Some Limit Theorems in Queuing Theory

by A. P. Lalchandani

DEPARTMENT OF OPERATIONS RESEARCH

COLLEGE OF ENGINEERING
CORNELL UNIVERSITY
ITHACA, NEW YORK 14850

SOME LIMIT THEOREMS IN QUEUING THEORY

by

Atma P. Lalchandani

TECHNICAL REPORT NO. 29

August 1967

Prepared under Contract Nonr-401(55)(NR-047-055)
for the
Office of Naval Research

DEPARTMENT OF OPERATIONS RESEARCH
CORNELL UNIVERSITY
ITHACA, NEW YORK

ACKNOWLEDGMENTS

The author wishes to express his sincere gratitude and appreciation to Prof. N. U. Prabhu for his unfailing assistance, encouragement and guidance throughout the course of this work. During this period Prof. Prabhu was supported by the National Science Foundation Grant GP 3783.

The author is also deeply indebted to Prof. D. L. Iglehart for his numerous suggestions during the course of this thesis. The author wishes to thank him for his patient understanding of the author's diverse problems.

The author wishes to thank Prof. J. Kiefer for being a member of his committee and always being ready to help.

The thesis was supported by the Office of Naval Research, Contract Nonr-401(55)(NR-047-055).

TABLE OF CONTENTS

CHAPTER		Page
I.	INTRODUCTION AND SUMMARY	1
II.	LUCHAK'S QUEUING MODEL	7
	2.1. THE PROCESS $Y_n(t)$	11
	2.2. SOME PROPERTIES OF THE $X_n(t)$ PROCESS	25
	2.3. CONVERGENCE IN DISTRIBUTION OF $X_n(t)$	33
	2.4. SOME EXTENSIONS	40
III.	THEOREMS FOR THE MAXIMUM OF PARTIAL SUMS	42
IV.	SOME QUEUING SYSTEMS	58
	4.1. THE SYSTEM $GI\|G\|1$	59
	4.2. THE SYSTEM $GI\|E_s\|1$	64
	4.3. THE SYSTEM $E_s\|G\|1$	68
V.	DIRECTIONS FOR FUTURE RESEARCH	74
	BIBLIOGRAPHY	75

CHAPTER I

INTRODUCTION AND SUMMARY

In this thesis, we obtain limit theorems for some stochastic processes arising in single server queuing systems. These systems are described in terms of the arrival and service mechanisms besides the priority rule used for the servicing order of the customers. The ratio of the mean service time to the mean interarrival time of the customers, called the traffic intensity and denoted by ρ, plays an important role in the analysis of a queuing system. When $\rho < 1$, the stochastic processes (queue length, waiting time) associated with the system, converge in distribution to non-degenerate limiting distributions; this fact is expressed by saying that the system attains a steady state. On the other hand, no such steady state exists in the case $\rho \geq 1$. Of course, one could introduce some mechanism, such as finite waiting room or customer impatience, which would keep the system stable even when $\rho \geq 1$. However, these mechanisms do not have any relevance to this thesis. Also, throughout this work, we deal solely with the first-come-first-served priority rule.

Our objective is to investigate the case $\rho \geq 1$ by exhibiting some properties of the behavior of the aforementioned stochastic processes. We shall demonstrate that these processes, suitably translated and scaled, possess non-degenerate limit distributions. The limit theorems are obtained by considering limiting operations on the two variables, traffic intensity and the time parameter. We give below a brief account of the work done in this area.

The area of research in the case of so-called heavy traffic, namely, when the traffic intensity ρ is less than but close to unity, was initiated by Kingman [15] in 1961. He considers the distribution of the waiting time in the single server queue with general independent input and general service time; this system is usually denoted by the symbol $GI|G|1$. Let us denote the service time of the n^{th} customer by v_n and the interval between the arrival times of the n^{th} and $(n+1)^{st}$ customers by u_n. The sequences of independent and identically distributed (i.i.d.) random variables, $\{u_n\}$ and $\{v_n\}$, are assumed to be independent of each other. Consequently, the random variables defined by $X_n = v_n - u_n$, form an i.i.d. sequence. Let $\sigma^2 = \text{Var}\{X_n\} < \infty$, and set

$$\alpha = -\frac{E(X_n)}{\sigma} \quad . \tag{1}$$

Letting W_n denote the waiting time of the n^{th} customer, Lindley [18] has shown that, if $W_1 = 0$, then

$$\Pr\{W_{n+1} \leq x\} = \Pr\{\max(S_1,S_2,\ldots,S_n) \leq x\} \quad , \tag{2}$$

where $S_j = \sum_{i=1}^{j} X_i$. Starting from this result, Kingman's result states that†

$$\lim_{\alpha \downarrow 0} \Pr\{\frac{\alpha}{\sigma} W(\alpha) \geq x\} = \begin{cases} e^{-2x} & , \ x > 0 \\ 1 & \ x \leq 0 \end{cases} , \qquad (3)$$

where $W(\alpha) = \lim_{n \to \infty} W_n(\alpha)$, α being used explicitly in the symbol $W_n(\alpha)$ for the waiting time. Stated in words, this means that for small, but positive α, the limiting waiting time has approximately the negative exponential distribution with mean $\sigma/2\alpha$. The traffic intensity and the parameter α are connected by the relation

$$\rho = \frac{E(v_n)}{E(u_n)} = \frac{E(X_n)}{E(u_n)} + 1 \quad . \qquad (4)$$

Thus, $\alpha = 0$ corresponds to $\rho = 1$, and α small but positive corresponds to ρ close to but less than unity.

Subsequent to 1961, a number of authors have obtained more general results in heavy traffic theory. Notable contributions have been made by Prokhorov [24] and Borovkov [2,3]. Prokhorov's work exhibits the interplay between the two limiting operations, $n \to \infty$ and $\rho \to 1$ (equivalently, $\delta = 1-\rho \to 0$). He exhibits a number of possible limit theorems depending on whether $n\delta^2$ converges to zero, a positive constant or plus infinity. The

---

† $x \downarrow c$ means that x approaches c from above. Similarly, $x \uparrow c$ denotes the approach from below.

limiting distributions are obtained in terms of the probabilities of Brownian motion. Kingman's result follows from the last of the abovementioned three cases.

In [2], the author considers batch arrivals in the $GI|G|1$ system. He furthermore generalizes this system by letting the interarrival time and the service time depend on $\delta = 1-\rho$. For such a system, the author obtains estimates of the remainder terms in the asymptotic exponential behavior of the waiting time. In his second paper [3], Borovkov considers general multiple channel queues and shows that the results in this case are of the same form as those obtained for single channel queues, e.g. by Prokhorov.

Among other contributions to heavy traffic theory are the following. Samandorov [26], who considers Poisson arrivals, single server systems. Pressman [23] considers multi-channel systems with exponential service times. Viskov [28] obtains some results based on Prokhorov's work. Brody [4] and Iglehart [12] consider the case when $\rho \equiv 1$ in an $M|G|1$ queue (single server, Poisson arrivals and general service time). Iglehart obtains limit distributions for certain functionals on the queue length and the waiting time processes. For an expository paper on heavy traffic, the reader is referred to Kingman [17].

A feature of the above results is that they provide limit theorems when the time parameter goes to infinity. Also, in most cases the stochastic process considered is discretely indexed by time. Our aim will be to consider a stochastic process that has a continuous index (time $t \geq 0$) and show the convergence of its distribution at each time point. As far as the author is aware, the only work done in this area in queuing theory is by Iglehart [11]. He considers a $M|M|n$ system and shows that the queue length process, suitably translated and scaled, converges weakly to the Ornstein-Uhlenbeck process as the interarrival time goes to zero and the number of servers grows large, the traffic intensity being fixed at a value less than one. A feature of this process is that it is a birth and death process (B-D process). Further, the author exhibits a similar convergence for a B-D process arising in the machine repairman problem.

Now, we are ready to give an outline of the work done in this thesis. It is organized into the following chapters. In Chapter II, we consider the continuous time phase length process in a Poisson input, generalized Erlangian service queuing model (Luchak's model). This is a Markov process with a denumerable state space; this process has a lower barrier at 0 and it takes upward jumps of random magnitude and downward jumps of 1 (B-D process takes upward and downward jumps of magnitude 1). We look at a sequence of these Luchak systems and exhibit the convergence of the distribution of the phase length process to that of Brownian motion (at each time point) as the traffic intensity goes to unity (from above and below). This convergence is obtained by considering the corresponding convergence of the jump chain associated with the phase length Markov process. This approach of considering the jump chain has been used in genetics by Karlin and McGregor [13] and Kimura [14].

In Chapter III, some theorems for the maximum of a sequence of partial sums are proved. Two cases are considered depending on whether the mean of the basic random variables constituting the partial sums is zero or positive. We utilize concepts from fluctuation theory and ladder variables. The random variable, maximum of partial sums, occurs frequently in studying imbedded Markov chains in queuing systems. We find that the two cases mentioned above correspond to the cases when the traffic intensity is equal to or greater than unity.

In Chapter IV, the results obtained in Chapter III are used to obtain limit theorems for some single server queuing systems. The queue length and the waiting time are two of the processes considered when the traffic intensity is fixed at a value greater than or equal to one.

Finally, in Chapter V, we mention some possible directions for future research.

CHAPTER II

LUCHAK'S QUEUING MODEL

Luchak's queuing model is described as follows. Customers arrive according to a Poisson process at rate $\lambda > 0$. Each customer demands N phases of service, where N is a random variable with the distribution

$$\Pr\{N = n\} = c_n \qquad (n \geq 1) \quad . \tag{5}$$

The service time for each phase has the negative exponential distribution $\mu e^{-\mu v}\, dv$ $(0 < v < \infty)$. Thus, the service time for each customer has the density function $b(v)$, where

$$b(v) = \sum_{n=1}^{\infty} c_n \mu^n e^{-\mu v} \frac{v^{n-1}}{(n-1)!} \qquad (0 < v < \infty) \quad , \tag{6}$$

which is commonly called the general Erlangian density. The queue discipline is first-come-first-served.

This system has been studied by Luchak [19], who considers the transient behavior of the phase length process; this process is the number of phases present in the system, waiting to be served. Prabhu and Lalchandani [22] have analysed the transient state of this system via the bivariate process $\{Q(t), R(t)\}$; $Q(t)$ is the queue length at time t and $R(t)$ is the residual number of phases present at the service counter at time t.

In this chapter, we shall study the behavior of the phase length process as the traffic intensity approaches unity. We assume the existence of the first two moments of the random variable N and denote them by

$$a = \sum_{n=1}^{\infty} n\, c_n < \infty \;, \quad b = \sum_{n=1}^{\infty} n^2\, c_n < \infty \quad . \tag{7}$$

The moment generating function (m.g.f.) of N is denoted by

$$C(z) = \sum_{n=1}^{\infty} c_n\, z^n \qquad (0 \le z \le 1) \quad . \tag{8}$$

The traffic intensity of the system is given by

$$\rho = \frac{\lambda C'(1)}{\mu} = \frac{\lambda a}{\mu} \quad . \tag{9}$$

We define a sequence of Luchak systems indexed by n $(n \geq 1)$; the parameters for the n^{th} system are λ_n, μ_n and $\rho_n = \lambda_n a/\mu_n$. The number of phases present in the n^{th} system at time t is denoted by $Q_n(t)$ $(n \geq 1, t \geq 0)$; $\{Q_n(t); t \geq 0\}$ is a continuous time Markov process with the denumerable state space $S = \{0,1,2,\ldots\}$.

The method of approach used in this chapter is as follows.

We choose the parameters λ_n and μ_n so that $\rho_n \to 1$ as $n \to \infty$. Also, we suitably translate and scale the process $Q_n(t)$ so that the distribution of the normalized process converges to a non-degenerate distribution, for all t. The limit distribution is the Brownian motion process $X(t)$ with infinitesimal variance

$$\sigma^2 = \frac{1}{a} + \frac{b}{a^2} \quad . \tag{10}$$

The characteristic function of the process $X(t)$ is defined as

$$\begin{aligned} \psi(s;t) &= E(e^{isX(t)} | X(0) = y) \\ &= e^{-\frac{s^2\sigma^2 t}{2} + isy} \qquad (s \text{ real}, \ t \geq 0) \quad . \end{aligned} \tag{11}$$

For Sections 2.1 through 2.3 we choose

$$\lambda_n = n \ , \ \mu_n = a(n+\sqrt{n}) \ , \quad (n \geq 1) \quad , \tag{12}$$

so that

$$\rho_n = \frac{\lambda_n a}{\mu_n} = \frac{n}{n+\sqrt{n}} \uparrow 1 \quad \text{as} \quad n \to \infty \quad . \tag{13}$$

Also, the normalized process will be defined by

$$X_n(t) = \frac{Q_n(t)-n}{a\sqrt{n}} \qquad (n \geq 1, \ t \geq 0) \quad . \tag{14}$$

To study the convergence of distribution of the $X_n(t)$ process as $n \to \infty$, we define the sequence $\{\tilde{X}_n(k);\ k = 0,1,\dots\}$ $(n \geq 1)$ of Markov chains, called the jump chain of the Markov process $X_n(t)$, as follows:

$$\tilde{X}_n(k) = \frac{\tilde{Q}_n(k)-n}{a\sqrt{n}} \qquad (k = 0,1,\dots;\ n = 1,2,\dots) \tag{15}$$

where $\tilde{Q}_n(k)$ is the Markov chain with the following transitions:

$$\tilde{Q}_n(k+1) = \begin{cases} \tilde{Q}_n(k) + j & \text{w.p.}^{\dagger} \quad g_n\, c_j \quad (j \geq 1) \\ \tilde{Q}_n(k) - 1 & \text{w.p.} \quad h_n \end{cases} \text{ if } \tilde{Q}_n(k) > 0$$
$$\qquad\qquad \tilde{Q}_n(k) + j \quad \text{w.p.} \quad 1 \quad (j \geq 1) \ \text{ if } \tilde{Q}_n(k) = 0$$

$$(k = 0,1,\dots;\ n = 1,2,\dots) \quad , \tag{16}$$

$$g_n = \frac{\lambda_n}{\lambda_n+\mu_n} \qquad (n \geq 1) \quad , \tag{17}$$

and

$$h_n = 1 - g_n = \frac{\mu_n}{\lambda_n+\mu_n} \qquad (n \geq 1) \quad . \tag{18}$$

We also define the continuous time process[††]

---

† w.p. means "with probability."

†† $[x]$ denotes the largest integer less than or equal to x.

$$Y_n(t) = \tilde{X}_n([(\lambda_n+\mu_n)t]) \quad (n \geq 1,\ t \geq 0) \quad . \qquad (19)$$

$Y_n(t)$ is the step process obtained from $\tilde{X}_n(k)$.

To economize on notation we shall restrict our analysis to the case† $t \in [0,1]$. The results obtained in this chapter hold for $t \in [0,T]$ as well, for all finite T.

We shall organize our work into four sections. In Section 2.1, we show the convergence of the distributions of the sequence $Y_n(t)$ to the distribution of $X(t)$. In Section 2.2, we study some properties of the process $X_n(t)$. These enable us to show that the limiting distributions of $X_n(t)$ and $Y_n(t)$ are identical as $n \to \infty$ (Section 2.3). In Section 2.4, we consider a different set of values for λ_n and μ_n. They are chosen so that $\rho_n \uparrow 1$ as $n \to \infty$.

2.1. THE PROCESS $Y_n(t)$

To obtain the convergence of the distributions of the sequence $\{Y_n(t)\}$ $(n \geq 1)$ (Theorem 2.3), we need the asymptotic behavior of a certain probability associated with the Markov chain $\{\tilde{Q}_n(k)\}$. This is done in two consecutive steps (Theorems 2.1 and 2.2). For this purpose, we introduce the following notation.

Let the single step transition probabilities of the Markov chain $\{\tilde{Q}_n(k)\}$ be

† $[a,b]$ denotes the set of all points x such that $a \leq x \leq b$.

$$P_{ij}^{(k)}(n) = \Pr\{\tilde{Q}_n(k) = j \mid \tilde{Q}_n(0) = i\}$$

$$(k \geq 0,\ j \geq 0,\ i \geq 0,\ n \geq 1) \quad . \tag{20}$$

We shall denote the m.g.f. of $P_{ij}^{(k)}(n)$ by

$$H_k(z) = \sum_{j=0}^{\infty} z^j P_{ij}^{(k)}(n) \qquad (0 \leq z \leq 1),\ (k \geq 0) \quad , \tag{21}$$

and its power series by

$$G(z,w) = \sum_{k=0}^{\infty} w^k H_k(z) \qquad (0 \leq w < 1) \tag{22}$$

$$= \sum_{k=0}^{\infty} w^k \sum_{j=0}^{\infty} z^j P_{ij}^{(k)}(n) \quad . \tag{23}$$

From (20) and (21) it follows that

$$H_0(z) = z^i \quad . \tag{24}$$

We shall be interested in the behavior of $P_{i0}^{(k)}(n)$ as $n \to \infty$ and to do so, we study its power series in the following.

<u>Theorem 2.1</u>. <u>Let</u> $0 < w < 1$. <u>Then</u>

$$\sum_{k=0}^{\infty} w^k P_{i0}^{(k)}(n) = \frac{\lambda_n}{\mu_n} \frac{[\zeta_n(w)]^{i+1}}{w-\zeta_n(w)} \qquad (n \geq 1,\ i \geq 0) \quad ,$$

<u>where</u> $z = \zeta_n(w)$ <u>is the unique root in</u> $0 \leq z < 1$,

of the equation

$$f(z) = z - wh_n - wg_n z\, C(z) = 0 \quad . \tag{25}$$

Proof

The Chapman-Kolmogorov equations of the Markov chain $\{\tilde{Q}_n(k)\}$, which follow from (16), are

$$P_{ij}^{(k)}(n) = h_n\, P_{i,j+1}^{(k-1)}(n) + c_j\, P_{i0}^{(k-1)}(n) + g_n \sum_{r=1}^{j-1} P_{ir}^{(k-1)}(n)\, c_{r-i} \quad (j \geq 2)$$

$$P_{i1}^{(k)}(n) = h_n\, P_{i2}^{(k-1)}(n) + c_1\, P_{i0}^{(k-1)}(n)$$

$$P_{i0}^{(k)}(n) = h_n\, P_{i1}^{(k-1)}(n) \qquad \text{for all} \quad k \geq 1 \quad . \tag{26}$$

Upon simplifying the set of Equations (26), we obtain, by using (22)

$$H_k(z) = \{\frac{h_n}{z} + g_n\, C(z)\}H_{k-1}(z) + \{C(z) - g_n\, C(z) - \frac{h_n}{z}\}P_{i0}^{(k-1)}(n) \qquad (k \geq 1) \quad . \tag{27}$$

Using (23), (24) and (27), we obtain

$$G(z,w) = \sum_{k=0}^{\infty} w^k P_{i0}^{(k)}(n) + z \frac{z^i - \{1-wC(z)\} \sum_{k=0}^{\infty} w^k P_{i0}^{(k)}(n)}{z - wh_n - wg_n z\, C(z)} . \tag{28}$$

Now, consider the function

$$f(z) = z - wh_n - wg_n z\, C(z) . \tag{29}$$

We have $f(0) = -wh_n < 0$, $f(1) = 1-wh_n-wg_n = 1-w > 0$, for $0 < w < 1$. Also,

$$\frac{\partial^2 f(z)}{\partial z^2} = -wg_n[zC''(z) + 2C'(z)] < 0 ,$$

$$\text{for } 0 < z,\ w < 1 .$$

Hence, $f(z)$ is concave for $0 < z < 1$ and thus $f(z) = 0$ has a unique root $z = \zeta_n(w)$ in $0 < z < 1$ for a fixed w, $0 < w < 1$.

Now, since $G(z,w)$ converges in the region $0 < z,\ w < 1$, the roots of the numerator and the denominator in the last term in (28) must coincide. Thus,

$$\sum_{k=0}^{\infty} w^k P_{i0}^{(k)}(n) = \frac{[\zeta_n(w)]^i}{1-w\, C(\zeta_n(w))} . \tag{30}$$

From (29), we have by substituting $z = \zeta_n(w)$,

$$\zeta_n(w) - wh_n - wg_n \zeta_n(w) C(\zeta_n(w)) = 0 \quad , \tag{31}$$

or alternatively,

$$1 - w C(\zeta_n(w)) = \frac{h_n}{g_n} \frac{w-\zeta_n(w)}{\zeta_n(w)} \quad . \tag{32}$$

The theorem thus follows from (17), (18), (30) and (32).

Before we proceed to Theorem 2.2, we shall need the following lemma.

Lemma 2.1. Let $q_n \geq 0$ and suppose that

$$Q(s) = \sum_{n=0}^{\infty} q_n s^n < \infty \quad \text{for} \quad 0 \leq s < 1 \quad .$$

If $L(.)$ is slowly varying at infinity† and $0 \leq \rho < \infty$, then each of the two relations

† A positive function $f(x)$ defined on $(0,\infty)$ is said to vary slowly at infinity if

$$\lim_{x \to \infty} \frac{f(cx)}{f(x)} = 1 \quad , \quad \text{for all} \quad c > 0 \quad .$$

$$Q(s) \sim (1-s)^{-\rho} \, L(\tfrac{1}{1-s}) \;, \quad s \uparrow 1$$

and

$$q_0 + q_1 + \ldots + q_n \sim \frac{1}{\Gamma(\rho+1)} \, n^{\rho} \, L(n) \;, \quad n \to \infty$$

implies the other†.

This lemma is due to Karmata, and a simplified proof is given in Feller [8], p. 423. A modification of this lemma is needed in Chapter III and is presented there as Lemma 3.1.

Now, we prove the following.

Theorem 2.2. Let $t \in [0,1]$. Then

$$\lim_{n \to \infty} \frac{\sqrt{n}}{\lambda_n + \mu_n} \sum_{k=0}^{[(\lambda_n+\mu_n)t]-1} P_{i0}^{(k)}(n) = \frac{t}{a+1} \quad .$$

Proof

We shall consider the power series

$$Q_n(w) = \sum_{k=0}^{\infty} w^k \sqrt{n} \, P_{i0}^{(k)}(n) \quad , \quad (n \geq 1, \; 0 \leq w < 1) \quad . \tag{33}$$

From Theorem 2.1, we see that

---

† $a(x) \sim b(x)$, $x \to c$ implies that

$$\lim_{x \to c} \frac{a(x)}{b(x)} = 1 \quad .$$

$$Q_n(w) = \frac{1}{1-w} A_n(w) \quad , \quad (0 < w < 1) \quad , \tag{34}$$

where

$$A_n(w) = \frac{\lambda_n}{\mu_n} \cdot \frac{\sqrt{n}\,[\zeta_n(w)]^{i+1}(1-w)}{w-\zeta_n(w)} \quad , \quad (0 < w < 1) \quad . \tag{35}$$

By differentiating Equation (31) with respect to w, we get

$$\zeta_n'(w) - h_n - g_n\{\zeta_n(w)\, C(\zeta_n(w)) + w\zeta_n'(w)\, C(\zeta_n(w))$$
$$+ w\zeta_n(w)\, C'(\zeta_n(w))\, \zeta_n'(w)\} = 0 \quad .$$

Since $\zeta_n(w) \to 1$ as $w \uparrow 1$, and $C'(1) = a$, we get

$$\lim_{w \uparrow 1} \zeta_n'(w) = \frac{1}{h_n - g_n a} = \frac{\lambda_n + \mu_n}{\mu_n - \lambda_n a} \quad . \tag{36}$$

Hence, from (35) we have

$$\lim_{w \uparrow 1} A_n(w) = \frac{\lambda_n}{\mu_n} \cdot \sqrt{n} \lim_{w \uparrow 1} \frac{1-w}{w-\zeta_n(w)}$$

$$= \frac{\lambda_n}{\mu_n} \sqrt{n} \lim_{w \uparrow 1} \frac{-1}{1-\zeta_n'(w)} \text{ , using L'Hospital's rule}$$

$$= \frac{1}{a+1} \cdot \frac{\sqrt{n}}{\sqrt{n}+1} \text{ , using (12) and (36)} \quad .$$

Letting

$$A_n = \frac{1}{a+1} \frac{\sqrt{n}}{\sqrt{n}+1} \quad , \tag{37}$$

it follows from (34) that

$$Q_n(w) \sim (1-w)^{-1} A_n \quad , \quad \text{as} \quad w \uparrow 1 \ , \quad \text{for all} \quad n \quad .$$

Since $\sqrt{n}\, P_{i0}^{(k)}(n) \geq 0$ and $Q_n(w) < \infty$ for $0 \leq w < 1$, the hypothesis of Lemma 2.1 are satisfied by $Q_n(w)$, with $\rho = 1$. Hence, it follows that

$$\sum_{k=0}^{\infty} \sqrt{n}\, P_{i0}^{(k)}(n) \sim K A_n \quad \text{as} \quad K \to \infty \ , \tag{38}$$

for each fixed n.

Now, fix $\varepsilon > 0$. Due to (37), we can find a N_1 such that

$$\frac{1}{a+1} - \frac{\varepsilon}{4} < A_n \leq \frac{1}{a+1} \quad \text{for} \quad n > N_1 \quad . \tag{39}$$

Since

$$\frac{(\lambda_n+\mu_n)t-2}{\lambda_n+\mu_n} \leq \frac{[(\lambda_n+\mu_n)t]-1}{\lambda_n+\mu_n} \leq \frac{(\lambda_n+\mu_n)t}{\lambda_n+\mu_n} \quad ,$$

we can find a N_2 such that

$$t - \frac{\varepsilon}{4} \leq \frac{[(\lambda_n+\mu_n)t]-1}{\lambda_n+\mu_n} \leq t \quad \text{for} \quad n > N_2 \quad . \tag{40}$$

Because of (38), we can find a N_3 such that

$$1 - \frac{\varepsilon}{4} < \frac{\sum_{k=0}^{[(\lambda_n+\mu_n)t]-1} \sqrt{n}\; P_{i0}^{(k)}(n)}{\{[(\lambda_n+\mu_n)t]-1\}A_n} < 1 + \frac{\varepsilon}{4} \quad \text{for} \quad n > N_3 \quad . \tag{41}$$

Now, let $N = \max(N_1,N_2,N_3)$. From (41), it follows that

$$(1-\frac{\varepsilon}{4})\; A_n\{\frac{[(\lambda_n+\mu_n)t]-1}{\lambda_n+\mu_n}\} < \frac{\sum_{k=0}^{[(\lambda_n+\mu_n)t]-1} \sqrt{n}\; P_{i0}^{(k)}(n)}{\lambda_n+\mu_n}$$

$$< (1+\frac{\varepsilon}{4})\; A_n\{\frac{[(\lambda_n+\mu_n)t]-1}{\lambda_n+\mu_n}\} \quad \text{for} \quad n > N \quad .$$

Using (39) and (40), it follows that

$$(1-\frac{\varepsilon}{4})(\frac{1}{a+1} - \frac{\varepsilon}{4})(t-\frac{\varepsilon}{4}) < \frac{\sum_{k=0}^{[(\lambda_n+\mu_n)t]-1} \sqrt{n}\; P_{i0}^{(k)}(n)}{\lambda_n+\mu_n}$$

$$< (1+\frac{\varepsilon}{4})\; \frac{1}{a+1}\, t \quad \text{for} \quad n > N \quad .$$

By some simplification, we get

$$\frac{t}{a+1} - \varepsilon < \frac{\sum\limits_{k=0}^{[(\lambda_n+\mu_n)t]-1} \sqrt{n}\, P_{i0}^{(k)}(n)}{\lambda_n+\mu_n} < \frac{t}{a+1} + \varepsilon$$

for $n > N$.

Since ε was arbitrary, the theorem follows.

Now, we are ready to prove the convergence of the distributions of the sequence $Y_n(t)$. We shall accomplish it by showing the convergence of the corresponding characteristic functions and then appealing to the Levy continuity theorem. We define the following notation.

The characteristic functions of $Y_n(t)$, $\tilde{X}_n(k)$ and N are denoted by

$$\psi_n(s,t) = E\{e^{is\, Y_n(t)}\} \quad (n \geq 1,\ t \geq 0,\ s \quad \text{real}) \quad , \qquad (42)$$

$$\psi_n(s,k) = E\{e^{is\, \tilde{X}_n(k)}\} \quad (n \geq 1,\ k \geq 0,\ s \quad \text{real}) \quad , \qquad (43)$$

and

$$\phi(s) = E\{e^{isN}\} \qquad (s \quad \text{real}) \quad . \qquad (44)$$

We prove the following.

Theorem 2.3. Let $t \in [0,1]$ and $-\infty < y < \infty$. Then

$$\lim_{n\to\infty} \Pr\{Y_n(t) \leq x | Y_n(0) = y\} = \Pr\{X(t) \leq x | X(0) = y\}$$

$$= \frac{(2\pi t)^{-\frac{1}{2}}}{\sigma} \int_{-\infty}^{x} e^{-\frac{(z-y)^2}{2\sigma^2 t}} dz \quad ,$$

$$(-\infty < x < \infty) \quad .$$

Proof

From (15) and (43), it follows that

$$\tilde{\psi}_n(s,k+1) = E\{e^{is\frac{\tilde{Q}_n(k)-n}{a\sqrt{n}}}\} \ , \ (k \geq 0) \quad .$$

Using (16), we have

$$\tilde{\psi}_n(s,k+1) = \tilde{\psi}_n(s,k) \ \{g_n \Pr\{\tilde{Q}_n(k) > 0\} \ \phi(\frac{s}{a\sqrt{n}})$$

$$+ h_n \Pr\{\tilde{Q}_n(k) > 0\} e^{-\frac{is}{a\sqrt{n}}} + \Pr\{\tilde{Q}_n(k) = 0\} \ \phi(\frac{s}{a\sqrt{n}})\}$$

$$= \tilde{\psi}_n(s,k) \ G_n(s,k) \quad , \quad (k \geq 0) \qquad (45)$$

where, using (17), we have

$$G_n(s,k) = (\lambda_n+\mu_n)^{-1} \{\lambda_n \, \phi(\frac{s}{a\sqrt{n}}) + \mu_n e^{-\frac{is}{a\sqrt{n}}}$$

$$+ \mu_n \Pr\{\tilde{Q}_n(k) = 0\} \{\phi(\frac{s}{a\sqrt{n}}) - e^{-\frac{is}{a\sqrt{n}}}\}\}$$

$$(k \geq 0) \quad . \quad (46)$$

From (45), we obtain by recursion,

$$\tilde{\psi}_n(s,k) = \{\prod_{j=0}^{k-1} G_n(s,j)\} \, \tilde{\psi}_n(s,0) \quad , \quad (k \geq 1) \quad . \qquad (47)$$

Substituting the values of λ_n and μ_n from (12) in (46), we get, by using the Taylor series development of the characteristic function, $\phi(s/a\sqrt{n})$, of N†,

$$\{(a+1)n + a\sqrt{n}\} \, G_n(s,k) = n \, \phi(\frac{s}{a\sqrt{n}}) + a(n+\sqrt{n})e^{-\frac{is}{a\sqrt{n}}}$$

$$+ a(n+\sqrt{n}) \Pr\{\tilde{Q}_n(k) = 0\} \{\phi(\frac{s}{a\sqrt{n}}) - e^{-\frac{is}{a\sqrt{n}}}\}$$

$$= n \, \{1 + \frac{is}{\sqrt{n}} - \frac{s^2}{2a^2n} b + o(\frac{1}{n})\}$$

$$+ a(n+\sqrt{n}) \, \{1 - \frac{is}{a\sqrt{n}} - \frac{s^2}{2a^2n} + o(\frac{1}{n})\} + a(n+\sqrt{n}) \Pr\{Q_n(k) = 0\}$$

$$\{\frac{is}{\sqrt{n}} (1+\frac{1}{a}) - \frac{s^2}{2a^2n} (b+1) + o(\frac{1}{n})\} \quad .$$

† $o(\frac{1}{n})$ means $n \, o(\frac{1}{n}) \to 0$ as $n \to \infty$.

Further simplification gives†

$$\{(a+1)n + a\sqrt{n}\}\ G_n(s,k) = (a+1)n + a\sqrt{n} - is - \frac{s^2}{2}\left(\frac{b}{a^2} + \frac{1}{a}\right)$$

$$+ is(a+1)(\sqrt{n}+1)\ \Pr\{\tilde{Q}_n(k) = 0\}$$

$$- \frac{s^2}{2}(b+1)\ \frac{(\sqrt{n}+1)}{\sqrt{n}}\ \Pr\{\tilde{Q}_n(k) = 0\} + O(1) \quad .$$

Dividing by $((a+1)n + a\sqrt{n})$ and using the Taylor expansion for $\ell n(1+z)$ as $z \to 0$, we have

$$\ell n\ G_n(s,k) \sim - \frac{(is + \frac{s^2}{2}(\frac{b}{a^2} + \frac{1}{a}))}{(a+1)n + a\sqrt{n}} + \frac{is(a+1)(\sqrt{n}+1)}{(a+1)n + a\sqrt{n}}\ \Pr\{\tilde{Q}_n(k) = 0\}$$

$$- \frac{s^2}{2}\ \frac{(b+1)(\sqrt{n}+1)}{\{(a+1)n + a\sqrt{n}\}\sqrt{n}}\ \Pr\{\tilde{Q}_n(k) = 0\}$$

$$+ \frac{O(1)}{(a+1)n + a\sqrt{n}} \quad \text{as} \quad n \to \infty \quad . \tag{48}$$

Taking logarithms of Equation (47), it follows that

$$\ell n\ \tilde{\psi}_n(s,k) = \ell n\ \tilde{\psi}_n(s,0) + \sum_{j=0}^{k-1} \ell n\ G_n(s,j) \quad . \tag{49}$$

† $a_n = O(1)$ means a_n is bounded by 1 as $n \to \infty$.

Using (19), (42) and (47), we get

$$\ell n\ \psi_n(s,t) = \ell n\ \psi_n(s,[(\lambda_n+\mu_n)t])$$

$$= \ell n\ \psi_n(s,0) + \sum_{j=0}^{[(\lambda_n+\mu_n)t]-1} \ell n\ G_n(s,j) \quad . \tag{50}$$

Taking $Y_n(0) = y$ and substituting from (48) in (50),

$$\ell n\ \psi_n(s,t) \sim isy - (is + \frac{s^2}{2}(\frac{b}{a^2} + \frac{1}{a})) \sum_{j=0}^{[(\lambda_n+\mu_n)t]-1} \frac{1}{(a+1)n + \sqrt{n}}$$

$$+ \frac{is(a+1)(\sqrt{n}+1)}{(a+1)n + a\sqrt{n}} \sum_{j=0}^{[(\lambda_n+\mu_n)t]-1} \Pr\{\tilde{Q}_n(j) = 0\}$$

$$- \frac{s^2}{2} \frac{(b+1)(\sqrt{n}+1)}{\{(a+1)n + a\sqrt{n}\}\sqrt{n}} \sum_{j=0}^{[(\lambda_n+\mu_n)t]-1} \Pr\{\tilde{Q}_n(j) = 0\}$$

$$+ \frac{O(1)}{(a+1)n + \sqrt{n}} \{[(\lambda_n+\mu_n)t]-1\} \quad \text{as} \quad n \to \infty \quad . \tag{51}$$

Now, using Theorem 2.2, we have from (51)

$$\lim_{n\to\infty} \psi_n(s,t) = isy - \frac{s^2}{2}(\frac{b}{a^2} + \frac{1}{a})t \quad . \tag{52}$$

Now, (11) and (52), along with the Levy continuity theorem, completes the proof of the theorem.

2.2. SOME PROPERTIES OF THE $X_n(t)$ PROCESS

In this section, we shall study asymptotic properties of the random variable $N_n(t)$ defined as

$$N_n(T) = \text{number of jumps of the } X_n(t) \text{ process in the time interval } [0,T],\ T \text{ finite} \quad . \tag{53}$$

Since the processes $Q_n(t)$ and $X_n(t)$ are just translates of each other, we could replace $X_n(t)$ by $Q_n(t)$ in (53). Let us denote the mean and the variance of $N_n(t)$ by

$$U_n(t) = E\{N_n(t) | Q_n(0) = [n + a\sqrt{n}y]\}$$

$$V_n(t) = Var\{N_n(t) | Q_n(0) = [n + a\sqrt{n}y]\} \quad , \tag{54}$$

respectively. We show (Theorem 2.5) that $\{U_n(t)-(\lambda_n+\mu_n)t\} \to 0$ as $n \to \infty$; similarly for $V_n(t)$. Before we can prove Theorem 2.5, we need an initial result that is contained in Theorem 2.4. For this, we need a result of Luchak [19] presented in the following.

Lemma 2.2. We have

$$\int_0^\infty e^{-\theta t} E\{z^{Q_n(t)} | Q_n(0) = i\} dt$$

$$= \frac{z^{i+1} - \frac{(1-z)}{(1-\zeta_n)} \zeta_n^{i+1}}{(\theta+\lambda_n+\mu_n)z - \lambda_n z\, C(z) - \mu_n} \qquad (0 \leq z < 1,\ \theta > 0) \quad ,$$

where $z = \zeta_n(\theta) \equiv \zeta_n$ is the unique root of the equation

$$(\theta+\lambda_n+\mu_n)z - \lambda z\, C(z) - \mu_n = 0 \quad \text{in} \quad 0 \leq z \leq 1 \quad .$$

The proof can be found in Luchak's paper and shall not be given here. Based on this lemma, we have the following.

Theorem 2.4. For any T finite,

$$\lim_{n \to \infty} n \Pr\{Q_n(t) = 0 | Q_n(0) = [n + a\sqrt{n}y]\} = 0 \quad ,$$

almost everywhere on the set $\{t | 0 \leq t \leq T\}$.

Proof

The time transform of $n \Pr\{Q_n(t) = 0 | Q_n(0)\}$ is obtained by substituting $z = 0$ and multiplying by n in the result of Lemma 2.2. It is thus

$$\int_0^\infty e^{-\theta t}\; n\, \Pr\{Q_n(t) = 0 \mid Q_n(0) = [n + a\sqrt{n}y]\}dt$$

$$= \frac{n\, \zeta_n^{[n + a\sqrt{n}y]}}{\mu_n(1-\zeta_n)}\quad , \quad \theta > 0 \quad . \tag{55}$$

We shall show that this transform converges to 0 as $n \to \infty$. To do so, we first show that

$$\zeta_n(\theta) < 1 - \frac{1}{\sqrt{n}} \quad \text{for all} \quad \theta > 2a + \frac{b}{2} + 2, \text{ and large } n\ . \tag{56}$$

Consider the function

$$f(z) = (\theta+\lambda_n+\mu_n)z - \lambda z\, C(z) - \mu_n \tag{57}$$

at the point $1 - 1/\sqrt{n}$. We have, using (12),

$$f(1-\frac{1}{\sqrt{n}}) = (\theta+(a+1)n+a\sqrt{n})(1-\frac{1}{\sqrt{n}}) - n(1-\frac{1}{\sqrt{n}})\ C(1-\frac{1}{\sqrt{n}}) - a(n+\sqrt{n})$$

$$= \theta(1-\frac{1}{\sqrt{n}}) + n - (a+1)\sqrt{n} - a + (\sqrt{n}-n)\ C(1-\frac{1}{\sqrt{n}})$$

$$= \theta(1-\frac{1}{\sqrt{n}}) + n - (a+1)\sqrt{n} - a + (\sqrt{n}-n)(1-\frac{a}{\sqrt{n}} + \frac{b}{2n} + o(\frac{1}{n}));$$

by expanding $C(1-\frac{1}{\sqrt{n}})$ about 1 for large n,

$$= \theta(1-\frac{1}{\sqrt{n}}) - (2a + \frac{b}{2}) + o(n^{-\frac{1}{2}}) \quad \text{for large } n$$

$$> \theta(1-\frac{1}{\sqrt{n}}) - (2a + \frac{b}{2} + 1)$$

$$> 1 - \frac{2a + \frac{b}{2} + 2}{\sqrt{n}} \quad \text{for } \theta > 2a + \frac{b}{2} + 2$$

$$> 0 \quad \text{for } n > (2a + \frac{b}{2} + 2)^2 \quad . \tag{58}$$

Since $f(\zeta_n) = 0$, $f(0) = -\mu_n < 0$, $f(1) = \theta > 0$ and ζ_n is the unique root of $f(z) = 0$ in $0 < z < 1$, (56) follows from (58). Thus we have for $\theta > 2a + b/2 + 2$,

$$\frac{n\,\zeta_n^{[n + a\sqrt{n}y]}}{\mu_n(1-\zeta_n)} = \frac{\zeta_n^{[n + a\sqrt{n}y]}}{a(1+\frac{1}{\sqrt{n}})(1-\zeta_n)} \text{ , using (12)}$$

$$\leq \frac{\sqrt{n}(1-\frac{1}{\sqrt{n}})^{[n + a\sqrt{n}y]}}{a(1+\frac{1}{\sqrt{n}})} \quad \text{for large } n,$$

$$\text{using (56)} \quad . \qquad (59)$$

Now, taking logarithms, we get

$$\ell n\{\sqrt{n}(1-\frac{1}{\sqrt{n}})^{n + a\sqrt{n}y}\} = \ell n\sqrt{n} + (n + a\sqrt{n}y)\ \ell n(1-\frac{1}{\sqrt{n}})$$

$$= \ell n\sqrt{n} - (n + a\sqrt{n}y)(\frac{1}{\sqrt{n}} + \frac{1}{2n} + o(\frac{1}{n}))$$

$$= \ell n\sqrt{n} - \sqrt{n} - (\frac{1}{2} + ay + O(1))$$

$$\to -\infty \quad \text{as} \quad n \to \infty \quad .$$

This means that

$$\sqrt{n}(1-\frac{1}{\sqrt{n}})^{n + a\sqrt{n}y} \to 0 \quad \text{as} \quad n \to \infty \quad . \qquad (60)$$

Hence from (59), we obtain

$$\lim_{n\to\infty} \frac{n\,\zeta_n^{[n+a\sqrt{n}y]}}{\mu_n(1-\zeta_n)} \le \lim_{n\to\infty} \frac{n\,\zeta_n^{n+a\sqrt{n}y-1}}{\mu_n(1-\zeta_n)}$$

$$= \lim_{n\to\infty} \frac{n\left(1-\frac{1}{\sqrt{n}}\right)^{n+a\sqrt{n}y-1}}{a\left(1+\frac{1}{\sqrt{n}}\right)}$$

$$= 0 \text{ , using (60)} \quad . \qquad (61)$$

Since $n\,\zeta_n^{[n+a\sqrt{n}y]}/\mu_n(1-\zeta_n) \geq 0$ for all n, (55) and (61) show that

$$\lim_{n\to\infty} \int_0^\infty e^{-\theta t}\, n\, \Pr\{Q_n(t) = 0 \mid Q_n(0) = [n + a\sqrt{n}y]\}dt = 0$$

$$\text{for } \theta > 2a + \frac{b}{2} + 2 \quad .$$

Hence, it follows from the extended continuity theorem for Laplace transforms (see Feller [8], p. 410) that

$$\lim_{n\to\infty} \int_0^T n\, \Pr\{Q_n(t) = 0 \mid Q_n(0) = [n + a\sqrt{n}y]\}dt = 0$$

for all finite T. The theorem is then an immediate consequence.

Now we are in a position to exhibit the following asymptotic behavior of $U_n(t)$ and $V_n(t)$.

<u>Theorem 2.5</u>. <u>Let</u> $0 \leq T < \infty$. <u>Then</u>

$$\lim_{n \to \infty} \{U_n(t) - (\lambda_n + \mu_n)t\} = 0 \quad \text{for} \quad t \in [0,T] \quad . \tag{62}$$

<u>Also</u>

$$\lim_{n \to \infty} \{V_n(t) - (\lambda_n + \mu_n)t\} = 0 \quad \text{for} \quad t \in [0,T] \quad . \tag{63}$$

<u>Proof</u>

We denote the probability of a single jump of the $Q_n(t)$ process (or equivalently, the $X_n(t)$ process) in a small time interval by[†]

$$P_n(t,h) = \Pr\{\text{one jump in } (t,t+h)\} \quad .$$

From the definition of $Q_n(t)$, we have

$$\begin{aligned} P_n(t,h) & \\ &= \Pr\{Q_n(t) = 0\}.\{\lambda_n h + o(h)\} + \Pr\{Q_n(t) > 0\}.\{(\lambda_n + \mu_n)h + o(h)\} \\ &= (\lambda_n + \mu_n)h - h\mu_n \Pr\{Q_n(t) = 0\} + o(h) \quad . \end{aligned} \tag{64}$$

Here, we have suppressed the initial state $Q_n(0) = [n + a\sqrt{n}y]$ and shall do so the same throughout this proof.

† The set (a,b) denotes all points x such that $a < x < b$.

For small $h > 0$, using (53), we can write

$$N_n(t+h) = N_n(t)\ \{1-P_n(t,h)\} + \{N_n(t)+1\}\ P_n(t,h) \quad .$$

Taking expectations and simplifying, we have

$$\frac{U_n(t+h) - U_n(t)}{h} = \frac{P_n(t,h)}{h} \quad . \tag{65}$$

Substituting from (64) and taking limits as $h \to 0$,

$$\begin{aligned} \frac{d}{dt}\{U_n(t)\} &= (\lambda_n+\mu_n) - \mu_n \Pr\{Q_n(t) = 0\} \\ &= (\lambda_n+\mu_n) - a(n+\sqrt{n}) \Pr\{Q_n(t) = 0\} \quad . \end{aligned} \tag{66}$$

Transposing and using Theorem 2.4, we get

$$\lim_{n \to \infty} \{\frac{d}{dt}\{U_n(t)\} - (\lambda_n+\mu_n)\} = 0 \ , \quad t \in [0,T] \ , \tag{67}$$

almost everywhere.

The first part of the theorem follows from (63) by using the Lebesque dominated convergence theorem (see Rudin [25], pp. 246-7) and noting that $U_n(0) = 0$.

For the second part of the theorem, we have by definition

$$V_n(t) = E\{(N_n(t) - U_n(t))^2\} = E\{(N_n(t))^2\} - U_n^2(t) \ . \tag{68}$$

Also by definition of $N_n(t)$, we have

$$(N_n(t+h))^2 = (N_n(t))^2 \{1-P_n(t,h)\} + \{N_n(t)+1)\}^2 P_n(t,h) \quad . \tag{69}$$

From (68) and (69), it follows that

$$\frac{V_n(t+h) - V_n(t)}{h} = \frac{U_n^2(t) - U_n^2(t+h) + 2P_n(t,h)\, U_n(t) + P_n(t,h)}{h}$$

$$= \frac{- P_n^2(t,h) + P_n(t,h)}{h} \quad ,$$

using (65) and simplifying.

Using (64) and taking limits as $h \to 0$, we obtain

$$\frac{d}{dt}\{V_n(t)\} = (\lambda_n+\mu_n) - \mu_n \Pr\{Q_n(t) = 0\} \quad . \tag{70}$$

Noting the similarity of (70) to (66), the second assertion of the theorem is similarly proved.

2.3. CONVERGENCE IN DISTRIBUTION OF $X_n(t)$

In this section, we show (Theorem 2.6) that the distribution of $X_n(t)$ converges to that of $X(t)$ as $n \to \infty$. Besides using the results of Theorems 2.3 and 2.5, we need the continuity (in t) of the distribution function of $X(t)$. This follows readily since the distribution function of $X(t)$ is given by

$$\Pr\{X(t) \le x \mid X(0) = y\} = \frac{(2\pi t)^{-\frac{1}{2}}}{\sigma} \int_{-\infty}^{x} \exp\{-\frac{(z-y)^2}{2\sigma^2 t}\} dz \quad ,$$

$$(-\infty < x,\ y < \infty) \quad . \qquad (71)$$

Before proving Theorem 2.6, we mention the following relation that follows from the definition of the processes.

$$\Pr\{X_n(t) \le x \mid N_n(t) = j\} = \Pr\{\tilde{X}_n(j) \le x\} \quad , \qquad (72)$$

for all x, t and $n \ge 1$.

Theorem 2.6. Let $t \in [0,1]$ and $-\infty < y < \infty$. Then

$$\lim_{n \to \infty} \Pr\{X_n(t) \le x \mid X_n(0) = y\} = \Pr\{X(t) \le x \mid X(0) = y\} \ ,$$

for $-\infty < x < \infty$.

Proof†

We have, using definition of conditional probability,

† We shall omit the initial state y while writing the probabilities, to economize on notation. Note that as $n \to \infty$, $X_n(0) = y \iff Q_n(0) = [n + a\sqrt{n}y]$, from (14). Also as $n \to \infty$, $\tilde{X}_n(0) = y \iff \tilde{Q}_n(0) = [n + a\sqrt{n}y]$, from (15).

$$\Pr\{X_n(t) \le x\} - \Pr\{X(t) \le x\}$$

$$= \sum_{j=0}^{\infty} [\Pr\{X_n(t) \le x | N_n(j) = j\} - \Pr\{X(t) \le x\}] \Pr\{N_n(t) = j\}$$

$$= \sum_{j=0}^{\infty} \{\Pr\{\tilde{X}_n(j) \le x\} - \Pr\{\tilde{X}(t) \le x\}\} \Pr\{N_n(t) = j\} \quad ,$$

using (72);

$$= \sum_{j=0}^{\infty} [\Pr\{\tilde{X}_n(j) \le x\} - \Pr\{\tilde{X}_n([(\lambda_n+\mu_n)t]) \le x\}] \Pr\{N_n(t) = j\}$$

$$+ \Pr\{Y_n(t) \le x\} - \Pr\{X(t) \le x\} \quad , \tag{73}$$

using the definition (19).

Fix x, $t \in [0,1]$ and $\varepsilon > 0$. By Theorem 2.5, we can choose a N_1 such that for $n > N_1$,

$$|\Pr\{Y_n(t) \le x\} - \Pr\{X(t) \le x\}| < \frac{\varepsilon}{2} \quad . \tag{74}$$

Now, choose K and N_2 such that for $n > N_2$,

$$\Pr\{|\frac{N_n(t) - (\lambda_n+\mu_n)t}{\{(\lambda_n+\mu_n)t\}^{\frac{1}{2}}}| > K\} < \frac{\varepsilon}{8} \quad . \tag{75}$$

This is possible because of the following argument.

Choose N_3 such that (Theorem 2.5)

$$|U_n(t) - (\lambda_n+\mu_n)t| < 1 \quad \text{for} \quad n > N_3 \quad . \tag{76}$$

Choose N_4 such that (Theorem 2.5)

$$\left|\left\{\frac{V_n(t)}{(\lambda_n+\mu_n)t}\right\}^{\frac{1}{2}}\right| < \frac{3}{2} \quad \text{for} \quad n > N_4 \quad . \tag{77}$$

Choose N_5 such that

$$V_n(t) > 1 \quad \text{for} \quad n > N_5 \quad . \tag{78}$$

Let $N_2 = \max(N_3,N_4,N_5)$ and let K be such that

$$\left(\frac{2K}{3} - 1\right)^2 > \frac{8}{\varepsilon} \quad . \tag{79}$$

Now, for $n > N_2$ and K, we have

$$\Pr\{|\frac{N_n(t) - (\lambda_n+\mu_n)t}{\{(\lambda_n+\mu_n)t\}^{\frac{1}{2}}}| > K\}$$

$$\leq \Pr\{\frac{|N_n(t) - (\lambda_n+\mu_n)t|}{\{V_n(t)\}^{\frac{1}{2}}} > \frac{2K}{3}\} \text{ , due to (77) ,}$$

$$\leq \Pr\{\frac{|N_n(t) - U_n(t)| + |U_n(t) - (\lambda_n+\mu_n)t|}{\{V_n(t)\}^{\frac{1}{2}}} > \frac{2K}{3}\}$$

$$\leq \Pr\{\frac{|N_n(t) - U_n(t)|}{\{V_n(t)\}^{\frac{1}{2}}} > \frac{2K}{3} - 1\} \text{ , due to (76) and (78) ,}$$

$$\leq \frac{1}{(\frac{2K}{3} - 1)^2} \text{ , by Chebyshev's inequality ,}$$

$$< \frac{\varepsilon}{8} \text{ , by (79) .}$$

Thus (75) is proved.

Now, we are in a position to show that for large n, the first term on the right hand side of (78) is less than $\varepsilon/2$. To do so, we shall define the set

$$J = \{j|\ |\frac{j - (\lambda_n+\mu_n)t}{((\lambda_n+\mu_n)t)^{\frac{1}{2}}}| > K\} \quad . \tag{80}$$

Then, we can write

$$\left|\sum_{j=0}^{\infty} [\Pr\{X_n(j) \leq x\} - \Pr\{X_n([(\alpha_n+\mu_n)t]) \leq x\}] \Pr\{N_n(t) = j\}\right|$$

$$\leq \sum_{j \in J} 2.\Pr\{N_n(t) = j\}$$

$$+ \sum_{j \notin J} |\Pr\{\tilde{X}_n(j) \leq x\} - \Pr\{\tilde{X}_n([(\alpha_n+\mu_n)t]) \leq x\}| \Pr\{N_n(t) = j\} . \tag{81}$$

But, due to continuity (in t) of $\Pr\{X(t) \leq x\}$ and Theorem 2.5, we can choose a N_6 such for $n > N_6$ and $j \notin J$,

$$|\Pr\{\tilde{X}_n(j) \leq x\} - \Pr\{\tilde{X}_n([(\alpha_n+\mu_n)t]) \leq x\}| < \frac{\varepsilon}{4} . \tag{82}$$

Now, we are ready to put the pieces together. Let $N = \max(N_1, N_2, N_6)$. From (73), we have for $n > N$,

$$|\Pr\{X_n(t) \leq x\} - \Pr\{X(t) \leq x\}|$$

$$\leq |\sum_{j=0}^{\infty} \Pr\{\tilde{X}_n(j) \leq x\} - \Pr\{\tilde{X}_n([(\lambda_n+\mu_n)t]) \leq x\}\} \Pr\{N_n(t) = j\}|$$

$$+ |\Pr\{Y_n(t) \leq x\} - \Pr\{X(t) \leq x\}|$$

$$\leq \sum_{j \in J} 2.\Pr\{N_n(t) = j\}$$

$$+ \sum_{j \notin J} |\Pr\{\tilde{X}_n(j) \leq x\} - \Pr\{\tilde{X}_n([(\lambda_n+\mu_n)t]) \leq x\}| \Pr\{N_n(t) = j\}$$

$$+ \frac{\varepsilon}{2} \quad , \text{ due to (75) and (81)} \quad ,$$

$$< \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{\varepsilon}{2} \quad , \text{ due to (75) and (82)} \quad ,$$

$$= \varepsilon \quad . \tag{83}$$

Since ε is arbitrary, the theorem follows from (83).

Thus, in Theorem 2.6, we have shown that, given $X_n(0) = y$, the distribution of $X_n(t)$ converges to that of $X(t)$ as $n \to \infty$. Since $\rho_n = n/n+\sqrt{n}$, we can say, loosely speaking, that the phase length process at time t is approximately distributed as

$$\{(\frac{\rho}{1-\rho})^2 + a \frac{\rho}{1-\rho} X(t)\}$$

for ρ close to but less than unity. Since as mentioned before, the results hold for $t \in [0,T]$, T finite, the statement remains true for all finite t.

It should be mentioned that the $M|M|1$ and the Erlangian $M|E_k|1$ systems are special cases of Luchak's model. Also, in $M|M|1$, the phase length process is the queue length process.

2.4. SOME EXTENSIONS

If we choose the set of parameters λ_n and μ_n to be

$$\lambda_n = n + \sqrt{n} \quad , \quad \mu_n = a\, n \quad , \tag{84}$$

and the process $X_n(t)$ defined in (14), an analysis, similar to the one done up to Section 2.3, can be carried out. With some changes in detail of procedure, a result identical to Theorem 2.6 is obtained. So we can say that the phase length process, at any finite time t, is approximately distributed as $\{(\rho-1)^2 + a(\rho-1)\, X(t)\}$ for ρ close to but greater than unity.

We also note that the phase length process in Luchak's queuing system is identical to the queue length process in the following bulk arrival system: customers arrive according to a Poisson process with parameter λ, in bulks of random size N. Each customer has a service time which is exponentially distributed with parameter μ. Hence the results of this chapter apply to the queue length process of this bulk arrival system.

It seems that the technique of studying the convergence of a continuous time process via the convergence of the jump chain as done in our work, can probably be used to study more general situations.

CHAPTER III

THEOREMS FOR THE MAXIMUM OF PARTIAL SUMS

In this chapter, we shall present some theorems that will be used to establish the limit theorems in the different queuing systems considered in the next chapter. These theorems are concerned with the maximum of a sequence of partial sums of independent and identically distributed (i.i.d.) random variables. We shall use the following notation.

Let $\{X_k;\ k = 1,2,\ldots\}$ be a sequence of i.i.d. random variables with mean and variance given by

$$E(X_k) = \mu \quad , \quad Var(X_k) = \sigma^2 \qquad , \tag{85}$$

both of which we assume to be finite. Let us denote the partial sums of $\{X_k\}$ by

$$S_0 \equiv 0 \quad , \quad S_n = X_1 + X_2 + \ldots + X_n \qquad (n \geq 1) \quad . \tag{86}$$

Let the maximum of this sequence be denoted by

$$M_n = \max(S_0, S_1, \ldots, S_n) \quad . \tag{87}$$

We are interested in the limiting behavior of M_n as $n \to \infty$, in the case $\mu \geq 0$. Spitzer [27] has shown in this case, that, with probability one, $M_n \to \infty$ as $n \to \infty$. We shall be interested in suitably norming M_n to obtain non-degenerate limit distributions as $n \to \infty$. For the case $\mu > 0$, we shall show similarity in the asymptotic behavior of M_n and the partial sums S_n (Theorem 4.1). As a consequence, a central limit theorem will be shown to hold for M_n (Theorem 4.2). The case $\mu = 0$ shall then be considered in Theorems 4.3 and 4.4, utilizing concepts from ladder variable theory.

Now, we are in a position to prove the following.

Theorem 3.1. Let $0 < \mu < \infty$, $0 < \sigma^2 < \infty$ and $\{c_n; n \geq 1\}$ be a sequence of constants such that $c_n > 0 (n \geq 1)$, $c_n \to \infty$ as $n \to \infty$, and $c_{n+1}/c_n > 1 (n \geq 1)$. Then, with probability one

$$\frac{M_n - S_n}{c_n} \to 0 \quad \text{as} \quad n \to \infty \quad .$$

Proof

By the definition of M_n in (87), we have†

$$\begin{aligned} M_n - S_n &= \max(0, S_1, S_2, \ldots, S_n) - S_n \\ &= \max(-S_n, S_1 - S_n, \ldots, S_{n-1} - S_n, 0) \\ &\sim \max(0, -S_1, -S_2, \ldots, -S_n) \ , \text{ for all } n \quad . \qquad (88) \end{aligned}$$

† For two random variables X and Y, we say $X \sim Y$ if X and Y have the same distribution.

We note the fact that $-S_n$'s are the partial sums of $-X_i$'s and $E(-X_i) = -\mu < 0$. This enables us to say that with probability one

$$\sup_{n \geq 0} (-S_n) < \infty \quad ;$$

this result is due to Spitzer and a convenience reference is Prabhu [20], p. 220. This means that

$$\Pr\{\lim_{n \to \infty} \max(0,-S_1,\ldots,-S_n) < \infty\} = 1 \quad .$$

Therefore, from (88),

$$\Pr\{\lim_{n \to \infty} (M_n - S_n) < \infty\} = 1 \quad . \tag{89}$$

Now, fix $\varepsilon > 0$, $\eta > 0$. Also, let A be a fixed constant, however large. Due to (89), we can find a $\bar{N}_1$, such that, for $n > \bar{N}_1$,

$$\Pr\{M_n - S_n > A, M_{n+1} - S_{n+1} > A, \ldots\} < \eta \quad . \tag{90}$$

Let $\bar{N}_2$ be the smallest index n of c_n for which $c_n > A/\varepsilon$. This is possible by the hypotheses on c_n. Let $N = \max(\bar{N}_1, \bar{N}_2)$. Then, we have

$$\Pr\left\{\frac{M_N - S_N}{c_N} > \varepsilon, \frac{M_{N+1} - S_{N+1}}{c_{N+1}} > \varepsilon, \ldots\right\}$$

$$= \Pr\left\{M_N - S_N > \varepsilon c_N, M_{N+1} - S_{N+1} > \varepsilon c_N \cdot \frac{c_{N+1}}{c_N}, \ldots\right\}$$

$$\leq \Pr\left\{M_N - S_N > A, M_{N+1} - S_{N+1} > A \cdot \frac{c_{N+1}}{c_N}, \ldots\right\}$$

$$\text{since} \quad \varepsilon c_N > A,$$

$$\leq \Pr\{M_N - S_N > A, M_{N+1} - S_{N+1} > A, \ldots\}$$

$$\text{since} \quad \frac{c_{N+j}}{c_N} > 1 \quad \text{for all} \quad j \geq 1,$$

$$< \eta, \text{ by } (90).$$

Since ε, η were arbitrary and $M_n - S_n \geq 0$ for all n, the theorem follows.

Now, we prove the asymptotic behavior of M_n in the following.

<u>Theorem 3.2</u>. <u>Let</u> $0 < \mu < \infty$, $0 < \sigma^2 < \infty$. <u>Then</u>†

$$\lim_{n \to \infty} \Pr\left\{\frac{M_n - n\mu}{\sigma\sqrt{n}} \leq x\right\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \exp\left(-\frac{y^2}{2}\right) dy$$

$$(-\infty < x < \infty) \quad .$$

---

† While this thesis was under preparation, a paper by Heyde [10] came to the author's attention. That paper deals with a similar theorem for the general case when the X_k's belong to the domain of attraction of a stable law.

Proof

Let us denote

$$U_n = \frac{M_n - n\mu}{\sigma\sqrt{n}} \quad , \quad V_n = \frac{S_n - n\mu}{\sigma\sqrt{n}} \quad .$$

Choosing $c_n = \sigma\sqrt{n}$ in Theorem 3.1, we find that

$$U_n - V_n \xrightarrow{p} 0 \quad \text{as} \quad n \to \infty \quad ; \tag{91}$$

(actually a stronger result holds). The central limit theorem for the partial sums S_n implies that the distribution function (d.f.), $G_n(.)$, of V_n converges to $G(.)$, the d.f. of the standard normal distribution. It remains to show that the d.f., $F_n(.)$, of U_n converges to $G(.)$. This fact follows from (91) and has been implicitly used in the literature, but was proved recently by Feller [8], p. 247. His proof is reproduced here.

We have, for $\varepsilon > 0$,

$$\begin{aligned} \Pr\{U_n \leq x\} &= \Pr\{U_n \leq x, U_n - V_n > -\varepsilon\} \\ &\quad + \Pr\{U_n \leq x, U_n - V_n \leq -\varepsilon\} \\ &\leq \Pr\{V_n \leq x+\varepsilon\} + \Pr\{U_n - V_n \leq -\varepsilon\} \quad . \end{aligned} \tag{92}$$

† $X_n \xrightarrow{p} c$ as $n \to \infty$ implies that the sequence of random variables $\{X_n\}$ converge stochastically to the constant c as $n \to \infty$.

Since we have $U_n - V_n \xrightarrow{p} 0$, the last probability in (92) can be made $< \varepsilon$ for large n. Hence we have

$$F_n(x) \leq G_n(x+\varepsilon) + \varepsilon \quad \text{for all } n \text{ sufficiently large.}$$

The same argument leads to an analagous inequality in the opposite direction. Hence $F_n(.)$ converges to $G(.)$ as $n \to \infty$. The theorem is therefore proved.

When the basic random variables X_k have zero mean and finite variance, the distribution of the random variable $M_n/\sqrt{n}$ converges to the normal distribution truncated at zero. This has been shown by Erdös and Kac [6] by using an invariance principle (see also Donsker [5]). We shall present an alternative approach, mainly for its intrinsic interest. The proof uses the concept of ladder variables of the sequence S_n. The following is a brief sketch of the theory. For details, see Feller [8] and Prabhu [20].

We define the sequence $\{N_k; k = 0,1,2,...\}$ of random variables, called the ladder epochs of $\{S_n\}$, as follows:

$$N_0 \equiv 0$$
$$N_k = \min\{n > N_{k-1} \mid S_n - S_{N_{k-1}} > 0\} \quad (k \geq 1) \quad .$$

The successive ladder heights are defined as

$$Z_k = S_{N_k} - S_{N_{k-1}} \quad (k \geq 1) \quad ;$$

$\{N_k, S_{N_k}\}$ is a two dimensional renewal sequence. Let the distribution and the generating function of N_1 be denoted by

$$p_n = \Pr\{N_n = n\} \quad (n \geq 1) \tag{93}$$

and

$$P(t) = \sum_{n=1}^{\infty} p_n t^n \quad , \quad |t| < 1 \tag{94}$$

respectively. We also define

$$N(n) = \max\{k \mid N_k \leq n\} \quad ;$$

$N(n)$ is the number of ladder epochs in $(0,n]$. We have

$$\begin{aligned} M_n &= Z_1 + Z_2 + \ldots + Z_{N(n)} \quad , & N(n) > 0 \\ &= 0 & N(n) = 0 \quad . \end{aligned} \tag{95}$$

To describe the asymptotic behavior of M_n, we have, therefore, to consider the behavior of $N(n)$ as $n \to \infty$, and also the behavior of the partial sums $Z_1 + Z_2 + \ldots + Z_n$. Considering first $N(n)$, we note that the occurrence of a ladder epoch is a recurrent event E (Feller [7]), whose recurrence times have the distribution $\{p_n\}$. The number of times E occurs in an interval $(0,n]$ is given by the random variable $N(n)$. When $\mu = 0$, it is known that with probability one, $N(n) \to \infty$ as $n \to \infty$. In Theorem 3.3 we prove that in this case, $N(n)$, suitably normed, has a limiting distribution. In order to prove this, we need the following.

Lemma 3.1. Let the sequence $\{q_n\}$ be monotonic and such that $q_n \geq 0$ and

$$Q(s) = \sum_{n=0}^{\infty} q_n s^n < \infty \quad \text{for} \quad 0 \leq s < 1 \quad .$$

If $L(.)$ varies slowly at infinity and $0 < \beta < \infty$, then the relation†

$$Q(s) \sim (1-s)^{-\beta} L(\tfrac{1}{1-s}) \qquad s \uparrow 1$$

is equivalent to

$$q_n \sim \frac{1}{\Gamma(\beta)} \lambda^{\beta-1} L(n) \qquad n \to \infty \quad .$$

This lemma is due to Karmata, and a simplified proof is given in Feller [7], p. 423.

Now, we prove the following.

† $f(x) \sim g(x)$ as $x \to c$ implies that $\lim_{x \to c} \frac{f(x)}{g(x)} = 1$.

Theorem 3.3. Let $\mu = 0$ and $0 < \sigma^2 < \infty$. Then

$$\lim_{n \to \infty} \Pr\{\frac{N(n)}{\sqrt{n}} \geq x\} = G_{\frac{1}{2}}(\frac{\pi e^{2c}}{x^2}) \quad , \quad x > 0$$

$$= 1 \quad , \quad x \leq 0$$

where

$$c = \sum_{n=1}^{\infty} \frac{1}{n}[\Pr\{S_n > 0\} - \frac{1}{2}] < \infty$$

and

$$G_{\frac{1}{2}}(x) = 2[1-\phi((\frac{\pi}{2x})^{\frac{1}{2}})] \quad , \quad x > 0$$

$$= 0 \quad , \quad x \leq 0 \tag{96}$$

is the stable law with index $\frac{1}{2}$. Here $\phi(x)$ is the distribution function of the standard normal random variable.

Proof

From the distribution of N_1, given by (93), we define the monotonic sequence

$$q_0 = 1 \ , \quad q_n = p_n + p_{n+1} + \ldots \qquad (n \geq 1)$$

Then, using (94), we have the generating function of the sequence $\{q_n\}$ as

$$Q(s) = \sum_{n=0}^{\infty} q_n s^n = \frac{1-P(s)}{1-s} \quad , \quad |s| < 1 \quad . \tag{97}$$

The generating function of $\{p_n\}$ has been obtained by Sparre Anderson [1] (referenced in Prabhu [20]) and is given as

$$p(s) = 1 - \exp[-\sum_{n=1}^{\infty} \frac{s^n}{n} \Pr\{S_n > 0\}] \quad . \tag{98}$$

From (97) and (98), we have

$$Q(s) = \frac{\exp[-\sum_{n=1}^{\infty} \frac{s^n}{n}\{\Pr\{S_n > 0\}-\frac{1}{2}\} - \frac{1}{2}\sum_{n=1}^{\infty} \frac{s^n}{n}]}{1-s}$$

$$= (1-s)^{-\frac{1}{2}} \exp[-\sum_{n=1}^{\infty} \frac{s^n}{n}\{\Pr\{S_n > 0\} - \frac{1}{2}\}] \quad . \tag{99}$$

Now letting

$$C(s) = \exp[-\sum_{n=1}^{\infty} \frac{s^n}{n} \Pr\{S_n > 0\} - \frac{1}{2}\}] \quad . \tag{100}$$

the expression (99) reduces to

$$Q(s) = (1-s)^{-\frac{1}{2}} C(s) \quad .$$

Now, we set

$$c = \sum_{n=1}^{\infty} \frac{1}{n}\{\Pr\{S_n > 0\} - \frac{1}{2}\} \quad ,$$

the series being at least conditionally convergent in the case $\mu = 0,\ 0 < \sigma^2 < \infty$ (see Feller [8], p. 575), which is a part of our hypothesis. Using this fact in (99), we get

$$\lim_{s \uparrow 1} C(s) = e^{-c} \quad .$$

To apply Lemma 3.1, set

$$L(x) = C(1-\frac{1}{x}) \quad .$$

We have

$$\lim_{x \to \infty} L(x) = \lim_{s \uparrow 1} C(s) = e^{-c} > 0 \quad . \tag{101}$$

Thus $L(x)$ is, trivially, a slowly varying function. So, $Q(s)$ satisfies the hypotheses of the lemma and, using (101), it follows that

$$q_n \sim \frac{1}{\Gamma(\frac{1}{2})} n^{\frac{1}{2}-1} e^{-c} \quad \text{as} \quad n \to \infty \quad ,$$

or, by simplifying

$$q_n \sim \frac{e^{-c}}{\sqrt{n\pi}} \quad \text{as} \quad n \to \infty \quad . \tag{102}$$

Now, from the theory of recurrent events as presented in Feller [7], the result (102) enables us to write the following asymptotic behavior of the random variable $N(n)$:

$$\lim_{n \to \infty} \Pr\{M(n) \geq \frac{y}{q_n}\} = G_{\frac{1}{2}}(y^{-2}) \quad . \tag{103}$$

Using (102) and letting $x = y\sqrt{x}\, e^c$, we get the required result.

It should be remarked here that in [8], p. 399, Feller has considered the power series $[1-P(s)]^{-1}$ and has used the above technique to study the asymptotic behavior of its coefficients. Thus we could have obtained (102) from his results also. Furthermore, if the basic random variables X_k have a symmetric distribution, $\Pr\{S_n > 0\} = \frac{1}{2}$ for all n and hence (99) would reduce to

$$Q(s) = (1-s)^{-\frac{1}{2}} \quad .$$

Thus,

$$q_n = \binom{-\frac{1}{2}}{n}(-1)^n = \binom{2n}{n} 2^{-2n} \qquad (n \geq 1) \quad .$$

So we have an exact expression for q_n and by Stirling's approximation for $n!$, we get that q_n is approximately $(n\pi)^{-\frac{1}{2}}$ for large n (compare with (102)).

Now we consider the random variable M_n in the following.

<u>Theorem 3.4</u>. <u>Let</u> $\mu = 0$, $0 < \sigma^2 < \infty$. <u>Then</u>

$$\lim_{n \to \infty} \Pr\{\frac{M_n}{\sigma\sqrt{n}} \leq x\} = \sqrt{\frac{2}{\pi}} \int_0^x \exp(-\frac{y^2}{2})dy \quad , \quad x > 0$$

$$= 0 \quad , \quad x \leq 0 \quad .$$

Proof

From (95), we have

$$\frac{M_n}{\sqrt{n}} = \frac{Z_1+Z_2+\ldots+Z_{N(n)}}{N(n)} \cdot \frac{N(n)}{\sqrt{n}} \quad . \tag{104}$$

Now, the first term on the right hand side in (104) can be written as

$$\frac{Z_1+Z_2+\ldots+Z_{N(n)}}{N_1+N_2+\ldots+N_{N(n)}} \cdot \frac{N_1+N_2+\ldots+N_{N(n)}}{N(n)} \quad .$$

The first term in the above expression converges to $E(X_1)$ (see Feller [7], p. 380) and the second term converges to $E(N_1)$, since $N(n) \to \infty$ as $n \to \infty$ and the law of large numbers. Thus

$$\frac{Z_1+Z_2+\ldots+Z_{N(n)}}{N(n)} \xrightarrow{P} E(X_1)\ E(N_1) \quad \text{as} \quad n \to \infty$$

$$= E(Z_1), \text{ by use of Wald's inequality} \quad .$$

From Feller [8], p. 575 we find that, in the case $\mu = 0$, $0 < \sigma^2 < \infty$, this is given by

$$E(Z_1) = \frac{\sigma}{2}\ e^{-c} > 0 \quad . \tag{105}$$

Thus, we find that the first term in (104) converges stochastically to a positive constant; also, by Theorem 3.3, the sequence of random variables $N(n)/\sqrt{n}$ converge in distribution to the stable law with index $\frac{1}{2}$. Thus, using the result for the limiting distribution of the product of two sequences of random variables (see Fisz [9]), we have,

$$\lim_{n \to \infty} \Pr\{\frac{M_n}{\sqrt{n}} \geq x\}$$

$$= \lim_{n \to \infty} \Pr\{\frac{Z_1+Z_2+\ldots+Z_{N(n)}}{N(n)} \cdot \frac{N(n)}{\sqrt{n}} \geq x\}$$

$$= G_{\frac{1}{2}}(\pi e^{2c} \cdot \frac{\sigma^2 e^{-2c}}{2x^2})$$

$$= G_{\frac{1}{2}}(\frac{\pi\sigma^2}{2x^2})$$

$$= 2[1 - \phi((\frac{\pi}{2} \cdot \frac{2x^2}{\pi\sigma^2})^{\frac{1}{2}})] \quad , \quad \text{from (96)}$$

$$= 2[1 - \phi(\frac{x}{\sigma})]$$

$$= \sqrt{\frac{2}{\pi}} \int_{\frac{x}{\sigma}}^{\infty} \exp(-\frac{y^2}{2})dy \quad , \quad x > 0 \quad . \qquad (106)$$

For $x \leq 0$, we have

$$\lim_{n \to \infty} \Pr\{\frac{M_n}{n} \geq x\} = 1 \quad . \qquad (107)$$

Taking complements of one in the Equations (106) and (10), the proof of the theorem is completed.

With an eye towards the next chapter, we shall need the following modification of M_n, defined as

$$M_n^i = \max(S_0, S_1, \ldots, S_{n-1}, i+S_n) \tag{108}$$

for all n and $i \geq 0$. In the following theorem, we shall show that the asymptotic behavior of M_n^i is identical to that of M_n.

Theorem 3.5. Let $0 < \mu < \infty$, $0 < \sigma^2 < \infty$. Then

$$\lim_{n \to \infty} \Pr\left\{\frac{M_n^i - n\mu}{\sigma\sqrt{n}} \leq x\right\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \exp\left(-\frac{y^2}{2}\right)dy \qquad (-\infty < x < \infty) \quad .$$

Also, when $\mu = 0$, $0 < \sigma^2 < \infty$, we have

$$\lim_{n \to \infty} \Pr\left\{\frac{M_n^i}{\sigma\sqrt{n}} \leq x\right\} = \sqrt{\frac{2}{\pi}} \int_0^x \exp\left(-\frac{y^2}{2}\right)dy \quad , \quad x > 0$$

$$= 0 \quad , \quad x \leq 0 \quad .$$

Proof

From (108), we can write

$$M_n^i = \max(M_{n-1}, i+S_n) \quad , \quad \text{for} \quad n \geq 1 \quad . \tag{109}$$

We now claim that

$$0 \le M_n^i - M_n \le i, \quad \text{all} \quad n, \quad i \ge 0 \quad . \tag{110}$$

To show (110), we consider the three mutually exclusive and exhaustive cases:

1. $\underline{M_{n-1} \le S_n}$: Here, $M_n^i = i + S_n$ and $M_n = S_n$. Thus $M_n^i - M_n = i$.

2. $\underline{S_n < M_{n-1} \le i + S_n}$: Here, $M_n^i = i + S_n$ and $M_n = M_{n-1}$. Thus, $M_n^i - M_n = i + S_n - M_{n-1} \ge 0$ and $< i$, since $S_n - M_{n-1} < 0$.

3. $\underline{i + S_n < M_{n-1}}$: Here, $M_n^i = M_{n-1}$ and $M_n = M_{n-1}$. Thus, $M_n^i - M_n = 0$.

Thus, (110) holds for all cases. Now, since $i/\sqrt{n} \to 0$ as $n \to \infty$, we have

$$\frac{M_n^i}{\sqrt{n}} - \frac{M_n}{\sqrt{n}} \overset{p}{\to} 0 \quad . \tag{111}$$

Now, using an argument similar to the one used after Equation (91), Theorems 3.2 and 3.4 complete the proof of this theorem.

CHAPTER IV

SOME QUEUEING SYSTEMS

In this chapter, we shall study some of the Markov chains imbedded in different queuing systems. It is known that these imbedded chains can be written as the maximum of partial sums of mutually i.i.d. random variables. Hence, we use the results of the previous chapter to obtain limit theorems in the cases where the traffic intensity equals one or is greater than one. It will be shown that the former corresponds to the case $\mu = 0$ and the latter to the case $\mu > 0$, in the terminology of Chapter III. We shall consider the systems $GI|G|1$, $GI|E_s|1$, and $E_s|G|1$ (these shall be defined explicitly as we go along). The analysis in this chapter follows along lines similar to those used in the book by Prabhu [21].

We recall that the distribution function of the standard normal distribution is denoted by

$$\phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \exp(-\frac{y^2}{2})dy \quad , \quad -\infty < x < \infty \quad . \tag{112}$$

Also, from Equation (96), we see that

$$\begin{aligned} 1 - G_{\frac{1}{2}}(\frac{\pi}{2x^2}) &= \sqrt{\frac{2}{\pi}} \int_0^x \exp(-\frac{y^2}{2})dy \; , \; x > 0 \\ &= 0 \qquad\qquad\qquad\qquad\quad , \; x \le 0 \; . \end{aligned} \tag{113}$$

We are now ready to start the analysis of the system $GI|G|1$.

4.1. THE SYSTEM GI|G|1

This is a single server queuing system with the first-come-first-served (FCFS) queue-discipline. Let u_n denote the interval between the arrival times of the n^{th} and the $(n+1)^{st}$ customer; v_n is the service time of the n^{th} customer $(n \geq 0)$ (the customer arriving at time 0 is labeled the zeroeth customer). $\{u_n\}$ and $\{v_n\}$ are independent sequences of positive i.i.d. random variables with their respective distribution functions given by

$$\Pr\{u_n \leq x\} = A(x) \qquad (x \geq 0)$$

$$\Pr\{v_n \leq x\} = B(x) \qquad (x \geq 0) \quad . \tag{114}$$

We assume the two distributions to have finite means denoted by

$$\mu_a = E(u_n) \text{ and } \mu_b = E(v_n) \quad . \tag{115}$$

We also assume that they possess finite non-zero variances

$$\sigma_a^2 = \mathrm{Var}(u_n) \text{ and } \sigma_b^2 = \mathrm{Var}(v_n) \tag{116}$$

The traffic intensity of the system is then given by

$$\rho = \frac{\mu_b}{\mu_a} \quad . \tag{117}$$

We define the sequence

$$X_n = v_n - u_n \qquad (n \geq 0) \tag{118}$$

and its partial sums by

$$S_0 = 0 \ , \ S_n = X_0+X_1+\ldots+X_{n-1} \qquad (n \geq 1) \quad . \tag{119}$$

Since the arrival stream and the servicing mechanism are independent of each other, the X_n's form a sequence of i.i.d. random variables. We have

$$\mu = E(X_n) = \mu_b - \mu_a = \mu_a(\rho-1) \quad , \tag{120}$$

so that

$$\begin{aligned} \mu > 0 &\iff \rho > 1 \\ \mu = 0 &\iff \rho = 1 \quad . \end{aligned} \tag{121}$$

Also,

$$\sigma^2 = \mathrm{Var}(X_n) = \sigma_a^2 + \sigma_b^2 \quad , \tag{122}$$

and it is finite and non-zero.

Let W_n be the waiting time of the n^{th} customer, i.e. the time from his arrival to the time when he commences service

$(n \geq 0)$. We have $W_0 = u \geq 0$, the waiting time of the zeroeth customer. We then have the following recursive relation:

$$W_{n+1} = \max(0, W_n + v_n - u_n) \qquad (n = 0,1,2,\ldots)$$
$$= \max(0, W_n + X_n) \;;$$

from this we obtain successively

$$W_1 = \max(0, u + X_0)$$
$$W_2 = \max(0, W_1 + X_1) = \max(0, X_1, X_1 + X_0 + u)$$
$$W_3 = \max(0, W_2 + X_2)$$
$$= \max(0, X_2, X_2 + X_1, X_2 + X_1 + X_0 + u)$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$W_n = \max[0, X_{n-1} + X_{n-2} + \ldots + X_{n-r} \qquad (1 \leq r \leq n-1)$$
$$X_{n-1} + X_{n-2} + \ldots + X_0 + u] \quad (n \geq 1) \quad . \qquad (123)$$

From (119), we see that $X_{n-1} + X_{n-2} + \ldots + X_{n-r} = S_n - S_{n-r}$; hence we can write (123) as

$$W_n = \max[S_n - S_{n-r} \; (0 \leq r \leq n-1), u + S_n] \qquad .$$

Since $S_n - S_{n-r} = X_{n-1} + X_{n-2} + \ldots + X_{n-r}$, and the X_n are identically distributed, it follows that $S_n - S_{n-r}$ has the same distribution as S_r. Therefore, we can write

$$W_n \sim \max[0,S_1,S_2,\ldots,S_{n-1},u+S_n] \quad . \tag{124}$$

When $\mu \geq 0$, the waiting time $W_n \to \infty$ with probability one. We see from (108) and (124) that

$$W_n \sim M_n^u \quad ,$$

and hence an easy consequence of Theorem 3.5 is Theorem 4.1†.

Theorem 4.1. Let $\rho = 1$. Then

$$\lim_{n \to \infty} \Pr\{\frac{W_n}{\sigma\sqrt{n}} \leq x\} = 1 - G\,(\frac{\pi}{2x^2}) \quad , \quad \text{all } x \quad .$$

Also, when $\rho > 1$, we have

$$\lim_{n \to \infty} \Pr\{\frac{W_n - n\mu}{\sigma\sqrt{n}} \leq x\} = \phi(x) \quad , \quad \text{all } x \quad .$$

Here, μ and σ are defined in (120) and (122).

Now, we shall consider the first passage time defined as

$$T(u) = \min\{k|W_k = 0, W_0 = u > 0\} \quad . \tag{125}$$

† The part of the theorem for the case $\rho > 1$ has been obtained by Kingman [16], using different methods.

$T(u)$ represents the number of customers served in the interval of time between the commencement of a busy period initiated by $W_0 = u$, and the next busy period. This random variable satisfies the following relation:

$$\Pr\{T(u) > n\} = \Pr\{u+S_0 > 0, u+S_1 > 0, \ldots, u+S_n > 0\}$$

$$= \Pr\{\max(-S_0, -S_1, \ldots, -S_n) < u\} \quad .$$

$$(n \geq 1). \quad (126)$$

From (136), it follows that

$$\Pr\{T(u\sqrt{n}) > n\} = \Pr\left\{\frac{\max(-S_0, -S_1, \ldots, -S_n)}{\sqrt{n}} < u\right\} \quad . \qquad (127)$$

Note that $-S_n$ is the partial sum of $-X_i$'s. Hence, when $\mu = E(X_n) = 0$ ($\Rightarrow \rho = 1$, from (121)), $-S_n$ is the partial sum of i.i.d. random variables with zero mean. Thus, from Theorem 3.4, we have Theorem 4.2.

Theorem 4.2. Let = 1. Then

$$\lim_{n \to \infty} \Pr\{T(u\sqrt{n}) > n\} = 1 - G_{\frac{1}{2}}\left(\frac{\pi\sigma^2}{2u^2}\right) \quad , \quad u > 0 \quad .$$

Here, σ is as defined in (122).

4.2. THE SYSTEM $GI|E_s|1$

This system is a particular case of the system $GI|G|1$, wherein the service time has the gamma distribution

$$dB(x) = \frac{e^{-\lambda x}\lambda^s x^{s-1}}{(s-1)!}\,dx \qquad (0 < x < \infty) \quad . \tag{128}$$

The mean and variance of this distribution is

$$\mu_b = \frac{s}{\lambda} \quad , \quad \sigma_b^2 = \frac{s}{\lambda^2} \quad . \tag{129}$$

We denote the Laplace-Stieltjes transform (L.S.T.) of the interarrival distribution by

$$\psi(\theta) = \int_0^\infty e^{-\theta x}\,dA(x) \quad , \quad (\theta > 0) \quad . \tag{130}$$

Thus, the mean and variance of this distribution are, respectively,

$$\mu_a = -\psi'(0) \quad , \quad \sigma_a^2 = \psi''(0) - [\psi'(0)]^2 \quad . \tag{131}$$

The traffic intensity of this system is thus given by

$$\rho = s[-\lambda\psi'(0)]^{-1} \quad .$$

We can consider the servicing of customers as being accomplished in s consecutive phases, the time required for each phase having the exponential distribution $\lambda e^{-\lambda t}$ $(0 < t < \infty)$ independently of the others. Each customer adds s phases to the system and we shall study the imbedded chain obtained from this phase length process. Let $t_0, t_1, t_2, \ldots$ be the instants of successive arrivals and Q_n denote the number of phases present in the system at time $t = t_n - 0$. Let X_n be the number of phases of service completed during $(t_{n-1}, t_n - 0)$ $(n = 1,2,\ldots)$; then $X_1, X_2, \ldots$ are mutually independent random variables with the common distribution

$$k_j = \Pr\{X_n = j\} = \int_0^\infty e^{-\lambda t} \frac{(\lambda t)^j}{j!} \, dA(t) \qquad (j \geq 0) \quad . \tag{132}$$

For its generating function, we have

$$K(z) = \sum_{j=0}^{\infty} k_j z^j = \psi(\lambda - \lambda z) \qquad (0 < z < 1) \quad . \tag{133}$$

Hence, the mean and variance of X_n are, respectively,

$$E(X_n) = K'(z)\Big|_{z=1} = -\lambda\psi'(0) = \frac{s}{\rho} \quad , \tag{134}$$

and

$$\begin{aligned} \mathrm{Var}(X_n) &= E(X_n^2) - [E(X_n)]^2 \\ &= K''(z)\Big|_{z=1} + K'(z)\Big|_{z=1} - \frac{s^2}{\rho^2} \\ &= \lambda^2\psi''(0) + \frac{s}{\rho} - \frac{s^2}{\rho^2} \quad . \end{aligned} \tag{135}$$

It is known that Q_n is a Markov chain and satisfies the following recursive relation:

$$Q_n = \max(0, Q_{n-1}+s-X_n) \quad (n \geq 1) \quad . \qquad (136)$$

We define

$$Y_n = s - X_n \quad (n \geq 1) \qquad (137)$$

and its partial sums by

$$S_0 = 0 \quad , \quad S_n = Y_1+Y_2+\ldots+Y_n \quad (n \geq 1) \quad . \quad (138)$$

If the initial number of phases at time t_0 is $Q_0 = i \geq 0$, then (136) gives us

$$\begin{aligned} Q_n &= \max(0, Q_{n-1}+Y_n) \\ &= \max(0, Y_n, Y_n+Y_{n-1}+Q_{n-2}) \\ &\quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ &= \max[0, S_n-S_{n-r} \ (1 \leq r \leq n-1), i+S_n] \\ &\sim \max[S_r \ (0 \leq r \leq n-1), i+S_n] \quad . \qquad (139) \end{aligned}$$

From (134), (135) and (137), it follows that

$$E(Y_n) = \frac{s}{\rho}(\rho-1) \tag{140}$$

and

$$\sigma^2(\rho) = \mathrm{Var}(Y_n) = \lambda^2\psi''(0) + \frac{s}{\rho} - \frac{s^2}{\rho^2} \quad , \tag{141}$$

where ρ is used to denote the dependence of the $\mathrm{Var}(Y_n)$ on the traffic intensity.

When $\rho \geq 1$, (140) shows that $E(Y_n) \geq 0$; hence from (139), we see that, with probability one, $Q_n \to \infty$ as $n \to \infty$. By comparing with (108), it follows that

$$Q_n \sim M_n^i \quad . \tag{142}$$

Thus, an upshot of Theorem 3.5 is the following.

Theorem 4.3. Let $\rho = 1$. Then

$$\lim_{n\to\infty} \Pr\left\{\frac{Q_n}{\sigma(1)\sqrt{n}} \leq x\right\} = 1 - G_{\frac{1}{2}}\left(\frac{\pi}{2x^2}\right) \;, \quad \text{all } x \quad .$$

Moreover, when $\rho > 1$, we have

$$\lim_{n\to\infty} \Pr\left\{\frac{Q_n - ns\left(\frac{\rho-1}{\rho}\right)}{\sigma(\rho)\sqrt{n}} \leq x\right\} = \phi(x) \;, \quad \text{all } x \quad .$$

Here, $\sigma(\rho)$ is as defined in (141).

Now, we shall define the random variable

$$N_i = \min\{k \mid i + S_k < 0\} \quad , \tag{143}$$

which represents the number of phases served in a busy period initiated by i phases. The similarity between N_i and the random variable $T(u)$ as defined in (155) is obvious, except that here the partial sums S_n are defined by (138). Consequently, a theorem similar to Theorem 4.2 follows, and is stated here without proof.

Theorem 4.4. Let $\rho = 1$. Then

$$\lim_{n \to \infty} \Pr\{N_{i\sqrt{n}} > n\} = 1 - G_{\frac{1}{2}}\left(\frac{\pi\sigma^2(1)}{2i^2}\right) \quad , \quad i > 0 \quad .$$

Here, $\sigma(1)$ is defined in (141).

It should be noted here that the $GI|M|1$ queuing system, i.e. the system with exponential service times, is a special case of the above system $GI|E_s|1$, when $s = 1$. In this case, the phase length process is the queue-length process since each customer demands one phase of service. The analysis of this section thus holds for $GI|M|1$, with the word queue-length replaced for phase length, and $s = 1$.

4.3. THE SYSTEM $E_s|G|1$

This system is the dual of the system $GI|E_s|1$, in the sense that here the interarrival times have the gamma distribution whereas the service times possess an arbitrary distribution. Here,

$$dA(x) = \frac{e^{-\lambda} \lambda^s x^{s-1}}{(s-1)!} dx \qquad (0 < x < \infty) \quad , \qquad (144)$$

and the L.S.T. of the service time distribution is

$$\psi(\theta) = \int_0^\infty e^{-\theta x} dB(x) \qquad (\theta > 0) \quad . \qquad (145)$$

Thus, the mean and variance of the two distributions are, respectively,

$$\mu_a = \frac{s}{\lambda} \quad , \quad \sigma_a^2 = \frac{s}{\lambda^2} \qquad (146)$$

$$\mu_b = -\psi'(0) \quad , \quad \sigma_b^2 = \psi''(0) - [\psi'(0)]^2 \quad . \qquad (147)$$

The traffic intensity is then

$$\rho = -\frac{\lambda\psi'(0)}{s} \quad . \qquad (148)$$

In this system the customers can be assumed to pass through s different stages, the duration of the stages being mutually independent random variables with the distribution $\lambda e^{-\lambda t}dt$ $(0 < t < \infty)$. Let $Q(t)$ be the number of stages completed by the customers at time t; then except in the special case $E_s|M|1$, the process $Q(t)$ is non-Markovian. We shall study the imbedded chain $\{Q_n\}$, where $Q_n = Q(t_n+0)$ $(n = 0,1,2,\ldots)$ and $t_0,t_1,t_2,\ldots$ are the instants of departure of the successive customers. To define Q_n, we shall define X_n to be the number of stages completed during $(t_{n-1}+0, t_n)$ $(n = 1,2,\ldots)$; $X_1, X_2, \ldots$ are mutually independent random variables with the common distribution

$$k_j = \Pr\{X_n = j\} = \int_0^\infty e^{-\lambda t} \frac{(\lambda t)^j}{j!} \, dB(t) \qquad (j \geq 0) \quad . \tag{149}$$

For its generating function, we have

$$K(z) = \sum_{j=0}^{\infty} k_j z^j = \psi(\lambda - \lambda z) \qquad (0 < z < 1) \quad . \tag{150}$$

Thus, the mean and variance of X_n are, respectively,

$$E(X_n) = K'(z)\big|_{z=1} = -\lambda\psi'(0) = s\rho < \infty \tag{151}$$

and

$$\begin{aligned} Var(X_n) &= K''(z)\big|_{z=1} + K'(z)\big|_{z=1} - s^2\rho^2 \\ &= \lambda^2\psi''(0) + s\rho - s^2\rho^2 \quad . \end{aligned} \tag{152}$$

It is known that Q_n is a Markov chain and satisfies the recurrence relation

$$Q_{n+1} = \begin{cases} Q_n + X_{n+1} - s & \text{if } \; Q_n > s \\ X_{n+1} & \text{if } \; Q_n \leq s \quad . \end{cases} \tag{153}$$

We define the sequence

$$Y_n = X_n - s \qquad (n \geq 1) \tag{154}$$

and its partial sum by

$$S_0 = 0, \quad S_n = Y_1+Y_2+\ldots+Y_n \qquad (n \geq 1) \quad . \tag{155}$$

From (151), (152) and (154), it follows that

$$E(Y_n) = s(\rho-1) \tag{156}$$

and

$$\sigma^2(\rho) = \mathrm{Var}(Y_n) = \psi''(0) + s\rho - s^2\rho^2 \quad . \tag{157}$$

Now, if the initial number of stages completed at time t_0 is $Q_0 = i \geq 0$, then (153) gives us

$$\begin{aligned} Q_n &= \max(X_n, Q_{n-1}+X_n-s) \\ &= X_n + \max(0, Q_{n-1}-s) \end{aligned}$$
$$(n = 1,2,\ldots) \quad . \tag{158}$$

Transposing X_n, (158) reduces to

$$Q_n - X_n = \max(0, Q_{n-1}-X_{n-1}+Y_{n-1}) \quad . \tag{159}$$

Proceeding as in (139), we get

$$Q_n - X_n \sim \max[S_r\ (0 \le r \le n-1), i+S_n]$$

and hence

$$Q_n \sim X_n + \max[0, S_1, \ldots, S_{n-1}, i+S_n] \quad . \qquad (160)$$

When $\rho \geq 1$, (156) shows that $E(Y_n) \geq 0$; hence (160) implies that, with probability one, $Q_n \to \infty$ as $n \to \infty$. Comparing (160) with (108), it follows that

$$Q_n \sim X_n + M_n^i \quad ,$$

or alternatively

$$Q_n \sim X_1 + M_n^i \quad . \qquad (161)$$

Since $E(X_1) = s\rho < \infty$, for arbitrary $\varepsilon > 0$, and $\delta > 0$, we can find a N such that

$$\Pr\left\{\frac{X_1}{\sqrt{n}} > \delta\right\} < \varepsilon \qquad (n > N) \quad .$$

Hence

$$\frac{X_1}{\sqrt{n}} \xrightarrow{p} 0 \text{ as } n \to \infty \quad . \qquad (162)$$

The following theorem thus follows from (161), (162) and Theorem 3.5.

Theorem 4.5. Let $\rho = 1$. Then

$$\lim_{n \to \infty} \Pr\left\{\frac{Q_n}{\sigma(1)\sqrt{n}} \le x\right\} = 1 - G_{\frac{1}{2}}\left(\frac{\pi}{2x^2}\right) \quad , \quad \text{all } x. \quad .$$

Moreover, when $\rho > 1$, we have

$$\lim_{n \to \infty} \Pr\left\{\frac{Q_n - ns(\rho-1)}{\sigma(\rho)\sqrt{n}} \le x\right\} = \phi(x) \quad , \quad \text{all } x \quad .$$

Here, $\sigma(\rho)$ is as defined in (156).

As in the previous section, we can consider the special case $M|G|1$ of the system $E_s|G|1$ and consider the queue length process instead of the phase length process by taking $s = 1$.

CHAPTER V

DIRECTIONS FOR FUTURE RESEARCH

One possible extension of the work in Chapter II would be to consider the dual of Luchak's system, i.e., the system where service times are exponential and the interarrival times have a general Erlangian distribution. The appropriate process to consider would be the number of stages completed by the arriving stream of customers.

Another possibility would be to consider a Markov process with a continuous state space, e.g. waiting time process in a $M|G|1$ queue. Maybe the concept of jump chain could be extended to this case.

Finally, one could study the convergence in distribution of a denumerable state space, non-Markovian process (at each time point), as opposed to the Markovian process that we have studied. An example of such a process is the queue length process in a $GI|G|1$ system.

BIBLIOGRAPHY

[1] Andersen, E. S. (1954). "On the fluctuations of sums of random variables,II." Math. Scand., 2, pp. 195-223.

[2] Borovkov, S. P. (1964). "Some limit theorems in the theory of mass service." Teor. Verojatnost. i Priminen, 9, pp. 550-565 (English translation).

[3] Borovkov, S. P. (1965). "Some limit theorems in the theory of mass service, II, multiple channel systems." Teor. Verojatnost. i Priminen, 10, pp. 375-400 (English translation).

[4] Brody, S. M. (1963). "On a limiting theorem of the theory of queues." Ukrain. Mat. Z., 15, pp. 76-79.

[5] Donsker, M. (1951). "An invariance principle for certain probability theorems." Mem. Amer. Math. Soc., 6.

[6] Erdos, P. and Kac, M. (1946). "On certain limit theorems of the theory of probability." Bull. Amer. Math. Soc., 52, pp. 292-302.

[7] Feller, W. (1949). "Fluctuation theory of recurrent events." Trans. Amer. Math. Soc., 67, pp. 98-119.

[8] Feller, W. (1966). An Introduction to Probability Theory and its Applications. Vol. II, John Wiley, New York, 626 pp.

[9] Fisz, M. (1963). Probability Theory and Mathematical Statistics. 3rd ed., John Wiley, New York, 677 pp.

[10] Heyde, C. C. (1967). "A limit theorem for random walks with drift." Journal of Appl. Prob., 4, pp. 144-150.

[11] Iglehart, D. L. (1965). "Limiting diffusion approximations for the many server queue and the machine repairman problem." Journal of Appl. Prob., 2, pp. 429-441.

[12] Iglehart, D. L. (1965). "Limit theorems for queues with traffic intensity one." Annals Math. Stat., 36, pp. 1437-1449.

[13] Karlin, S. and McGregor, J. L. (1964). "On some stochastic models in genetics." Stochastic Models in Medicine and Biology. J. Gurland (ed.). University of Wisconsin Press, Madison, Wisconsin, pp. 245-271.

[14] Kimura, M. (1964). Diffusion Models in Population Genetics. Methuen's Review Series in Applied Probability, Vol. 2.

[15] Kingman, J. F. C. (1961). "The single server queue in heavy traffic." Proc. Cambridge Philos. Soc., 57, pp. 902-904.

[16] Kingman, J. F. C. (1962). "Some inequalities for the queue GI|G|1." Biometrika, 49, pp. 315-324.

[17] Kingman, J. F. C. (1964). "The heavy traffic approximation in the theory of queues." Ch. 6 of Congestion Theory, North Carolina Press.

[18] Lindley, D. V. (1952). "Theory of queues with a single server." Proc. Cam. Phil. Soc., 48, pp. 277-289.

[19] Luchak, G. (1958). "The continuous time solution of the equations of the single channel queue with a general class of service time distributions by the method of generating functions." J. Roy. Stat. Soc., B20, pp. 176-187.

[20] Prabhu, N. U. (1965). Stochastic Processes. Macmillan, New York, 233 pp.

[21] Prabhu, N. U. (1965). Queues and Inventories. John Wiley, New York, 275 pp.

[22] Prabhu, N. U. and Lalchandani, A. P. (1966). "Markov process analysis of Luchak's queuing model." Tech. Report No. 16, Dept. of I. E. and O. R., Cornell University.

[23] Pressman, E. L. (1965). "On waiting time for many-server queuing systems." Teor. Verojatnost. i Priminen, 10, pp. 63-73.

[24] Prokhorov, Y. V. (1963). "Transition behaviour in queuing processes." Lit. Mat. Sb. III, pp. 199-206 (in Russian).

[25] Rudin, W. (1964). Principles of Mathematical Analysis. McGraw Hill, 270 pp.

[26] Samandarov, E. G. (1963). "Service systems in heavy traffic." Teor. Verojatnost. i Priminen, 8, pp. 327-330 (English translation).

[27] Spitzer, F. (1956). "A combinatorial lemma and its application to probability theory." Trans. Amer. Math. Soc., 82, pp. 323-339.

[28] Viskov, O. V. (1964). "Two asymptotic formulae of the theory of mass service." Teor. Verojatnost. i Priminen, 9, pp. 177-178.

UNCLASSIFIED
Security Classification

DOCUMENT CONTROL DATA - R&D

(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)

1. ORIGINATING ACTIVITY (Corporate author) Cornell University Department of Operations Research Ithaca, New York 14850	2a. REPORT SECURITY CLASSIFICATION UNCLASSIFIED 2b GROUP

3. REPORT TITLE

SOME LIMIT THEOREMS IN QUEUING THEORY

4. DESCRIPTIVE NOTES (Type of report and inclusive dates)

Technical Report, August 1967

5. AUTHOR(S) (Last name, first name, initial)

Lalchandani, Atma P.

6. REPORT DATE	7a. TOTAL NO. OF PAGES	7b. NO. OF REFS
August 1967	76	28

8a. CONTRACT OR GRANT NO. Nonr-401(55) b. PROJECT NO. NR-047-055 c. d.	9a. ORIGINATOR'S REPORT NUMBER(S) Technical Report No. 29 9b. OTHER REPORT NO(S) (Any other numbers that may be assigned this report)

10. AVAILABILITY/LIMITATION NOTICES

Distribution of this document is unlimited

11. SUPPLEMENTARY NOTES	12. SPONSORING MILITARY ACTIVITY Logistics and Mathematical Statistics Branch Office of Naval Research Washington, D. C.

13. ABSTRACT

Stochastic processes arising in queuing systems with traffic intensity greater than or equal to unity are studied. Convergence of distributions of continuous time as well as discrete time stochastic processes is investigated. The former utilize concepts involving convergence of distributions of jump chains associated with Markov processes; the latter, discrete time chains, use elements of ladder variable theory.

DD FORM 1 JAN 64 1473

Unclassified
Security Classification

Unclassified

Security Classification

14 KEY WORDS	LINK A		LINK B		LINK C	
	ROLE	WT	ROLE	WT	ROLE	WT
Limit theorems Waiting time Queue length Luchak's model Jump chain Partial sums Traffic intensity Heavy traffic						

INSTRUCTIONS

1. ORIGINATING ACTIVITY: Enter the name and address of the contractor, subcontractor, grantee, Department of Defense activity or other organization *(corporate author)* issuing the report.

2a. REPORT SECURITY CLASSIFICATION: Enter the overall security classification of the report. Indicate whether "Restricted Data" is included. Marking is to be in accordance with appropriate security regulations.

2b. GROUP: Automatic downgrading is specified in DoD Directive 5200.10 and Armed Forces Industrial Manual. Enter the group number. Also, when applicable, show that optional markings have been used for Group 3 and Group 4 as authorized.

3. REPORT TITLE: Enter the complete report title in all capital letters. Titles in all cases should be unclassified. If a meaningful title cannot be selected without classification, show title classification in all capitals in parenthesis immediately following the title.

4. DESCRIPTIVE NOTES: If appropriate, enter the type of report, e.g., interim, progress, summary, annual, or final. Give the inclusive dates when a specific reporting period is covered.

5. AUTHOR(S): Enter the name(s) of author(s) as shown on or in the report. Enter last name, first name, middle initial. If military, show rank and branch of service. The name of the principal author is an absolute minimum requirement.

6. REPORT DATE: Enter the date of the report as day, month, year; or month, year. If more than one date appears on the report, use date of publication.

7a. TOTAL NUMBER OF PAGES: The total page count should follow normal pagination procedures, i.e., enter the number of pages containing information.

7b. NUMBER OF REFERENCES: Enter the total number of references cited in the report.

8a. CONTRACT OR GRANT NUMBER: If appropriate, enter the applicable number of the contract or grant under which the report was written.

8b, 8c, & 8d. PROJECT NUMBER: Enter the appropriate military department identification, such as project number, subproject number, system numbers, task number, etc.

9a. ORIGINATOR'S REPORT NUMBER(S): Enter the official report number by which the document will be identified and controlled by the originating activity. This number must be unique to this report.

9b. OTHER REPORT NUMBER(S): If the report has been assigned any other report numbers *(either by the originator or by the sponsor)*, also enter this number(s).

10. AVAILABILITY/LIMITATION NOTICES: Enter any limitations on further dissemination of the report, other than those imposed by security classification, using standard statements such as:

(1) "Qualified requesters may obtain copies of this report from DDC."

(2) "Foreign announcement and dissemination of this report by DDC is not authorized."

(3) "U. S. Government agencies may obtain copies of this report directly from DDC. Other qualified DDC users shall request through ______."

(4) "U. S. military agencies may obtain copies of this report directly from DDC. Other qualified users shall request through ______."

(5) "All distribution of this report is controlled. Qualified DDC users shall request through ______."

If the report has been furnished to the Office of Technical Services, Department of Commerce, for sale to the public, indicate this fact and enter the price, if known.

11. SUPPLEMENTARY NOTES: Use for additional explanatory notes.

12. SPONSORING MILITARY ACTIVITY: Enter the name of the departmental project office or laboratory sponsoring *(paying for)* the research and development. Include address.

13. ABSTRACT: Enter an abstract giving a brief and factual summary of the document indicative of the report, even though it may also appear elsewhere in the body of the technical report. If additional space is required, a continuation sheet shall be attached.

It is highly desirable that the abstract of classified reports be unclassified. Each paragraph of the abstract shall end with an indication of the military security classification of the information in the paragraph, represented as *(TS), (S), (C), or (U)*.

There is no limitation on the length of the abstract. However, the suggested length is from 150 to 225 words.

14. KEY WORDS: Key words are technically meaningful terms or short phrases that characterize a report and may be used as index entries for cataloging the report. Key words must be selected so that no security classification is required. Identifiers, such as equipment model designation, trade name, military project code name, geographic location, may be used as key words but will be followed by an indication of technical context. The assignment of links, rules, and weights is optional