders were obtained from Ref. 11.

A convenient check on the choice of the multiples of σ and the order of approximation is to calculate P_α for $\alpha = -\infty$ which should be 1, because P_α is a probability function. This calculation also gives a rather loose but useful check on the accuracy that can be expected for finite values of α. Clearly, the area under the total curve can be no more accurate than the area under some subset of the curve for the functions involved and for a given order of approximation. The program automatically calculates $P_{-\infty}$, and prints the value obtained.

C. RESULTS

The greatest number of stages attempted with the program was three. Therefore a quadruple integral was the most complicated evaluation to be done. Satisfactory results were obtained using a multiple of nine for all infinite limits. The order of approximation at each level of integration was the following:

Level	Integral
1	48 (innermost)
2	24
3	24
P_α	16 (outermost)

These orders of approximation gave $P_{-\infty}$ accurate to two significant figures. They gave $P_{1.5}, \ldots, P_5$ accurate to no worse than four significant figures. This accuracy was determined by performing the following approximation:

Level	Integral
1	48 (innermost)
2	48
3	48
P_α	48 (outermost)

The two results are then compared for $\alpha = 1.5$. The accuracy of P_α for $\alpha = 2, \ldots, 5$ will be no worse than the accuracy of P_α for $\alpha = 1.5$.

D. DESCRIPTION OF DATA

Table I describes the inputs to the control card.

Table I. Control Card, Data Set 1

Program Symbol	Mathematical Symbol	Definition
NZ	N_z	Number of stages ≤ 8
NA	N_α	Number of entries in ALPHA vector ≤ 20
I0	I_0	Order of approximation to first level integrals
I1	I_1	Order of approximation to second level integrals
I2	I_2	Order of approximation to third level integrals
I3	I_3	Order of approximation to fourth level integrals
I4	I_4	Order of approximation to fifth level integrals
I5	I_5	Order of approximation to sixth level integrals
I6	I_6	Order of approximation to seventh level integrals
I7	I_7	Order of approximation to eighth level integrals
I8	I_8	Order of approximation to P_α level integral

Specific values of IJ from 1 to 9 in the order given in Table I are furnished in Table II.

Table II. Values of IJ (J = 0, . . . , 8), Data Set 1

IJ	Order
1	2
2	4
3	8
4	16
5	20
6	24
7	32
8	40
9	48

The information in Table III shows the case to be run has three stages, 10 ALPHA's; the order of integration of level one integrals is 48, of level two integrals is 24, of level three integrals is 24, and of the P_α level integral is 16.

Table III. Format of Card, Data Set 1

Column	Item	Example
1 - 3	NZ	003
4 - 6	NA	010
6 - 7	I0	09
8 - 9	I1	08
10 - 11	I2	08

Table III. Format of Card, Data Set 1 (Continued)

Column	Item	Example
12 - 13	I3	not used when NZ = 3
14 - 15	I4	not used when NZ = 3
16 - 17	I5	not used when NZ = 3
18 - 19	I6	not used when NZ = 3
20 - 21	I7	not used when NZ = 3
22 - 23	I8	04

Table IV. Vector Data, Data Set 2

$i = 1, \ldots, N_z$ and $j = 1, \ldots, N_\alpha$

Program Symbol	Mathematical Symbol	Definition
NT	N_{T_i}	Multiple of σ_{T_i} for computing infinite limits for T_i integrals. If $N_{T_i} \geq 0$, $c_i = c_i$ input. If $N_{T_i} < 0$, $c_i = c_i$ input $/ \sigma_{T_i}$. $\lvert N_{T_i} \rvert$ is always used in the calculations.
NX	N_{x_i}	Multiple of σ_{x_i} for computing infinite limit for x_i integrals.
BETA	β_i	Bias on a random variable.
SIGL	σ_{λ_i}	Standard deviation of s.
DELØ	δ_{o_i}	Oxidizer outage limit on s.
DELF	δ_{F_i}	Fuel outage limit on s.
RB	R_{B_i}	Burning ratio.

Table IV. Vector Data, Data Set 2 (Continued)

Program Symbol	Mathematical Symbol	Definition
NL	N_{λ_i}	Multiple of σ_{λ_i} for computing infinite limit for Z_i integrals.
SW	σ_{w_i}	Standard deviation of w_i. If $\sigma_{w_i} \geq 0$, $\sigma_{w_i} = \sigma_{w_{i\ input}}$ if $\sigma_{w_i} < 0$, $\sigma_{w_i} = \|\sigma_{w_i}\| \cdot \sigma_{z_i}$.
c	c_i	Multiple of σ_{T_i} for computing cutoff point for T_i integrals.
SCAL1	k_{1_i}	Input scale factor. If $k_{1_i} > 0$, $k_{2_i} = k_{1_i}/\sigma_{w_i}$; If $k_{1_i} = 0$, $k_{2_i} = 1$; if $k_{1_i} < 0$, $k_{2_i} = \|k_{1_i}\|$.
ALPHA	α_j or L_{α_j}	If ALPHA ≥ 0, magnitude of ALPHA $= \alpha_j$; if ALPHA < 0, magnitude of ALPHA $= L_{\alpha_j}$.

The card format for each of the quantities in Table IV is as follows (each vector may be continued on more than one card):

Table V. Format of Card, Data Set 2

Card Column	Entry (all quantities are right justified)
1 - 20	± .XXXXXXXXE±XX
21 - 40	± .XXXXXXXXE±XX
41 - 60	± .XXXXXXXXE±XX
61 - 80	± .XXXXXXXXE±XX

The order of the deck of vectors shown in Table VI is the same as the order of the defining table (Table IV).

Table VI. Data Deck Setup (one case)

Number	Data Item
1	Control Card
2	N_T Vector
3	N_Z Vector
4	BETA Vector
5	SIGL Vector
6	DELØ Vector
7	DELF Vector
8	RB Vector
9	NL Vector
10	SW Vector
11	C Vector
12	SCAL1 Vector
13	ALPHA Vector

For the CDC 6600 Chippewa version 1.1 FORTRAN IV Compiler the deck setup is shown in Table VII.

Table VII. Production Deck Setup

Number	Item
1	System accounting cards (Aerospace only)
2	RUN(S) Card
3	ØTAN2. Card
4	7 8 Card 9

Table VII. Production Deck Setup (Continued)

Number	Item
5	FORTRAN Source Deck
6	7 8 Card 9
7	Data deck
8	6 7 8 Card 9

As many cases as desired can be run with one pass on the computer.

E. <u>DESCRIPTION OF OUTPUTS</u> (see Table IX for sample outputs)

All vector quantities are printed in the order left to right, top to bottom along the page.

Output set 1 - Inputs

All the input cards are printed directly under the work INPUTS on the first output page. The order and format of these quantities is the same as the card order and format.

The output quantities in Table VIII are printed directly under the word OUTPUTS on the second output page.

Table VIII. Outputs, Output Set 2

$i = 1, \ldots, N_z$ and $j = 1, \ldots, N_\alpha$

Data Item	Mathematical Symbol	Definition
1	$\overline{z}_i$	Mean of z_i
2	σ_{z_i}	Standard deviation of z_i
3	k_i	Scale factor for z_i
4	$P_{z_{i_{-\infty}}}$	Area under total $f_{z_i}(z)$ curve + $F_{z_i}(0)$
5	σ_{w_i}	Standard deviation of w_i
6	$\overline{T}_i$	Mean of T_i
7	σ_{T_i}	Standard deviation of T_i
8	∞_{T_i}	Infinite upper limit for T_i integrals
9	$-\infty_{T_i}$	Infinite lower limit for T_i integrals
10	$\overline{T}_i + c_i \sigma_{T_i}$	Cutoff point for T_i integrals
11	$P_{-\infty}$	Area under $f_T(T)$
12	L_{α_j}	Lower limit for evaluating $P_{\alpha j}$
13	P_{α_j}	$P(T_N > \overline{T}_{N_{\alpha_j}} + \sigma_{T_{N_{\alpha_j}}} N_{T_{\alpha j}})$

Each data item is ordered from left to right, top to bottom. When the item consists of a vector it has the same number of entries as the order of the vector. The first item (item number one) is the $\overline{z}_i$ vector, etc.

Table IX illustrates the output format.

Table IX. Output Format

INPUTS
4 1 3 3 4 4 4 4 4 4 4

5.00000000E+00	5.00000000E+00	5.00000000E+00	5.00000000E+00
5.00000000E+00	5.00000000E+00	5.00000000E+00	5.00000000E+00
0.	0.	0.	0.
1.00000000E+00	1.00000000E+00	1.00000000E+00	1.00000000E+00
0.	0.	0.	0.
0.	0.	0.	0.
0.	0.	0.	0.
5.00000000E+00	5.00000000E+00	5.00000000E+00	5.00000000E+00
1.00000000E+00	1.00000000E+00	1.00000000E+00	1.00000000E+00
-5.00000000E+00	-5.00000000E+00	-5.00000000E+00	-5.00000000E+00
0.	0.	0.	0.

0.

OUTPUTS

1	7.978819028906017E-01	7.978819028906017E-01	7.978819028906017E-01
1	7.978819028906017E-01		
2	6.027865455230135E-01	6.027865455230135E-01	6.027865455230135E-01
2	6.027865455230135E-01		
3	1.000000000000000E+00	1.000000000000000E+00	1.000000000000000E+00
3	1.000000000000000E+00		
4	1.000000816991182E+00	1.000000816991182E+00	1.000000816991182E+00
4	1.000000816991182E+00		
5	1.000000000000000E+00	1.000000000000000E+00	1.000000000000000E+00
5	1.000000000000000E+00		
6	7.978819028906017E-01	1.570471922070205E+00	2.576162567167771E+00
6	3.405654581328690E+00		
7	1.167626489706173E+00	1.684922203591590E+00	1.950734873180153E+00
7	2.285572303296846E+00		
8	6.636014351421437E+00	9.995082940028112E+00	1.232983693306846E+01
8	1.483351609781289E+01		
9	-5.040250545640220E+00	-6.854139095887717E+00	-7.177511708732051E+00
9	-8.022206935155509E+00		
10	-5.040250545640220E+00	-6.854139095887717E+00	-7.177511708732051E+00
10	-8.022206935155509E+00		
11	1.033881814360640E+00		
12	3.405654581328690E+00		
13	5.072428624196270E-01		

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13. ABSTRACT

The distribution of performance errors in bipropellant rockets is determined as a function of engine and propellant loading parameters, and contributing errors. The distribution is found to be highly skewed, so that the probability of exceeding three sigma can be significantly greater than it would be if the distribution were normal. Thus, three sigma as a minimum performance limit is not justified. A detailed analysis is required in each case to determine the relationship between performance margin and probability of mission success. Also, a realistic method of optimizing the propellant loading bias is developed. The methods previously used have resulted in fuel biases that are usually much lower than the optimum. Computer programs have been developed to perform the analyses described in this report, both for the single-stage and the multistage case.

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Abstract (Continued)