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NOrd 8993
Cornell Aeronautical Laboratory
August 1, 1946

CM 341
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To: Carl L. Frederick

From: Mark G. Foster

Subject: A Theory of Approximations in Simulator Behavior

References:
(a) Keller Report Curtiss-Wright ID-362-B-1, December 31, 1945
(b) CM-189 Electro-Hydraulic Proportional Servo by Servo Mechanisms Group, University of Virginia
(c) CM-310 Roll Simulator for Guided Missile by O. O. Haglund, Cornell Aeronautical Laboratory

Abstract

A mathematical expression is written down for the changes in transient frequency and damping which result when the feedback branch of a single loop system is altered slightly. The result is applied in a general discussion of the accuracy of aerodynamic simulators with numerical examples of roll simulation, and suggested experiments to evaluate the approximations involved in practice.

Details

A. **Theoretical**

A simulator, when used to test a missile guiding control, completes a feedback loop by supplying the β (feedback) part of a Mu-Beta ($\mu \cdot \beta$) system, where μ represents the amplification in the forward transmission path. The transient response of such a system to any disturbing influence is a set of oscillations $\sum A_i e^{p_i t}$ at complex frequencies $p_i = \lambda_i + j\omega_i$, where p_i are the roots of the expression $(1 - \mu\beta)$, i.e. solutions to the equation $\mu\beta = 1 + j0$. When all values of λ_i are negative, the system is said to be stable, in that transient oscillations will decay with time after they are initiated. The problem in control design is to make all the λ_i sufficiently negative so that the transient component oscillations will decay rapidly, and quiescence will be restored sufficiently soon after any disturbance.

Special consideration must be given to the effect of a simulator which does not provide exactly the same β as the dynamical system it simulates, but only approximately so. The purpose of simulation experiments is to determine the important λ 's and ω 's of the $\mu\beta$ system which will obtain when the missile is in flight, and it will be shown below that the precision obtained for λ_i and ω_i on a percentage basis is not at all the same as that obtained for the absolute value, $|\beta|$, on a percentage basis. An expression will be developed for small p_i deviations from desired flight values in terms of β variations from actual flight behavior.

Let $\beta_\epsilon = \xi + j\eta$ be a small perturbation in β so that β_s equals feedback given by simulator, β_o equals feedback required by the equation of motion. Then $\beta_\epsilon = \beta_o - \beta_s$ and observed transient frequencies will be found for $1-\mu\beta_s = 0$ or for $1-\mu\beta_o = -\mu\beta_\epsilon$ instead of zero.

The values p_i' corresponding to $1-\mu\beta_o = -\mu\beta_\epsilon$ can be found by observing that

$$(1-\mu\beta_o)\Big]_{p_i'} = (1-\mu\beta_o)\Big]_{p_i} + \frac{d}{dp}(1-\mu\beta_o)\Big]_{p_i}(p_i'-p_i) + ----$$

as long as p_i is not a singular point of the $1-\mu\beta_o$ function.

Thus

$$-\mu\beta_\epsilon\Big]_{p_i'} = 0 + \frac{d}{dp}(1-\mu\beta_o)\Big]_{p_i}(p_i'-p_i) + ----$$

Since

$$(1-\mu\beta_o)\Big]_{p_i} \equiv 0$$

Also $\mu\beta_\epsilon\big]_{p_i} = \frac{\beta_\epsilon}{\beta_0}\Big]_{p_i}$, i.e. $(1-\mu\beta_0)\big]_{p_i} = 0$

hence
$$p_i' \doteq p_i + \frac{\frac{\beta_\epsilon}{\beta_0}\Big]_{p_i}}{\frac{d}{dp}(\mu\beta_0)\Big]_{p_i}}$$

for $\left|\frac{\beta_\epsilon}{\beta_0}\right|$ sufficiently small, i.e. $|p_i' - p_i|$ sufficiently small. Strictly speaking, $\frac{\beta_\epsilon}{\beta_0}$ should be evaluated at the observed "frequency" p_i' in this equation, but it is doubtful if the refinement of distinguishing between $\beta_\epsilon\big]_{p_i}$ and $\beta_\epsilon\big]_{p_i'}$ will be worth while in view of the other uncertainties in the situation, therefore this distinction is ignored henceforth.

An example of the foregoing can be found in the design and use of a roll simulator. The equation of motion of a missile in roll is

$$A\ddot{\varphi} + G\dot{\varphi} = -H\alpha$$

where
- φ = angle of roll
- α = angle of control fin
- A = rotational inertia
- G = damping coefficient
- H = fin effectiveness coefficient

The β is $\frac{\varphi}{\alpha}$ computed from this equation, and the μ is $\frac{\alpha}{\varphi}$ computed from the internal control system. The transient oscillations are given by

$$\beta = -\frac{H}{Ap^2 + Gp} \quad \text{and} \quad \mu\beta = 1 + j0$$

Thus $p_i = -\frac{G}{2A} \pm j\sqrt{\frac{\mu H}{A} - \left(\frac{G}{2A}\right)^2}$ for an idealized control where μ = constant independent of p.

Two examples will be treated quantitatively for roll simulation:

Case I Constant μ (the idealized case, useful for simulator calibration)

Since for roll simulation $\beta_o = \frac{-H}{Ap^2+Gp}$ then

$$\frac{d}{dp}(\mu\beta_o) = \frac{+\mu H(2Ap+G)}{(Ap^2+Gp)^2} \quad \text{and} \quad p_i' = p_i + \frac{\mu\beta_\epsilon(Ap^2+Gp)^2}{\mu H(2Ap+G)}$$

evaluated at $\quad p_i = \frac{-G}{2A} \pm j\sqrt{\frac{\mu H}{A} - \left(\frac{G}{2A}\right)^2}$

This leads to $\quad p_i' = p_i + \frac{\mu^2 H\beta_\epsilon}{2Aj\sqrt{\frac{\mu H}{A} - \left(\frac{G}{2A}\right)^2}}$

at $\quad p_i = \frac{-G}{2A} + j\sqrt{\frac{\mu H}{A} - \left(\frac{G}{2A}\right)^2}$

and $\quad p_i' = \frac{p_i + \mu^2 H\beta_\epsilon}{-2Aj\sqrt{\frac{\mu H}{A} - \left(\frac{G}{2A}\right)^2}}$

at $\quad p_i = \frac{-G}{2A} - j\sqrt{\frac{\mu H}{A} - \left(\frac{G}{2A}\right)^2}$

The observed damping constant, λ_i, is accordingly reduced in the ratio $1 - \frac{\mu^2 H}{\omega_i G}\eta$ and the observed frequency, ω_i, of transient oscillations by $1 - \frac{\mu^2 H}{2A\omega_i^2}\xi$ from their accurate values, due to the approximation $\xi + j\eta = \beta_\epsilon + 0$

For the following numerical values:

[redacted]

μ = 1/6

A = 3440 lb.in.^{2} = 8.93 dynamic units

H = 36900 lb.in./radian

G = 401 lb.in./rad./sec.

it is found that $\omega_i = \sqrt{\frac{\mu H}{A} - \left(\frac{G}{2A}\right)^2} = 13.45 \text{ sec.}^{-1}$

$$\lambda_i = -\frac{G}{2A} = -22.5 \text{ sec.}^{-1}$$

For $\xi + j\eta \neq 0$

$$\lambda_i' = \lambda_i\left(1 - \frac{\mu^2 H}{\omega_i G}\eta\right) = \lambda_i\left(1 - \frac{36900}{36 \times 13.5 \times 401}\eta\right)$$

$$= \lambda_i(1 - .188\eta)$$

$$\omega_i' = \omega_i\left(1 - \frac{\mu^2 H \xi}{2A\omega_i^2}\right) = \omega_i\left(1 - \frac{36900\xi}{36 \times 2 \times 8.93 \times (13.5)^2}\right)$$

$$= \omega_i(1 - .310\xi)$$

Showing that for λ_i' and ω_i' to be correct within 5% we must have $|\xi| < \frac{.05}{.310} = .161$ and $|\eta| < \frac{.05}{.188} = .266$. Since $\beta]_{p = p_i} = 6 + j0$, the percentage error in $|\beta|$ must be less than $\frac{0.362}{6}$ = 5%, but this applies for μ = 1/6 only. For damping less than about 70% critical, the percentage error resulting from the use of an approximate simulator varies with μ, so that for μ = 1/2 (a value which has been suggested) the precision in $|\beta|$ must be three times as good as above, in spite of the smaller value of $\beta]_{p_i}$ which obtains for the larger μ.

A portion of the $\mu\beta$ plane for this example is plotted in Figure 1, with the lines at ω = 13.45 and λ = -22.5 specially darkened. The branch point at $\mu\beta$ = 1.36 + $j0$ is connected with the critical damping characteristics of the system, and cannot be located precisely without knowing the analytic form of $\mu\beta$ as a function of p.

Case II University of Virginia Type of Internal Control

Reference (a) gives an analytic expression for the dynamic behavior of the University of Virginia servo as a third degree equation in p . This equation is obtained by analyzing the dynamics of the control motor, but ignoring inertia lag in the hydraulic system. While the later data of Reference (b) can be fitted much better by a fourth degree equation in p (which includes a small correction term for hydraulic fluid inertia), the cubic expression of Reference (a) will be used in this illustration.

Thus, for a control whose internal behavior is specified by

$$\mu = \frac{\mu_0}{p^3 + 144p^2 + (120)^2 p + 3.02 \times 10^5}$$

(See Reference (a) pp. 23 and 24)

and whose external feedback is given by

$$\beta = -\frac{4130}{p^2 + 45p} \qquad \text{(Same as Case I)}$$

we have

$$\mu\beta = -\frac{2.08 \times 10^8}{p(p+45)(p+26.8)(p+58 \text{ [illegible]}}$$

if the linkages are so proportioned that, for very slow variations, [illegible]

This numerical scale corresponds to that of Case I above.

The transient oscillation "frequencies" are given by the roots of

$$\text{[illegible]} + 1.89 \times 10^2 p^4 + 2.09 \times 10^4 p^3 + 4.56 \times 10^5 p^2 + 1.36 \times \text{[illegible]}\, p + \text{[illegible]}$$

These roots are $p = -58.4 \pm 59.2$, $-5.8 \pm j$ [illegible]

and $-.6$ [illegible]

The transient term of most importance is the oscillation mode at frequency $\frac{16.1}{2}$ = 2.5 cps, since not only are the higher order roots characterised by much greater damping, but the amplitudes of the oscillations at frequencies of larger [illegible] are much less than that of the lowest mode in gust-initiated transients.

[redacted]

The value of

$$\frac{d}{dp}(\mu\beta)\Big]_{\mu\beta=1} \text{ is } \frac{5(p^4+1.512\times 10^2 p^3+1.253\times 10^4 p^2+3.8\times 10^5 p+2.7\times 10^6)}{2.08\times 10^8}$$

evaluated at each of the five roots above

Therefore at $p_i = -6.7+16.1j$, $\frac{d}{dp}(\mu\beta) = -4.3\times 10^{-2} + 7.7\times 10^{-2}j$.

Hence $p_i' = p_i + \frac{\beta_e}{\beta_o}\frac{1}{-.043+.077j} = p_i - \frac{\beta_e}{\beta_o}(5.5+9.9j)$

at $p_i = -6.7+16.1j$, $\beta_o = \frac{-4130}{p^2+45p} = \frac{1}{.125-.123j}$

$\therefore p_i' = p_i - \beta_e(192+0\,56j) = -6.7+16\,1j - \beta_e(1.92+0\,56j)$

But since

$\beta_e = \xi + j\eta$, $p_i' = -6.7(1+.28\xi-.083\eta)+16.1j(1-.035\eta-.119\xi)$

and the percentage error involved is

$28\xi - 8.3\eta$ in the damping factor and

$-3.5\xi - 11.9\eta$ in frequency

In Figure 2 is plotted a portion of the $\mu\beta$ plane in the region around $p = -6.7+16.1j$. It is not consistent with the title of this paper to explore the irregularities around the branch points at $-36.5+j0$, $-10.46+j0$, $-23.48\pm 17.5j$ since these are not close to the "resonant" points given by $\mu\beta = 1+j0$

B. <u>Design of Experiments</u>

The mathematical treatment in Section A of this report has been carried somewhat beyond the margin of utility for the case of the University of Virginia control, in view of the imperfect approximation to its dynamic behavior furnished by a characteristic equation of degree no higher than the third. It is the purpose of this section to suggest experiments which will give a reasonable basis for estimating the β_ϵ and $p_i' - p_i$ appropriate to a given experimental setup for any control of the general types heretofore suggested.

In Section A, the observed value of transient complex frequency p_i', the accurate value p_i, and the simulator error β_ϵ are shown to be related by

$$p_i' = p_i + \frac{\mu\beta_\epsilon\Big]_{p_i'}}{\frac{d}{dp}(\mu\beta_0)\Big]_{p_i}}$$

as long as $|p_i' - p_i|$ and $|\beta_\epsilon|$ are small enough so that the first two terms of the Taylor series form a sufficient approximation to the $\mu\beta$ function in the neighborhood of the Nyquist point. To the same degree of approximation the difference can be written

$$p_i' = p_i + \frac{\mu\beta_\epsilon\Big]_{p_i'}}{\frac{d}{dp}(\mu\beta_s)\Big]_{p_i'}} \quad \text{or} \quad p_i = p_i' - \frac{\mu\beta_\epsilon\Big]_{p_i'}}{\frac{d}{dp}(\mu\beta_s)\Big]_{p_i'}}$$

and it is to be observed that p_i is thus given in terms of experimentally determin[ed] quantities.

The value of p_i' can be observed directly by measuring frequency and logarithmic decrement of transient oscillations initiated by a step-function signal fed into the system. By varying the numerical coupling factor between the control and the simulator, there results a change in scale of the $\mu\beta_s$ plot, so that the effect is one of mo[v]ing the Nyquist point. If the numerical magnitude of μ is thus changed by the ratio k, the resulting p_i' corresponds to that obtained from the original $\mu\beta_s$ plot at the point $\frac{1}{k} + j0$ hence the observed $\Delta p_i'$ corresponds to a $d(\mu\beta_s)$ of $1 - \frac{1}{k}$, or

$$\frac{dp'}{d(\mu\beta_s)} = \frac{1}{\frac{d(\mu\beta_s)}{dp'}} = \frac{k(\Delta p \text{ observed})}{k-1}$$

The value of $\left.\beta_c\right]_{p_i}$ can best be determined by computing (from the motion equations) the β appropriate to the observed p_i, and comparing it to the observed $\frac{\text{missile movement}}{\text{control surface motion}}$ ratio from the oscillograms of the simulator-control system. The value of $\left.\mu\right]_{p_i}$ will be given by $\left.\frac{1}{\beta}\right]_{p_i}$

It is appropriate to remark that the values of frequency and logarithmic decrement are not very easy to measure accurately on the oscillograms in the neighborhood of critical damping. It will usually happen, however, that considerable errors in $p_i' - p_i$ can be tolerated in the neighborhood of critical damping just because the change thereby introduced in the measurable behavior of the system will not be important.

A more fundamental objection is that the mathematical treatment of this paper is based on the assumption that $\mu\beta$ exists as defined, and is an analytic function of p. The usual remarks about linear approximations to non-linear systems can be made, however, since the analysis refers only to a differential region around the Nyquist point, hence is an appropriate approximation for any system which is continuous enough in its dynamic behavior to warrant analysis by differential means at all. The extent by which $\mu\beta$ departs from being analytic can probably be explored by introducing phase shift as well as magnitude shift in changing the scale of the $\mu\beta$ plot (see page 8.), thus evaluating $\frac{\Delta p}{\Delta(\mu\beta)}$ for other than real values of $\Delta(\mu\beta)$.

MGF:Y

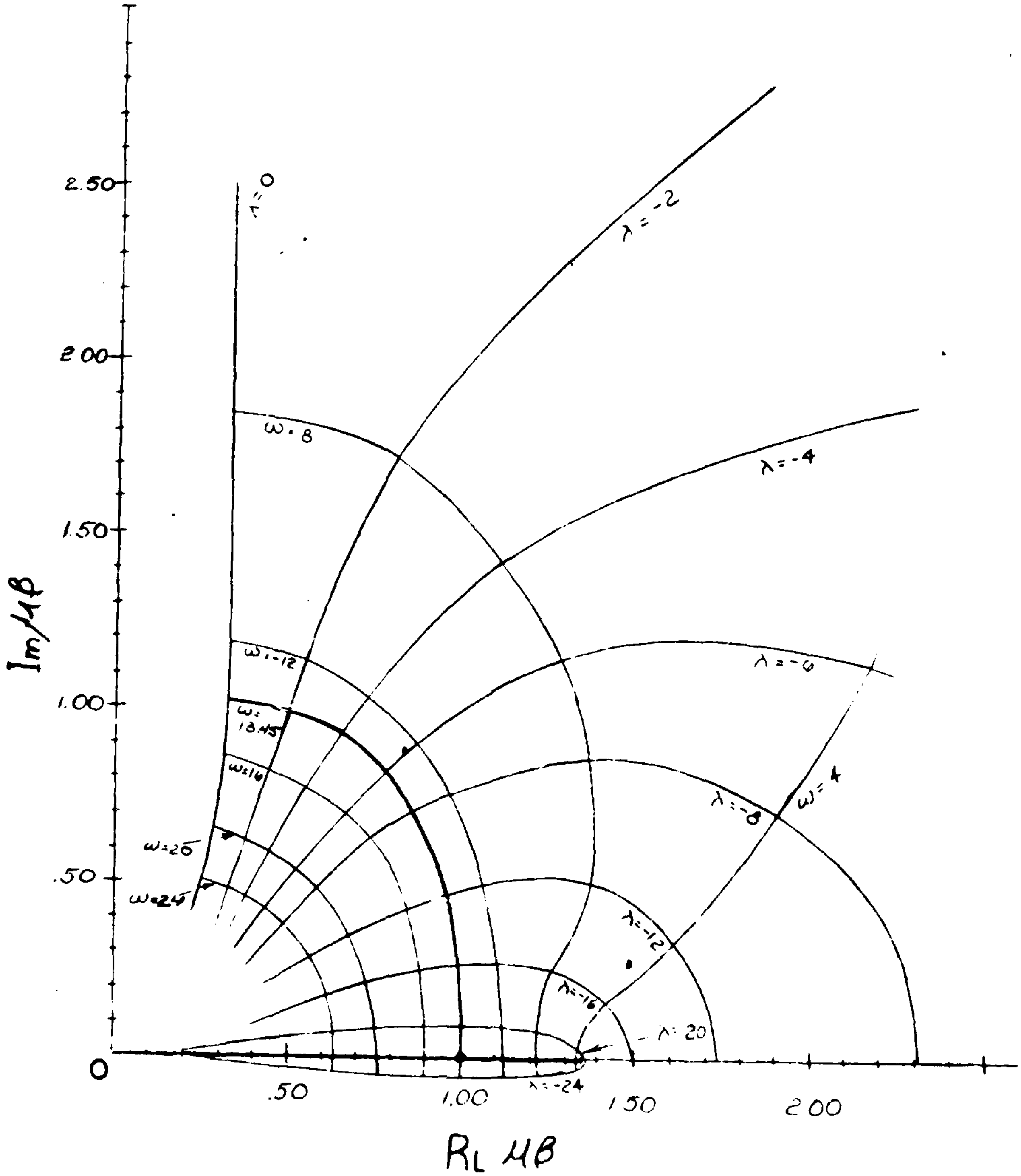

FIGURE 1
μβ PLANE FOR
CONSTANT $\mu = \frac{1}{6}$
$p = \lambda + j\omega$

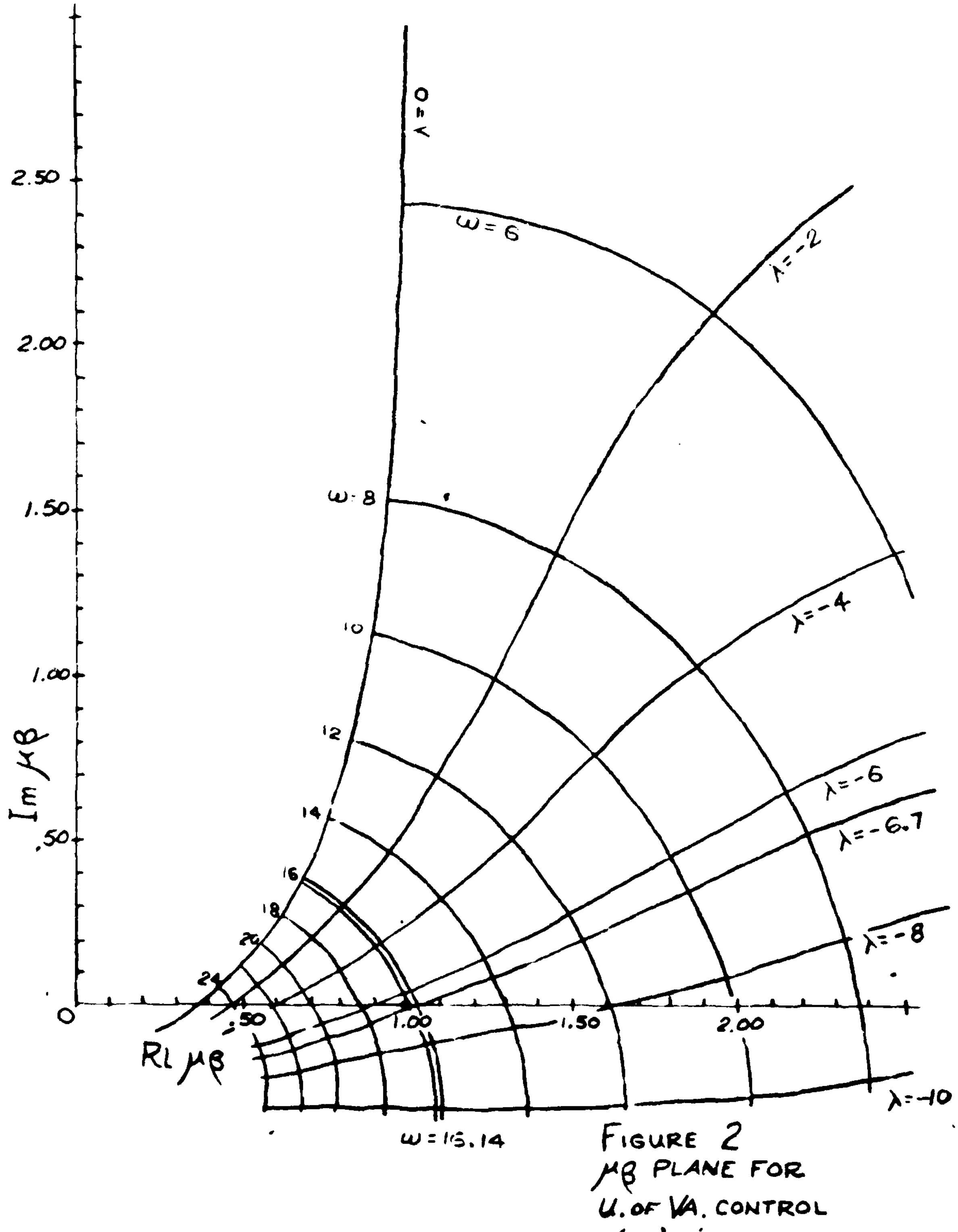

FIGURE 2
μβ PLANE FOR
U. OF VA. CONTROL
$p = \lambda + j\omega$