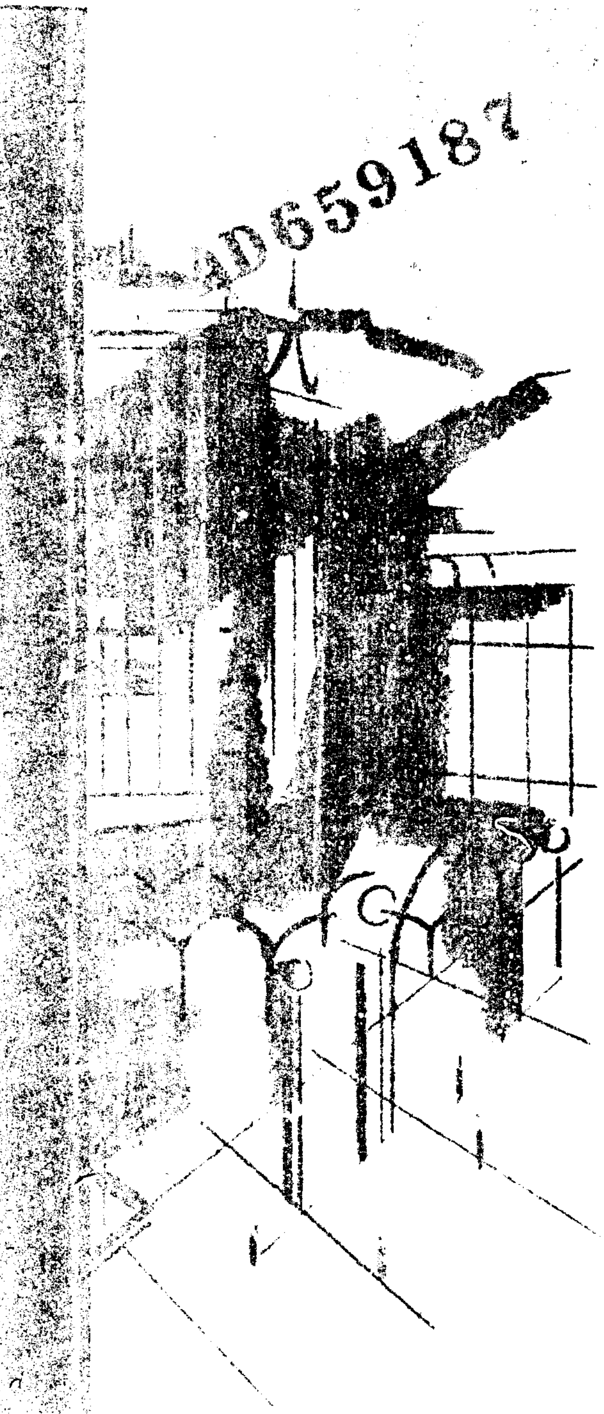

FINITE DEFORMATIONS OF
INCOMPRESSIBLE SIMPLE SOLIDS

by

M. M. Carroll

Report No. AM-67-6
Contract Nonr-222(69), Project NR-064-436
Division of Applied Mechanics
July 1967

COLLEGE OF ENGINEERING
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SUMMARY

Exact solutions of several problems which involve finite deformations of homogeneous isotropic incompressible simple solids are obtained by the inverse method. The deformations involved include bending, stretching, shearing, torsion and both cylindrical and spherical inflation. The corresponding stresses are expressed in terms of the functionals which describe the material response to homogeneous plane deformations. The results obtained apply, in particular, to incompressible viscoelastic solids.

1. INTRODUCTION

A general theory of the nonlinear mechanical behaviour of materials with memory was developed in a series of papers by Green and Rivlin [1, 3]* and Green, Rivlin and Spencer [2], and in an alternative form by Noll [4]. The theory applies equally to simple solids and to simple fluids and there has been considerable subsequent development, by several workers, in each area. A comprehensive account of the work prior to 1965 is contained in the book by Truesdell and Noll [5].

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*Numbers in square brackets indicate references at the end of the paper.

Despite the extensive theoretical development, a body of exact solutions comparable to that obtained in finite elasticity theory is not presently available for materials with memory. In fact, the only area which has proved to be at all fruitful in this regard is that of steady viscometric flow of incompressible simple fluids. The simplicity of such flows compensates sufficiently for the complicated nature of the material response, which is modified by the incompressibility assumption, so that several exact solutions have been obtained (see [5], also the book by Coleman, Markovitz and Noll [6] and the review paper by Rivlin [7]).

Rivlin [8] first observed the great simplification which results in a nonlinear mechanical theory when the material is assumed to be incompressible. This observation has led to the discovery of the above-mentioned viscometric solutions and of several exact solutions in finite elasticity theory (Rivlin [8, 9, 10], Green and Shield [11], Adkins, Green and Shield [12], Ericksen and Rivlin [13], Ericksen [14], Klingbeil and Shield [15], Singh and Pipkin [16])[*]. These solutions involve deformations which are controllable for homogeneous isotropic incompressible elastic materials. A deformation is said to be controllable for materials of a certain type if it can be supported without body forces in every material of that type. Problems involving controllable deformations can be solved by the inverse method, in which the deformation is specified precisely at the outset.

The above-mentioned deformations of incompressible elastic materials are exact solutions of the equations of equilibrium. Each deformation involves some constant parameters, which may be replaced by functions of time (amplitude

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[*] Singh and Pipkin [17] have obtained corresponding exact solutions for an incompressible elastic dielectric.

functions) to give quasi-static solutions, i.e., time-dependent solutions in a theory in which inertial effects are not considered. The resulting deformations can then be specialized to yield exact solutions of the equations of motion (quasi-equilibrated motions), by requiring the amplitude functions to be such that the acceleration can be expressed as the gradient of a single-valued scalar potential (Truesdell [18]). Two special quasi-equilibrated motions, radial oscillations of cylindrical tubes and of spherical shells, have been considered by Knowles [19] and by Guo Zhong-Heng and Solecki [20], respectively.

It was shown in a previous paper (Carroll [21]) that every deformation which is presently known to be controllable for homogeneous isotropic incompressible elastic materials is controllable also for homogeneous isotropic incompressible simple materials. Consequently, the inverse method may be applied to the solution of problems involving such deformations and this is done in the present paper, mainly with regard to simple solids. Homogeneous motions of incompressible simple solids have been discussed by Coleman and Truesdell [22], and the torsion of a cylinder of isotropic incompressible simple material with fading memory was discussed by Christensen [23].

The constitutive equations for homogeneous isotropic incompressible simple materials are presented in Section 2. These equations are specialized, in Section 3, for a class of deformations for which the (time-dependent) strain matrix may be expressed as a linear combination of four time-independent matrices with special properties. The material response to such deformations is determined by three functionals of four argument functions.

Following a brief discussion of the equations of motion, in Section 4, the class of homogeneous plane deformations, i.e., homogeneous plane strain superimposed on uniform normal extension, is considered in Section 5. Such deformations are shown to belong to the class of deformation discussed in Section 3, and it is shown that the material response to homogeneous plane deformations is determined by two independent material functionals of four argument functions. Two special cases are considered, namely pure homogeneous deformations (the material response being determined by one material functional of two argument functions) and simple shear (the material response being determined by three material functionals of one argument function).

It is shown, in subsequent sections, that knowledge of the response of a homogeneous isotropic incompressible simple material to every homogeneous plane deformation suffices to determine the material response to every deformation which is presently known to be controllable for such materials. Consequently, the stresses corresponding to every such deformation can be expressed in terms of the material functionals which characterize the material response to homogeneous plane deformation, and this is carried out in Sections 6-10.

Extensional, flexural and shearing deformations of a rectangular block, and of an annular wedge, are discussed in Sections 6 and 7, respectively) (these deformations represent slight generalizations of the corresponding deformations considered in elasticity theory). The extension, inflation, bending, torsion and shearing of cylinders and wedges is discussed in Section 8, and particular cases are considered in some detail. The inflation of a spherical shell is discussed in Section 9, and the extension, bending, and azimuthal shearing of an annular wedge is discussed in Section 10.

The stresses corresponding to each of these deformations are expressed in terms of the material functionals which characterize the response to homogeneous plane deformation. Resultant tractions are calculated in some cases, but specific deformation histories are not considered. In particular, it is shown that (a) the material response to pure homogeneous deformation determines the response to extension, inflation and bending of cylinders, (b) the response to simple extension determines the response to spherical inflation and (c) the response to simple shear determines the response to pure torsion, and to both axial and azimuthal shear.

An alternative method of obtaining the solutions discussed in Sections 6-10 is outlined in Section 11. This method involves consideration of the local deformation and of the balance of forces acting on a suitably chosen volume element. Finally, the various material functionals which characterize the material response to homogeneous plane deformations are calculated, in the Appendix, for a particular simple material.

It should be emphasized that, while the deformations considered are time-dependent, inertial effects are not considered, and in this respect the solutions obtained are not exact. However, exact solutions of the dynamic equations (quasi-equilibrated motions) may be obtained by the procedure outlined above in the elastic case. The appropriate restrictions on the amplitude functions for the deformations considered in Sections 6-9 are listed in [5], and those for the deformation considered in Section 10 are listed in [21]. The equations which govern radial oscillations of cylindrical tubes and spherical shells of incompressible simple material are displayed in Sections 8 and 9.

The emphasis in this paper is on deformations of simple solids and the solutions obtained apply, in particular, to incompressible viscoelastic solids. However, many of the results apply also to simple fluids. For example, it is evident that several inhomogeneous flows, which are independent of material properties, are possible — for example, radial (cylindrical or spherical) flow with spatially uniform velocity.

2. CONSTITUTIVE EQUATIONS

The motion of a body can be described by specifying the coordinates $x_i(X_A,\tau)$ at time τ of the generic particle whose coordinates in an undeformed reference state were X_A, all coordinates being measured with respect to the same fixed rectangular Cartesian frame x. Upper and lower case Latin indices have the range 1,2,3 and the usual summation convention is adopted. It will sometimes be convenient to suppress dependence on the refarence coordinates (for example, to write $x_i(\tau)$ for $x_i(X_A,\tau)$) and also to suppress dependence on the present time t, thus x_i denotes $x_i(t)$.

For a compressible simple material, the stress σ_{ij} at a given particle at time t is determined by the values of the displacement gradients $x_{i,A}(\tau)$ $(= \partial x_i(\tau)/\partial X_A)$ at that particle at all times up to and including the time t. For an incompressible material the displacement gradients must satisfy the incompressibility condition

$$\left| x_{i,A}(\tau) \right| = 1 \quad . \tag{2.1}$$

For such materials, the stress is determined only to within an arbitrary pressure, so that

$$\sigma_{ij} = - p\, \partial_{ij} + S_{ij} \quad , \tag{2.2}$$

where p is arbitrary, ∂_{ij} denotes the Kronecker delta and the extra stress S_{ij} is determined by the history of the deformation gradients, thus

$$S_{ij} = \mathcal{S}_{ij}\{x_{i,A}(\tau)\}_{\tau=-\infty}^{t} \quad . \tag{2.3}$$

The condition that arbitrary superposed time-dependent rigid rotations of the body shall leave the stress unaltered with respect to the body leads to ([1], [4])

$$S_{ij} = x_{i,A}\, x_{j,B}\, \mathcal{S}_{AB}\, [E_{PQ}(\tau)]_{\tau=-\infty}^{t} \quad , \tag{2.4}$$

where

$$2E_{PQ}(\tau) = x_{i,P}(\tau)\, x_{i,Q}(\tau) - \delta_{PQ} \quad . \tag{2.5}$$

If the material is homogeneous, then the form of the functionals $\mathcal{S}_{AB}$ in the constitutive equation (2.4) is the same for all particles. If the material is isotropic in the undeformed reference state, then (2.4) has the form ([1], Wineman and Pipkin [24])

$$\underline{S} = \underline{F}\, \underline{\mathcal{S}}\, [\underline{E}(\tau)]_{\tau=-\infty}^{t}\, \underline{F}^{T} \quad , \tag{2.6}$$

with

$$\underline{\mathcal{S}}[\underline{E}(\tau)]_{\tau=-\infty}^{t} = \sum_{\alpha=0}^{5} \underline{\mathcal{S}}^{(\alpha)} \Big[\underline{f}^{(\alpha)}(\tau_1,\dots,\tau_\alpha)\Big|_{\tau_1,\dots,\tau_\alpha=-\infty}^{t};\ \operatorname{tr} \underline{f}^{(\beta)}(\tilde{\tau}_1,\dots,\tilde{\tau}_\beta)\Big|_{\tilde{\tau}_1,\dots,\tilde{\tau}_\beta=-\infty}^{t}\Big] \quad (\beta=1,\dots,6) \quad . \tag{2.7}$$

$$\underline{f}^{(0)} = \underline{I} \ , \quad 2\underline{f}^{(\beta)}(\tau_1,\dots,\tau_\beta) = \underline{E}(\tau_1)\dots\underline{E}(\tau_\beta) + \underline{E}(\tau_\beta)\dots\underline{E}(\tau_1)$$

Here $\underline{S}$, $\underline{F}$, $\underline{E}(\tau)$ and $\underline{I}$ are 3X3 matrices defined by $\underline{S} = \| S_{ij} \|$, $\underline{F} = \| \partial x_i/\partial X_A \|$, $\underline{E}(\tau) = \| E_{PQ}(\tau) \|$ and $\underline{I} = \| \delta_{ij} \|$, $\underline{F}^T$ denotes the transpose of $\underline{F}$, tr $\underline{f}$ denotes the trace of $\underline{f}$ and the matrix functionals $\underline{\mathcal{S}}^{(\alpha)}$ are linear in their matrix arguments $f^{(\alpha)}(\tau_1,\dots,\tau_\alpha)$ $(\alpha=0,1,\dots,5)$.

3. CONSTITUTIVE EQUATIONS FOR A SPECIAL CLASS OF DEFORMATIONS

Consider the class of deformations for which the strain matrix $\underline{E}(\underline{X},\tau)$ can be expressed in the form

$$2\underline{E}(\underline{X},\tau) = k(\underline{X},\tau)\, \underline{K}(\underline{X}) + \ell(\underline{X},\tau)\, \underline{L}(\underline{X}) + m(\underline{X},\tau)\, \underline{M}(\underline{X}) + n(\underline{X},\tau)\, [\underline{N}(\underline{X}) + \underline{N}(\underline{X})^T] \quad , \tag{3.1}$$

where the time-independent matrices $\underline{K}$, $\underline{L}$, $\underline{M}$ and $\underline{N}$ have the following properties:

(i) the set

$$\underline{K}\ ,\ \underline{L}\ ,\ \underline{M}\ ,\ \underline{N}\ ,\ \underline{N}^T\ ,\ \underline{0} \tag{3.2}$$

is closed under matrix multiplication ($\underline{0}$ is the null matrix), with multiplication table

$$\begin{array}{c|ccccc}
 & \underline{K} & \underline{L} & \underline{M} & \underline{N} & \underline{N}^T \\
\hline
\underline{K} & \underline{K} & \underline{0} & \underline{0} & \underline{0} & \underline{0} \\
\underline{L} & \underline{0} & \underline{L} & \underline{0} & \underline{N} & \underline{0} \\
\underline{M} & \underline{0} & \underline{0} & \underline{M} & \underline{0} & \underline{N}^T \\
\underline{N} & \underline{0} & \underline{0} & \underline{N} & \underline{0} & \underline{L} \\
\underline{N}^T & \underline{0} & \underline{N}^T & \underline{0} & \underline{M} & \underline{0}
\end{array} \tag{3.3}$$

(ii)

$$\text{tr}\ \underline{K} = \text{tr}\ \underline{L} = \text{tr}\ \underline{M} = 1\quad ,\quad \text{tr}\ \underline{N} = 0 \tag{3.4}$$

(iii)

$$\underline{K} + \underline{L} + \underline{M} = \underline{I}\quad . \tag{3.5}$$

By virtue of these three properties, substitution for $\underline{E}(\underline{X},\tau)$ from (3.1) in the constitutive equation $(2.7)_1$ gives

$$\mathcal{S}(\underline{X},t) = \mathcal{K}(\underline{X},t)\underline{K}(\underline{X}) + \mathcal{L}(\underline{X},t)\underline{L}(\underline{X}) + \mathcal{M}(\underline{X},t)\underline{M}(\underline{X}) + \mathcal{N}(\underline{X},t)\ [\underline{N}(\underline{X}) + \underline{N}(\underline{X})^T]\quad , \tag{3.6}$$

where

$$
\begin{aligned}
\mathcal{K}(\underline{X},t) &= \mathcal{K}[k(\underline{X},\tau)\ ;\ \ell(\underline{X},\tau)\ ;\ m\ (\underline{X},\tau)\ ;\ n(\underline{X},\tau)]_{\tau=-\infty}^{t} \quad , \\
\mathcal{L}(\underline{X},t) &= \mathcal{L}[k(\underline{X},\tau)\ ;\ \ell(\underline{X},\tau)\ ;\ m\ (\underline{X},\tau)\ ;\ n(\underline{X},\tau)]_{\tau=-\infty}^{t} \quad , \\
\mathcal{M}(\underline{X},t) &= \mathcal{M}[k(\underline{X},\tau)\ ;\ \ell(\underline{X},\tau)\ ;\ m\ (\underline{X},\tau)\ ;\ n(\underline{X},\tau)]_{\tau=-\infty}^{t} \quad , \\
\mathcal{N}(\underline{X},t) &= \mathcal{N}[k(X,\tau)\ ;\ \ell(\underline{X},\tau)\ ;\ m\ (\underline{X},\tau)\ ;\ n(\underline{X},\tau)]_{\tau=-\infty}^{t} \quad ,
\end{aligned}
\tag{3.7}
$$

The form of these scalar functionals is determined by the form of the matrix functionals $\underline{\mathcal{L}}^{(\alpha)}$ ($\alpha=0,1,---,5$) of (2.7).

The following symmetry relations between the functionals $\mathcal{K}$, $\mathcal{L}$, $\mathcal{M}$ and $\mathcal{N}$ are evident from (3.3), (3.4) and (3.5):

$$
\begin{aligned}
\mathcal{K}[k\ ;\ \ell\ ;\ m\ ;\ n] &= \mathcal{K}[k;\ m\ ;\ \ell\ ;\ n] \quad , \\
\mathcal{L}[k\ ;\ \ell\ ;\ m\ ;\ n] &= \mathcal{M}[k\ ;\ m\ ;\ \ell\ ;\ n] \quad , \\
\mathcal{N}[k\ ;\ \ell\ ;\ m\ ;\ n] &= \mathcal{N}[k\ ;\ m\ ;\ \ell\ ;\ n] \quad ,
\end{aligned}
\tag{3.8}
$$

$$
\mathcal{K}[k\ ;\ \ell;\ m\ ;\ 0] = \mathcal{L}[m\ ;\ k\ ;\ \ell\ ;\ 0] = \mathcal{M}[\ell\ ;\ m\ ;\ k\ ;\ 0] \ , \tag{3.9}
$$

$$
\mathcal{N}[k\ ;\ \ell\ ;\ m\ ;\ 0] = 0 \quad . \tag{3.10}
$$

It follows from $(3.8)_2$ that only three of the functionals (3.7) are independent. It is also evident from (3.3), (3.4) and (3.5) that $\mathcal{K}$, $\mathcal{L}$ and $\mathcal{M}$ are even functionals of $n(\tau)$, while $\mathcal{N}$ is an odd functional of $n(\tau)$.

The response of any homogeneous isotropic incompressible simple material* to deformations such that the associated strain matrix can be expressed in the form (3.1) is thus determined by three scalar functionals of four argument

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*In order to avoid tedious repetition, the words "simple material" will henceforth be used to mean "homogeneous isotropic simple material".

functions. It will be shown in subsequent sections that the strain matrices associated with a class of homogeneous deformations, and with all of the inhomogeneous deformations which are presently known to be controllable for incompressible simple materials, have the form (3.1).

It follows from (3.9) that the material response to strains of the form (3.1), with $n(\underline{X},\tau) = 0$, is determined by one scalar functional of three argument functions.

4. EQUATIONS OF MOTION

In the absence of body forces, the equations of motion are

$$\sigma_{ij,j} = \rho\, a_i \quad , \quad \sigma_{ij} = \sigma_{ji} \quad , \tag{4.1}$$

where ρ is the density of the material and a_i are the components of acceleration. The symmetry condition $(4.1)_2$ was used in writing the constitutive equation (2.7). For an incompressible material $(4.1)_1$ becomes

$$S_{ij,j} = p_{,i} + \rho\, a_i \quad . \tag{4.2}$$

In a quasi-static theory, this equation is replaced by

$$S_{ij,j} = p_{,i} \quad . \tag{4.3}$$

Most of the deformations considered here are quasi-static, i.e., solutions of (4.3). However, Truesdell [18] has shown that such solutions can be specialized so as to yield dynamic solutions by requiring that the acceleration have the form

$$a_i = - \zeta_{,i} \quad , \tag{4.4}$$

where ζ is a single-valued function, in which case the inertia term in (4.2) can be incorporated in the arbitrary pressure gradient $p_{,i}$. Some special dynamic solutions are considered in Sections 8 and 9.

The tractions T_i on a surface with outward normal n_i at time t are

$$T_i = \sigma_{ij} n_j = -p\, n_i + S_{ij}\, n_j \quad . \tag{4.5}$$

5. HOMOGENEOUS PLANE DEFORMATIONS

The deformation described by

$$\begin{gathered} x(\tau) = A(\tau)X \ , \quad y(\tau) = C(\tau)Y + D(\tau)Z \ , \quad z(\tau) = E(\tau)Y + F(\tau)Z \ ; \\ A(\tau)\ [\ C(\tau)\ F(\tau) - D(\tau)\ E(\tau)\ ] = 1 \ , \end{gathered} \tag{5.1}$$

where $x(\tau)$, $y(\tau)$, $z(\tau)$ and X, Y, Z are rectangular Cartesian coordinates, represents a homogeneous plane deformation, i.e., a homogeneous plane strain superimposed on uniform normal extension. The deformation (5.1) carries the rectangular block bounded in the reference state by the planes

$$X = X' \ , \ X'' \ ; \ Y = \pm Y' \ , \ Z = \pm Z' \tag{5.2}$$

into the parallelepiped bounded at time t by the planes

$$x = AX' \ , \ AX'' \ ; \quad Fy - Dz = \pm Y'/A \ ; \quad Ey - Cz = \mp Z'/A \quad . \tag{5.3}$$

The matrices $\underline{K}$, $\underline{L}$, $\underline{M}$ and $\underline{N}$, defined by

$$\underline{K} = \left\| \begin{matrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} \right\| , \qquad \underline{L} = \left\| \begin{matrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{matrix} \right\| ,$$

$$\underline{M} = \left\| \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{matrix} \right\| \qquad \underline{N} = \left\| \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{matrix} \right\| \tag{5.4}$$

have the properties (i), (ii) and (iii) listed in Section 3, and the strain matrix associated with the deformation (5.1) is given by

$$2\underline{E}(\tau) = k(\tau)\underline{K} + \ell(\tau)\underline{L} + m(\tau)\underline{M} + n(\tau)\,(\underline{N} + \underline{N}^T) \quad , \tag{5.5}$$

with

$$\begin{aligned} k(\tau) &= A(\tau)^2 - 1 \quad , & \ell(\tau) &= C(\tau)^2 + E(\tau)^2 - 1 \quad , \\ m(\tau) &= D(\tau)^2 + F(\tau)^2 - 1 \quad , & n(\tau) &= C(\tau)\, D(\tau) + E(\tau)\, F(\tau) \quad . \end{aligned} \tag{5.6}$$

Consequently, substitution from (5.5) in the constitutive equation $(2.7)_1$ leads to an expression of the form (3.6), and substitution in (2.6) gives the extra stress matrix $\underline{S}$. The physical components at time t are

$$\begin{aligned} S_{xx} &= A^2\mathcal{K} \quad , & S_{yz} &= CE\mathcal{L} + (CF + DE)\mathcal{N} + DF\mathcal{M} \; , \\ S_{yy} &= C^2\mathcal{L} + 2CD\mathcal{N} + D^2\mathcal{M} \quad , & S_{zx} &= 0 \quad , \\ S_{zz} &= E^2\mathcal{L} + 2EF\mathcal{N} + F^2\mathcal{M} \quad , & S_{xy} &= 0 \quad , \end{aligned} \tag{5.7}$$

where $\mathcal{K}$, $\mathcal{L}$, $\mathcal{M}$ and $\mathcal{N}$ are functionals of the argument functions (5.6). Since the extra stress components are independent of position, the equations of motion (4.3) are satisfied, p being an arbitrary function of t.

The extra stress components (5.7) may be expressed as functionals of the amplitude functions $C(\tau)$, $D(\tau)$, $E(\tau)$ and $F(\tau)$, the normal stretch function $A(\tau)$ being determined by the incompressibility condition $(5.1)_4$, thus

$$\begin{aligned} S_{xx} &= \mathcal{S}_{11}\,[C(\tau);\ D(\tau),\ E(\tau);\ F(\tau)]_{\tau=-\infty}^{t} \equiv A^2\mathcal{K} \quad , \\ S_{yy} &= \mathcal{S}_{22}\,[C(\tau);\ D(\tau);\ E(\tau);\ F(\tau)]_{\tau=-\infty}^{t} \equiv C^2\mathcal{L} + 2CD\mathcal{N} + D^2\mathcal{M} \quad , \\ S_{zz} &= \mathcal{S}_{33}\,[C(\tau);\ D(\tau);\ E(\tau);\ F(\tau)]_{\tau=-\infty}^{t} \equiv E^2\mathcal{L} + 2EF\mathcal{N} + F^2\mathcal{M} \quad , \\ S_{yz} &= \mathcal{S}_{23}\,[C(\tau);\ D(\tau);\ E(\tau);\ F(\tau)]_{\tau=-\infty}^{t} \equiv CE\mathcal{L} + (CF + DE)\mathcal{N} + DF\mathcal{M} \quad , \end{aligned} \tag{5.8}$$

where $\mathcal{K}$, $\mathcal{L}$, $\mathcal{M}$ and $\mathcal{N}$ are functionals of the argument functions (5.6). The stress components at time t are given by

$$\begin{aligned} T_{xx} = -p + S_{xx} \ , \quad T_{yy} = -p + S_{yy} \ , \quad T_{zz} = -p + S_{zz} \quad , \\ T_{yz} = S_{yz} \ , \quad T_{zx} = T_{xy} = 0 \ . \end{aligned} \tag{5.9}$$

Because of the arbitrary pressure p in $(5.9)_{1,2,3}$ it is evident that the material response to homogeneous plane deformation may be described completely by the functional expressions for two normal stress differences ($T_{yy} - T_{xx}$ and $T_{zz} - T_{xx}$, say) and for the shear stress T_{yz}. These functionals $\mathcal{Y}_{22}$, $\mathcal{Y}_{33}$ and $\mathcal{Y}_{23}$, which may be called material functionals, are defined by

$$\begin{aligned}
\mathcal{Y}_{22}[C(\tau);\ D(\tau);\ E(\tau);\ F(\tau)]_{\tau=-\infty}^{t} &\equiv \mathcal{S}_{22}[C(\tau);\ D(\tau);\ E(\tau);\ F(\tau)]_{\tau=-\infty}^{t} - \\
&\quad - \mathcal{S}_{11}[C(\tau);\ D(\tau);\ E(\tau);\ F(\tau)]_{\tau=-\infty}^{t} \quad , \\
\mathcal{Y}_{33}[C(\tau);\ D(\tau);\ E(\tau);\ F(\tau)]_{\tau=-\infty}^{t} &\equiv \mathcal{S}_{33}[C(\tau);\ D(\tau);\ E(\tau);\ F(\tau)]_{\tau=-\infty}^{t} - \\
&\quad - \mathcal{S}_{11}[C(\tau);\ D(\tau);\ E(\tau);\ F(\tau)]_{\tau=-\infty}^{t} \quad , \\
\mathcal{Y}_{23}[C(\tau);\ D(\tau);\ E(\tau);\ F(\tau)]_{\tau=-\infty}^{t} &\equiv \mathcal{S}_{23}[C(\tau);\ D(\tau);\ E(\tau);\ F(\tau)]_{\tau=-\infty}^{t} \quad .
\end{aligned} \tag{5.10}$$

By suitable choice of the pressure p, the planes x=constant may be rendered free of traction, in which case the stress components (5.9) become

$$T_{xx} = 0 \ , \quad T_{yy} = \mathcal{Y}_{22} \ , \quad T_{zz} = \mathcal{Y}_{33} \ , \quad T_{yz} = \mathcal{Y}_{23} \ , \quad T_{zx} = T_{xy} = 0 \quad . \tag{5.11}$$

Thus, the material functionals (5.10) give the nonvanishing stress components corresponding to a homogeneous plane deformation, when the fundamental plane is free of traction.

The following symmetry relations, which are evident from physical considerations, may be verified from (3.8), (5.6), (5.8) and (5.10):

$$\begin{aligned}
\mathcal{S}_{11}[C(\tau);\ D(\tau);\ E(\tau);\ F(\tau)]_{\tau=-\infty}^{t} &= \mathcal{S}_{11}[F(\tau);\ E(\tau);\ D(\tau);\ C(\tau)]_{\tau=-\infty}^{t} \quad , \\
\mathcal{S}_{22}[C(\tau);\ D(\tau);\ E(\tau);\ F(\tau)]_{\tau=-\infty}^{t} &= \mathcal{S}_{33}[F(\tau);\ E(\tau);\ D(\tau);\ C(\tau)]_{\tau=-\infty}^{t} \quad , \\
\mathcal{S}_{23}[C(\tau);\ D(\tau);\ E(\tau);\ F(\tau)]_{\tau=-\infty}^{t} &= \mathcal{S}_{23}[F(\tau);\ E(\tau);\ D(\tau);\ C(\tau)]_{\tau=-\infty}^{t} \quad ,
\end{aligned} \tag{5.12}$$

$$\begin{aligned}
\mathcal{S}_{11}[C(\tau);\ 0;\ 0;\ F(\tau)]_{\tau=-\infty}^{t} &= \mathcal{S}_{22}[1/C(\tau)F(\tau);\ 0;\ 0;\ C(\tau)]_{\tau=-\infty}^{t} = \\
&= \mathcal{S}_{33}[F(\tau);\ 0;\ 0;\ 1/C(\tau)F(\tau)]_{\tau=-\infty}^{t}
\end{aligned} \tag{5.13}$$

and

$$\mathcal{S}_{23}[C(\tau);\ 0;\ 0;\ F(\tau)]^{t}_{\tau=-\infty} = 0 \quad . \qquad (5.14)$$

The corresponding symmetry relations satisfied by the material functionals can be read off from (5.10), (5.12), (5.13) and (5.14). In particular,

$$\mathcal{Y}_{22}[C(\tau);\ D(\tau);\ E(\tau);\ F(\tau)]^{t}_{\tau=-\infty} = \mathcal{Y}_{33}[F(\tau),\ E(\tau),\ D(\tau);\ C(\tau)]^{t}_{\tau=-\infty} \quad , \qquad (5.15)$$

so that only two of the material functionals (5.10) are independent.

It will be shown that *knowledge of the response of an incompressible simple material to homogeneous plane deformation suffices to determine the material response to every inhomogeneous deformation which is presently known to be controllable for such materials*. Consequently, the stresses corresponding to these inhomogeneous deformations can be expressed in terms of the material functionals (5.10), and this will be done in subsequent sections.

Two special cases of the deformation (5.1) will now be considered.

Case 1. PURE HOMOGENEOUS DEFORMATION

$$x(\tau) = A(\tau)X \ , \quad y(\tau) = C(\tau)Y \ , \quad z(\tau) = F(\tau)Z \ ; \quad A(\tau)\ C(\tau)\ F(\tau) = 1 \quad . \qquad (5.16)$$

The extra stress components corresponding to the pure homogeneous deformation (5.16) are obtained by appropriate specialization of (5.8), and may be expressed in terms of a single functional $\mathcal{S}$ of the principal stretch functions, thus

$$S_{xx} = \mathcal{S}[C(\tau);\ F(\tau)]^{t}_{\tau=-\infty} \ , \quad S_{yy} = \mathcal{S}[F(\tau);\ A(\tau)]^{t}_{\tau=-\infty} \ , \quad S_{zz} = \mathcal{S}[A(\tau);\ C(\tau)]^{t}_{\tau=-\infty} \ ,$$

$$S_{yz} = S_{zx} = S_{xy} = 0 \quad , \qquad (5.17)$$

where

$$\mathcal{S}[C(\tau);\ F(\tau)]_{\tau=-\infty}^{t} = \mathcal{S}_{11}[C(\tau);\ 0,\ 0;\ F(\tau)]_{\tau=-\infty}^{t} \qquad (5.18)$$

is symmetric in its argument functions. The symmetry relations (5.12), (5.13) and (5.14) were used in deriving (5.17). The corresponding normal stress differences can be expressed in the form

$$T_{yy} - T_{xx} = \mathcal{Y}[C(\tau);\ F(\tau)]_{\tau=-\infty}^{t}\ , \quad T_{zz} - T_{xx} = \mathcal{Y}[F(\tau);\ C(\tau)]_{\tau=-\infty}^{t}\ , \qquad (5.19)$$

where

$$\mathcal{Y}[C(\tau);\ F(\tau)]_{\tau=-\infty}^{t} = \mathcal{Y}_{22}[C(\tau);\ 0;\ 0;\ F(\tau)]_{\tau=-\infty}^{t}\ . \qquad (5.20)$$

Thus, the material functional $\mathcal{Y}$, defined by (5.20), completely describes the material response to pure homogeneous deformation. Observe that $\mathcal{Y}[C(\tau);\ F(\tau)]_{\tau=-\infty}^{t}$ gives the difference between the principal stress in the direction with stretch function $C(\tau)$ and that in the direction with stretch function $1/C(\tau)\ F(\tau)$.

The special case when two of the principal stretch functions are the same ($A(\tau) = C(\tau) = F(\tau)^{-\frac{1}{2}}$, say) corresponds to uniform extension in the z-direction with stretch function $F(\tau)$ The pressure p may be chosen so that the lateral boundaries are free of traction, in which case the only nonvanishing stress component at time t is

$$T_{zz} = \mathcal{Y}[F(\tau);\ F(\tau)^{-\frac{1}{2}}]_{\tau=-\infty}^{t} \equiv \hat{\mathcal{Y}}[F(\tau)]_{\tau=-\infty}^{t}\ , \qquad (5.21)$$

say. The functional expression $\hat{\mathcal{Y}}$ for the tension T_{zz} in terms of the stretch function $F(\tau)$, may be called the <u>tension functional</u> for the material

Case 2. SIMPLE SHEAR

$$x(\tau) = X \ , \quad y(\tau) = Y + D(\tau)Z \ , \ z(\tau) = Z \quad . \tag{5.22}$$

The normal stress differences and the shear stress, corresponding to the deformation (5.22), may be written as

$$\begin{aligned} T_{yy} - T_{xx} &= \mathscr{G}_2\,[D(\tau)]^t_{\tau=-\infty} = \mathscr{Y}_{22}[1;\ D(\tau);\ 0;\ 1]^t_{\tau=-\infty} \quad , \\ T_{zz} - T_{xx} &= \mathscr{G}_3\,[D(\tau)]^t_{\tau=-\infty} = \mathscr{Y}_{33}[1;\ D(\tau);\ 0;\ 1]^t_{\tau=-\infty} \quad , \\ T_{yz} &= \mathscr{G}[D(\tau)]^t_{\tau=-\infty} = \mathscr{Y}_{23}[1;\ D(\tau);\ 0;\ 1]^t_{\tau=-\infty} \quad . \end{aligned} \tag{5.23}$$

The material functionals $\mathscr{G}_2$, $\mathscr{G}_3$ and $\mathscr{G}$, defined by (5.23), completely describe the material response to simple shear. Similar functionals were used by Colemen and Noll [25] to discuss viscometric flow of incompressible simple fluids. It is easily verified that $\mathscr{G}_2$ and $\mathscr{G}_3$ are even functionals of their argument function, while $\mathscr{G}$ is an odd functional of its argument function.

The pressure p may be chosen so that the plane of shear is free of traction, in which case the stress components at time t are

$$\begin{aligned} &T_{xx} = 0 \ , \quad T_{yy} = \mathscr{G}_2[D(\tau)]^t_{\tau=-\infty} \ , \quad T_{zz} = \mathscr{G}_3[D(\tau)]^t_{\tau=-\infty} \quad , \\ &T_{yz} = \mathscr{G}[D(\tau)]^t_{\tau=-\infty} \ , \quad T_{zx} = T_{xy} = 0 \quad . \end{aligned} \tag{5.24}$$

The functional expression $\mathscr{G}$ for the shear stress T_{yz} in terms of the amount of shear $D(\tau)$, may be called the <u>shear functional</u> for the material.

6. BENDING, STRETCHING, SHEARING AND TORSION OF A RECTANGULAR BLOCK

Bending and stretching of a rectangular block of elastic material was discussed by Rivlin [9], and was generalized by Ericksen [14] to include axial shearing. The general deformation described by

$$r(\tau)^2 = 2A(\tau)X , \quad \theta(\tau) = C(\tau)Y + D(\tau)Z , \quad z(\tau) = E(\tau)Y + F(\tau)Z \; ; \qquad (6.1)$$

$$A(\tau) \, [C(\tau)F(\tau) - D(\tau)E(\tau)] = 1 \; ,$$

where $r(\tau)$, $\theta(\tau)$, $z(\tau)$ are cylindrical polar coordinates, carried the block bounded in the reference state by the planes

$$X = X' \; , \; X'' \; ; \quad Y = \pm Y' \; ; \quad Z = \pm Z' \qquad (6.2)$$

into the solid bounded at time t by the cylinders and helicoidal surfaces

$$r = \sqrt{2AX'} \; , \; \sqrt{2AX''} \; ; \quad F\theta - Dz = \pm Y'/A \; ; \quad E\theta - Cz = \mp Z'/A \; .^{*} \qquad (6.3)$$

SPECIAL CASES

(i) $D = E = 0$. This corresponds to bending and stretching of the block into the annular wedge bounded at time t by the cylinders and planes

$$r = \sqrt{2AX'} \; , \; \sqrt{2AX''} \; ; \quad \theta = \pm CY' \; ; \quad z = \pm FZ \; . \qquad (6.4)$$

(ii) $D = 0$, $E \neq 0$. This corresponds to a simple shearing of the annular wedge (6.4) such that the bounding surfaces at time t are

$$r = \sqrt{2AX'} \; , \; \sqrt{2AX''} \; ; \quad \theta = \pm CY' \; ; \quad z - E\theta/C = \pm FZ' \; . \qquad (6.5)$$

---

[*] The associated deformation considered in [14] does not contain the torsion term $D(\tau)$. For an elastic material, such a deformation field is not less general than that described by (6.1), since the form (6.1) can be recovered by a rotation of coordinates. However, a more general deformation of a rectangular block is obtained here, because the case $D(\tau) = 0$ corresponds to a special orientation of the block, in its undeformed reference state, relative to the deformation field.

(iii) $D \neq 0$, $E = 0$. This corresponds to torsion of the annular wedge (6.4) such that the bounding surfaces at time t are

$$r = \sqrt{2AX'} \ , \quad \sqrt{2AX''} \ ; \quad \theta - Dz/F = \pm BY' \ ; \quad z = \pm FZ' \quad . \tag{6.6}$$

The strain matrix associated with the deformation (6.1) is given by

$$2\underline{E}(X,\tau) = k(X,\tau)\underline{K} + \ell(X,\tau)\underline{L} + m(X,\tau)\underline{M} + n(X,\tau)\,(\underline{N} + \underline{N}^T) \quad , \tag{6.7}$$

where the matrices $\underline{K}$, $\underline{L}$, $\underline{M}$ and $\underline{N}$ are defined by (5.4), and where

$$\begin{aligned} k(X,\tau) &= A(\tau)/2X - 1 \quad , & \ell(X,\tau) &= 2A(\tau)C(\tau)^2X + E(\tau)^2 - 1 \quad , \\ m(X,\tau) &= 2A(\tau)D(\tau)^2X + F(\tau)^2 - 1 \quad , & n(X,\tau) &= 2A(\tau)C(\tau)\,D(\tau)X + E(\tau)\,F(\tau) \quad . \end{aligned} \tag{6.8}$$

Consequently, substitution from (6.7) in the constitutive equation $(2.7)_1$ leads to an expression of the form (3.6), and substitution in (2.6) gives the extra stress matrix $\underline{S}$. The physical components of extra stress in the cylindrical system, at time t , are

$$\begin{aligned} S_{rr} &= A^2\mathcal{K}/r^2 \quad , & S_{\theta z} &= [CE\mathcal{L} + (CF + DE)\mathcal{N} + DF\mathcal{M}]r \quad , \\ S_{\theta\theta} &= (C^2\mathcal{L} + 2CD\mathcal{N} + D^2\mathcal{M})r^2 \quad , & S_{zr} &= 0 \quad , \\ S_{zz} &= E^2\mathcal{L} + 2EF\mathcal{N} + F^2\mathcal{M} \quad , & S_{r\theta} &= 0 \quad , \end{aligned} \tag{6.9}$$

where $\mathcal{K}$, $\mathcal{L}$, $\mathcal{M}$ and $\mathcal{N}$ are functionals of the argument functions (6.8). Setting $\tau = t$ in $(6.1)_1$ gives $X = r^2/2A$, so that the extra stress components (6.9) may be regarded as functions of r and t . The equations of motion (4.3) are satisfied if p is also a function of r and t , of the form

$$p = S_{rr} + \int^r (1/r)(S_{rr} - S_{\theta\theta})\,dr \quad . \tag{6.10}$$

Observe that replacing the set of amplitude functions

$$A(\tau)\ ,\ C(\tau)\ ,\ D(\tau)\ ,\ E(\tau)\ \text{and}\ F(\tau)\quad , \tag{6.11}$$

in the argument functions (5.6) and in the components of extra stress (5.7), by the set

$$A(\tau)/\sqrt{2A(\tau)X}\ ,\quad C(\tau)\sqrt{2A(\tau)X}\ ,\quad D(\tau)\sqrt{2A(\tau)X}\ ,\ E(\tau)\ \text{and}\ F(\tau)\quad , \tag{6.12}$$

respectively, yields the argument functions (6.8) and the physical components of extra stress (6.9). Consequently, the extra stress components (6.9) are given by

$$\begin{aligned}
S_{rr} &= \mathcal{S}_{11}[C(\tau)r(\tau);D(\tau)r(\tau);E(\tau);F(\tau)]_{\tau=-\infty}^{t}\ ,\ S_{\theta z}=\mathcal{S}_{23}[C(\tau)r(\tau);D(\tau)r(\tau);E(\tau);F(\tau)]_{\tau=-\infty}^{t}\ ,\\
S_{\theta\theta} &= \mathcal{S}_{22}[C(\tau)r(\tau);D(\tau)r(\tau);E(\tau);F(\tau)]_{\tau=-\infty}^{t}\ ,\ S_{zr} = 0\quad ,\\
S_{zz} &= \mathcal{S}_{33}[C(\tau)r(\tau);D(\tau)r(\tau);E(\tau),F(\tau)]_{\tau=-\infty}^{t}\ ,\ S_{r\theta} = 0\quad ,
\end{aligned} \tag{6.13}$$

where $r(\tau) = \sqrt{2A(\tau)X}$, and the functionals $\mathcal{S}_{11}$, $\mathcal{S}_{22}$, $\mathcal{S}_{33}$ and $\mathcal{S}_{23}$ are defined by (5.8). Since the substitution (6.11), (6.12) is invertible, the following obtains: knowledge of the response of an incompressible simple material to homogeneous plane deformation suffices to determine the response of a block of the material to bending, stretching, shearing and torsion, and conversely.

The physical components of stress, obtained from (6.10) and (6.13), are

$$\begin{aligned}
T_{rr} &= \int_{X''}^{X} (1/2X)\,\mathcal{Y}_{22}[C(\tau)r(\tau);D(\tau)r(\tau);E(\tau);F(\tau)]_{\tau=-\infty}^{t}dX\ ,\\
&\qquad\qquad T_{\theta z}=\mathcal{Y}_{23}[C(\tau)r(\tau);D(\tau)r(\tau);E(\tau);F(\tau)]_{\tau=-\infty}^{t}\\
T_{\theta\theta} &= T_{rr}+\mathcal{Y}_{22}[C(\tau)r(\tau);D(\tau)r(\tau);E(\tau);F(\tau)]_{\tau=-\infty}^{t}\ ,\ T_{zr} = 0\quad ,\\
T_{zz} &= T_{rr}+\mathcal{Y}_{33}[C(\tau)r(\tau);D(\tau)r(\tau);E(\tau);F(\tau)]_{\tau=-\infty}^{t}\ ,\ T_{r\theta} = 0\quad ,
\end{aligned} \tag{6.14}$$

where the material functionals $\mathcal{Y}_{22}$, $\mathcal{Y}_{33}$ and $\mathcal{Y}_{23}$ are defined by (5.10). The constant of integration in $(6.14)_1$ is chosen so that the plane $X = X''$ deforms into a cylinder free of traction. The surface tractions which are required to maintain the deformation (6.1) in the block can be calculated from $(4.5)_1$ and (6.14).

The physical components of stress corresponding to bending and stretching of the block (Case (i) above) are found by appropriate specialization of (6.14). They can be expressed in the form

$$\begin{aligned} T_{rr} &= \int_{X''}^{X} (1/2X)\,\mathcal{Y}[C(\tau)r(\tau);\ F(\tau)]_{\tau=-\infty}^{t}\,dX \quad , & T_{\theta z} &= 0 \quad , \\ T_{\theta\theta} &= T_{rr} + \mathcal{Y}[C(\tau)r(\tau);\ F(\tau)]_{\tau=-\infty}^{t} \quad , & T_{zr} &= 0 \quad , \\ T_{zz} &= T_{rr} + \mathcal{Y}[F(\tau);\ C(\tau)r(\tau)]_{\tau=-\infty}^{t} \quad , & T_{r\theta} &= 0 \quad , \end{aligned} \qquad (6.15)$$

where the material functional $\mathcal{Y}$ is defined by (5.20). Thus, knowledge of the material response to pure homogeneous deformation suffices to determine the response of a block of the material to bending and stretching, and conversely.

The following formulae for the resultant tractions on the plane ends $\theta = \pm\theta' = \pm CY'$, $r' = \sqrt{2AX'} \leq r \leq r'' = \sqrt{2AX''}$, and on the sides $z = \pm FZ'$, of a wedge are readily obtained from (4.5), together with the equations of motion (4.3) written in terms of the physical components of stress (see, for example, Truesdell and Noll [5], Sect. 56):

$$\begin{aligned} R &= -r'\,T_{rr}\,(r = r') \quad , \\ M &= -\frac{1}{2}\,r'^2\,T_{rr}\,(r = r') - \int_{r'}^{r''} r\,T_{rr}\,dr \quad , \\ N &= -r'^2\,\theta'\,T_{rr}\,(r = r') + \theta' \int_{r'}^{r''} (2T_{zz} - T_{rr} - T_{\theta\theta})\,r\,dr \end{aligned} \qquad (6.16)$$

Here R and M denote the resultant normal force and the resultant moment about the axis r = 0, respectively, per unit axial length on the ends, while N denotes the resultant normal force on the sides. The condition that the cylinder r = r' may also be free of traction is, from $(6.15)_1$,

$$\int_{X'}^{X''} (1/2X)\, \mathcal{Y}[C(\tau)r(\tau);\ F(\tau)]_{\tau=-\infty}^{t}\, dX = 0 \tag{6.17}$$

and the resultant force R on the ends vanishes whenever this condition is met. Thus, if the amplitude functions are such that the condition (6.17) is met for all times t , the bending can be effected by terminal couples, together with normal forces on the sides of the block.

7. STRAIGHTENING, STRETCHING AND SHEARING OF AN ANNULAR WEDGE

Straightening, stretching and shearing of an annular wedge of elastic material has been discussed by Ericksen [14]. The deformation described by

$$\begin{gathered} x(\tau) = \frac{1}{2} A(\tau)R^2 , \quad y(\tau) = C(\tau)\Theta + D(\tau)Z , \quad z(\tau) = E(\tau)\Theta + F(\tau)Z ; \\ A(\tau)\ [C(\tau)F(\tau) - D(\tau)E(\tau)] = 1 , \end{gathered} \tag{7.1}$$

carries the annular wedge bounded in the reference state by the cylinders and planes

$$R = R' , R'' ; \quad \Theta = \pm \Theta' ; \quad Z = \pm Z' \tag{7.2}$$

into the parallelpiped bounded at time t by the planes

$$x = \frac{1}{2}AR'^2\ ,\ \frac{1}{2}AR''^2\ ;\ \ Fy - Dz = \pm\,\Theta'/A\ ;\ Ey - Cz = \pm\,Z'/A\ \ .^{*} \tag{7.3}$$

SPECIAL CASES

(i) $D = E = 0$. This corresponds to straightening and stretching of the annular wedge into the rectangular block bounded at time t by the planes

$$x = \frac{1}{2}AR'^2\ ,\ \ \frac{1}{2}AR''^2\ ;\ \ y = \pm\,C\Theta'\ ;\ \ z = \pm\,FZ'\ \ . \tag{7.4}$$

(ii) $D = 0$, $E \neq 0$. This corresponds to a simple shearing of the block (7.4) such that the bounding planes at time t are

$$x = \frac{1}{2}AR'^2\ ,\ \frac{1}{2}AR''^2\ ;\ \ y = \pm\,C\Theta'\ ;\ \ z - Ey/C = \pm\,FZ'\ \ . \tag{7.5}$$

(iii) $D \neq 0$, $E = 0$. This corresponds to a simple shearing of the block (7.4) such that the bounding planes at time t are

$$x = \frac{1}{2}AR'^2\ ,\ \ \frac{1}{2}AR''^2\ ;\ \ y - Dz/F = \pm\,C\Theta'\ ;\ \ z = \pm\,FZ'\ \ . \tag{7.6}$$

The strain matrix associated with the deformation (7.1) is given by

$$\begin{aligned} 2\underline{E}(R,\ \Theta,\tau) &= k\ (R,\tau)\ \underline{K}(\Theta) + \ell(R,\tau)\ \underline{L}\ (\Theta) + \\ &+ m(R,\tau)\ \underline{M}(\Theta) + n(R,\tau)\ [\underline{N}(\Theta) + \underline{N}(\Theta)^{T}]\ \ , \end{aligned} \tag{7.7}$$

---

[*] The associated deformation considered in [14] has the form (7.1) with $D(\tau) = 0$, which may be obtained from (7.1) by a rotation of coordinates. However, a more general deformation of an annular wedge is obtained here, because the case $D(\tau) = 0$ corresponds to a special orientation of the wedge in its reference state relative to the deformation field (7.1).

where

$$k(R\,,\,\tau) = A(\tau)^2\,R^2 - 1\quad,\qquad \ell(R\,,\,\tau) = [C(\tau)^2 + E(\tau)^2]/R^2 - 1\quad,$$
$$m(R\,,\,\tau) = D(\tau)^2 + F(\tau)^2 - 1\quad,\qquad n(R\,,\,\tau) = [C(\tau)D(\tau) + E(\tau)F(\tau)]/R \tag{7.8}$$

and the time-independent matrices $\underline{K}(\theta)$, $\underline{L}(\theta)$, $\underline{M}(\theta)$ and $\underline{N}(\theta)$, defined by

$$\underline{K}(\theta) = \left\|\begin{matrix} \cos^2\theta & \sin\theta\cos\theta\;0 & 0 \\ \sin\theta\cos\theta & \sin^2\theta & 0 \\ 0 & 0 & 0 \end{matrix}\right\|\;,\quad \underline{L}(\theta) = \left\|\begin{matrix} \sin^2\theta & -\sin\theta\cos\theta & 0 \\ \sin\theta\cos\theta & \cos^2\theta & 0 \\ 0 & 0 & 0 \end{matrix}\right\|\;,$$
$$\underline{M}(\theta) = \left\|\begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix}\right\|\;,\quad \underline{N}(\theta) = \left\|\begin{matrix} 0 & 0 & -\sin\theta \\ 0 & 0 & \cos\theta \\ 0 & 0 & 0 \end{matrix}\right\| \tag{7.9}$$

have the properties (i), (ii) and (iii) listed in Section 3. Consequently, substitution from (7.7) in the constitutive equation $(2.7)_1$ yields an expression of the form (3.6) which, together with (2.6), gives the extra stress matrix $\underline{S}$. The physical components at time t are

$$S_{xx} = A^2R^2\mathcal{K}\quad,\qquad S_{yz} = CE\,\mathcal{L}/R^2 + (CF + DE)\mathcal{N}/R + DF\mathcal{M}\quad,$$
$$S_{yy} = C^2\mathcal{L}/R^2 + 2CD\mathcal{N}/R + D^2\mathcal{M}\quad,\qquad S_{zx} = 0\quad, \tag{7.10}$$
$$S_{zz} = E^2\mathcal{L}/R^2 + 2EF\mathcal{N}/R + F^2\mathcal{M}\quad,\qquad S_{xy} = 0\quad,$$

where $\mathcal{K}$, $\mathcal{L}$, $\mathcal{M}$ and $\mathcal{N}$ are functionals of the argument functions (7.8). Setting $\tau = t$ in $(7.1)_1$ gives $R^2 = 2x/A$, so that the extra stress components (7.10) may be regarded as functions of x and t The equations of motion (4.3) are satisfied if p is also a function of x and t ,of the form

$$p = S_{xx} + \hat{p} \quad , \qquad (7.11)$$

where $\hat{p}$ is an arbitrary function of t .

Comparison of (7.8) and (7.10) with (5.6) and (5.7) shows that the extra stress components (7.10) may be written as

$$\begin{array}{ll} S_{xx} = \mathcal{S}_{11}[C(\tau)/R;D(\tau);E(\tau)/R;F(\tau)]_{\tau=-\infty}^{t} , & S_{yz} = \mathcal{S}_{23}[C(\tau)/R;D(\tau);E(\tau)/R;F(\tau)]_{\tau=-\infty}^{t} , \\ S_{yy} = \mathcal{S}_{22}[C(\tau)/R;D(\tau);E(\tau)/R;F(\tau)]_{\tau=-\infty}^{t} , & S_{zx} = 0 \quad , \\ S_{zz} = \mathcal{S}_{33}[C(\tau)/R;D(\tau);E(\tau)/R;F(\tau)]_{\tau=-\infty}^{t} , & S_{xy} = 0 \quad , \end{array} \qquad (7.12)$$

the functionals $\mathcal{S}_{11}$, $\mathcal{S}_{22}$, $\mathcal{S}_{33}$ and $\mathcal{S}_{23}$ being defined by (5.8). Thus, knowledge of the response of an incompressible simple material to homogeneous plane deformation suffices to determine the response of an annular wedge of the material to straightening, stretching and shearing, and conversely. It follows from the results of the previous section that there is a similar reciprocal relationship between the families of deformations (6.1) and (7.1).

Planes x=constant may be rendered free of traction by setting $p = 0$ in (7.11), in which case the stress components at time t are

$$\begin{array}{ll} T_{xx} = 0 \quad , & T_{yz} = \mathcal{Y}_{23}[C(\tau)/R;D(\tau);E(\tau)/R;F(\tau)]_{\tau=-\infty}^{t} , \\ T_{yy} = \mathcal{Y}_{22}[C(\tau)/R;D(\tau);E(\tau)/R;F(\tau)]_{\tau=-\infty}^{t} , & T_{zx} = 0 \quad , \\ T_{zz} = \mathcal{Y}_{33}[C(\tau)/R;D(\tau);E(\tau)/R;F(\tau)]_{\tau=-\infty}^{t} , & T_{xy} = 0 \quad , \end{array} \qquad (7.13)$$

The surface tractions which are required to maintain the deformation (7.1) in the wedge can be calculated from $(4.5)_1$ and (7.13).

The stress components corresponding to straightening and stretching of the wedge (Case (i) above) are found by appropriate specialization of (7.13). They can be expressed in the form

$$T_{xx} = 0 \quad , \quad T_{yy} = \mathcal{Y}[C(\tau)/R;\ F(\tau)]_{\tau=-\infty}^{t} \quad , \quad T_{zz} = \mathcal{Y}[F(\tau);\ C(\tau)/R]_{\tau=-\infty}^{t} \quad ,$$
$$T_{yz} = T_{zx} = T_{xy} = 0 \quad , \tag{7.14}$$

where the material functional $\mathcal{Y}$ is defined by (5.20). Thus, knowledge of the material response to pure homogeneous deformation suffices to determine the response of an annular wedge of the material to straightening and stretching, and conversely.

The resultant normal force T_y and moment M about the z-axis on the ends $y = \pm C\Theta'$, and the resultant normal force T_z on the sides $z = \pm FZ'$, at time t , are given by substitution from (7.14) in

$$T_y = 2AFZ'\int_{R'}^{R''} T_{yy}\ RdR \ , \ M = A^2FZ'\int_{R'}^{R''} T_{yy}\ R^3dR \ , \ T_z = 2AC\Theta'\int_{R'}^{R''} T_{zz}\ RdR \quad .$$

In particular, the condition that the straightening can be effected without a resultant force on the ends of the wedge is

$$\int_{R'}^{R''} \mathcal{Y}[C(\tau)/R;\ F(\tau)]_{\tau=-\infty}^{t}\ RdR = 0 \quad . \tag{7.16}$$

8. EXTENSION, INFLATION, BENDING, TORSION AND SHEARING OF AN ANNULAR WEDGE

The extension, inflation, bending, torsion and shearing of elastic cylinders, tubes and wedges has been discussed by Rivlin [9, 10], Green and Shield [11], Adkins, Green and Shield [12] and Ericksen and Rivlin [13]. The general deformation is described by

$$r(\tau)^2 = A(\tau)R^2 + B(\tau) , \quad \theta(\tau) = C(\tau)\Theta + D(\tau)Z , \quad z(\tau) = E(\tau)\Theta + F(\tau)Z \; ;$$
$$A(\tau)\,[C(\tau)F(\tau) - D(\tau)E(\tau)] = 1 \tag{8.1}$$

The strain matrix associated with this deformation has the form (7.7), with

$$k(R,\tau) = A(\tau)^2R^2/r(\tau)^2 - 1 \quad , \qquad \ell(R,\tau) = [C(\tau)^2r(\tau)^2 + E(\tau)^2]/R^2 - 1 \quad ,$$
$$m(R,\tau) = D(\tau)^2r(\tau)^2 + F(\tau)^2 - 1 \quad , \qquad n(R,\tau) = [C(\tau)D(\tau)r(\tau)^2 + E(\tau)F(\tau)]/R \quad , \tag{8.2}$$

where $r(\tau)$ is given by $(8.1)_1$ and the matrices $\underline{K}(\Theta)$, $\underline{L}(\Theta)$, $\underline{M}(\Theta)$ and $\underline{N}(\Theta)$ are defined by (7.9). The extra stress matrix $\underline{S}$ may be obtained by a procedure similar to that outlined in the previous section. The physical components of extra stress at time t are

$$S_{rr} = A^2R^2\mathcal{K}/r^2 \qquad\qquad S_{\theta z} = [CE\mathcal{L}/R^2 + (CF + DE)\mathcal{N}/R + DF\mathcal{M}]r \quad ,$$
$$S_{\theta\theta} = (C^2\mathcal{L}/R^2 + 2CD\mathcal{N}/R + D^2\mathcal{M})r^2 \; , \qquad S_{zr} = 0 \quad , \tag{8.3}$$
$$S_{zz} = E^2\mathcal{L}/R^2 + 2EF\mathcal{N}/R + F^2\mathcal{M} \; , \qquad S_{r\theta} = 0 \quad ,$$

where $\mathcal{K}$, $\mathcal{L}$, $\mathcal{M}$ and $\mathcal{N}$ are functionals of the argument functions (8.2). Setting $\tau = t$ in $(8.1)_1$ gives $R^2 = (r^2 - B)/A$, so that the extra stress components (8.3) may be regarded as functions of r and t . The equations of motion (4.3) are satisfied if p is also a function of r and t , of the form (6.10), i.e.,

$$p = S_{rr} + \int^{r} (1/r)\,(S_{rr} - S_{\theta\theta})dr \tag{8.4}$$

Comparison of (8.2) and (8.3) with (5.6) and (5.7) shows that the extra stress components (8.3) may be written as

$$
\begin{aligned}
&S_{rr}=\mathcal{S}_{11}[C(\tau)r(\tau)/R;\ D(\tau)r(\tau);\ E(\tau)/R;\ F(\tau)]_{\tau=-\infty}^{t}\ ,\\
&\qquad\qquad S_{\theta z}=\mathcal{S}_{23}[C(\tau)r(\tau)/R;\ D(\tau)r(\tau);\ E(\tau)/R;\ F(\tau)]_{\tau=-\infty}^{t}\ ,\\
&S_{\theta\theta}=\mathcal{S}_{22}[C(\tau)r(\tau)/R;\ D(\tau)r(\tau);\ E(\tau)/R;\ F(\tau)]_{\tau=-\infty}^{t}\ ,\quad S_{zr}=0\ ,\\
&S_{zz}=\mathcal{S}_{33}[C(\tau)r(\tau)/R;\ D(\tau)r(\tau);\ E(\tau)/R;\ F(\tau)]_{\tau=-\infty}^{t}\ ,\quad S_{r\theta}=0
\end{aligned}
\tag{8.5}
$$

where $r(\tau)$ is given by $(8.1)_1$ and the material functionals $\mathcal{S}_{11}$, $\mathcal{S}_{22}$, $\mathcal{S}_{33}$ and $\mathcal{S}_{23}$ are defined by (5.8). Thus, knowledge of the response of an incompressible simple material to homogeneous plane deformation suffices to determine the response of a cylinder of the same material to extension, inflation, bending, torsion and shearing. The substitution which maps the argument functions in (5.8) onto the argument functions in (8.5) can be inverted in a number of ways, for example with $B(\tau)=0$ or with $C(\tau)=1$. Thus, knowledge of the material response to extension, torsion and shearing, together with either bending or inflation, suffices to determine the response to homogeneous plane deformation.

The physical components of stress at time t , obtained from (8.4) and (8.5), are

$$
\begin{aligned}
&T_{rr}=\int_{R''}^{R}(AR/r^2)\mathcal{Y}_{22}[C(\tau)r(\tau)/R;D(\tau)r(\tau);E(\tau)/R;F(\tau)]_{\tau=-\infty}^{t}\,dR\ ,\\
&\qquad\qquad T_{\theta z}=\mathcal{Y}_{23}[C(\tau)r(\tau)/R;D(\tau)r(\tau);E(\tau)/R;F(\tau)]_{\tau=-\infty}^{t}\ ,\\
&T_{\theta\theta}=T_{rr}+\mathcal{Y}_{22}[C(\tau)r(\tau)/R;D(\tau)r(\tau);E(\tau)/R;F(\tau)]_{\tau=-\infty}^{t}\ ,\quad T_{zr}=0\ ,\\
&T_{zz}=T_{rr}+\mathcal{Y}_{33}[C(\tau)r(\tau)/R;D(\tau)r(\tau);E(\tau)/R;F(\tau)]_{\tau=-\infty}^{t}\ ,\quad T_{r\theta}=0\ .
\end{aligned}
\tag{8.6}
$$

The constant of integration in $(8.6)_1$ is chosen so that the cylinder $R=R''$ deforms into a cylinder free of traction

Some special cases of the deformation (8.1) will now be considered.

Case 1. EXTENSION, INFLATION AND BENDING

$$r(\tau)^2 = A(\tau)R^2 + B(\tau) \quad , \quad \theta(\tau) = C(\tau)\Theta \quad , \quad z(\tau) = F(\tau)Z \; ; \quad A(\tau)C(\tau)D(\tau) = 1 \quad . \qquad (8.7)$$

The physical components of stress corresponding to the deformation (8.7), which are obtained by appropriate specialization of (8.6), may be expressed in the form

$$\begin{aligned} T_{rr} &= \int_{R''}^{R} [(AR/r^2)]\, \mathfrak{Y}[C(\tau)r(\tau)/R;\ F(\tau)]_{\tau=-\infty}^{t}\, dR \quad , & T_{\theta z} &= 0 \quad , \\ T_{\theta\theta} &= T_{rr} + \mathfrak{Y}[C(\tau)r(\tau)/R;\ F(\tau)]_{\tau=-\infty}^{t} \quad , & T_{zr} &= 0 \quad , \\ T_{zz} &= T_{rr} + \mathfrak{Y}[F(\tau);\ C(\tau)r(\tau)/R]_{\tau=-\infty}^{t} \quad , & T_{r\theta} &= 0 \quad , \end{aligned} \qquad (8.8)$$

where the material functional $\mathfrak{Y}$ is defined by (5.20). Thus, knowledge of the material response to pure homogeneous deformation suffices to determine the response to extension, inflation and bending. It can easily be seen that knowledge of the material response to extension and inflation, or to extension and bending, suffices to determine the response to pure homogeneous deformation.

The resultant tractions which are required to support the deformation (8.7) in an annular wedge can be obtained by substitution from (8.8) in (6.16). In particular, the condition that the cylinder $R = R'$ may also deform into a cylinder free of traction is

$$\int_{R'}^{R''} (AR/r^2)\, \mathfrak{Y}[C(\tau)r(\tau)/R;\ F(\tau)]_{\tau=-\infty}^{t}\, dR = 0 \qquad (8.9)$$

and when this condition is met, the resultant tractions on the ends $\theta = \pm C\Theta'$ of the wedge vanish. Thus, if the amplitude functions are such that the condition (8.9) is met for all times t, the extension, inflation and bending can be effected by terminal couples, together with normal forces on the sides $z = \pm FZ'$.

The deformation (8.7), with $B(\tau) = 0$, describes extension and bending without inflation. In this case, the material functional $\mathcal{G}$ in (8.8) is independent of position, and the physical components of stress at time t are given by

$$T_{rr} = \mathcal{G}[C(\tau)A(\tau)^{\frac{1}{2}} ; F(\tau)]_{\tau=-\infty}^{t} \log R/R'' , \quad T_{\theta\theta} = T_{rr} + \mathcal{G}[C(\tau)A(\tau)^{\frac{1}{2}} ; F(\tau)]_{\tau=-\infty}^{t} ,$$

$$T_{zz} = T_{rr} + \mathcal{G}[F(\tau); C(\tau)A(\tau)^{\frac{1}{2}}]_{\tau=-\infty}^{t} , \tag{8.10}$$

$$T_{\theta z} = T_{zr} = T_{r\theta} = 0 .$$

Case 2. RADIAL OSCILLATIONS OF A HOLLOW CYLINDER

Inflation, without extension, of a hollow cylinder is described by

$$r(\tau)^2 = R^2 + B(\tau) , \quad \theta(\tau) = \Theta , \quad z(\tau) = Z . \tag{8.11}$$

The corresponding physical components of stress at time t are obtained by appropriate specialization of (8.8). In particular, the pressure difference P between the inner and outer cylinders is given by

$$P = \int_{R'}^{R''} (R/r^2) \, \mathcal{G}[r(\tau)/R; 1]_{\tau=-\infty}^{t} \, dR , \tag{8.12}$$

with $r(\tau)$ given by $(8.11)_1$.

As mentioned previously, exact solutions of the dynamic equations $(4.1)_1$ are obtained from the quasistatic solutions considered thus far by requiring that the acceleration have the form (4.4). This requirement does not restrict the form of the inflation function $B(\tau)$ in (8.11). The equation which governs the radial oscillations of a hollow cylinder, subjected to a pressure difference $P(t)$, is obtained by including the appropriate inertia terms in (8.12). Analysis similar to that in [5] leads to the equation

$$x \log(1 + \gamma/x^2)\ddot{x} + [\log(1 + \gamma/x^2) - \gamma/(\gamma + x^2)]\dot{x}^2 + f(x,\gamma) = 2P/\rho R'^2 \quad , \tag{8.13}$$

where

$$x(\tau) = r'(\tau)/R' \ , \quad u(\tau) = r(\tau)^2/R^2 \ , \quad \gamma = R''^2/R'^2 - 1 \quad ,$$
$$f(x,\gamma) \equiv \frac{1}{\rho R'^2} \int_{\frac{\gamma+x^2}{\gamma+1}}^{x^2} [1/u(u-1)] \, \mathcal{Y}[u(\tau)^{\frac{1}{2}} \ ; \ 1]_{\tau=-\infty}^{t} \, du \tag{8.14}$$

and $r'(\tau)$ denotes the inner radius of the tube at time τ. Equations (8.13) and (8.14) are the counterpart of the equations obtained by Knowles [19] for radial oscillations of a tube of homogeneous isotropic incompressible elastic material. The relevant material property, in the elastic case, is the generalized shear modulus. It can easily be seen, from $(8.14)_4$, that radial oscillations of a tube of incompressible simple material are not, in general, determined by the shear functional for the material.

Case 3. PURE TORSION

$$r(\tau) = R \ , \quad \theta(\tau) = \Theta + D(\tau)Z \ , \quad z(\tau) = Z \tag{8.15}$$

The physical components of stress at time t , obtained from (8.6), may be written as

$$\begin{aligned} T_{rr} &= \int_{R''}^{R} (1/R) \, \mathcal{G}_2[D(\tau)R]_{\tau=-\infty}^{t} \, dR \ , & T_{\theta z} &= \mathcal{G}[D(\tau)R]_{\tau=-\infty}^{t} \ , \\ T_{\theta\theta} &= T_{rr} + \mathcal{G}_2[D(\tau)R]_{\tau=-\infty}^{t} \ , & T_{zr} &= 0 \ , \\ T_{zz} &= T_{rr} + \mathcal{G}_3[D(\tau)R]_{\tau=-\infty}^{t} \ , & T_{r\theta} &= 0 \ , \end{aligned} \tag{8.16}$$

where the material functionals $\mathcal{G}_2$, $\mathcal{G}_3$ and $\mathcal{G}$ are defined by (5.23). Thus, knowledge of the material response to simple shear suffices to determine the response to pure torsion, and conversely.

The torsional couple T and the resultant normal force N on the ends of a hollow cylinder $R' \leq R \leq R''$, at time t , are

$$T = 2\pi \int_{R'}^{R''} \mathscr{G}[D(\tau)R]_{\tau=-\infty}^{t} R^2 \, dR \quad ,$$

$$N = -\pi R'^2 \, T_{rr}(R=R') + \pi \int_{R'}^{R''} \left\{ 2\,\mathscr{G}_3[D(\tau)R]_{\tau=-\infty}^{t} - \mathscr{G}_2[D(\tau)R]_{\tau=-\infty}^{t} \right\} R dR \quad . \qquad (8.17)$$

In general, a pressure

$$-T_{rr}\,(R=R') = \int_{R'}^{R''} (1/R)\,\mathscr{G}_2[D(\tau)R]_{\tau=-\infty}^{t}\, dR \quad , \qquad (8.18)$$

on the inner cylinder $R=R'$, is required to support the deformation (8.15) Christensen [23] has discussed the torsion of a solid cylinder of incompressible simple material with fading memory, and he has observed that the normal forces N are not necessarily compressive.

Case 4. AXIAL SHEAR

$$r(\tau) = R \ , \quad \theta(\tau) = \Theta \ , \quad z(\tau) = E(\tau)\Theta + Z \quad . \qquad (8.19)$$

The physical components of stress corresponding to the deformation (8.19) are obtained by appropriate specialization of (8.6), and may be written as

$$T_{rr} = \int_{R''}^{R} (1/R)\ \mathscr{G}_3[E(\tau)/R]_{\tau=-\infty}^{t}\, dR \quad , \qquad T_{\theta z} = \mathscr{G}[E(\tau)/R]_{\tau=-\infty}^{t} \quad ,$$

$$T_{\theta\theta} = T_{rr} + \mathscr{G}_3[E(\tau)/R]_{\tau=-\infty}^{t} \quad , \qquad T_{zr} = 0 \quad , \qquad (8.20)$$

$$T_{zz} = T_{rr} + \mathscr{G}_2[E(\tau)/R]_{\tau=-\infty}^{t} \quad , \qquad T_{r\theta} = 0 \quad .$$

Thus, knowledge of the material response to simple shear suffices to determine the response to axial shear, and conversely.

9. INFLATION OF A SPHERICAL SHELL

The inflation of an elastic spherical shell has been discussed by Green and Shield [11]. This deformation is described by

$$r(\tau)^3 = R^3 + B(\tau) , \quad \theta(\tau) = \Theta , \quad \phi(\tau) = \Phi \quad , \tag{9.1}$$

where $r(\tau)$, $\theta(\tau)$, $\phi(\tau)$ and R , Θ, Φ denote spherical polar coordinates at time τ and in the reference state, respectively. The associated strain matrix is given by

$$2\underline{E}(R, \Theta , \Phi) = k(R,\tau)\, \underline{K}(\Theta,\Phi) + q(R,\tau)\, \underline{Q}(\Theta,\Phi) \quad , \tag{9.2}$$

where

$$k(R,\tau) = R^4/r(\tau)^4 - 1 , \quad q(R,\tau) = r(\tau)^2/R^2 - 1 \quad , \tag{9.3}$$

$r(\tau)$ being given by $(9.1)_1$, and

$$\underline{K}(\Theta,\Phi) = \left\| \begin{matrix} \sin^2\Theta \cos^2\Phi & \sin^2\Theta \sin\Phi \cos\Phi & \sin\Theta \cos\Theta \cos\Phi \\ \sin^2\Theta \sin\Phi \cos\Phi & \sin^2\Theta \sin^2\Phi & \sin\Theta \cos\Theta \sin\Phi \\ \sin\Theta \cos\Theta \cos\Phi & \sin\Theta \cos\Theta \sin\Phi & \cos^2\Theta \end{matrix} \right\| , \quad \underline{Q}(\Theta,\Phi) = \underline{I} - \underline{K}(\Theta,\Phi) \quad . \tag{9.4}$$

It is easily verified that

$$\underline{K}^2 = \underline{K} \quad , \quad \mathrm{tr}\, \underline{K} = 1 \tag{9.5}$$

and hence

$$\underline{Q}^2 = \underline{Q} \quad , \quad \underline{K}\,\underline{Q} = \underline{Q}\,\underline{K} = 0 \ , \quad \mathrm{tr}\, \underline{Q} = 2 \tag{9.6}$$

It is evident from $(9.4)_2$, (9.5) and (9.6) that the matrices $\underline{K}$ and $\underline{Q}$, the latter being identified with the matrix $\underline{L} + \underline{M}$, have the properties (i), (ii) and (iii) listed in Section 3. Consequently, substitution from (9.2) in the constitutive equation $(2.7)_1$ yields an expression of the form (3.6) which, together with

(2 6), gives the extra stress matrix $\underline{S}$. The physical components in the spherical system at time t , are

$$S_{rr} = R^4\mathcal{K}/r^4 , \quad S_{\theta\theta} = S_{\phi\phi} = r^2\mathcal{L}/R^2 , \quad S_{\theta\phi} = S_{\phi r} = S_{r\theta} = 0 , \qquad (9.7)$$

where $\mathcal{K}$ and $\mathcal{L}$ are functionals of the argument functions

$$k(R,\tau) = R^4/r(\tau)^4 - 1 , \quad \ell(R,\tau) = m(R,\tau) = r(\tau)^2/R^2 - 1 , \quad n(R,\tau) = 0 , \qquad (9.8)$$

and $r(\tau)$ is given by $(9.1)_1$. Setting $\tau = t$ in $(9.1)_1$ gives $R = r^3 - B)^{\frac{1}{3}}$, so that the extra stress components (9.7) may be regarded as functions of r and t. The equations of motion (4.3) are satisfied if p is also a function of r and t, of the form

$$p = S_{rr} + 2\int^{r} (1/r)\,(S_{rr} - S_{\theta\theta})\,dr . \qquad (9.9)$$

Comparison of (9.8) and (9.7) with (5.6) and (5.7) shows that the extra stress components may be written as

$$S_{rr} = \mathcal{S}[r(\tau)/R;\ r(\tau)/R]^{t}_{\tau=-\infty} , \quad S_{\theta\theta} = S_{\phi\phi} = \mathcal{S}[r(\tau)/R;\ R^2/r(\tau)^2]^{t}_{\tau=-\infty} ,$$
$$S_{\theta\phi} = S_{\phi r} = S_{r\theta} = 0 , \qquad (9.10)$$

where the material functional $\mathcal{S}$ is defined by (5.18). The physical components of stress at time t , obtained from (9.9) and (9.10), may be written as

$$T_{rr} = -2\int_{r''}^{r} (1/r)\,\hat{\mathcal{Y}}[R^2/r(\tau)^2]^{t}_{\tau=-\infty}\,dr , \quad T_{\theta\theta} = T_{\phi\phi} = T_{rr} - \hat{\mathcal{Y}}[R^2/r(\tau)^2]^{t}_{\tau=-\infty} ,$$
$$T_{\theta\phi} = T_{\phi r} = T_{r\theta} = 0 , \qquad (9.11)$$

where $\hat{\mathcal{Y}}$ is the tension functional defined by (5.21) and the constant of integration in $(9.11)_1$ is chosen so that the sphere $r=r''$ is free of traction. The identities

$$\mathcal{Y}[f(\tau);\ f(\tau)]_{\tau=-\infty}^{t} = -\mathcal{Y}[f(\tau)^{-2};\ f(\tau)]_{\tau=-\infty}^{t} \equiv -\hat{\mathcal{Y}}[f(\tau)^{-2}]_{\tau=-\infty}^{t} \quad , \tag{9.12}$$

which are evident from the definitions of the material functionals $\mathcal{Y}$ and $\hat{\mathcal{Y}}$, were used in deriving (9.11). It follows from (9.11) that knowledge of the response of an incompressible simple material to uniform extension suffices to determine the reponse of a spherical shell of the material to inflation, and conversely.

The pressure difference P , at time t , between the inner and outer surfaces r=r' and r=r" of a spherical shell subjected to the deformation (9.1) is given by

$$P = -2\int_{r'}^{r''} (1/r)\ \hat{\mathcal{Y}}\,[R^2/r(\tau)]_{\tau=-\infty}^{t}\, dr \quad . \tag{9.13}$$

The requirement that the associated acceleration have the form (4.4) does not restrict the form of the inflation function B(τ) in (9.1). The equation which governs the radial oscillations of a spherical shell, subjected to a pressure difference P(t) , is obtained by including the appropriate inertia terms in (9.13). Analysis similar to that of Guo Zhong-Heng and Solecki [20] leads to the equation

$$\frac{d}{dx}\left\{ [1 - (1+\gamma/x^3)^{-\frac{1}{3}}]x^3\ \dot{x}^2 \right\} + x^2\, g(x,\gamma) = 2Px^2/\rho R'^2 \quad , \tag{9.14}$$

where

$$x(\tau) = r'(\tau)/R' \ , \quad u(\tau) = r(\tau)^3/R^3 \ , \quad \gamma = R''^3/R'^3 - 1 \quad ,$$

$$g(x,\gamma) = -\frac{4}{3\,\rho R'^2}\int_{\frac{\gamma+x^3}{\gamma+1}}^{x^3} [1/u(u-1)]\ \hat{\mathcal{Y}}\,[u(\tau)^{-\frac{2}{3}}]_{\tau=-\infty}^{t}\, du \quad . \tag{9.15}$$

Here R' and R" denote the inner and outer radii of the shell in its undeformed state, and r'(τ) denotes the inner radius at time τ. Equations (9.13) and (9.14) are the counterpart of the equations obtained in [20] for radial oscillations of a spherical shell of homogeneous isotropic incompressible elastic material.

10. EXTENSION, BENDING AND AZIMUTHAL SHEARING

Azimuthal shearing of an elastic annular wedge has been discussed by Klingbeil and Shield [15], and generalized by Singh and Pipkin [16] to include extension and bending. Such a deformation may be described by

$$r(\tau) = A(\tau)R\ , \quad \theta(\tau) = B(\tau)\log R + C(\tau)\Theta\ , \quad z(\tau) = F(\tau)Z; \quad A(\tau)^2C(\tau)F(\tau) = 1\ . \qquad (10.1)$$

The associated strain matrix may be expressed in the form

$$2\underline{E}(\Theta,\tau) = k(\tau)\underline{K}(\Theta) + \ell(\tau)\underline{L}(\Theta) + m(\tau)\underline{M}(\Theta) + n(\tau)[\underline{N}(\Theta) + \underline{N}(\Theta)^T]\ , \qquad (10.2)$$

with

$$\begin{aligned} k(\tau) &= F(\tau)^2 - 1\ , \qquad & \ell(\tau) &= A(\tau)^2[1 + B(\tau)^2] - 1\ , \\ m(\tau) &= A(\tau)^2\, C(\tau)^2 - 1\ , \qquad & n(\tau) &= -A(\tau)^2\, B(\tau)C(\tau)\ , \end{aligned} \qquad (10.3)$$

and

$$\underline{K}(\Theta) = \begin{Vmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{Vmatrix}\ , \qquad \underline{L}(\Theta) = \begin{Vmatrix} \cos^2\Theta & \sin\Theta\cos\Theta & 0 \\ \sin\Theta\cos\Theta & \sin^2\Theta & 0 \\ 0 & 0 & 0 \end{Vmatrix}\ ,$$

$$\underline{M}(\Theta) = \begin{Vmatrix} \sin^2\Theta & -\sin\Theta\cos\Theta & 0 \\ -\sin\Theta\cos\Theta & \cos^2\Theta & 0 \\ 0 & 0 & 0 \end{Vmatrix}\ , \qquad \underline{N}(\Theta) = \begin{Vmatrix} \sin\Theta\cos\Theta & -\cos^2\Theta & 0 \\ \sin^2\Theta & -\sin\Theta\cos\Theta & 0 \\ 0 & 0 & 0 \end{Vmatrix}\ . \qquad (10.4)$$

Observe that the set of matrices $\underline{K}$, $\underline{L}$, $\underline{M}$ defined by $(10.4)_{1,2,3}$ is a permutation of the set $\underline{K}$, $\underline{L}$, $\underline{M}$ defined by $(7.9)_{1,2,3}$.

The matrices defined by (10.4) have the properties (i), (ii) and (iii) listed in Section 3. Consequently, substitution from (10.2) in the constitutive equation $(2.7)_1$ yields an expression of the form (3.6) which, together with (2.6), gives the extra stress matrix $\underline{S}$. The physical components at time t are

$$\begin{aligned} S_{rr} &= A^2\mathcal{L} , & S_{\theta z} &= 0 , \\ S_{\theta\theta} &= A^2 (B^2\mathcal{L} - 2BC\mathcal{N} + C^2\mathcal{M}) , & S_{zr} &= 0 , \\ S_{zz} &= F^2\mathcal{K} , & S_{r\theta} &= A^2 (B\mathcal{L} - C\mathcal{N}) ; \end{aligned} \tag{10.5}$$

where $\mathcal{K}$, $\mathcal{L}$, $\mathcal{M}$ and $\mathcal{N}$ are functionals of the argument functions (10.3) Since the extra stress components (10.5) are independent of position, it is easily seen that the equations of motion (4.3) are satisfied if p is a function of r , θ and t , given by

$$p = (S_{rr} - S_{\theta\theta}) \log r + 2\theta S_{r\theta} + \hat{p} , \tag{10.6}$$

where $\hat{p}$ is an arbitrary function of t .

Comparison of (10.3) and (10.5) with (5.6) and (5.7) shows that the extra stress components (10.5) may be expressed in the form

$$\begin{aligned} S_{rr} &= \mathcal{S}_{22}[A(\tau);0; -A(\tau)B(\tau);A(\tau)C(\tau)]_{\tau=-\infty}^{t} , & S_{\theta z} &= 0 , \\ S_{\theta\theta} &= \mathcal{S}_{33}[A(\tau);0; -A(\tau)B(\tau);A(\tau)C(\tau)]_{\tau=-\infty}^{t} , & S_{zr} &= 0 , \\ S_{zz} &= \mathcal{S}_{11}[A(\tau);0; -A(\tau)B(\tau);A(\tau)C(\tau)]_{\tau=-\infty}^{t} , & S_{r\theta} &= -\mathcal{S}_{23}[A(\tau);0;-A(\tau)B(\tau);A(\tau)C(\tau)]_{\tau=-\infty}^{t} , \end{aligned} \tag{10.7}$$

where the functionals $\mathcal{S}_{11}$, $\mathcal{S}_{22}$, $\mathcal{S}_{33}$ and $\mathcal{S}_{23}$ are defined by (5.8). The physical components of stress at time t , obtained from (10.6) and (10.7), are given by

$$T_{rr} = -(\mathcal{Y}_{22} - \mathcal{Y}_{33}) \log r + 2\theta \mathcal{Y}_{23} - p^* , \quad T_{\theta\theta} = T_{rr} - \mathcal{Y}_{22} + \mathcal{Y}_{33} , \quad T_{zz} = T_{rr} - \mathcal{Y}_{22} ,$$
$$T_{\theta z} = T_{zr} = 0 , \quad T_{r\theta} = -\mathcal{Y}_{23} , \tag{10.8}$$

where p^* $(= p - S_{rr})$ is an arbitrary function of t, and the argument functions of the material functionals $\mathcal{Y}_{22}$, $\mathcal{Y}_{33}$ and $\mathcal{Y}_{23}$ are

$$A(\tau);\ 0;\ -A(\tau)B(\tau);\ A(\tau)C(\tau) \quad . \tag{10.9}$$

Thus, knowledge of the response of an incompressible simple material to a restricted class of homogeneous plane deformations (described by (5.1), with $D(\tau) = 0$) suffices to determine the material response to extension, bending and azimuthal shearing.

Since the pressure p in (10.6) is a function of both r and θ, the tractions which are required to support the deformation (10.1) are, in general, rather complicated. For example, normal and tangential tractions, which depend on θ, are required on any cylinder r = constant, and the arbitrary function p^* can be chosen so that the normal traction vanishes on a generator θ = constant.

The special case of (10.1) when $B(\tau) = 0$ was discussed in Section 8. The special case

$$r(\tau) = R , \quad \theta(\tau) = B(\tau) \log R + \Theta , \quad z(\tau) = Z \tag{10.10}$$

corresponds to azimuthal shearing without extension or bending. The physical components of stress at time t, corresponding to the deformation (10.10), are obtained by specialization of (10.8) and (10.9) and may be expressed in the form

$$T_{rr} = \left\{ \mathcal{G}_2[B(\tau)]^t_{\tau=-\infty} - \mathcal{G}_3[B(\tau)]^t_{=-\infty} \right\} \log r - 2\theta \mathcal{G}[B(\tau)]^t_{\tau=-\infty} - p , \quad T_{\theta z} = 0 ,$$
$$T_{\theta\theta} = T_{rr} + \mathcal{G}_2[B(\tau)]^t_{\tau=-\infty} - \mathcal{G}_3[B(\tau)]^t_{\tau=-\infty} , \qquad T_{zr} = 0 , \tag{10.11}$$
$$T_{zz} = T_{rr} - \mathcal{G}_3[B(\tau)]^t_{\tau=-\infty} , \qquad T_{r\theta} = \mathcal{G}[B(\tau)]^t_{\tau=-\infty} ,$$

where the material functionals $\mathcal{G}_2$, $\mathcal{G}_3$ and $\mathcal{G}$ are defined by (5.23). Thus, knowledge of the material response to simple shear suffices to determine the response to azimuthal shear, and conversely.

11. AN ALTERNATIVE PROCEDURE

A description of the material response to several inhomogeneous deformations in terms of the response to homogeneous plane deformations was effected, in previous sections, by demonstrating that each of these deformations belongs to the special class defined in Section 3. This description may also be effected by a more elementary procedure, based on consideration of the associated local deformations.

Every deformation is locally inhomogeneous, in the sense that the deformation of a material element which is small in all of its dimensions is homogeneous, when terms of the second order of smallness are neglected. Since the mechanical response of a simple material is a local action, it is not surprising that knowledge of the response to homogeneous deformations suffices to determine the response completely (see Truesdell and Noll [5]). Each of the deformations discussed in Sections 6-10 is locally equivalent to a homogeneous plane deformation, and the response to each deformation can therefore be expressed in terms of the material functionals which characterize the response to homogeneous plane deformations.

Consider, for example, the spherical inflation described by

$$r(\tau)^3 = R^3 + B(\tau) \ , \quad \theta(\tau) = \Theta \ , \quad \phi(\tau) = \Phi \ . \qquad (11.1)$$

This deformation carries the elementary cuboid bounded in the reference state by the coordinate surfaces

$$R\ ,\ R + dR\ ;\quad \Theta\ ,\ \Theta + d\Theta\ ;\quad \Phi\ ,\ \Phi + d\Phi\ , \tag{11.2}$$

into the cuboid bounded at time τ by the coordinate surfaces

$$r(\tau)\ ,\ r(\tau) + dr(\tau)\ ;\quad \theta(\tau)\ ,\ \theta(\tau) + d\theta(\tau)\ ;\quad \phi(\tau)\ ,\ \phi(\tau) + d\phi(\tau)\ . \tag{11.3}$$

The radial, longitudinal and latitudinal lengths of the cuboid in the reference state, and at time τ , are

$$dR\ ,\quad R\ d\Theta\ ,\quad R\ \sin\Theta\ d\Phi \tag{11.4}$$

and

$$dr(\tau) = (R^2/r(\tau)^2)dR\ ,\quad r(\tau)d\theta(\tau) = (r(\tau)/R)Rd\Theta\ ,\quad r(\tau)\sin\theta(\tau)d\phi(\tau) = (r(\tau)/R)R\ \sin\Theta\ d\Phi\ , \tag{11.5}$$

respectively. Thus, the local deformation consists of a radial stretch $R^2/r(\tau)^2$ and equal longitudinal and latitudinal stretches $r(\tau)/R$, i.e., spherical inflation is locally equivalent to simple extension.*

From this point of view, it is evident that the material response to spherical inflation is determined by the response to simple extension. The extra stresses (9.10) may be written directly and the pressure may be evaluated by considering the balance of forces (including inertial forces, if desired) acting on the elementary cuboid at time t .

The various controllable deformations may be discussed in this manner, and the method of approach is particularly effective for those deformations which are locally equivalent to pure homogeneous deformation, or to simple shear. A similar procedure was employed in the recent report by Rivlin [26] on steady viscometric flows.

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*This may be seen more readily by considering the deformation of the material element bounded in the reference state by the spheres R , R + dR and by the cone $d\Theta$.

12. APPENDIX: DETERMINATION OF THE MATERIAL FUNCTIONALS FOR A PARTICULAR MATERIAL

The various solutions presented in this paper are expressed in terms of the material response to homogeneous plane deformation. The material functionals which characterize this response may be determined from the general constitutive functionals (2.7), with the analysis of Sections 3 and 5. This calculation will now be carried out for a particular incompressible simple material, with constitutive equations given by

$$\underline{\sigma} = - p\underline{I} + \underline{F}\,\underline{\mathcal{S}}\,[\underline{E}(\tau)]_{\tau=-\infty}^{t}\,\underline{F}^{T} \quad ,$$

$$\underline{\mathcal{S}}[E(\tau)]_{\tau=-\infty}^{t} = \int_{-\infty}^{t} \phi(t-\tau)\dot{\underline{E}}(\tau)d\tau + \int_{-\infty}^{t}\int_{-\infty}^{t} \psi(t-\tau_1, t-\tau_2)\dot{\underline{E}}(\tau_1)\dot{\underline{E}}(\tau_2)d\tau_1\, d\tau_2 \quad , \qquad (12.1)$$

where the kernel function ψ is symmetric in its arguments. Pipkin [27] has shown that (12.1) represents the second-order approximation in a Green-Rivlin expansion, i.e., the integral expansion of (2.7) in which terms of the third and higher orders in the strain rate matrix $\dot{\underline{E}}(\tau)$ are neglected. An experimental program involving homogeneous strains, for the determination of the kernel functions in (12.1), was described by Lockett [28].

It will be useful to introduce the notation

$$f = f(\tau_\alpha) \ , \quad \dot{f}_\alpha = f(\tau_\alpha) \ ; \quad (\alpha = 1,2) \qquad (12.2)$$

Substitution from (5.5) in (12.1), and use of the multiplication table (3.3) together with (3.6) gives

$$\begin{aligned}
\mathcal{K}[k_1;\ell_1;m_1;n_1]_{\tau_1=-\infty}^{t} &= \frac{1}{2}\int_{-\infty}^{t}\phi(t-\tau_1)\dot{k}_1 d\tau_1 + \frac{1}{4}\int_{-\infty}^{t}\int_{-\infty}^{t}\psi(t-\tau_1,t-\tau_2)\dot{k}_1\dot{k}_2 d\tau_1 d\tau_2 \quad ,\\
\mathcal{L}[k_1;\ell_1;m_1;n_1]_{\tau_1=-\infty}^{t} &= \frac{1}{2}\int_{-\infty}^{t}\phi(t-\tau_1)\dot{\ell}_1 d\tau_1 + \frac{1}{4}\int_{-\infty}^{t}\int_{-\infty}^{t}\psi(t-\tau_1,t-\tau_2)(\dot{\ell}_1\dot{\ell}_2 + \dot{n}_1\dot{n}_2)d\tau_1 d\tau_2 \quad ,\\
\mathcal{M}[k_1;\ell_1;m_1;n_1]_{\tau_1=-\infty}^{t} &= \frac{1}{2}\int_{-\infty}^{t}\phi(t-\tau_1)\dot{m}_1 d\tau_1 + \frac{1}{4}\int_{-\infty}^{t}\int_{-\infty}^{t}\psi(t-\tau_1,t-\tau_2)(\dot{m}_1\dot{m}_2 + \dot{n}_1\dot{n}_2)d\tau_1 d\tau_2 \quad ,\\
\mathcal{N}[k_1;\ell_1;m_1;n_1]_{\tau_1=-\infty}^{t} &= \frac{1}{2}\int_{-\infty}^{t}\phi(t-\tau_1)\dot{n}_1 d\tau_1 + \frac{1}{4}\int_{-\infty}^{t}\int_{-\infty}^{t}\psi(t-\tau_1,t-\tau_2)(\dot{\ell}_1+\dot{m}_1)\dot{n}_2 d\tau_1 d\tau_2 \quad .
\end{aligned} \qquad (12.3)$$

Substitution from (5.6) in (12.3) leads to

$$\begin{aligned}
\mathcal{K} &= \frac{1}{2}\int_{-\infty}^{t}\phi(t-\tau_1)\frac{d}{d\tau_1}(A_1^2)d\tau_1 + \frac{1}{4}\int_{-\infty}^{t}\int_{-\infty}^{t}\psi(t-\tau_1,t-\tau_2)\frac{d}{d\tau_2}(A_1^2)\frac{d}{d\tau_2}(A_2^2)\,d\tau_1 d\tau_2 \quad ,\\
\mathcal{L} &= \frac{1}{2}\int_{-\infty}^{t}\phi(t-\tau_1)\frac{d}{d\tau_1}(C_1^2+E_1^2)d\tau_1 + \frac{1}{4}\int_{-\infty}^{t}\int_{-\infty}^{t}\psi(t-\tau_1,t-\tau_2)\Big[\frac{d}{d\tau_1}(C_1^2+E_1^2)\frac{d}{d\tau_2}(C_2^2+E_2^2) +\\
&\qquad + \frac{d}{d\tau_1}(C_1D_1+E_1F_1)\frac{d}{d\tau_2}(C_2D_2+E_2F_2)\Big]\,d\tau_1 d\tau_2 \quad ,\\
\mathcal{M} &= \frac{1}{2}\int_{-\infty}^{t}\phi(t-\tau_1)\frac{d}{d\tau_1}(D_1^2+F_1^2)d\tau_1 + \frac{1}{4}\int_{-\infty}^{t}\int_{-\infty}^{t}\psi(t-\tau_1,t-\tau_2)\Big[\frac{d}{d\tau_1}(D_1^2+F_1^2)\frac{d}{d\tau_2}(D_2^2+F_2^2) +\\
&\qquad + \frac{d}{d\tau_1}(C_1D_1+E_1F_1)\frac{d}{d\tau_2}(C_2D_2+E_2F_2)\Big]\,d\tau_1 d\tau_2 \quad ,\\
\mathcal{N} &= \frac{1}{2}\int_{-\infty}^{t}\phi(t-\tau_1)\frac{d}{d\tau_1}(C_1D_1+E_1F_1)d\tau_1 + \frac{1}{4}\int_{-\infty}^{t}\int_{-\infty}^{t}\psi(t-\tau_1,t-\tau_2)\frac{d}{d\tau_1}(C_1^2+D_1^2+E_1^2+F_1^2)\frac{d}{d\tau_2}(C_2D_2+E_2F_2)\\
&\qquad d\tau_1 d\tau_2 \quad .
\end{aligned} \qquad (12.4)$$

The material functionals $\mathcal{Y}_{22}$, and $\mathcal{Y}_{23}$ are given (see (5.8) and (5.10)) by substitution from (12.4) in

$$\begin{aligned} \mathcal{Y}_{22}[C_1;\ D_1;\ E_1;\ F_1]_{\tau_1=-\infty}^{t} &= C^2\mathcal{L} + 2CD\mathcal{N} + D^2\mathcal{M} - A^2\mathcal{K} \quad , \\ \mathcal{Y}_{23}[C_1;\ D_1;\ E_1;\ F_1]_{\tau_1=-\infty}^{t} &= CE\mathcal{L} + (CF + DE)\mathcal{N} + DF\mathcal{M} \quad , \end{aligned} \qquad (12.5)$$

and $\mathcal{Y}_{33}$ is then given by (5.15).

The material functional $\mathcal{Y}$, which characterizes the reponse to pure homogeneous deformation, is defined by (5.20) and may be determined by specialization of (12.4) and (12.5). It is easily seen that

$$\begin{aligned} \mathcal{Y}[C_1;\ F_1]_{\tau_1=-\infty}^{t} &= \int_{-\infty}^{t} \phi(t-\tau_1)(C^2C_1C_1 - A^2A_1\dot{A}_1)\ d\tau_1 + \\ &+ \int_{-\infty}^{t}\int_{-\infty}^{t} \psi(t-\tau_1, t-\tau_2)(C^2C_1C_2\dot{C}_1\dot{C}_2 - A^2A_1A_2\dot{A}_1\dot{A}_2)\ d\tau_1 d\tau_2 \quad , \end{aligned} \qquad (12.6)$$

where $A(\tau) = 1/C(\tau)F(\tau)$. In particular, the tension functional $\hat{\mathcal{Y}}$, defined by (5.21), is

$$\begin{aligned} \hat{\mathcal{Y}}[F(\tau)]_{\tau=-\infty}^{t} &= \int_{-\infty}^{t} \phi(t-\tau_1)(F^2F_1 + 1/2FF_1^2)\dot{F}_1 d\tau_1 + \\ &+ \int_{-\infty}^{t}\int_{-\infty}^{t} \psi(F^2F_1F_2 - 1/4FF_1^2F_2^2)\dot{F}_1\dot{F}_2 d\tau_1 d\tau_2 \quad . \end{aligned} \qquad (12.7)$$

The material functionals $\mathcal{G}_2$, $\mathcal{G}_3$ and $\mathcal{G}$ characterize the reponse to simple shear and are defined by (5.23). Appropriate specialization of (12.4) and (12.5) gives

$$\begin{aligned} \mathcal{G}_2[D(\tau)]_{\tau=-\infty}^{t} &= \int_{-\infty}^{t} \phi(t-\tau_1)(1 + DD_1)\dot{D}D_1 d\tau_1 + \frac{1}{4}\int_{-\infty}^{t}\int_{-\infty}^{t} \psi(t-\tau_1, t-\tau_2)[1 + 4DD_1 + \\ &\qquad + D^2(1 + 4D_1D_2)]\dot{D}_1\dot{D}_2\ d\tau_1 d\tau_2 \quad , \\ \mathcal{G}_3[D(\tau)]_{\tau=-\infty}^{t} &= \int_{-\infty}^{t} \phi(t-\tau_1)D_1\dot{D}_1 d\tau_1 + \frac{1}{4}\int_{-\infty}^{t}\int_{-\infty}^{t} \psi(t-\tau_1, t-\tau_2)(1 + 4D_1D_2)\dot{D}_1D_2 d\tau_1 d\tau_2 \quad , \\ \mathcal{G}[D(\tau)]_{\tau=-\infty}^{t} &= \frac{1}{2}\int_{-\infty}^{t} \phi(t-\tau_1)(1 + 2DD_1)\dot{D}_1 d\tau_1 + \frac{1}{4}\int_{-\infty}^{t}\int_{-\infty}^{t} \psi(t-\tau_1, t-\tau_2)[2D_1 + \\ &\qquad + D(1 + 4D_1D_2)]\dot{D}_1\dot{D}_2\ d\tau_1 d\tau_2 \quad . \end{aligned} \qquad (12.8)$$

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REFERENCES

[1] Green, A.E. and R.S. Rivlin, Arch. Rational Mech. Anal. 1, 1 (1957).

[2] Green, A.E., R.S. Rivlin and A.J.M. Spencer, Ibid., 3, 82 (1959).

[3] Green, A.E. and R.S. Rivlin, Ibid., 5, 387 (1961).

[4] Noll, W., Ibid., 2, (1958).

[5] Truesdell, C. and W. Noll, *Handbuch der Physik* III 3. Berlin: Springer 1965.

[6] Coleman, B.D., H. Markovitz and W. Noll, Viscometry of Simple Fluids. Berlin: Springer 1965.

[7] Rivlin, R.S., Frontiers of Research in Fluid Dynamics, G. Temple and R. Seeger, eds. New York: Interscience 1965.

[8] Rivlin, R.S., Phil. Trans. Roy. Soc. London A241, 379 (1948).

[9] Rivlin, R.S., Proc. Roy. Soc. London A195, 463 (1949).

[10] Rivlin, R.S., Phil. Trans. Roy. Soc. London A242, 173 (1949).

[11] Green, A.E. and R.T. Shield, Proc. Roy. Soc. London A202, 407 (1950)

[12] Adkins, J.E., A.E. Green and R.T. Shield, Phil. Trans. Roy. Soc. London A246, 181 (1953).

[13] Ericksen, J.L. and R.S. Rivlin, J. Rational Mech. Anal. 3, 281 (1954).

[14] Ericksen, J.L., ZAMP 5, 466 (1954).

[15] Klingbeil, W.W. and R.T. Shield, Ibid. 17, 489 (1966).

[16] Singh, M. and A.C. Pipkin, Ibid. 16, 706 (1965).

[17] Singh, M. and A.C. Pipkin, Arch. Rational Mech. Anal. 21, 169 (1966)

[18] Truesdell, C., Ibid. 11, 106 (1962).

[19] Knowles, J.K., Quart. Appl. Math. 18, 71 (1960).

[20] Guo Zhong-Heng and R. Solecki, Arch. Mech. Stosow. 15, 427 (1963).

[21] Carroll, M. M., Int. Jour. Engng. Sci., in press.

[22] Coleman, B.D. and C. Truesdell, ZAMM 45, 547 (1965).

[23] Christensen, R.M., forthcoming.

[24] Wineman, A.S. and A.C. Pipkin, Arch. Rational Mech. Anal. 17, 184 (1964).

[25] Coleman, B.D. and W. Noll, Ann. N.Y. Acad. Sci. 89, 672 (1961).

[26] Rivlin, R.S. Brown University Technical Report Nonr 562(40)/19 (1966).

[27] Pipkin, A.C., Rev. Modern Phys. 36, 1034 (1964).

[28] Lockett, F.J., Int. J. Engng. Sci. 3, 59 (1965)

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13. ABSTRACT

Exact solutions of several problems which involve finite deformations of homogeneous isotropic incompressible simple solids are obtained by the inverse method. The deformations involved include bending, stretching, shearing, torsion and both cylindrical and spherical inflation. The corresponding stresses are expressed in terms of the functionals which describe the material response to homogeneous plane deformations. The results obtained apply, in particular, to incompressible viscoelastic solids.

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