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HYPERSONIC VISCOUS INTERACTION ON A SLENDER BODY OF REVOLUTION WITH SURFACE MASS TRANSFER

T. Y. Li and J. F. Gross

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PREFACE

Slender re-entry vehicles at high altitude and high velocity possess a flow field which results from the interaction between the boundary layer formed near the surface and the shock wave originating from the leading edge or tip. This Memorandum studies the analytical conditions necessary for the solution of a set of simplified equations which describe the interaction flow field. Surface mass transfer, which occurs in the case of an ablating vehicle, is also considered. The results of this Memorandum will be useful in the interpretation of experimental data and the implementation of numerical analyses. The Memorandum is part of a continuing study for the Advanced Research Projects Agency in re-entry aerodynamics.

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ABSTRACT

An analytical formulation of the problem of hypersonic viscous interaction on a very slender body of revolution with a thick boundary layer and mass transfer is attempted. Particular attention is directed to the establishment of the mathematical restrictions necessary to ensure the existence of similar solutions of the laminar boundary layer equations for this class of problems. Yasuhara's analysis is extended to include surface mass transfer and binary mixture effects, and it is shown that similar solutions are possible only for three-quarter-power-law bodies of revolution. With regard to the problem of hypersonic vis ous interaction on a slender body of revolution with a thick boundary layer and surface mass transfer, it is shown that the criterion for the existence of similar solutions is that $r_w v_w x^{\frac{1}{2}}$ must govern the body surface shape and the surface injection velocity.

CONTENTS

SYMBOLS

A = constant defined in Eq. (72)

A^* = constant defined in Eq. (77)

a^* = constant defined in Eqs. (86) and (133)

a_1^* = constant defined in Eqs. (118) and (134)

B = constant defined in Eq. (81)

b = dimensional constant defined in Eq. (96)

C_i = concentration

C_p = specific heat at constant pressure

c = constant defined in Eq. (119)

D = constant of integration defined in Eq. (127)

D_{12} = diffusion coefficient

d = constant defined in Eq. (120)

C = dimensionless enthalpy = H/He

H = total enthalpy

h = specific enthalpy

I = constant defined by definite integral given in Eq. (136)

j = effective number of degrees of freedom

K' = dimensionless velocity in x-direction = u/u_e

k = thermal conductivity

L = characteristic dimension

Le = Lewis number

M = Mach number

m = constant exponent defined in Eq. (96)

m_i = molecular weight

Pr = Prandtl number

p = pressure

R = universal gas constant

$\bar{R}$ = mixture gas constant

r = radial distance

Sc = Schmidt number

s = similarity coordinate defined in Eq. (12)

s_k = similarity coordinate defined in Eq. (10)

s^* = similarity coordinate defined in Eq. (14)

u = flow velocity component in x-direction

v = flow velocity component in y-direction

$\left.\begin{matrix} x \\ y \end{matrix}\right\}$ = coordinates as shown in Fig. 1

Z_i = dimensionless concentration C_i/C_{iw}

$(\ )_e$ = edge of the boundary layer

$(\ )_i$ = i^{th} component

$(\ )_k$ = generalized coordinate

$(\ )_w$ = wall

$(\ )_\infty$ = undisturbed stream

α = angle defined in Fig. 1

β = pressure gradient parameter defined in Eq. (33)

γ_e = specific heat ratio C_p/C_v

δ = boundary layer thickness

δ^* = displacement thickness of boundary layer

ε = hypersonic parameter defined in Eq. (71)

η = dimensionless similarity coordinate defined in Eq. (13)

η_k = dimensionless similarity coordinate defined in Eq. (11)

η^* = dimensional similarity coordinate defined in Eq. (15)

θ_o = constant defined in Eq. (72)

λ = function defined in Eq. (24)

μ = viscosity

ρ = density

ψ = stream function

Ω = parameter defined in Eq. (143)

I. INTRODUCTION

The purpose of this Memorandum is to present an analytical formulation of the problem of hypersonic viscous interaction on a very slender body of revolution with a thick boundary layer and surface mass transfer. Particular attention is directed to establishing the mathematical restrictions necessary to ensure the existence of similar solutions of the laminar boundary layer equations for this class of problems. The determination of similarity conditions is important because similar solutions are amenable to comparatively simple analysis[1] and find diverse application through the use of the local similarity concept. Recently, Yasuhara[2] demonstrated that similar solutions can be determined for a slender, three-quarter-power-law body of revolution when the boundary layer thickness is of the same order as the body radius. In this Memorandum we extend Yasuhara's analysis to include surface mass transfer and binary mixture effects, and show that in the extended case similar solutions are possible only for three-quarter-power-law bodies of revolution. It is further shown that these similar solutions are characterized by the restrictive condition that r_e/r_w = constant.* This requirement for similar solutions is particularly pertinent in the regime where $r_e/r_w = 0(1)$ or $y_e/r_w = 0(1)$.

The equations which describe the boundary-layer growth on a body of revolution contain the transverse curvature expression (r^2/r_w^2). For similar solutions to exist, this expression must be a function of the similarity variable η. Particular attention is directed to the regime where $r_w^2/r_e^2 \ll 1$ or $y_e/r_w \gg 1$. We derive a new system of equations that possesses the same analytical features as the equations studied previously by Stewartson,[3] Glauert-Lighthill,[4] and Mark.[5] The possibility that this new system has similar solutions is then examined under appropriate boundary conditions, including the injection of gas at the body surface. The results indicate that similar solutions characterized by $\rho r v$-similarity in the boundary layer can be obtained. Finally, with regard to the problem of hypersonic interaction

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*See List of Symbols

on a very slender body of revolution with a thick boundary layer and surface mass transfer, it is shown that the criterion for the existence of similar solutions is that $r_w v_w \sim x^{1/2}$ must govern the body surface shape and the surface injection velocity.

II. CLASSIFICATION OF FLOW REGIMES

Viscous flow regimes may be usefully classified with respect to the ratio of the characteristic body radius to the thickness of the boundary layer. Three such regimes described below, may be distinguished for the purposes of our problem.

THE M REGIME

This regime comprises cases in which $y_e/r_w = O(\epsilon)$ where $\epsilon \ll 1$. Here, Mangler's transformation[6] permits a reduction of axisymmetric flow boundary layer equations to two-dimensional boundary layer equations. Transverse curvature effects are, of course, unimportant in this regime.

THE P-E-Y REGIME

This regime comprises cases in which $y_e/r_w = O(1)$ or $r_e/r_w = O(1)$. The appropriate compressible boundary layer equations with mass transfer are considered by Yasuhara[2] and Probstein and Elliott,[7] who show the importance of the effect of transverse curvature in this regime.

THE G-L REGIME

In this regime the boundary layer thickness is large compared with the radius of the body of revolution; that is, $y_e/r_w \gg 1$. Using the criterion stated above, we can characterize this regime by $(r_w/r_e)^2 = O(\epsilon)$, where $\epsilon \ll 1$. Stewartson[3] and Glauert and Lighthill[4] dealt with incompressible flow problems in this regime. Mark[5] and Steiger and Bloom[8] also considered certain classes of compressible boundary layer flows in this regime. Their results demonstrated the extreme importance of transverse curvature effects. The boundary layer equations for the G-L regime possess a basic structure which leads to a logarithmic velocity profile distribution near the body surface in the case of zero mass transfer.

III. BINARY BOUNDARY LAYER EQUATIONS AND BOUNDARY CONDITIONS

The no-slip condition must apply at the body surface. This requires the use of the boundary layer equations in the region adjacent to the body surface. A gas is injected normal to the surface such that a binary boundary layer flow results. The binary boundary layer equations for axisymmetric flow may be written:

$$\frac{\partial}{\partial x}(\rho u r) + \frac{\partial}{\partial y}(\rho v r) = 0 \tag{1}$$

$$\rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} = -\frac{\partial p}{\partial x} + \frac{1}{r}\frac{\partial}{\partial y}\left(\mu r \frac{\partial u}{\partial y}\right) \tag{2}$$

$$\frac{\partial p}{\partial y} = 0 \tag{3}$$

$$\rho u \frac{\partial H}{\partial x} + \rho v \frac{\partial H}{\partial y} = \frac{1}{r}\frac{\partial}{\partial y}\left[r \frac{\mu}{Pr}\frac{\partial H}{\partial y}\right] + \frac{1}{r}\frac{\partial}{\partial y}\left[r\mu\left(1 - \frac{1}{Pr}\right)\frac{\partial}{\partial y}\left(\frac{u^2}{2}\right)\right]$$

$$+ \frac{1}{r}\frac{\partial}{\partial y}\left[r\mu \frac{Le}{Pr}\left(1 - \frac{1}{Le}\right)\sum_i h_i \frac{\partial C_i}{\partial y}\right] \tag{4}$$

$$\rho u \frac{\partial C_i}{\partial x} + \rho v \frac{\partial C_i}{\partial y} = \frac{1}{r}\frac{\partial}{\partial y}\left(r\rho D_{12}\frac{\partial C_i}{\partial y}\right) \tag{5}$$

where

C_{p_i} = specific heat at constant pressure of the i^{th} component

C_p = $\sum_i C_i C_{p_i}$

C_i = mass fraction of i^{th} component

D_{12} = binary diffusion coefficient

H = total enthalpy = $h + (1/2)u^2$

h = $\sum_i C_i h_i$

h_i = enthalpy of the i^{th} component

k = thermal conductivity of the fluid

Le = Lewis number, $\rho D_{12} C_p/k$

Pr = Prandtl number, $\mu C_p/k$

p = fluid pressure

r = $r_w + y \cos \alpha$

r_w = $r_w(x)$ defines the shape of the body surface

Sc = Schmidt number, $\mu/\rho D_{12}$

u,v = flow velocities in the x,y directions, respectively

x,y = coordinates, as shown in Fig. 1

μ = viscosity coefficient of the fluid

ρ = fluid density

Within the boundary layer we have, from Eq. (3),

$$p_e(x) = \bar{R}\rho T \qquad (6)$$

where

$$\bar{R} = \sum_i C_i R_i = \sum_i C_i \frac{R}{m_i} \qquad (7)$$

m_i = molecular weight of i-gas

R = universal gas constant

R_i = component gas constant

$\bar{R}$ = mixture gas constant

The boundary conditions at the wall are characterized by the no-slip condition, constant wall temperature, the concentration of the injectant gas, and the Eckert condition, which specifies that the normal mass flow velocity of the main component (i = 2) in the gas mixture vanishes at the body surface.[9]

$$\text{when} \quad y = 0; \qquad u = 0 \tag{8a}$$

$$C_1 = C_{1w} = \text{constant} \tag{8b}$$

$$h = H_w = \text{constant} \tag{8c}$$

$$(\rho v)_w = -\frac{P_w}{C_{2w}} D_{12} \left(\frac{\partial C_1}{\partial y}\right)_w \tag{8d}$$

At the edge of the boundary layer, the properties of the flow must match those given in the inviscid flow.

$$\text{when} \quad y = \delta, \qquad u = u_e \tag{9a}$$

$$C_1 = 0 \tag{9b}$$

$$H = H_e \tag{9c}$$

TRANSFORMATION OF BINARY BOUNDARY LAYER EQUATIONS

We shall introduce a coordinate transformation to transform, under certain prescribed conditions, the partial differential equations describing the axially symmetric laminar boundary layer into a set of nonlinear coupled ordinary differential equations. The transformation is quite general so that it can be applied to all the regimes described above. The boundary layer coordinates (x,y) are transformed into the similarity coordinates (s_k, η_k) as follows:

$$s_k = \int_0^x C\rho_e u_e \mu_e r_k^2 \, dx \tag{10}$$

$$\eta_k = \frac{\rho_e u_e}{\sqrt{2s_k}} \int_0^y \frac{\rho}{\rho_e} r \, dy \tag{11}$$

where C = the Chapman-Rubesin constant

r_k = characteristic dimension

The transformation can be applied to the P-E-Y and the G-L regimes as follows:

In the P-E-Y regime, $r_k = r_w$, thus

$$s_k = s = \int_0^x C\rho_e\mu_e u_e r_w^2 \, dx \tag{12}$$

$$\eta_k = \eta = \frac{\rho_e u_e}{\sqrt{2s}} \int_0^y \frac{\rho}{\rho_e} r \, dy \tag{13}$$

In the G-L regime, $r_k = r_e$, thus

$$s_k = s^* = \int_0^x C\rho_e\mu_e u_e r_e^2 \, dx \tag{14}$$

$$\eta_k = \eta^* = \frac{\rho_e u_e}{\sqrt{2s^*}} \int_0^y \frac{\rho}{\rho_e} r \, dy \tag{15}$$

We shall discuss later these forms of the transformation and their significance. Their immediate application is to the boundary layer equations which can now be rewritten in the new variables (s_k, η_k). The stream function ψ is defined by:

$$\frac{\partial\psi}{\partial x} = -\rho v r \tag{16}$$

$$\frac{\partial\psi}{\partial y} = \rho u r \tag{17}$$

We consider those flows where u, H, and C_i are expressed as follows:

$$\frac{u}{u_e} = K'(\eta_k, s_k) \tag{18}$$

$$\frac{H}{H_e} = G(\eta_k, s_k) \tag{19}$$

$$\frac{C_i}{C_{iw}} = Z_i(\eta_k, s_k) \tag{20}$$

where the prime denotes $\partial/\partial\eta_k$. Equations (2), (4), and (5) may now be written:

$$\left(\lambda \frac{r^2}{r_k^2} K''\right)' + KK'' = \frac{2s_k}{u_e}\frac{du_e}{ds_k}\left(K'^2 - \frac{\rho_e}{\rho}\right)$$

$$+ 2s_k\left[\frac{\partial K'}{\partial s_k} K' - \frac{\partial K}{\partial s_k} K''\right] \tag{21}$$

$$\left(\lambda \frac{r^2}{r_k^2} \frac{G'}{Pr}\right)' + KG' = \frac{u_e^2}{H_e}\left[\lambda \frac{r^2}{r_K^2}\left(\frac{1}{Pr} - 1\right) K'K''\right]$$

$$+ 2s_k\left[K' \frac{\partial G}{\partial s_k} - \frac{\partial K}{\partial s_k} G'\right] \tag{22}$$

$$+ \frac{C_{1w}}{H_e}\left[\lambda \frac{r^2}{r_k^2} \frac{1}{Sc}\left(\frac{1}{Le} - 1\right)\left(h_1 - h_2\right) Z_1'\right]'$$

$$\left(\lambda \frac{r^2}{r_k^2} \frac{1}{Sc} Z_1'\right)' + KZ_1' = 2s_k\left(\frac{\partial Z_1}{\partial s_k} K' - \frac{\partial K}{\partial s_k} Z_1'\right) \tag{23}$$

where

$$\lambda = \frac{1}{C} \frac{\rho\mu}{\rho_e \mu_e} \tag{24}$$

In the new coordinate system, the boundary conditions are expressed as follows:

when $\eta_k = 0$

$$K'(0) = 0 \tag{25}$$

$$Z_1(0) = 1 \tag{26a}$$

$$G(0) = \frac{H_w}{H_e} \tag{26b}$$

$$2s_k \frac{\partial K}{\partial s_k}(0) + K(0) = \left(\frac{\lambda}{Sc}\right)_w \frac{C_{1w}}{1 - C_{1w}} Z_1'(0)\left(\frac{r_w}{r_k}\right)^2 \tag{27}$$

when $\eta_k = \eta_{k_e}$

$$K'(\eta_e) = 1 \tag{28}$$

$$Z_1(\eta_{k_e}) = 0 \tag{29a}$$

$$G(\eta_{k_e}) = 1 \tag{29b}$$

IV. SIMILAR SOLUTIONS OF THE HYPERSONIC, BINARY AXISYMMETRIC BOUNDARY LAYER

By neglecting terms of order $(1/M_e^2)$ in a hypersonic boundary layer, we obtain the following expressions from Eqs. (18) and (19).

$$\left(K'^2 - \frac{\rho_e}{\rho}\right)\frac{2s_k}{u_e}\frac{du_e}{ds_k} = \frac{\gamma_e - 1}{2\gamma_e}\,\beta F_1(\eta_k, s_k)\,(G - K'^2) \tag{30}$$

$$\frac{u_e^2}{H_e} = 2 \tag{31}$$

$$C_{1w}\frac{h_1 - h_2}{H_e} = F_2(\eta_k, s_k)(G - K'^2) \tag{32}$$

where

$$\beta = 2\,\frac{d\,\ell n\,p_e}{d\,\ell n\,s_k} \tag{33}$$

$$F_1(\eta_k, s_k) = \frac{1 + C_{1w}\left(\frac{m_2}{m_1} - 1\right)Z_1(\eta_k, s_k)}{1 + C_{1w}\left(\frac{j_1 + 2}{j_2 + 2}\,\frac{m_2}{m_1} - 1\right)Z_1(\eta_k, s_k)} \tag{34}$$

$$F_2(\eta_k, s_k) = \frac{C_{1w}\left(\frac{j_1 + 2}{j_2 + 2}\,\frac{m_2}{m_1} - 1\right)}{1 + C_{1w}\left(\frac{j_1 + 2}{j_2 + 2}\,\frac{m_2}{m_1} - 1\right)Z_1(\eta_k, s_k)} \tag{35}$$

The lowing relationships were used in the derivation of Eqs. (34) and (35):

$$\frac{\bar{R}}{R_e} = \frac{\Sigma C_i R_i}{(\Sigma C_i R_i)_e} = 1 + \left(\frac{R_1}{R_2} - 1\right) C_{1w} Z_1(\eta_k, s_k) \tag{36}$$

$$C_{p_i} = \frac{j_i + 2}{2} R_i \tag{37}$$

The j_i is defined as the effective number of degrees of freedom of the i^{th} component.[11] In this study, $j = 5$ for diatomic gases and $j = 3$ for monatomic gases. Substitution of Eqs. (30) to (35) into Eqs. (21) to (23) gives:

$$\left(\lambda \frac{r^2}{r_k^2} K''\right)' + KK'' = \frac{\gamma_e - 1}{2\gamma_e} \beta F_1(\eta_k, s_k)(G - K'^2)$$

$$+ 2s_k\left[\frac{\partial K'}{\partial s_k} K' - \frac{\partial K}{\partial s_k} K''\right] \tag{38}$$

$$\left(\frac{1}{Pr} \lambda \frac{r^2}{r_k^2} G'\right)' + KG' = 2\left[\lambda \frac{r^2}{r_k^2}\left(\frac{1}{Pr} - 1\right) KK''\right]'$$

$$+ 2s_k\left[K' \frac{\partial G}{\partial s_k} - \frac{\partial K}{\partial s_k} G'\right]$$

$$+ \left[\lambda \frac{r^2}{r_k^2} \frac{1}{Sc}\left(\frac{1}{Le} - 1\right) F_2(\eta_k, s_k)(G - K'^2) Z_1'\right]' \tag{39}$$

$$\left(\lambda \frac{r^2}{r_k^2} \frac{1}{Sc} Z_1'\right)' + KZ_1' = 2s_k\left[\frac{\partial Z_1}{\partial s_k} K' - \frac{\partial K}{\partial s_k} Z_1'\right] \tag{40}$$

For the problem of hypersonic viscous flow past a very slender body of revolution with injection of coolant gas at the surface, these equations can be treated under the boundary conditions given in Eqs. (25) to (29).

If similar solutions are to be obtained, the following conditions must be satisfied:

(a) $$K = K(\eta_k)$$

$$G = G(\eta_k)$$

$$Z_1 = Z_1(\eta_k)$$

Hence, Eqs. (38) to (40) become:

$$\left(\lambda \frac{r^2}{r_k^2} K''\right)' + KK'' = \frac{\gamma_e - 1}{2\gamma_e} \beta F_1(\eta_k)(G - K'^2) \tag{38a}$$

$$\left(\frac{1}{Pr} \lambda \frac{r^2}{r_k^2} G'\right)' + KG' = 2\left[\lambda \frac{r^2}{r_k^2} \left(\frac{1}{Pr} - 1\right)KK''\right]$$

$$+ \left[\lambda \frac{r^2}{r_k^2} \frac{1}{Sc} \left(\frac{1}{Le} - 1\right) F_2(\eta_k)(G - K'^2) Z_1'\right] \tag{39a}$$

$$\left(\lambda \frac{r^2}{r_k^2} \frac{1}{Sc} Z_1'\right)' + KZ_1' = 0 \tag{40a}$$

These equations are to be solved under the boundary conditions given in Eqs. (25) to (29), whence Eq. (27) becomes

$$K(0) = \left(\frac{\lambda}{Sc}\right)_w \frac{C_{1w}}{1 - C_{1w}} Z_1'(0) \left(\frac{r_w}{r_k}\right)^2 \tag{27a}$$

(b)
$$\lambda = \lambda(\eta_k) \quad (41)$$

$$Sc = Sc(\eta_k) \quad (42)$$

$$Pr = Pr(\eta_k) \quad (43)$$

(c) The transverse curvature parameter r^2/r_k^2 must be a function of η_k; that is,

$$\frac{r^2}{r_k^2} = f_1(\eta_k) \quad (44)$$

(d) The pressure gradient parameter, as defined in Eq. (33), must be constant:

$$\beta = 2 \frac{d \, \ell n \, p_e}{d \, \ell n \, s_k} \quad (45)$$

(e) The surface boundary conditions are defined such that C_{1w}, H_w/H_e, and $(r_w/r_k)^2$ are constants.

THE ANALYTIC CHARACTERISTICS OF THE G-L EQUATIONS

Glauert and Lighthill[4] studied the problem of the incompressible axisymmetric boundary layer of a long cylinder. Their primary purpose was to examine a very thick boundary layer on a slender body; that is, where $r_e/r_w >> 1$. In our notation, their equation for the boundary layer flow is

$$[\eta^* K'']' + \frac{1}{2} KK'' = 0 \quad (46)$$

Suppression of the pressure gradient term in Eq. (38a) and taking $\lambda = 1$ gives

$$\left[\left(\frac{r}{r_e}\right)^2 K''\right]' + KK'' = 0 \tag{47}$$

For incompressible flow, it will be shown later that $(r/r_e)^2 = \eta^*/\eta_e^*$. Hence, Eq. (47) can be written

$$\left[\frac{\eta^*}{\eta_e^*} K''\right]' + KK'' = 0 \tag{48}$$

The similarity between Eqs. (46) and (48) demonstrates that the system of equations presented here is entirely consistent with the Glauert and Lighthill equation.

SIMILARITY CONDITIONS FOR THE NORMAL VELOCITY

In seeking similar solutions for problems relating to mass transfer in the boundary layer, it is essential to ensure that the normal velocity at the surface, which governs the axial rate of mass transfer, is a function of the similarity variable only. This can be shown by first combining Eqs. (11) and (17):

$$u = u_e \frac{1}{\sqrt{2s_k}} \frac{\partial\psi}{\partial\eta_k} \tag{49}$$

Combining Eqs. (18) and (49) gives, in the case of similar solutions,

$$\psi = \sqrt{2s_k}\, K(\eta_k) \tag{50}$$

From Eq. (16), one obtains

$$-\rho v r = \sqrt{2s_k} \frac{\partial\eta_k}{\partial x} K' + K \frac{\partial}{\partial x} (\sqrt{2s_k}) \tag{51}$$

Now

$$\frac{\partial \eta_k}{\partial x} = - \frac{\partial \eta_k}{\partial y} \frac{\partial y}{\partial s_k} \frac{\partial s_k}{\partial x} \tag{52}$$

Inversion of Eq. (11) gives

$$y = \frac{2s_k}{\rho_e u_e r_k} \int_0^{\eta_k} \frac{\rho_e r_k}{\rho r} d\eta \tag{53}$$

which by differentiation, yields

$$\frac{\partial y}{\partial s_k} = \frac{1}{\rho_e u_e r_k} \frac{1}{\sqrt{2s_k}} \left\{1 - \frac{2s_k}{\rho_e u_e r_k} \frac{\partial}{\partial s_k} (\rho_e u_e r_k)\right\} \int_0^{\eta_k} \frac{\rho_e r_k}{\rho r} d\eta \tag{54}$$

Combining Eqs. (52) and (54) yields

$$\frac{\partial \eta_k}{\partial x} = - \frac{1}{2s_k} \frac{\partial s_k}{\partial x} \left\{1 - \frac{2s_k}{\rho_e u_e r_k} \frac{\partial}{\partial s_k} (\rho_e u_e r_k)\right\} \frac{\rho r}{\rho_e r_k} \int_0^{\eta_k} \frac{\rho_e r_k}{\rho r} d\eta \tag{55}$$

This result is substituted into Eq. (51) to yield

$$- \frac{\rho v r}{\frac{1}{\sqrt{2s_k}} \frac{\partial s_k}{\partial x}} = K + \left\{\left[\frac{2s_k}{\rho_e u_e r_k} \frac{\partial(\rho_e u_e r_k)}{\partial s_k} - 1\right]\left[\int_0^{\eta_k} \frac{\rho_e r_k}{\rho r} d\eta\right] \frac{\rho r}{\rho_e r_k}\right\} K' \tag{56}$$

From Eqs. (30) and (33), we obtain, respectively

$$\frac{2s_k}{u_e} \frac{du_e}{ds_k} = - \frac{\beta}{\gamma_e M_e^2} \tag{57}$$

$$\frac{\rho_e}{\rho} = \frac{\gamma_e - 1}{2} M_e^2 F_1 \left(G - K'^2\right) + K'^2 \tag{58}$$

Hence, Eq. (56) can be written

$$- \frac{\rho v r}{\frac{1}{\sqrt{2s_k}} \frac{\partial s_k}{\partial x}} = K - \left\{1 - \frac{2s_k}{\rho_e u_e r_k} \frac{\partial}{\partial s_k} (\rho_e u_e r_k)\right\} \frac{r}{r_k} F_3 \left(\eta_k, \frac{r_k}{r}\right) K' \tag{59}$$

where

$$F_3\left(\eta_k, \frac{r_k}{r}\right) = \frac{\int_0^{\eta_k} \left[K'^2 + \frac{\gamma_e - 1}{2} M_e^2 F_1 (G - K'^2)\right] \frac{r_k}{r} d\eta}{K'^2 + \frac{\gamma_e - 1}{2} M_e^2 F_1 (G - K'^2)} \tag{60}$$

In hypersonic boundary layers the terms of $O(1/M_e^2)$ are neglected, and Eqs. (59) and (60) reduce to

$$- \frac{\rho v r}{\frac{1}{\sqrt{2s_k}} \frac{\partial s_k}{\partial x}} = K - \left\{1 - \frac{2s_k}{\rho_e u_e r_k} \frac{\partial}{\partial s_k} (\rho_e u_e r_k)\right\} \frac{r}{r_k} F_4\left(\eta_k, \frac{r_k}{r}\right) K' \tag{61}$$

where

$$F_4\left(\eta_k, \frac{r_k}{r}\right) = \frac{\int_0^{\eta_k} F_1 (G - K'^2) \frac{r_k}{r} d\eta}{F_1 (G - K'^2)} \tag{62}$$

It should be pointed out that the right-hand side of Eq. (61) is a function of η_k only, provided that

(a) $$\frac{r}{r_k} = f(\eta_k) \tag{63}$$

and

(b)
$$\frac{2s_k}{\rho_e u_e r_k} \frac{\partial}{\partial s_k} (\rho_e u_e r_k) = \text{const.} \tag{64}$$

Now condition (b) can be written as follows:

$$\frac{2s_k}{\rho_e u_e r_k} \frac{\partial}{\partial s_k} (\rho_e u_e r_k) = - \frac{\beta}{\gamma_e M_e^2} \left\{ 1 + \frac{\frac{2s_k}{\rho_e} \frac{\partial \rho_e}{\partial s_k}}{\frac{2s_k}{u_e} \frac{\partial u_e}{\partial s_k}} + \frac{\frac{2s_k}{r_k} \frac{\partial r_k}{\partial s_k}}{\frac{2s_k}{u_e} \frac{\partial u_e}{\partial s_k}} \right\} \tag{65}$$

where

$$- \frac{\beta}{\gamma_e M_e^2} = \frac{2s_k}{u_e} \frac{du_e}{ds_k}$$

as defined in Eq. (57). In the hypersonic adiabatic free stream, the total enthalpy is constant so that $dh_e + u_e \, du_e = 0$. Indeed, in the hypersonic boundary layer, $\delta \approx \delta^*$ so that very little mass flows into the boundary layer. Hence, the specific entropy is constant along the edge of the boundary layer and it follows that

$$\frac{1}{\rho_e} dp_e + u_e \, du_e = 0 \tag{66}$$

which leads to

$$\frac{u_e}{\rho_e} \frac{\partial \rho_e}{\partial u_e} + M_e^2 = 0 \tag{67}$$

and

$$\frac{\dfrac{1}{\rho_e}\dfrac{\partial \rho_e}{\partial s_k}}{\dfrac{1}{u_e}\dfrac{\partial u_e}{\partial s_k}} = - M_e^2 \tag{68}$$

We therefore have

$$\frac{2s_k}{\rho_e u_e r_k}\frac{\partial}{\partial s_k}(\rho_e u_e r_k) = - \frac{\beta}{\gamma_e M_e^2}(1 - M_e^2) + \frac{2s_k}{r_k}\frac{\partial r_k}{\partial s_k} \tag{69}$$

This shows that

$$\frac{2s_k}{r_k}\frac{\partial r_k}{\partial s_k} = \text{const.} \tag{70}$$

is sufficient for condition (b); this is a geometrical constraint on the problem.

BOUNDARY LAYER EFFECTS ON THE MATCHING CONDITIONS BETWEEN INVISCID AND VISCOUS FLOW REGIONS

The axis of symmetry of the body lies along the direction of the undisturbed free stream. At the body surface, a particular rate of mass transfer due to coolant gas injection is specified. The disturbed flow region surrounding the body is separated from the undisturbed flow by the leading-edge shock wave. The region between the shock wave and the body surface can be divided into an inviscid flow region and a viscous boundary layer region. We shall match the normal velocity and the pressure on the common boundary of these inviscid and viscous flow regions.

Stewartson[11] and Oguchi[12] have obtained similar solutions of the hypersonic inviscid flow region using the small-perturbation theory. These results can be summarized in the following equations for the normal velocity and pressure in the flow field:

$$\frac{v_e}{u_\infty} = n\epsilon\theta_o \left(\frac{x}{L}\right)^{n-1} \tag{71}$$

$$\frac{p_e}{\gamma_e M_\infty^2 \epsilon^2 p_\infty} = (2)^{\frac{2n-1}{n}} \frac{n^2}{\gamma_e^2 + 1} \left(\frac{\gamma_e - 1}{\gamma_e + 1}\right)^{\gamma_e} \frac{1}{\theta_o} \left(A^{\frac{1}{N}} N\right)^{\gamma_e} \left(\frac{x}{L}\right)^{2(n-1)} \tag{72}$$

where A, N, and θ_o are constants. These values v_e and p_e will be matched with the corresponding viscous flow solution at the edge of the boundary layer. These matching conditions will determine n and ϵ in Eqs. (71) and (72).

The value of v_e/u_e can be obtained from the boundary layer analysis by evaluating Eq. (56) at $y = \delta$. This result has also been obtained by the authors using a direct integration of the continuity equation,(13) and gives, for hypersonic boundary layers,

$$\frac{v_e}{u_e} = \frac{\rho_w v_w r_w}{\rho_e u_e r_e} + \frac{d\delta^*}{dx} \tag{73}$$

V. SIMILAR SOLUTIONS IN THE P-E-Y REGIME

We note that in the P-E-Y regime, $r_k = r_w$ and (s_k, η_k) become (s, η), as defined in Eqs. (12) and (13). The conditions for similar solutions in this regime are:

(a)
$$\lambda = \lambda(\eta) \tag{74a}$$

$$Sc = Sc(\eta) \tag{74b}$$

$$Pr = Pr(\eta) \tag{74c}$$

(b) The transverse curvature parameter r^2/r_w^2 must be a function of η:

$$\frac{r^2}{r_w^2} = f_1(\eta) \tag{75}$$

(c) The pressure gradient parameter as defined in Eq. (33) must be constant:

$$\beta = 2\,\frac{d\,\ell n\,p_e}{d\,\ell n\,s} \tag{76}$$

(d) The surface boundary conditions are defined such that C_{1w} and H_w/H_e are constant. This requires uniform distribution of the wall temperature and the concentration of the injectant gas.

(e) The boundary layer effects given in Eq. (73) must be included in the matching condition.

The above condition (c) implies that

$$s^{\beta/2} = A^* p_e \tag{77}$$

where A^* is a dimensional constant. From Eq. (12), one obtains

$$s = \int_0^x \frac{\mu}{\bar{R}T} \frac{u_e}{\lambda} p_e r_w^2 \, dx \tag{78}$$

In the hypersonic flow regime, $u_e \simeq u_\infty$. In Eq. (78), λ can be treated as a constant evaluated at some reference temperature and composition, $\lambda = \lambda_m$. To the same approximation, then,

$$\frac{\mu}{\bar{R}T} = \left(\frac{\mu}{\bar{R}T}\right)_m = \text{const.} \tag{79}$$

and hence

$$s = B \int_0^x p_e r_w^2 \, dx \tag{80}$$

where

$$B = \left(\frac{\mu}{\bar{R}T\lambda}\right)_m u_\infty = \text{const.} \tag{81}$$

Combining Eqs. (77) and (80) yields

$$\frac{\int_0^x p_e r_w^2 \, dx}{p_e^{2/\beta}} = \frac{(A^*)^{2/\beta}}{B} = \text{const.} \tag{82}$$

Consider now condition (b), which requires that r^2/r_w^2 be a function of η only. From geometry

$$\left(\frac{r}{r_w}\right)^2 = 1 + \frac{2 \cos \alpha}{r_w^2} \int_0^y r \, dy \tag{83}$$

or

$$\left(\frac{r}{r_w}\right)^2 = 1 + \frac{(2 \cos \alpha) \sqrt{2s}}{\rho_e u_e r_w^2} \int_0^{\eta} \frac{\rho_e}{\rho} d\eta \tag{84}$$

Equations (58), (80), and (84) together give

$$\left(\frac{r}{r_w}\right)^2 = 1 + (\sqrt{B}\sqrt{2}\, u_e \cos \alpha) \left(\frac{\gamma_e - 1}{\gamma_e}\right) a^* \int_0^{\eta} F_1(G - K'^2)\, d\eta \tag{85}$$

where terms of $0(1/M_\infty^2)$ have been neglected, $\cos \alpha \simeq 1$, and a^* is defined as

$$a^* = \frac{\left[\int_0^x p_e r_w^2\, dx\right]^{1/2}}{p_e r_w^2} \tag{86}$$

Now a^* must be a constant in order that r^2/r_w^2 be a function of η only.

Consider finally the matching condition. From Eqs. (71) and (73), we have

$$n\epsilon\theta_o\left(\frac{x}{L}\right)^{n-1} = \left(\frac{\rho_w v_w r_w}{\rho_e u_e r_e} + \frac{d\delta^*}{dx}\right) + \frac{dr_w}{dx} \tag{87}$$

$$\begin{pmatrix}\text{Effective body}\\ \text{angle}\end{pmatrix} \begin{pmatrix}\text{viscous effects}\\ \text{on body surface}\end{pmatrix} \begin{pmatrix}\text{local inviscid}\\ \text{body angle}\end{pmatrix}$$

In matching the viscous and inviscid solutions, we have included the boundary layer effects. Since

$$r_e = r_w + \delta \cos \alpha \tag{88}$$

$$\simeq r_w + \delta^* \tag{89}$$

Equation (87) may be written

$$n\epsilon\theta_o\left(\frac{x}{L}\right)^{n-1} = \frac{\rho_w v_w r_w}{\rho_e u_e r_e} + \frac{dr_e}{dx} \tag{90}$$

Since, by Eq. (51),

$$\frac{\rho_w v_w r_w}{\rho_e u_e r_e} = -\frac{u_e K(0)}{\gamma_e M_e^2}\frac{1}{p_e r_e}\frac{\partial}{\partial x}\left(\sqrt{2s}\right) \tag{91}$$

we may write

$$\frac{\rho_w v_w r_w}{\rho_e u_e r_e} = -\sqrt{\frac{\mu_\infty}{2\rho_\infty u_\infty}}\;\frac{M_\infty^2}{M_e^2}\sqrt{p_\infty}\;\frac{K(0)}{a^*}\;\frac{1}{p_e r_e}\sqrt{\frac{B_e}{B}} \tag{92}$$

where $B_e = (\mu u/\bar{R}T\lambda)_e$. The final expression for the matching condition is therefore

$$n\epsilon\theta_o\left(\frac{x}{L}\right)^{n-1} = -\sqrt{\frac{\mu_\infty}{2\rho_\infty u_\infty}}\;\frac{M_\infty^2}{M_e^2}\;\frac{\sqrt{p_\infty}}{a^*}\;\frac{K(0)}{p_e r_e}\sqrt{\frac{B_e}{B}} + \frac{dr_e}{dx} \tag{93}$$

Similar solutions in the P-E-Y regime must satisfy the conditions given in Eqs. (82), (86), and (93). Equation (72) can be rewritten

$$p_e = dx^{2(n-1)} \tag{94}$$

$$d = (2)^{\frac{2n-1}{n}}\, n^2\, \frac{\gamma_e}{\gamma_e^2 + 1}\left(\frac{\gamma_e - 1}{\gamma_e + 1}\right)^{\gamma_e}\frac{1}{\theta_o}\left(A^{\frac{1}{N}} N\right)^{\nu_e} L^{2(1-n)}\, M_\infty^2 \epsilon^2 p_\infty \tag{95}$$

Let

$$r_w = bx^m \tag{96}$$

where b is a dimensional constant. From Eqs. (82), (86), and (93), one obtains

$$1 + \frac{2m + 1}{2(n - 1)} = \frac{2}{\beta} \tag{97}$$

$$n + m = \frac{3}{2} \tag{98}$$

$$n = m \tag{99}$$

These equations yield $n = m = 3/4$, $\beta = -1/2$. Equations (38a), (39a), and (40a) may now be written

$$\left(\lambda \frac{r^2}{r_w^2} K''\right)' + KK'' = \frac{\gamma_e - 1}{4\gamma_e} F_1(K'^2 - G) \tag{100}$$

$$\left(\frac{1}{Pr} \lambda \frac{r^2}{r_w^2} G'\right)' + KG' = 2\left[\lambda \frac{r^2}{r_w^2}\left(\frac{1}{Pr} - 1\right) KK''\right]' \tag{101}$$

$$+ \left[\lambda \frac{r^2}{r_w^2} \frac{1}{Sc}\left(\frac{1}{Le} - 1\right) F_2(G - K'^2) Z_1'\right]'$$

$$\left(\lambda \frac{r^2}{r_w^2} \frac{1}{Sc} Z_1'\right)' + KZ_1' = 0 \tag{102}$$

The boundary conditions for this set are:

$$\eta = 0; \qquad K'(0) = 0 \tag{103a}$$

$$Z_1(0) = 1 \tag{103b}$$

$$G(0) = \frac{H_w}{H_e} \tag{103c}$$

$$K(0) = \frac{\lambda}{Sc} \quad \frac{C_{1w}}{C_{2w}} Z_1'(0) \tag{103d}$$

$$\eta = \eta_e; \quad K'(\eta_e) = 1 \tag{104a}$$

$$Z_1(\eta_e) = 0 \tag{104b}$$

$$G(\eta_e) = 1 \tag{104c}$$

The geometrical constraint for similarity given by Eq. (70) will now be examined for this regime. Equation (80) reduces to

$$\frac{1}{s} \frac{\partial s}{\partial x} = \frac{p_e r_w^2}{\int_0^x p_e r_w^2 \, dx} \tag{105}$$

By Eqs. (94) and (96), we obtain

$$\frac{\frac{1}{r_w} \frac{\partial r_w}{\partial x}}{\frac{1}{s} \frac{\partial s}{\partial x}} = \frac{s}{r_w} \frac{\partial r_w}{\partial s} = \frac{m}{2(n - 1) + 2m + 1} = \text{const.} \tag{106}$$

Thus the condition of similarity is satisfied. The foregoing arguments show that similar solutions exist for the case of a slender body of revolution in hypersonic flow with surface mass transfer. The shape of the body must be

$$r_w = bx^{3/4} \tag{107}$$

For zero mass transfer, the result confirms the conclusions reached by Yasuhara.[2]

VI. SIMILAR SOLUTIONS IN THE G-L REGIME

In the G-L regime, $r_k = r_e$ and (s_k, η_k) is transformed into (s^*, η^*), as shown in Eqs. (14) and (15). The conditions for similarity in this regime are:

(a)
$$\lambda = \lambda(\eta^*) \tag{108a}$$

$$Sc = Sc(\eta^*) \tag{108b}$$

$$Pr = Pr(\eta^*) \tag{108c}$$

(b) The transverse curvature parameter r^2/r_e^2 must be a function of η^*:

$$\frac{r^2}{r_e^2} = f_2(\eta^*) \tag{109}$$

(c) The pressure gradient parameter as defined in Eq. (33) must be constant:

$$\beta = 2\,\frac{d\,\ell n\,p_e}{d\,\ell n\,s^*} \tag{110}$$

(d) The viscous term given in Eq. (73) must be included in the matching condition:

$$\frac{v_e}{u_e} = \frac{\rho_w v_w r_w}{\rho_e u_e r_e} + \frac{d\delta^*}{dx} \tag{73}$$

(e) The surface boundary conditions are defined such that C_{1w} and H_w/H_e are constant.

(f) The boundary condition [Eq. (27a)] relating to the mass transfer at the surface must be compatible with the requirements for a similar solution.

$$K(0) = \left(\frac{\lambda}{Sc}\right)_w \frac{C_{1w}}{1 - C_{1w}} Z_1'(0) \left(\frac{r_w}{r_e}\right)^2 \tag{111}$$

Condition (c) implies that

$$(s^*)^{\beta/2} = A_1^* p_e \tag{112}$$

where A_1^* is a dimensional constant. Equations (14), (81), and (112) give

$$\frac{\int_0^x p_e r_e^2 \, dx}{p_e^{2/\beta}} = \frac{(A_1^*)^{2/\beta}}{B} = \text{const.} \tag{113}$$

which is similar to Eq. (82) for the P-E-Y regime.

Condition (b) requires that the transverse curvature parameter r^2/r_e^2 must be a function of η^*. Since this study concerns a slender body of revolution, an approximation is made by assuming that the x-axis and the axis of symmetry are coincident; that is, $\cos \alpha \simeq 1$. Equation (15) then yields

$$\frac{\sqrt{2s^*}}{\rho_e u_e} \frac{\rho_e}{\rho} d\eta^* = r\,dy \cong r\,dr \tag{114}$$

It follows that

$$r^2 = 2\sqrt{2s^*} \frac{1}{\rho_e u_e} \int_0^{\eta^*} \frac{\rho_e}{\rho} d\eta^* \tag{114a}$$

$$r_e^2 = 2\sqrt{2s^*} \frac{1}{\rho_e u_e} \int_0^{\eta_e^*} \frac{\rho_e}{\rho} d\eta^* \tag{114b}$$

where η_e^* denotes the boundary layer thickness in the η^* variable. These equations show that, in the G-L regime,

$$\left(\frac{r}{r_e}\right)^2 = \frac{\int_0^{\eta^*} \frac{\rho_e}{\rho}\, d\eta^*}{\int_0^{\eta_e^*} \frac{\rho_e}{\rho}\, d\eta^*} \tag{115}$$

Combining Eqs. (58) and (115) gives

$$\left(\frac{r}{r_e}\right)^2 = \frac{\int_0^{\eta^*} F_1(G - K'^2)\, d\eta^*}{\int_0^{\eta_e^*} F_1(G - K'^2)\, d\eta^*} \tag{116}$$

where terms of $O(1/M_e^2)$ are neglected. Equation (116) shows that $(r/r_e)^2$ is a function of η^* within the present approximation. Indeed, $r/r_e \cong y/\delta$, so that r/r_e is a proper similarity variable in the physical coordinate of the boundary layer.

For this regime, the matching condition in Eq. (87) is

$$n\epsilon\theta_o\left(\frac{x}{L}\right)^{n-1} = -\sqrt{\frac{u_\infty B_e}{2\rho_\infty u_\infty B}}\; \frac{\sqrt{p_\infty}}{a_1^*} \left(\frac{M_\infty^2}{M_e^2}\right) \frac{K(0)}{p_e r_e} + \frac{dr_e}{dx} \tag{117}$$

where

$$a_1^* = \frac{\left[\int_0^x p_e r_e^2\, dx\right]^{1/2}}{p_e r_e^2} = \text{constant} \tag{118}$$

Therefore, similar solutions in the G-L regime must satisfy the conditions in Eqs. (113), (117), and (118). Next we assume that

$$r_e = cx^m \tag{119}$$

$$p_e = dx^{2(n-1)} \tag{120}$$

and determine that $m = n = 3/4$, $\beta = -1/2$. The boundary layer equations may be rewritten as:

$$\left(\lambda \frac{r^2}{r_e^2} K''\right)' + KK'' = \frac{\gamma_e - 1}{4\gamma_e} F_1(K'^2 - G) \tag{121}$$

$$\left(\frac{\lambda}{Pr} \frac{r^2}{r_e^2} G'\right)' + KG' = 2\left[\lambda \frac{r^2}{r_e^2}\left(\frac{1}{Pr} - 1\right) KK''\right]'$$

$$+ \left[\lambda \frac{r^2}{r_e^2} \frac{1}{Sc} \left\{\frac{1}{Le} - 1\right\} F_2(G - K'^2) Z_1'\right]' \tag{122}$$

$$\left(\lambda \frac{r^2}{r_e^2} \frac{1}{Sc} Z_1'\right)' + KZ_1' = 0 \tag{123}$$

The boundary conditions are

$$\eta^* = 0 \qquad K'(0) = 0 \tag{124a}$$

$$Z_1(0) = 1 \tag{124b}$$

$$G(0) = \frac{H_w}{H_e} \tag{124c}$$

$$K(0) = \left(\frac{\lambda}{Sc}\right) \frac{C_{1w}}{C_{2w}} Z_1'(0) \left(\frac{r_w}{r_e}\right)^2 \tag{124d}$$

and

$$\eta^* = \eta_e^* \qquad K'(\eta_e^*) = 1 \tag{125a}$$

$$Z_1(\eta_e^*) = 0 \tag{125b}$$

$$G(\eta_e^*) = 1 \tag{125c}$$

Condition (f) requires that

$$\left(\frac{r_w}{r_e}\right)^2 = \text{const.} \tag{126}$$

Treatment of the geometrical constraint given by Eq. (70) gives the same result for the G-L regime that was shown previously in Eq. (106). In the G-L regime, therefore, "exact" similar solutions are obtainable only for the case where $r_w \sim x^{3/4}$. For very slender bodies, the condition $(r_w/r_e)^2 = 0$ should be considered. This suggests an inconsistency, since $K(0) = 0$ for finite $Z_1'(0)$. To alleviate this difficulty the present authors, in Ref. 18, adopt the following condition:

$$K(0) = \frac{C_{1w}}{C_{2w}} D \tag{127}$$

where D is a finite and negative constant. When this alternative condition can be accepted, it replaces Eq. (124d) thus making then possible "approximate" similar solutions of the G-L regime. Within this approximation, it may be observed that

$$r_e \sim \delta \sim x^{3/4} \tag{128}$$

$$p_e \sim x^{-1/2} \tag{129}$$

in the hypersonic viscous interaction solutions of the G-L regime. This approximate result is completely analogous to the strong interaction solution on a hypersonic flat plate with surface mass transfer.[14] Pressure data on a solid cone in the strong interaction region have been analyzed by Yasuhara,[15] and these data are in general agreement with the present prediction. This agreement may be fortuitous, however, as the hypersonic viscous interaction problem for a slender impermeable cone yields rigorously only a nonsimilar solution.

VII. MATCHING THE INVISCID AND VISCOUS FLOW SOLUTIONS

Consideration of the above solutions and those of Mirels[17] for viscous and inviscid regions show that similar solutions exist both in the inviscid and viscous flow regions. It remains now to match these solutions to obtain the proper behavior in the strong interaction region.

Equations (83) and (85) yield

$$\left(\frac{r_e}{r_k}\right)^2 = \left(\frac{r_w}{r_k}\right)^2 + \frac{\gamma_e - 1}{\gamma_e} u_e \sqrt{2B}\, a_k I_k \cos\alpha \tag{130}$$

where

$$I_k = \int_0^{\eta_{k_e}} F_1(G - K'^2)\, d\eta_k \tag{131}$$

and

$$a_k = \frac{\left[\int_0^x p_e r_k^2\, dx\right]^{1/2}}{p_e r_k^2} \tag{132}$$

In the P-E-Y regime, $\eta_k = \eta$, $a_k = a^*$, $r_k = r_w$. In the G-L regime, $\eta_k = \eta^*$, $a_k = a_1^*$, $(r_w/r_e)^2 \to 0$, $r_k = r_e$. It can be shown that

$$a^* = \frac{1}{b\sqrt{2d}} \tag{133}$$

$$a_1^* = \frac{1}{c\sqrt{2d}} \tag{134}$$

b, c, and d being the same as defined in Eqs. (96), (110), and (94), respectively.

In the P-E-Y regime, Eq. (130) becomes ($\cos \alpha \simeq 1$)

$$c = b\left(1 + \frac{\gamma_e - 1}{\gamma_e} u_e \sqrt{\frac{B}{d}} \frac{I}{b}\right)^{1/2} \tag{135}$$

where

$$I = \int_0^{\eta_e} F_1(G - K'^2)\, d\eta \tag{136}$$

In the G-L regime, the analogous result is:

$$c = \frac{\gamma_e - 1}{\gamma_e} u_e \sqrt{\frac{B}{d}} I^* \tag{137}$$

where

$$I^* = \int_0^{\eta_e^*} F_1(G - K'^2)\, d\eta^* \tag{138}$$

If

$$\frac{\gamma_e - 1}{\gamma_e} u_e \sqrt{\frac{B}{d}} \frac{I}{b} \gg 1 \tag{139}$$

then Eq. (135) becomes approximately

$$c = \frac{\gamma_e - 1}{\gamma_e} u_e \sqrt{\frac{B}{d}} \left(I \frac{a_1^*}{a^*}\right) \tag{140}$$

which is formally identical to Eq. (137). Thus Eq. (139) provides a numerical estimate of the size of the body to which the G-L regime approximation applies; that is,

$$b \ll \frac{\gamma_e - 1}{\gamma_e} u_e \sqrt{\frac{B}{d}} I \qquad (141)$$

It also shows that the G-L regime may be treated in a manner similar to the general scheme of the P-E-Y regime. Therefore, only the P-E-Y regime will be examined in this section. Equation (135) may be written as

$$\frac{r_e^2}{r_w^2} = \left(\frac{c}{b}\right)^2 = 1 + \Omega \qquad (142)$$

where

$$\Omega = \sqrt{\frac{B}{B_e}} (\gamma_e - 1) \frac{1}{2^{1/3}} \left(\frac{4}{3}\right)\left(\frac{\gamma_e + 1}{\gamma_e}\right)^{\frac{1}{2}} \left(\frac{\gamma_e + 1}{\gamma_e - 1}\right)^{\gamma_e/2} \sqrt{\theta_o} \left(A^{\frac{1}{N}} N\right)^{-\gamma_e/2} \frac{I}{M_\infty \epsilon} \left(\frac{L^{1/4}}{bM_\infty}\right) \chi$$

$$(143)$$

and

$$\chi = \frac{M_\infty}{\sqrt{R_{e_\infty}}} \qquad R_{e_\infty} = \frac{\rho_\infty u_\infty L}{\mu_\infty} \qquad (144)$$

where χ is the hypersonic viscous interaction parameter. In the present problem, $b/L^{1/4}$ is nondimensional. In terms of this representation, Eq. (139) becomes $\Omega \gg 1$, which would be the condition for the G-L regime approximation. Values of A and θ_o to be used in Eq. (143) are given in the Appendix. From Eq. (92), one obtains

$$\frac{\rho_w v_w r_w}{\rho_e u_e r_e} = - \frac{\Omega}{\sqrt{1 + \Omega}} \frac{K(0)}{1} \frac{b}{M_e^2} \left(\frac{1}{\chi}\right)^{1/4} \qquad (145)$$

It can be further shown that

$$\epsilon\theta_o L^{1/4} = \left(1 - \frac{4}{3}\,\frac{\Omega}{1+\Omega}\,\frac{K(0)}{I}\,\frac{1}{M_e^2}\right) c \tag{146}$$

Define

$$\epsilon_{inv} = \frac{b}{L^{1/4}\theta_o} \tag{147}$$

then

$$\frac{\epsilon}{\epsilon_{inv}} = \sqrt{1+\Omega}\left(1 - \frac{4}{3}\,\frac{\Omega}{1+\Omega}\,\frac{K(0)}{I}\,\frac{1}{M_e^2}\right) \tag{148}$$

For very slender bodies of revolution, $M_e = 0(M_\infty)$; thus, neglecting $0(1/M_e^2)$ terms,

$$\frac{\epsilon}{\epsilon_{inv}} = \sqrt{1+\Omega} \tag{149}$$

provided $K(0)/I = 0(1)$. Equation (149) applies when the effect of the $\rho_w v_w r_w/\rho_e u_e r_e$ term is negligible and is of $0(1/M_e^2)$.

The expression for Ω in Eq. (143) contains ϵ; thus,

$$\Omega = \ell\,\frac{\chi}{(M_\infty\epsilon)(M_\infty\epsilon_{inv})} \tag{150}$$

where

$$\ell = \sqrt{\frac{B}{B_e}}\,(\gamma_e - 1)\,\frac{1}{2^{1/3}}\left(\frac{4}{3}\right)\left(\frac{\gamma_e + 1}{\gamma_e}\right)^{1/2}\left(\frac{\gamma_e + 1}{\gamma_e - 1}\right)^{\gamma_e/2}\frac{I}{\sqrt{\theta_o}}\left(A^{\frac{1}{N}}\,N\right)^{-\gamma_e/2} \tag{151}$$

These expressions can be introduced into Eq. (149) to yield

$$\left(\frac{\epsilon}{\epsilon_{inv}}\right)^2 = 1 + \left(\frac{\ell\chi}{M_\infty^2}\right)\frac{1}{\epsilon\epsilon_{inv}} \tag{152}$$

Thus ϵ can be determined and the hypersonic viscous interaction problem is solved. To compute the induced pressure due to viscous interaction, we define

$$d_{inv} = \frac{d}{1 + \Omega} \tag{153}$$

$$p_{inv} = d_{inv}\,\chi^{-1/2} \tag{154}$$

The surface pressure on the body of revolution in an inviscid hypersonic flow is p_{inv}. The induced pressure rise is due to viscous interaction

$$\begin{aligned} \Delta p &= p_e - p_{inv} = (d - d_{inv})\chi^{-1/2} \\ &= \Omega\, p_{inv} \end{aligned} \tag{155}$$

Therefore

$$\begin{aligned} \frac{\Delta p}{p_\infty} = 2^{1/3}\,\frac{3}{4}\left\{\left(\frac{\gamma_e - 1}{\gamma_e + 1}\right)^{\gamma_e/2} (\gamma_e - 1)\sqrt{\frac{\gamma_e}{\gamma_e + 1}}\right\} \\ \left\{\theta_c^{-3/2}\left(A^{\frac{1}{N}} N\right)^{\gamma_e/2} I\sqrt{\frac{B}{B_e}}\,\frac{M_\infty^3}{\sqrt{1 + \Omega}}\sqrt{\frac{\mu_\infty}{\rho_\infty u_\infty x}}\right\} \end{aligned} \tag{156}$$

VIII. CONCLUSIONS

We have obtained a similar solution for the case of a hypersonic laminar boundary layer on an axisymmetric body. The strong interaction between the boundary layer and the shock wave has been considered, and the analysis includes mass transfer from the surface. In the viscous flow regime, the boundary layer equations were transformed to a set of ordinary differential equations employing appropriate similarity variables. The inviscid region was treated using the well-known similarity transformation for the small-disturbance equations. These solutions were matched at the interface between the two regions, requiring that the pressure and normal velocity be continuous. The matching procedure yielded an analytical expression for the surface pressure and a similarity law for the normal velocity at the wall.

We have derived a set of analytic expressions that gives the relationship among the induced pressure at the wall, normal injection velocity at the wall, and other important wall variables, in terms of the parameters that describe the strong interaction flow.

A transformation appropriate to both the P-E-Y regime and the G-L regime has been derived from Eqs. (10) and (11). This transformation leads to the general set of equations, which can then be applied to the appropriate regime.

Within the restrictions indicated in the present study, similar solutions have been obtained for the case of a thick boundary layer on a very slender body of revolution. However, when the boundary layer thickness and the characteristic body dimension are of the same order of magnitude, it was shown, in agreement with Yashuara's[2] earlier conclusions, that rigorous similarity is possible only for three-quarter-power bodies.

For an axisymmetric body, the similarity condition for the normal velocity at the wall follows a $(\rho v r)$ law rather than the well-known (ρv) law in the two-dimensional case.

The restrictive nature of the similar solutions is adequately illustrated in the present study. In physical problems of interest, it may be necessary to deal with more complicated situations than those allowed by similar solutions. We have seen in Section VI that in hypersonic viscous interaction problems, similar solution may be applicable only in the outer viscous flow region and may not be applicable in the inner viscous flow region (cf. Eq. (124d)). In such cases, we strictly must obtain nonsimilar solutions of Eqs. (38) to (40), under the boundary conditions in Eqs. (25) to (29). The scheme of calculations used in Refs. 19 and 20 may be adopted for this purpose. In the present Memorandum, the surface condition in Eq. (127) is adopted to obtain "approximate" similar solutions of the G-L regime. Alternatively, for the G-L regime, the visoous flow region may be dealt with by a composite layer approach in order that all the physical boundary conditions are rigorously satisfied. Stewartson[16] has recently treated the hypersonic slender cone problems by this composite viscous layer representation.

Appendix

VALUES OF A AND θ_o

The similarity solutions for inviscid hypersonic flow over slender power-law bodies have been computed by Mirels.[17] In terms of Mirel's data for axisymmetric three-quarter-power law bodies, we have

γ_e	η_b	$F(\eta_b)$
1.4	.875	.696
1.67	.819	.634

These values of η_b and $F(\eta_b)$ are related to θ_o and A by the following formulas:

$$\theta_o = \eta_b$$

$$F(\eta_b) = \frac{1}{\gamma_e + 1} 2^{\frac{2n-1}{n}} \left(\frac{\gamma_e - 1}{\gamma_e + 1}\right)^{\gamma_e} \frac{1}{\theta_o} \left(A^{\frac{1}{N}} N\right)^{\gamma_e}$$

where

$$N = \frac{n\gamma_e}{n - 1 + n\gamma_e} > 0$$

$$n = \frac{3}{4}$$

REFERENCES

1. Li, T. Y., and H. T. Nagamatsu, "Similar Solutions of Compressible Boundary Layer Equations," J. Aeron. Sci., Vol. 22, No. 9, September 1955.

2. Yasuhara, M., "Axisymmetric Viscous Flow Past Very Slender Bodies of Revolution," J. Aeron. Sci., Vol. 29, June 1962, pp. 667-679.

3. Stewartson, K., "The Asymptotic Boundary Layer on a Circular Cylinder in Axial Incompressible Flow," Quart. Appl. Math., Vol. 13, No. 2, July 1955, pp. 113-122.

4. Glauert, M. B., and M. J. Lighthill, "The Axisymmetric Boundary Layer on a Long Thin Cylinder," Proc. Roy. Soc. London, Vol. 230, 1955, pp. 188-203.

5. Mark, R. M., "Laminar Boundary Layers on Slender Bodies of Revolution in Axial Flow," GALCIT Hypersonic Wind Tunnel Memo No. 21, July 1954.

6. Mangler, W., "Zusamnenhang zwischen ebenen und rotationsyymmetrischen Greuzschichten in Kompressiblen Flüssigkeiten," ZAMM, Vol. 28, No. 4, 1948, pp. 97-103.

7. Probstein, R. F., and D. Elliott, "The Transverse Curvature Effect in Compressible Axially Symmetric Laminar Boundary Layer Effect," J. Aeron. Sci., Vol. 23, March 1956, pp. 208-224.

8. Steiger, M. H., and M. H. Bloom, "Thick Boundary Layers Over Slender Bodies With Some Effects of Heat Transfer, Mass Transfer and Pressure Gradient," Int. J. Heat Mass Transfer, Vol. 5, 1962.

9. Eckert, E. R. G., and P. J. Schneider, "Effect of Diffusion in an Isothermal Boundary Layer," J. Aeron. Sci., Vol. 23, No. 4, April 1956.

10. Gazley, C., Jr., "Fundamentals of Nonequilibrium Flow Processes," Hypersonic Gas Dynamics Notes, 1958, pp. 10-13.

11. Stewartson, K., "On the Motion of a Flat Plate at High Speed in a Viscous Compressible Fluid. II: Steady Motion," J. Aeron. Sci., Vol. 22, 1955, p. 303.

12. Oguchi, H., "First Order Approach to a Strong Interaction Problem in Hypersonic Flow Over an Insulated Flat Plate," Aero. Res. Institute, University of Tokyo, Rep. 330, 1958.

13. Li, T. Y., and J. F. Gross, "On Transverse Curvature Effects in Axisymmetric Hypersonic Boundary Layers," AIAA Journal, October 1964; The RAND Corporation, P-2872, 1964.

14. Li, T. Y., and J. F. Gross, "Hypersonic Strong Viscous Interaction on a Flat Plate With Surface Mass Transfer," HTFMI Proc., June 1961; The RAND Corporation, RM-3000-PR, March 1963.

15. Yasuhara, M., "On the Hypersonic Viscous Flow Past Slender Bodies of Revolution, J. Phys. Soc., Japan, Vol. 11, No. 8, August 1956.

16. Stewartson, K., "Viscous Hypersonic Flow Past a Slender Cone," Phys. of Fluids, Vol. 7, No. 5, May 1964, pp. 667-675.

17. Mirels, H., "Similarity Solutions for Inviscid Flow Over Slender Power Law and Related Bodies," ARS Publication No. 1111-60, May 1960.

18. Li, T. Y., and J. F. Gross, "Hypersonic Viscous Interaction on a Slender Body of Revolution With Surface Mass Transfer," Zeitschrift für Flugwissenschaften, Vol. 14, No. 3, March 1966, pp. 129-139; The RAND Corporation, P-2977-1, 1964.

19. Wei, M. H., "Asymptotic Boundary Layer Over a Slender Body of Revolution in Axial Compressible Flow," AIAA Journal, May 1965, Vol. 3, No. 5, pp. 809-816.

20. Albacete, L. M., "The Effect of Fluid Injection on the Incompressible Laminar Boundary Layer Over a Slender Parabola of Revolution," NOLTR 65-226, February 25, 1966.

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10. ABSTRACT

An analytical formulation of the problem of hypersonic viscous interaction on a very slender body of revolution with a thick boundary layer and mass transfer. Particular attention is directed to the establishment of the mathematical restrictions necessary to ensure the existence of similar solutions of the laminar boundary-layer equations for this class of problems. Yasuhara's analysis is extended to include surface mass transfer and binary mixture effects. It is shown that when the boundary-layer thickness is of the same order of magnitude as the characteristic body dimension, rigorous similarity is possible only for three-quarter-power-law bodies. This is in agreement with Yasuhara's earlier conclusions. Similar solutions to the boundary-layer growth equations exist when the transverse curvature expression is a function of the similarity variable. The criterion for the existence of similar solutions for the problem of hypersonic viscous interaction is that there must be a definite relationship between body surface shape and surface injection velocity.

11. KEY WORDS

Ablation
Mass transfer
Hypersonic flow
Fluid mechanics
Reentry vehicles
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