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GENERAL SYSTEMS THEORY AND ITS MATHEMATICAL FOUNDATION*

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Abstract

Objective of this paper is to present the foundations for a mathematical theory of general systems, to discuss the limitations introduced by the use of the mathematical methods in the theory and finally to indicate the areas of application in engineering.

Two points are emphasized: A need for the simplicity and precision in the definition of the basic concepts so that the description of the complex systems does not become unmanageable. An explicit recognition of the goal-seeking approach as constituting an important aspect of the systems theory. In order to make the second point more specific a formalization of the notion of a goal-seeking system is developed in the last section.

The present article can be viewed as an updated version on an earlier paper since the basic problems considered and the viewpoints adopted here are the same as in [1].

1. What is the Mathematical Theory of General Systems?

We consider the mathematical theory of general systems to be a theory about mathematical models of real-life systems* such that the essential properties of these systems are revealed using a minimal mathematical structure (compatible with the intuitive interpretations of those properties). Such a general systems theory is currently under development in the framework of set-theory and the bordering branches of abstract mathematics [(1)(2)(3)(9)(10)(11)].

The selection of this basis for general systems theory stems from the point-of-view that the objects of study in mathematics (regardless of the special branch one has in mind) are essentially (and in the first place) sets and relations between sets and their elements. Various branches of mathematics differ primarily in what additional properties (i.e. "structure") the sets (and the relations) under consideration possess**.

* The term "real-life system" is used here only as a label to denote the physical, economical, (or even conceptual) etc., class of interrelated objects (or phenomena) defined within a given subject-matter. It is essentially an interpretation of the given mathematical model.

** It should be noticed that according to the established custom in mathematics the term set theory is used only to denote the study of relations between sets (and their elements) which have very little (if any) additional structure (such as e.g., ordering). The specialized fields of mathematics in which the sets have more properties are not considered as being part of set theory. For example, if the sets under consideration have some functions which map the elements of a set into itself one talks about (partial) algebras or if some set-valued functions are defined on the sets under consideration one talks about (general) topology, etc.

A distinction should be made between the objects of study in mathematics and the way this study is conducted. The formalization of the latter process is the domain of metamathematics, i.e. a formal theory of deductive reasoning in mathematical studies. Metamathematics, in turn, also uses various mathematical structures. These structures are selected on philosophical grounds since they reflect sufficiently closely what intuitively (and on the basis of philosophical logic) is considered to be deductive reasoning. However, they comprise a special subclass of the class of mathematical structures and from the purely abstract, formalistic point of view, they cannot be preferred for the development of a theory of the behavior of various real-life systems. Traditionally, the mathematical structures used in metamathematics were finitistic, but recently more powerful mathematical methods have been used [4]. The introduction of non-finitistic methods in metamathematics has resulted in greater simplicity and increased efficiency. This is why set theory (i.e., mathematics) rather than logic (i.e., metamathematics) is used as the basis for a general theory of systems.

Let us now turn, more specifically, to some of the basic concepts and problems of general systems theory.

Given a family of sets, $\bar{X} = \{X_1,\ldots,X_n\}$; *A (general) system* is defined to be a relation on $\bar{X}$, i.e., $S \subset X_1 \times \ldots \times X_n$, where $\times$ indicates the Cartesian product. The sets $X_1,\ldots,X_n$ which enter the relation are called *objects*. Each X_i represents the totality of all appearances of (or experiences with) an attribute of the real-life phenomena under consideration. Similarly, S represents the totality of all appearances of (experiments with) the real-life system.

There are two important types of problems in general systems theory:

1) Constructive specification: How to provide an efficient procedure for use in prediction; i.e., to determine some of the elements of the system when some other elements are given. Constructive specification as a basis for predicting the systems behavior is essential for the utility of the systems notions.

2) Systems properties: How to formalize certain properties of interest in the characterization of real-life systems and how these properties are related with constructive specification.

There are two basic ways to provide a constructive specification of systems: the *terminal approach* and the *goal-seeking approach*.

α) In the terminal approach the systems objects are partitioned into two classes, $X = X_1 \times \ldots \times X_m$, and, $Y = X_{m+1} \times \ldots \times X_n$, so that the system becomes $S \subset X \times Y$. The objects in X are called *inputs* (and represent the cause, stimulus, for the phenomenon under consideration) while Y are called *outputs* (and represent the effects, response). The constructive specification of terminal systems is arrived at by providing additional structure on the object-sets so that a simpler system can be defined (hopefully even of finite cardinality) that can be used to specify the original system, e.g., via a process of recursion or induction. Such simpler systems used for constructive specification are called *auxiliary functions* (2). Often, they require introduction of some new (auxiliary) objects in the description of systems, most notably the *state object* or the *state space* [2][3].

β) Constructive specification in (and indeed the definition of the concept of) the goal-seeking approach is achieved by introduction of the notion of a goal for a system and then by describing the behavior of the system in reference to that goal. The last section of this paper contains a formal definition of a goal-seeking system. It should be emphasized that the goal-seeking description of a system is needed not for philosophical or conceptual reasons but rather for the purely technical reasons of arriving at a constructive specification. It might even be considered as an alternative way for implicit definition of a function (or relation); namely, for a given class of systems one might not have available a constructive specification via the terminal approach but only in terms of goal-seeking. That does not mean, of course, that the basic concepts involved in the terminal description of such a system (e.g., the state object) cannot be defined; rather, it might be because the associated auxiliary functions are not available in an analytic or algorithmic form so that one has to resort to the goal-seeking approach [5].

To make the present discussion more complete the following should be noted:

a) The properties one is concerned with in systems theory refer, as a rule, to the information processing and decision-making aspects of the real-life phenomena rather than to the physical (or other) laws per se. In this sense, systems theory is a theory of information processing and decision-making.

b) There are two routes along which one can proceed to develop general systems theory. One can start from the class of real-life problems and proceed by formalizing their verbal descriptions. In this way one starts from the most abstract (least constrained) description and proceeds by introducing more structure and considering the consequences of each new assumption. This approach we shall refer to as *formalization*. The second approach which, we shall call the *generalization* approach, starts with two given classes of mathematical models and by generalization develops a broader class which preserves the properties and subsumes the original classes. We pursue the formalization approach. The generalization approach has been pursued, in particular, in connection with the unification of the control and automata theories (presented in the algebraic rather than logical framework) (6)(7)(15).

2. Limitations of a Mathematical Theory of General Systems

Before considering the potential benefits and usefulness of a mathematical theory of general systems it is only appropriate to consider its weaknesses. The theory can be criticized from two opposite sides:

1) Since it is essentially a mathematical theory it can put severe restrictions on the description of the behavior of the real-life systems, especially if one deals with complex biological or social situations.

2) Since the theory uses rather weak mathematical structures, it is not possible to "solve" too many problems or even to develop deep enough results to be useful.

Consider the first objection. It is certainly true that systems theory cannot be broader than its mathematical foundation. Yet, at this point one should not proceed further without investigating with some more care what this foundation really is. An appropriate starting point would be to consider more carefully the concept of a theory. For our purpose, the most appropriate notion of a theory is that advanced by H. Curry [(8)]. To arrive at the notion of a theory one starts from a class of (elementary) statements P about the subject matter of concern. It should be noticed that at this point one is dealing with an "intuitively comprehended collection of elements" which Curry calls a conceptual class. (Technically, we differ here from [(8)] in that the class of statements does not have to be definite.) A *theory*, then, is a subclass of statements $T \subseteq P$, which are asserted to be true. This assertion can be the result of experimentation or may reflect certain postulates about the behavior of the observed phenomenon. For further sharpening of the notion of a theory we have to recognize two aspects of a theory: the *informal*, concerned with the meaning, interpretation, of the statements; and the *formal*, concerned with the structural aspects of the observed phenomenon. We are concerned here only with the formal aspects of a theory. Linguistic analysis reveals two components of a statement: terms (nouns) denoting the objects of concern and the functors denoting the relation between the terms. The statements in the class P have some undetermined constituents; otherwise, their truth or falsity would not depend upon the theory T but will be decided on the contensive basis, i.e., by the content of the sentences themselves. The undecided elements could be assumed to take on values, in general, from some conceptual classes. We shall now make the assumption that these classes are sets in a given axiomatic set theory. Nouns, then, would designate certain sets and the functors certain relations.

Consider now a mathematical structure which is a relation on appropriately defined sets $S \subset X_1 \times \ldots \times X_n$. The structure S will be considered to be a *mathematical model for the theory* T if S is a valid interpretation for T, i.e., if every statement in T is true in S. Apparently, for a given theory T there is a multitude of valid mathematical models.* Notice that only validity relative

*It should be noted that for certain special types of formalized theories (such metamathematical systems as the propositional calculus, etc.) there is quite an active development of the mathematical theory of models (for these particular formalized theories). Our concern here, of course, is with a broader notion of both a theory and its model (since a theory does not have to be an inductive class of statements).

to T is required. Thus, some of the models can be incorrect in the sense that there exists a statement outside of the elementary theory T which is false in the subject-matter interpretation; yet, it is true in S. Therefore, one can select a model which is valid but incorrect. This is, in practice, too often the case since the correctness of a given model is not easy (or even possible) to check beforehand. A valid mathematical model S for a theory T is precisely the (general) system S as defined in Sec. 1 and represents the mathematical model of the real-life system about which the theory T has been formulated.

In summary, then, as soon as we have a theory it is possible to introduce the notion of a system. The restrictions which general systems theory (as defined here) imposes, follow basically from the requirement that the range of any undetermined term satisfy an axiomatic set-theory. This essentially amount to the requirement that the statements in the theory do not contain the well-known paradoxes (of referring to itself, etc.).

The important point that follows from the preceeding discussion is that the inadequacy of the model (system) is precisely due to the limitations in our theory of real-life phenomena and are not due to the introduction of the notion of a system. If the system has very little structure and in this sense does not reveal much, that is again due to the limitation of our knowledge about the behavior of the real-life system (i.e., theory T) rather than due to the application of formal, mathematical methods.

2) The second objection, that general systems theory might not have enough structure to yield useful results, can be fully answered only in time. In principle, of course, it depends upon how one defines "usefulness". However, it is too early in the development of general systems theory to try to provide a full scale argument to counteract that objection. If the development of general systems theory is needed and, indeed, is inevitable for further understanding of the complex phenomena (as I feel to be the case) then, perhaps, the most prudent thing to do at this point in time is to state one's conviction and let future developments provide the answer. However, some of the following papers [9][10][11] could serve to support the argument against this objection.

3. Complexity and Large-Scale Systems: Abstraction and Hierarchy

Consider the problem of complexity either in the operation or in the design of systems. Of course, there is no definition of complexity at the present; nor, in general, do we think one can be developed on formal grounds. However, there seems to be a consensus that a system appears to be "large scale" or "complex" if the computational, analytical, economical or any other combination of factors prevent the achievement of the objectives to a satisfactory degree. How can one approach these complex problems? The traditional way, of course, has been to use approximations. There are two new approaches which show some promise: 1) The *abstraction approach*; 2) The *hierarchical or multi-level approach.*

1) In the approach via abstraction one uses a mathematical model which is less structured and models only some of the dominant, "key", features of the problem. For example, suppose the system is described by a large number of partial differential equations. The study of the stability of such a system via the Liapunov method can be quite complex. However, if one recognizes the algebraic structure of the system's transformation one might consider the stability problem algebraically [12]; thereby using a less detailed representation of the system. This is, of course, where the mathematical theory of general systems should be of help. Other aspects and properties, like decomposition and structuring of systems, can also be studied in an algebraic framework.

The distinction between the approximation and abstraction approach should be noted. In the former, one uses the same mathematical structure and the simplification is achieved by the omission of some parts of the model that are considered as less important, e.g., a fifth order differential equation is substituted by a second order equation by considering only the two "dominant" state variables of the system. In the latter approach, however, one uses a different mathematical structure, one which is more abstract, and considers the system as a whole but from a less detailed viewpoint. The simplification is not achieved by the omission of the variables but by the suppression of some of the details considered unessential.

2) In the hierarchical approach one decomposes the problem and solves the subproblems independently. Partial solutions are then coordinated by a hierarchy of decision processes aiming at coming as close to the overall solution as possible. The decomposition and the coordination can be done either in time or in space. In the design or computational problems, one solves the subproblems sequentially in time and achieves the coordination by an iteration process. In the problems of complex operational systems, the overall system is decomposed into subsystems whose operation is simultaneous in time and the coordination is achieved by "on-line" intervention during the actual operation of the subsystems.

It might be of interest to point out that, in spite of apparent dissimilarities between general systems theory and the theory of multi-level (hierarchical) systems, the motivation for their development is quite similar: to deal with complex, large-scale problems. Actually, one should look at these two theories as being intended to deal with the same type of problems but from different starting points.

4. The Role of General Systems Theory in the Engineering Process

The mathematical theory of general systems can be of use in engineering in several different contexts and for several different purposes:

a) Complexity: First, there is the problem of complexity in large scale systems. There is not much we should add here to the discussion in Section 3. It suffices to point out that, as the problems we consider become more complex and the use of computers for simulation and problem-solving become more widespread, the need for a conceptual framework for both the explanation of a problem and its solution becomes more acute.

b) Model-Building and Structural Considerations: One of the most crucial steps in the engineering process is to select a structure for the system to be designed or, similarly, to analyze the structural considerations of the behavior and operation of a system. A detailed mathematical model, even when available, is not suitable for this purpose. Traditionally, engineers have used the block diagram basically for the purpose of grasping the overall composition of the system and the subsequent structural considerations. Of course, the principle attractiveness of block diagrams is their simplicity; while the major drawback is the lack of precision. General systems theory can be useful as a tool in basic structural considerations by preserving the simplicity of the block diagrams while introducing the precision of mathematics. Actually, the role of general systems theory in the engineering methodology can, perhaps, best be represented by the diagram in Fig. (1).

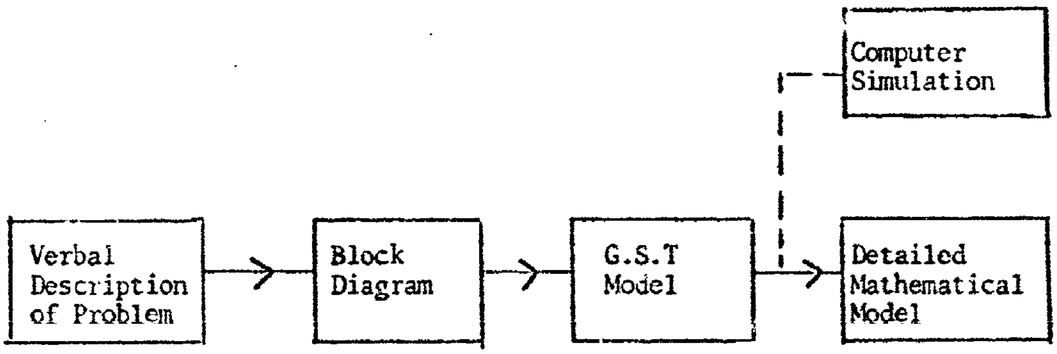

FIGURE 1

General systems models therefore lay between the block diagram representation and a detailed mathematical (or computer) model. Especially for complex systems, a general systems model represents a necessary step since the gulf between the block diagram and the detailed model can be too great. Of course, availability of certain general systems techniques to treat the problem (at least in a preliminary way) on the general systems level can significantly enhance the usefulness of adding this step in the process.

c) <u>Precise Definition of Concepts and Interdisciplinary Communication:</u>

General systems theory provides a framework for interdisciplinary communication since it is general enough so as not to introduce constraints of its own; yet, by its precision removes misunderstanding to a considerable degree. For example, different notions of adaptation used in the field of psychology, biology, engineering, etc. can be first formalized in general systems theory terms and then compared.

It is often stated that systems theory has to reflect the "invariant" aspects of different real-life systems which are true for structurally similar phenomena from different fields (disciplines). This can be accomplished only if the relevant concepts are defined with sufficient care and precision. Otherwise, the danger of confusion is too great. It might be appropriate then to consider the mathematical theory of general systems as a framework for the formalization of basic systems concepts. In this sense, it is quite basic for the "systems approach" in general and systems theory in particular. The important point in using general systems theory in this context is that, after introducing a concept in a precise manner, it is not crucial whether the definition is "correct" in any given interpretation but rather whether the concept is defined so that it can be examined, compared and subsequently changed if it fails to meet certain intuitive requirements. In other words, one has a basis for "objectively" evaluating the formalization of properties of the real-life systems. In this sense general systems theory offers a "language" for interdisciplinary communication. This usage of general systems theory might seem trivial from the purely mathematical standpoint but it is not so from the viewpoint of managing a large systems engineering effort in which a team of specialists from different disciplines is working on a complex problem.

5. Formalization of the Goal-Seeking Notion in General Systems Theory

As was mentioned in Section 1, the goal-seeking representation offers an approach to the study of real-life systems as important as, if not more important than, the terminal approach. In this section we shall define more precisely in general systems terms the notion of a goal-seeking system in order to illustrate both *the process of formalization and the kind of notions usually developed in general systems theory.*

Given a system

$$S \subset X \times Y, \tag{1}$$

to arrive at the notion of a goal-seeking representation for S we need two preliminary concepts; namely, the concept of a goal and the concept of a decision-maker. (For simplicity we shall view S as a function which implies that the members of X are input-state pairs.)

a) Goal

Let $X = M \times U$. A *goal* for S is defined then by a triplet of relations $\alpha = (G,T,R)$ defined in reference to a set V such that

$$\begin{aligned} G&: S \to V \\ T&: U \to V \\ R &\subset V \times V \end{aligned} \tag{2}$$

The set V represents the value or performance measure set. Under interpretation, G represents the *performance (or goal) function* that assigns a value $G(s) \in V$ to every appearance of the system, i.e., $s \in S$. T represents the *tolerance (reference) function*. For every $u \in U$, T gives the value $T(u) \in V$ that should be used to evaluate the performance of a given $y = S(m,u)$. Finally, R represents the *satisfaction relation*. For any $(m,u) \in M \times U$, the satisfaction with the behavior of the system will be evaluated with reference to $G(m,u),S(m,u)),T(u)$ and R.

Given a goal $\alpha = (G,T,R)$ for a system S we have then two notions relating the inputs with the goal.

The input $x \in X$ *achieves* the goal α if

$$(G(x,S(x)),T(u)) \in R \tag{3}$$

where $x = (m,u)$.

The input $m \in M$ *satisfies* the goal α relative to $U' \subset U$ if for all $u \in U'$ the input $x = (m,u)$ achieves the goal α, i.e., for all $u \in U'$

$$(G(m,u,S(m,u)),T(u)) \in R \tag{4}$$

The triplet $\beta = (S,U',\alpha)$ will be referred to as a *decision problem*. An input $m \in M$ *satisfies the decision problem* (S,U',α) if it satisfies the goal α relative to U'.

b) <u>Decision-maker (Decision-system)</u>

Given a system

$$S: M \times U \to Y, \tag{5}$$

informally, the system will be referred to as the decision-maker if a decision problem β is given such that for every $(m,u) \in M \times U$, the output $y = S(m,u)$ satisfies β (in a given sense).

More precisely, S will be termed a *decision-maker* if the following is given:

α) A pair of mappings

$$P: Y \times U \to M$$
$$W: U \to M \qquad (6)$$

such that

$$m = W(u) \leftrightarrow (S(m,u) = y) \wedge (P(y,u) = m) \qquad (7)$$

i.e., W is a composition of S and P as specified by (6).

β) A goal α for P such that for all $u \in U$, $S(w,W(u))$ satisfies the decision problem $\beta = (P,U',\alpha)$ where $U' \subset U$ is defined by a predefined set-valued mapping $F: U \to \pi(U)$.

Under interpretation, U represents the uncertainty set and the mapping F selects (e.g., by prediction) the subset U' for which the output of S should achieve the given goal.

c) <u>Goal-seeking System</u>

Finally, we are in a position to introduce the notion of a *goal-seeking system*.

Given a system

$$S: X \to Y, \qquad (8)$$

there are two ways how S can be defined as a goal-seeking system.

1) Let α be a goal for S. The system is considered as an *(open-loop) goal-seeking system* if every $x \in X$ satisfies the goal α.

2) S is considered as a *(feedback) goal-seeking system* if a set M is given together with a pair of mappings (D,P)

$$\begin{aligned} P&: M \times X \to Y \\ D&: X \times Y \to M \end{aligned} \tag{9}$$

such that

α) $y = S(x) \leftrightarrow (P(m,x) = y) \wedge (D(x,y) = m)$

β) D is a decision-maker relative to a goal α for a mapping P_M defined on $M \times U$ into Y, i.e.,

$$P_M: M \times U \to Y \tag{10}$$

Apparently, according to the second notion S is a goal-seeking system if there is given a pair of mappings (P,D) such that S is a (feedback) composition of P and D and, furthermore D is a decision-maker relative to a goal α defined for P_M.

Starting from the notion of a goal-seeking system, some other notions can be defined such as learning (adaptation), self-organization etc. For example, learning can be defined as a process aimed at the reduction of the uncertainty set U (13)(14) while self-organization can be defined as a process of changing the structure of the goal-seeking process, i.e., the functions defining a goal-seeking system (such as performance function, process model P_M, tolerance function, satisfaction relation etc.).

REFERENCES

1. Mesarovic, M.D., "Foundations for a General Systems Theory", in "Views on General Systems Theory", John Wiley, New York, 1964.

2. Mesarovic, M.D., "On the Auxiliary Functions and Constructive Specification of the General Time Systems", Report SRC 85-A-66-33, Case Institute of Technology.

3. Windeknecht, T.G., "Mathematical Systems Theory: Causality", MATHEMATICAL SYSTEMS THEORY Journal, Springer-Verlag, 1967.

4. Rasiowa and Sikorski, "Mathematics of Metamathematics", 1966.

5. Mesarovic, M.D., "Systems Theory and Biology", in the Proceedings of the Third Systems Symposium at Case, Springer-Verlag, 1968.

6. Zadeh, L., "The Concept of State in Systems Theory", in "Views on General Systems Theory", J. Wiley, New York, 1964.

7. Kalman, R., "Algebraic Aspects of the Theory of Dynamical Systems", in "Differential Equations and Dynamical Systems", Academic Press, 1967.

8. Curry, H., "Foundation of Mathematical Logic", McGraw Hill, New York, 1963.

9. Windeknecht, T.G., "An Axiomatic Theory of Systems", Proceedings 1967 Systems Science and Cybernetics Conference, Boston, Massachusetts, 1967.

10. Birta, L., "The Concept of Generativity in General Systems Theory", Proceedings 1967 Systems Science and Cybernetics Conference, Boston, Massachusetts, 1967.

11. Macko, D., "Hierarchical and Multi-Level Systems", Proceedings 1967 Systems Science and Cybernetics Conference, Boston, Massachusetts, 1967.

12. Windeknecht, T.G., Mesarovic, M.D., "On General Dynamical Systems and Finite Stability", in "Differential Equations and Dynamical Systems", Academic Press, 1967.

13. Mesarovic, M.D., "A Unified Theory of Learning and Information", Proc. of the 1st Comp. and Info. Sci. Symposium, Spartan, Washington, D.C., 1963.

14. Mesarovic, M.D., "Toward A Formal Theory of Problem Solving", Symposium on Comp. of Human Reasoning, Washington, D.C., Spartan Press, 1964.

15. Wymore, W., "Mathematical Theory of Systems Engineering", J. Wiley, 1967.