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THEORETICAL ANALYSIS OF A COAXIAL HALL ACCELERATOR

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1) Introduction

Performance of a magnetohydrodynamic accelerator is essentially dependent on the value of the magnetic interaction parameter

$$S = \frac{\sigma B^2 L}{\varrho V} \qquad \begin{array}{l} L \ldots \text{characteristic length} \\ V \ldots \text{characteristic velocity} \end{array}$$

Thus, to achieve efficient interaction while keeping the length of the device reasonably small. plasma accelerators should operate at high magnetic field strength and rather low densities. In this regime, however, the Hall effect becomes important, thus limiting the performance of devices based on the classical Faraday effect.

To overcome these difficulties several configurations have been suggested (see pg.458 in /1/ and /2/) in which the Hall component of the plasma currents is used for the desired interaction. Probably the most interesting one is the coaxial Hall accelerator shown in fig.1, which uses an applied magnetic field in the radial and an applied electric field in the axial direction. This device has been first suggested by Cann et al. /3/.

The applied magnetic field together with the axial electric field gives rise to Hall currents in the azimuthal direction which interact with the magnetic field to yield the desired acceleration in the axial direction. The current component parallel to the elctric field will tend to cause rotation.

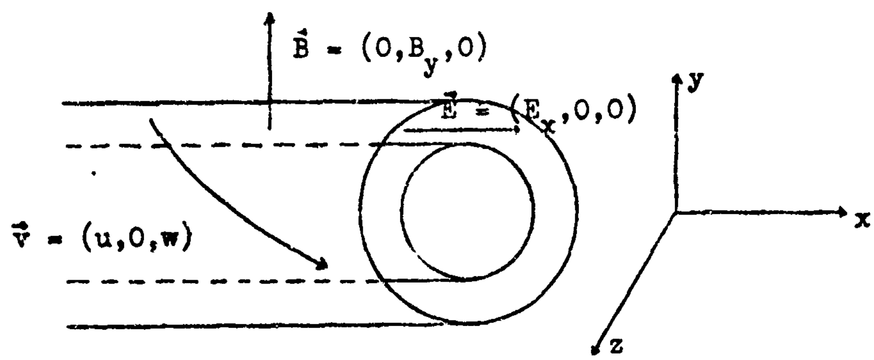

Fig. 1

Theoretical performance studies for this device have been carried out by other authors /1/, /4/, and especially by Cann himself /5/. The present analysis differs from the other ones in that the role of the magnetic interaction parameter is clearly pointed out, and that a closed analytical solution for the variation in the plasma parameters with axial distance is given.

2) Geometry and basic equations

The geometry of the device is shown in fig.1. If the diameter of the annular region is small or compared to the radius of both the outer and the inner cylinder, the problems can be analyzed in cartesian coordinates. From the rotational symmetry of the problem follows that $\frac{\partial}{\partial z} = 0$; by the same token also no z - component of the electric field will arise. The assumption $\frac{\partial}{\partial y} = 0$ is justified by the small ratio of $\frac{\text{diameter of the annulus}}{\text{radius}}$.

Summarized these assumptions read:

$$\vec{v} = (u, 0, w) \qquad \frac{\partial}{\partial_y} = \frac{\partial}{\partial_z} = 0 \qquad \vec{j} = (j_x, 0, j_z)$$

$$\vec{B} = (0, B_y, 0)$$

$$\vec{E} = (E_y, 0, 0)$$

Ohm's law is used in the form:

$$\vec{j} = \sigma(\vec{E} + \vec{v} \times \vec{B}) - \omega_e\tau_e\vec{j} \times \frac{\vec{B}}{B} + \frac{\omega_e\tau_e\omega_I\tau_I}{B^2} f^2 \left[(\vec{j} \times \vec{B}) \times \vec{B}\right] \quad (1)$$

(see pg.191 of /1/) where $f = \frac{n_N m_N}{m_I n_I + m_N n_N}$ (N = neutrals, I = ions, e = electrons).

In components equ.1 reads

$$j_x = \frac{1}{1+\omega_e\tau_e\omega_I\tau_I f^2}\left[\sigma(E_x - wB_y) + \omega_e\tau_e j_z\right]$$

$$j_z = \frac{1}{1+\omega_e\tau_e\omega_I\tau_I f^2}\left[\sigma u B_y - \omega_e\tau_e j_x\right]$$

Current continuity then demands that

$$\operatorname{div} \vec{j} = 0: \; j_x = \text{const.} = j_o.$$

The equations describing the gasdynamic behaviour of the plasma - equation of mass, momentum in the x and y direction and of entropy then read:

$$\rho u = \dot{m} = \text{const} \qquad (2)$$

$$\rho \cdot u \frac{dv}{dx} + \frac{dp}{dx} = - j_z \cdot B_y \tag{3}$$

$$\rho\, u \frac{dw}{dx} = j_x B_y \tag{4}$$

$$\rho\, uT \frac{dS}{dx} = \frac{\vec{j}^2}{\sigma} + \frac{\omega_e \tau_e \cdot \omega_I \tau_I \cdot f^2}{\sigma} \vec{j}^2 \tag{5a}$$

For a derivation of the right-hand side term in equ.(5a) giving the Ohmic heating see pg.121 of ref. /1/. By making use of the equation of state of an ideal gas (i.e. $p = \rho RT$ and $S = c_v \ln p - c_p \ln \rho + S_o$, where R, c_v and c_p are referred to one kg.) equ.(5a) can be converted into

$$u\left(\frac{1}{\gamma-1} \frac{dp}{dx} - \frac{\gamma}{\gamma-1} \frac{p}{\rho} \frac{d\rho}{dx}\right) = (1 + \omega_e \tau_e \omega_I \tau_I f^2)\, \vec{j}^2 \tag{5b}$$

3) Operating regimes (local analysis)

The total input of electrical energy into the plasma per m^3 and sec is given by

$$j_x \cdot E_x = j_x\left(\frac{j_x \beta}{\sigma} + wB_y - \frac{\omega_e \tau_e}{\beta} uB_y + \frac{\omega_e \tau_e}{\sigma\beta} j_x\right) \tag{6a}$$

$$\beta = (1 + \omega_e \tau_e \omega_I \tau_I \; f^2)$$

Neglecting for the moment rotation, and defining a dimensionless parameter E as

$$E = \frac{\omega_e \tau_e j_x}{\sigma u B_y} \tag{7a}$$

and calling $\frac{\omega_e \tau_e}{\beta} = \omega\tau$ (7b)

we can write this relation as

$$j_x \cdot E_x = \frac{\sigma u^2 B_y^2}{\beta} E\left[E\left(1 + \frac{1}{\omega^2 \tau^2}\right) - 1\right] \quad (6b)$$

Thus the device will operate as a generator - i.e. deliver energy to the outer circuit - if the above expression becomes negative. This is the case for

$$0 < E < \frac{\omega^2 \tau^2}{1 + \omega^2 \tau^2} \quad (8a)$$

Likewise we may speak of a motor if energy is fed into the plasma, i.e. if

$$E < 0 \quad \text{or} \quad E > \frac{\omega^2 \tau^2}{1 + \omega^2 \tau^2} \quad (8b)$$

Note that motor is not necessarily equivalent to accelerator, as the former definition includes also the case in which the power input is exclusively used for enthalpy increase. This can be seen in fig.2, where the criterium 8 is plotted in the E - $\omega\tau$ plane.

The definition of the dimensionless parameter E in equ. (7a) is especially suitable, as it allows for a formal analogy to the parameter E^* used in the theory of Faraday accelerators, which is given by

$$E^* = \frac{E_y}{u B_z} \quad (7c)$$

Written in E, all equations for the coaxial Hall accelerator become identical to those for the ideal (i.e. $\omega\tau = 0$) Faraday accelerator, if $\omega\tau \rightarrow \infty$ and E is substituted by E^*.

This may be regarded as an analogy between the ideal Hall accelerator ($\omega\tau \rightarrow \infty$) and the ideal Faraday accelerator ($\omega\tau \rightarrow 0$). Of course however ion slip will put an upper limit to the value of $\omega\tau$ actually achievable (see definition equ.7b).

3.2 Acceleration and deceleration

To determine the sign of the change of the flow parameters with axial distance x it is necessary to bring equ.(3) and (5) into canonic form, where use has to be made of the algebraic relations (1) and (2). (Equ.(4) can be separated from (3) and (5)). As a result one obtains

$$\frac{1}{u}\frac{du}{dx} = \frac{\dot{m}\beta}{\sigma B_o^2} \cdot \frac{M^2}{M^2-1} \left\{(E-1) - (\gamma-1)\left[(E-1)^2 + \frac{E^2}{\omega^2\tau^2}\right]\right\} \tag{9a}$$

$$\frac{1}{p}\frac{dp}{dx} = \frac{\dot{m}\beta}{\sigma B_o^2} \cdot \frac{\gamma M^2}{M^2-1} \left\{(E-1) - (\gamma-1)M^2\left[(E-1)^2 + \frac{E^2}{\omega^2\tau^2}\right]\right\} \tag{9b}$$

Thus to yield acceleration, the expression in braces at the right side of (9a) has to be larger than zero in the supersonic and smaller than zero in the subsonic case. Concentrating on the former ($M^2 > 1$) we obtain that for acceleration

$$\frac{\omega^2\tau^2}{\omega^2\tau^2+1}\left(\frac{2\gamma-1}{2\gamma-2} - \sqrt{\frac{1}{(2\gamma-2)^2} - \frac{\gamma}{(\gamma-1)\omega^2\tau^2}}\right) < E < \frac{\omega^2\tau^2}{\omega^2\tau^2+1}\left(\frac{2\gamma-1}{2\gamma-2} + \sqrt{\frac{1}{(2\gamma-2)^2} - \frac{\gamma}{(\gamma-1)\omega^2\tau^2}}\right) \tag{10a}$$

has to be valid. For $\omega\tau \rightarrow \infty$ we obtain

$$1 < E^* < \frac{\gamma}{\gamma-1}$$

the corresponding relation for the Faraday accelerator. Furthermore we can see from (10a) that for values of

$$\omega\tau < \sqrt{4\gamma(\gamma-1)} \qquad (10b)$$

acceleration cannot be achieved by any combination of the other parameters (in the supersonic case). Equ.9a may also be used to determine the value of E yielding maximum acceleration, if the other parameters are kept constant. As a result we get

$$E_{MAX} = \frac{\omega^2\tau^2}{1+\omega^2\tau^2}\left(\frac{2\gamma-1}{2\gamma-2}\right) \qquad (10c)$$

Both the boundaries of the accelerator regime, as well as E_{MAX} are shown in fig.2 for $\gamma = 1.67$. Only for large values of $\omega\tau$ the lower boundary of the accelerator regime coincides with the upper one for generator action. This also follows from the formal analogy to the ideal Faraday type device.

3.3 Pressure variation

Equation (9b) may be used to derive relation similar to (10a - c) for the variation in pressure. For supersonic flow, pressure will decrease downstream, if

$$\frac{\omega^2\tau^2}{\omega^2\tau^2+1}\left(\frac{2(\gamma-1)M^2+1}{2(\gamma-1)M^2} - \sqrt{\frac{1}{4(\gamma-1)^2M^4} - \frac{1}{\omega^2\tau^2} - \frac{1}{(\gamma-1)M^2\omega^2\tau^2}}\right) < E$$

$$E < \frac{\omega^2\tau^2}{\omega^2\tau^2+1}\left(\frac{2(\gamma-1)M^2+1}{2(\gamma-1)M^2} + \sqrt{\frac{1}{4(\gamma-1)^2M^4} - \frac{1}{\omega^2\tau^2} - \frac{1}{(\gamma-1)M^2\omega^2\tau^2}}\right) \qquad (11)$$

Otherwise the pressure will increase. Contrary to the criterium for velocity increase, the bounds for E given by relation (11a) depend explicitely on the Mach number. Maximum pressure decrease is achieved, when E is chosen to satisfy

$$E = \frac{\omega^2\tau^2}{1+\omega^2\tau^2}\left(1 + \frac{1}{2(\gamma-1)M^2}\right) \tag{11b}$$

3.4 Variation of Mach number

Making use of

$$M^2 = \frac{u^2}{a^2} = \frac{u^2\rho}{\gamma p} = \frac{um}{\gamma p}$$

which differentiated yields

$$\frac{dM}{M} = \frac{1}{2}\left(\frac{du}{u} - \frac{dp}{p}\right)$$

equations (9a) and (9b) can be combined to give

$$\frac{1}{M}\frac{dM}{dx} = \frac{\dot{m}\beta}{2\sigma B_0^2}\frac{M^2}{M^2-1}\left\{(\gamma-1)(E-1) - (\gamma-1)(\gamma M^2+1)\left[(E-1)^2 + \frac{E^2}{\omega^2\tau^2}\right]\right\} \tag{9c}$$

In supersonic flow, increase in Mach number thus occurs only, if E satisfies

$$\frac{\omega^2\tau^2}{\omega^2\tau^2+1}\left(\frac{2\gamma M^2(\gamma-1)+3\gamma-1}{2(\gamma-1)(\gamma M^2+1)} - \sqrt{\frac{(\gamma+1)^2}{4(\gamma-1)^2(\gamma M^2+1)^2} - \frac{\gamma M^2(\gamma-1)+2\gamma}{\omega^2\tau^2(\gamma-1)(\gamma M^2+1)}}\right) < E$$

(12a)

$$E < \frac{\omega^2\tau^2}{\omega^2\tau^2+1}\left(\frac{2\gamma M^2(\gamma-1)+3\gamma-1}{2(\gamma-1)(\gamma M^2+1)} + \sqrt{\frac{(\gamma+1)^2}{4(\gamma-1)^2(\gamma M^2+1)^2} - \frac{\gamma M^2(\gamma-1)+2\gamma}{\omega^2\tau^2(\gamma-1)(\gamma M^2+1)}}\right)$$

In fig.3 the regimes in which increase in Mach number occurs are shown for five values of M (full lines). The smallest value used is slightly above 1, as M = 1 for itself is a singular case (see equ.9c).

From (12a) then follows that for each Mach number a minimum value of the Hall parameter $\omega\tau$ exists, below which increase in Mach number can never be achieved (see fig.1). This value of $\omega\tau$ is given by

$$\omega\tau = \sqrt{\frac{4(\gamma-1)(\gamma M^2+1)}{\gamma + 1} \left(\frac{(\gamma-1)(\gamma M^2+1)}{\gamma + 1} + 1\right)} \qquad (12b)$$

Finally one can find that for each value of the Mach number M, maximum increase of M will be obtained for

$$E = \frac{\omega^2\tau^2}{1+\omega^2\tau^2} \left(1 + \frac{\gamma + 1}{2(\gamma-1)(\gamma M^2+1)}\right) \qquad (12c)$$

For M slightly above 1 and for M = 2 the corresponding curves are also given in fig.3 (dashed line).

3.5 Optimum power efficiency

The usual optimization criterium for propulsion applications is maximization of the ratio $\frac{\text{power input via Lorentz force}}{\text{total power input}}$. In our parameter this ratio η becomes $\dfrac{E - 1}{E\left(E\left(1+\frac{1}{\omega^2\tau^2}\right)-1\right)}$.

It thus has a maximum for

$$E = E_{opt} = 1 + \frac{1}{\sqrt{1+\omega^2\tau^2}} \qquad (13)$$

Also this relation is represented graphically in fig.2.

4) Total performance

To study the overall behaviour of a coaxial Hall accelerator, the governing equations (2) - (5) have to be integrated with respect to x over the whole accelerator length. For this purpose we introduce the following dimensionless variables:

$$\bar{u} = u/u_o, \; \bar{w} = w/w_o, \; p = p/p_o, \; \dot{m} = \varrho_o u_o, \; M_o^2 = \frac{u_o^2}{a_o^2} = \frac{\dot{m}u_o}{\gamma p_o} \qquad (14a-e)$$

$$dS = \frac{\sigma B_y^2}{\dot{m}\beta} dx \qquad (14f)$$

The introduction of the interaction parameter S as independent variable makes the following analysis applicable also to the case of variable electrical and of varying magnetic field strength without further modification (of course always under the condition that the assumptions concerning the geometry remain valid). On the basis of a kinetic argument given by Cann /5/ (his quantity is the reciprocal of the parameter E used in our analysis), it can be shown that E will remain constant over the accelerator length; provided the degree of ionization is constant too. To see this we insert the relations:

$$\sigma = \frac{n_\epsilon e^2}{m_\epsilon} \tau_\epsilon \text{ (see /1/ pg.190)} \qquad (15a)$$

$$j_x = \text{const (since div.}\vec{j} = 0) \qquad (15b)$$

$$\omega_\epsilon = \frac{eB_y}{m_\epsilon} \qquad (15c)$$

$$\alpha = \frac{n_I}{n} = \frac{n_\epsilon}{n} \text{ (assuming quasineutrality)} \qquad (15d)$$

$$n m_a u = \dot{m} \quad (m_a \text{ being the atomic mass}) \tag{15e}$$

into equation (7a), obtaining:

$$E = \frac{\omega_\epsilon \tau_e j_x}{\sigma u B_y} = \frac{m_e \omega_e \tau_e n m_a j_x}{n_e e^2 \tau_e B_y \dot{m}} = \frac{j_x m_a}{\dot{m} \alpha e} = \text{const}$$

The governing equations then read

$$\frac{d\bar{u}}{dS} + \frac{1}{\gamma M_o^2} \frac{d\bar{p}}{dS} = \bar{u}(E - 1) \tag{16a}$$

$$\bar{u} \frac{d\bar{p}}{dS} + \gamma p \frac{d\bar{u}}{dS} = \gamma(\gamma-1) M_o^2 \left[\left(\frac{E}{\omega\tau}\right)^2 + (E-1)^2 \right] \bar{u}^2 \tag{16b}$$

$$\frac{\text{[illegible]}}{dS} = \frac{E}{\omega\tau} \, \bar{u} \tag{16c}$$

After some manipulations equation (16a) and (16b) can be combined into the second order equation

$$\bar{u}^2 \bar{u}'' - a\bar{u}'^3 + b\bar{u}\bar{u}'^2 = 0 \tag{17}$$

where

$$a = \frac{\gamma + 1}{(\gamma-1)\left[\left(\frac{E}{\omega\tau}\right)^2 + (E-1)^2\right] - (E-1)}$$

$$b = \frac{\gamma(E - 1)}{(\gamma-1)\left[\left(\frac{E}{\omega\tau}\right)^2 + (E-1)^2\right] - (E-1)} - 2$$

By the transformation

$$\bar{u} = x \tag{18a}$$

$$\frac{d\bar{u}}{dS} = y \tag{18b}$$

equation (17) can be transformed into the first-order equation

$$\frac{dy}{dx} = \frac{a}{x^2} y^2 - \frac{b}{x} y \tag{19}$$

which is of the Bernoulli type and can be tranformed into a linear one by the substitution

$$y = \frac{1}{z} \tag{20a}$$

yielding

$$z' - \frac{b}{x} z = - \frac{a}{x^2} \tag{21}$$

with the solution

$$z = Cx^b + \frac{a}{b+1} \frac{1}{x} \tag{22}$$

After re-transformation and after carrying out the integration of (18b) we obtain the implicite solution

$$c'\bar{u}^{b+1} + \ln\bar{u} = \frac{b+1}{a}(S-S_o) \tag{23}$$

The constants c' and S_o have to be determined by the initial conditions

$$S = 0: \quad \bar{u} = 1$$
$$\bar{p} = 1$$

One then obtains

$$c'(u^{b+1} - 1) + \ln u = \frac{b+1}{a} S \tag{24a}$$

and

$$p = M_o^2 \left\{ (\gamma + 1)(c'^{b+1} + \frac{1}{b+1} u) + u \right\} \tag{24b}$$

where

$$c' = \frac{1 - M_o^2}{M_o^2(\gamma+1)} - \frac{1}{b+1}$$

Equation (16c) can now be integrated too, to give

$$w = \frac{E}{\omega\tau}\left[\frac{a}{b+2}\, c'(u^{b+2} - 1) + \frac{a}{b+1}\,(u - 1)\right] \tag{24c}$$

where use has been made of $\int f(u)dx = \int f(u)\frac{du}{u'}$, and the relation (18b) was substituted for u'. Similarly we obtain the total power input via Lorentz force as

$$E_L = \varrho_o u_o^3 \int (E-1)u^2 dS = \rho_o u_o^3\,(E-1)\left\{\frac{ac'}{b+3}\,u^{b+2} + \frac{a}{2(b+1)}\,u^2\right\} \tag{24d}$$

These results are illustrated on fig.5-8. Fig.5 is a plot of velocity versus interaction parameter for $\omega\tau = 6.0$, $\gamma = 1{,}67$ and for E = 2.0 (case _), 1.69 (case II), 1.16 (case III). The last two values correspond to E_{MAX} and E_{OPT} (equ.10c and 13). As pointed out already by Cann /5/ velocity is increasing without limiting value. The dashed line gives the analogous curve for E = 2.0 and $\omega\tau = 3.0$ (case IV).

Fig.6 shows the corresponding results for the variation in pressure, again for the above four cases.

Fig.7 illustrates the influence of the entrance Mach number on velocity (full line) and pressure variation (dashed line). Shown are the cases $M_o = 1.2$ (curve I) and $M_o = 3.0$ (curve II); E = 2.0 and $\omega\tau = 6.0$ in both cases.

Fig.8 is a plot of the interaction parameter S necessary for a tenfold increase in velocity versus Hall parameter. Curve I corresponds to E = 2.0, curve II to the case E_{MAX} (i.e. for each $\omega\tau$ the value E satisfying relation (10c) is taken), curve III to the case E_{OPT}.

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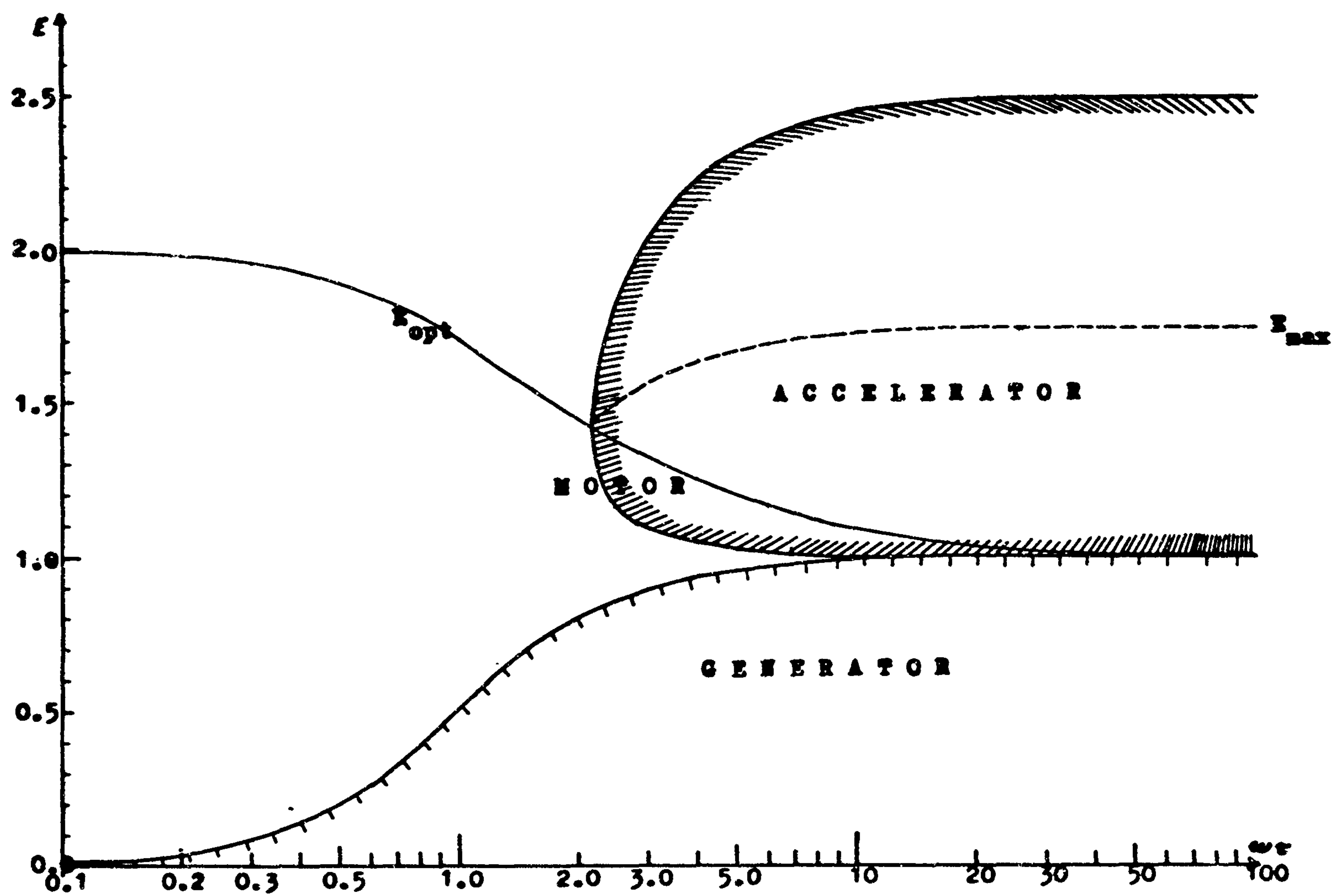

Fig. 2: Characteristic regimes

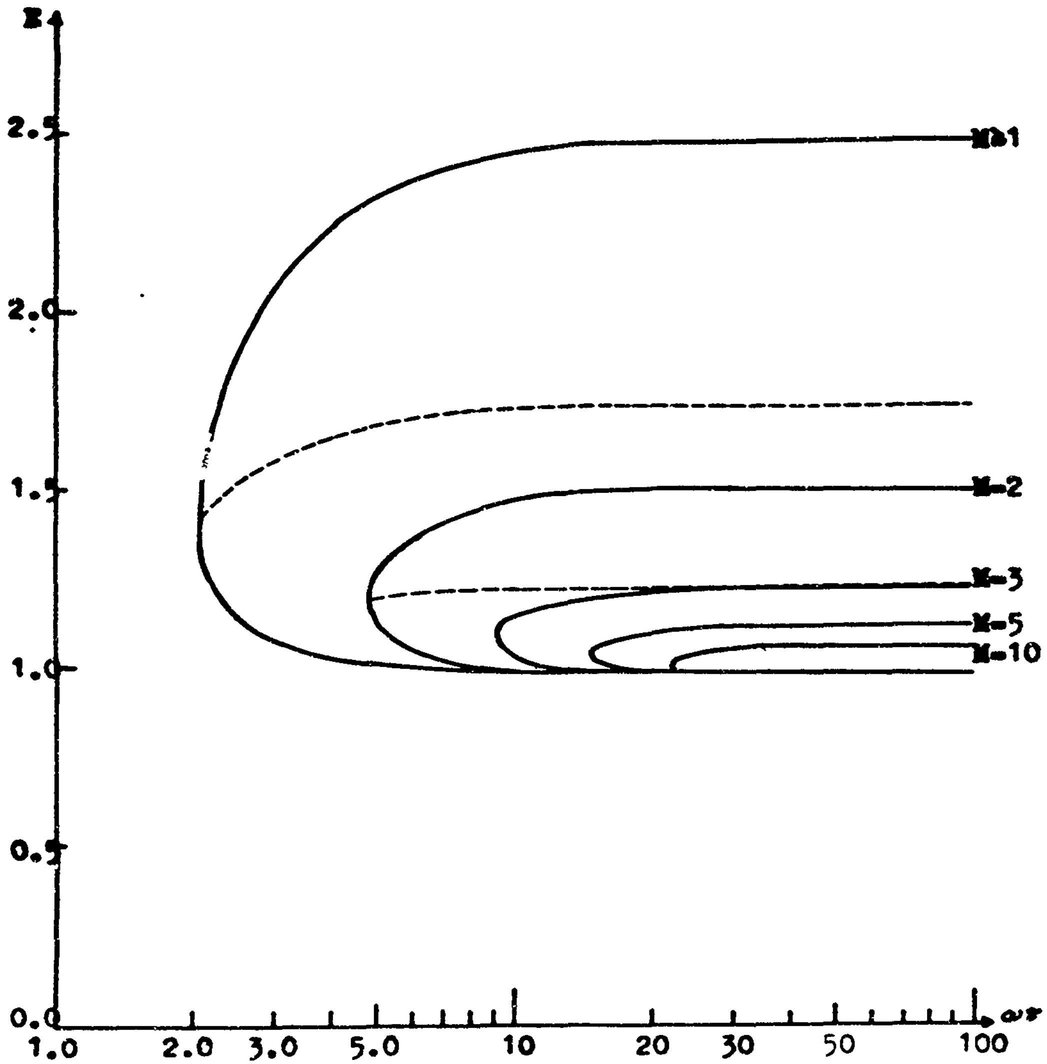

Fig. 3 : Regimes for increase in Mach number

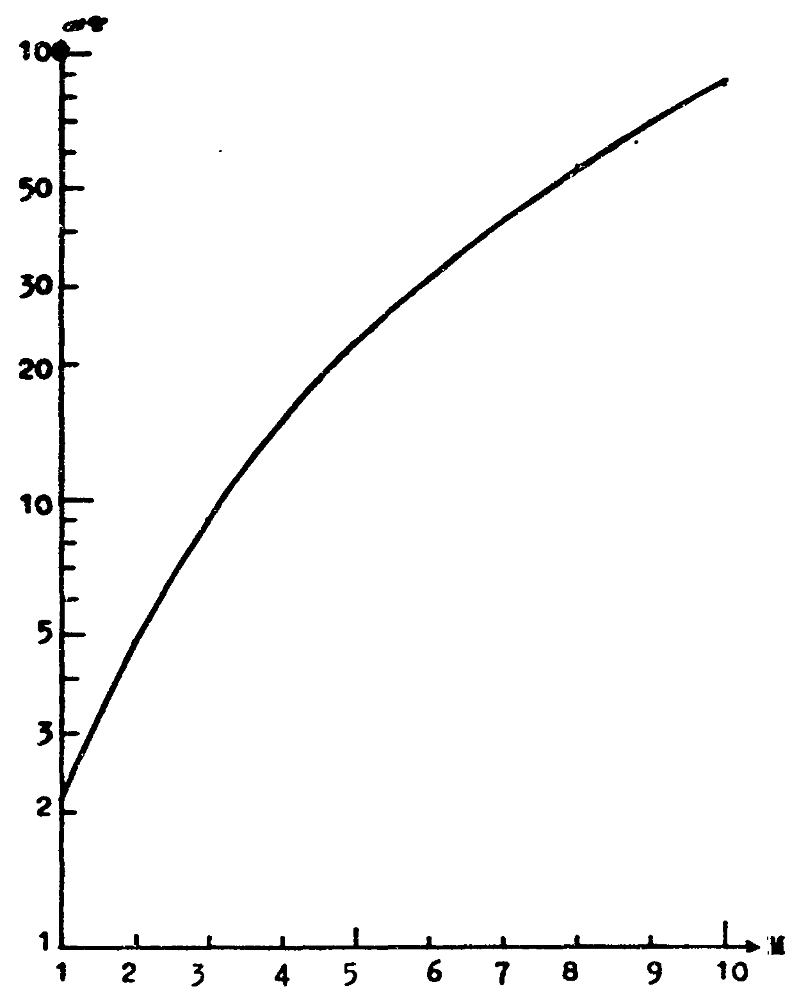

Fig. 4: Minimum $\omega\tau$ for increase in Mach number

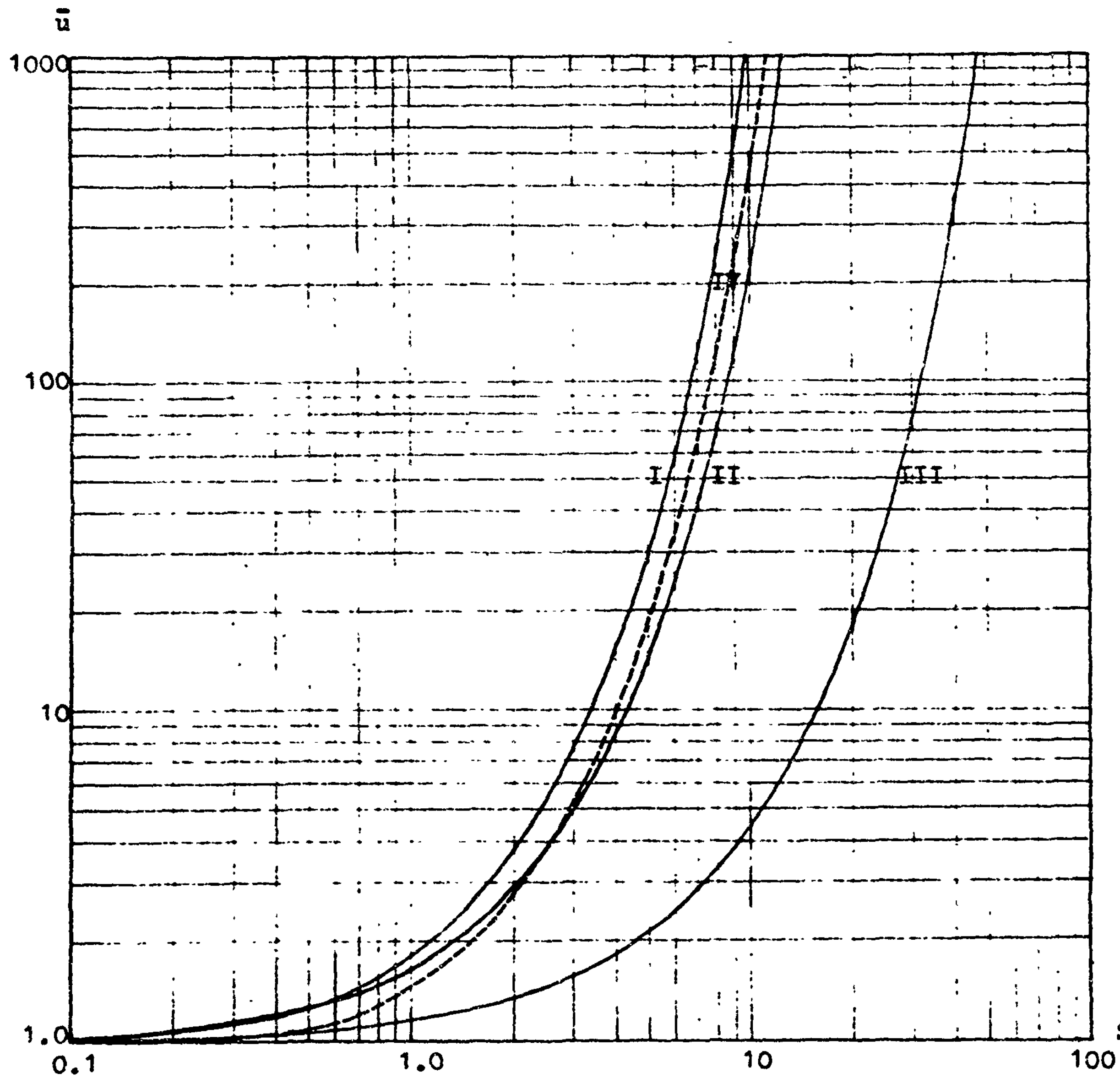

Fig. 5

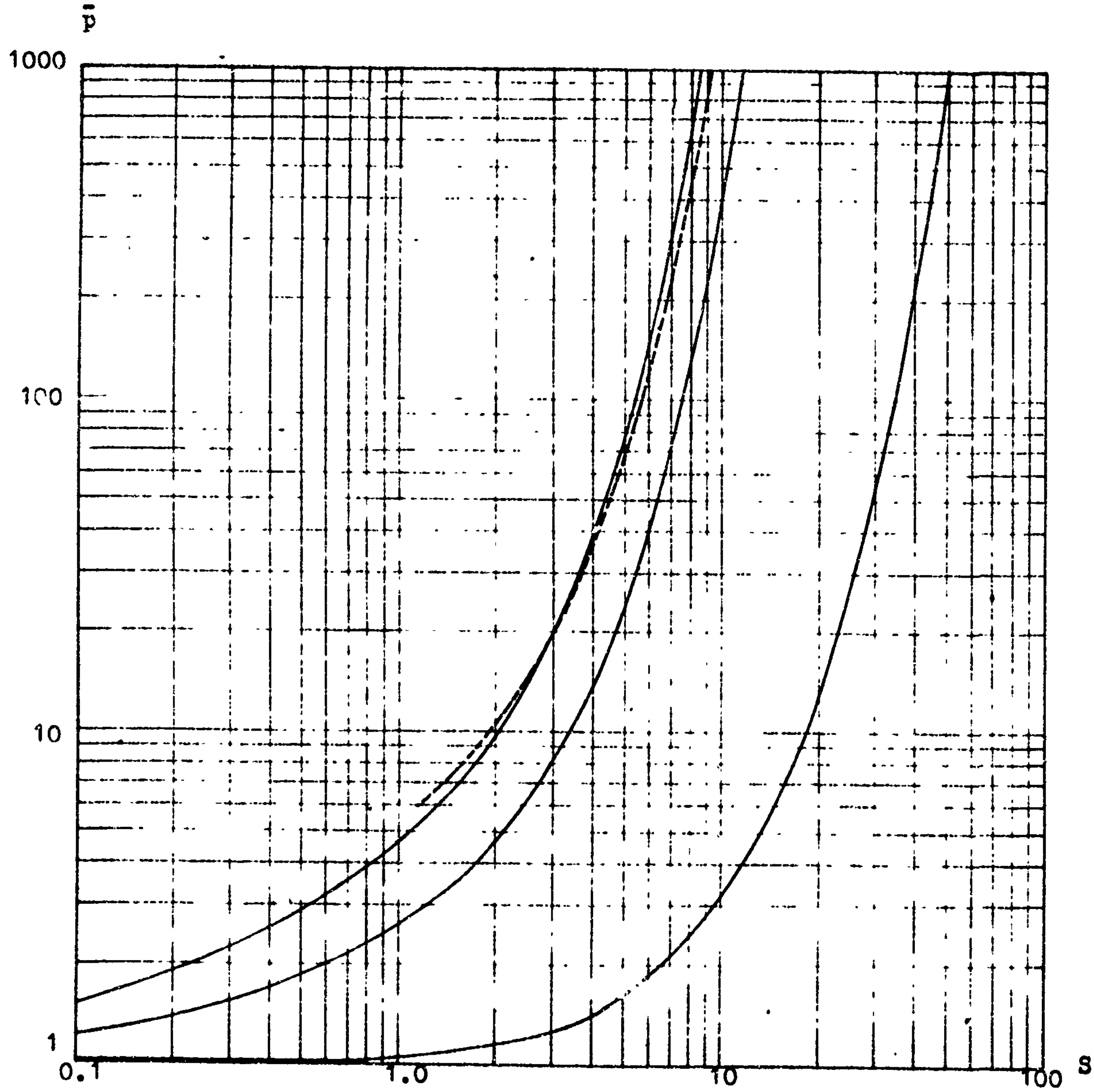

Figure 6

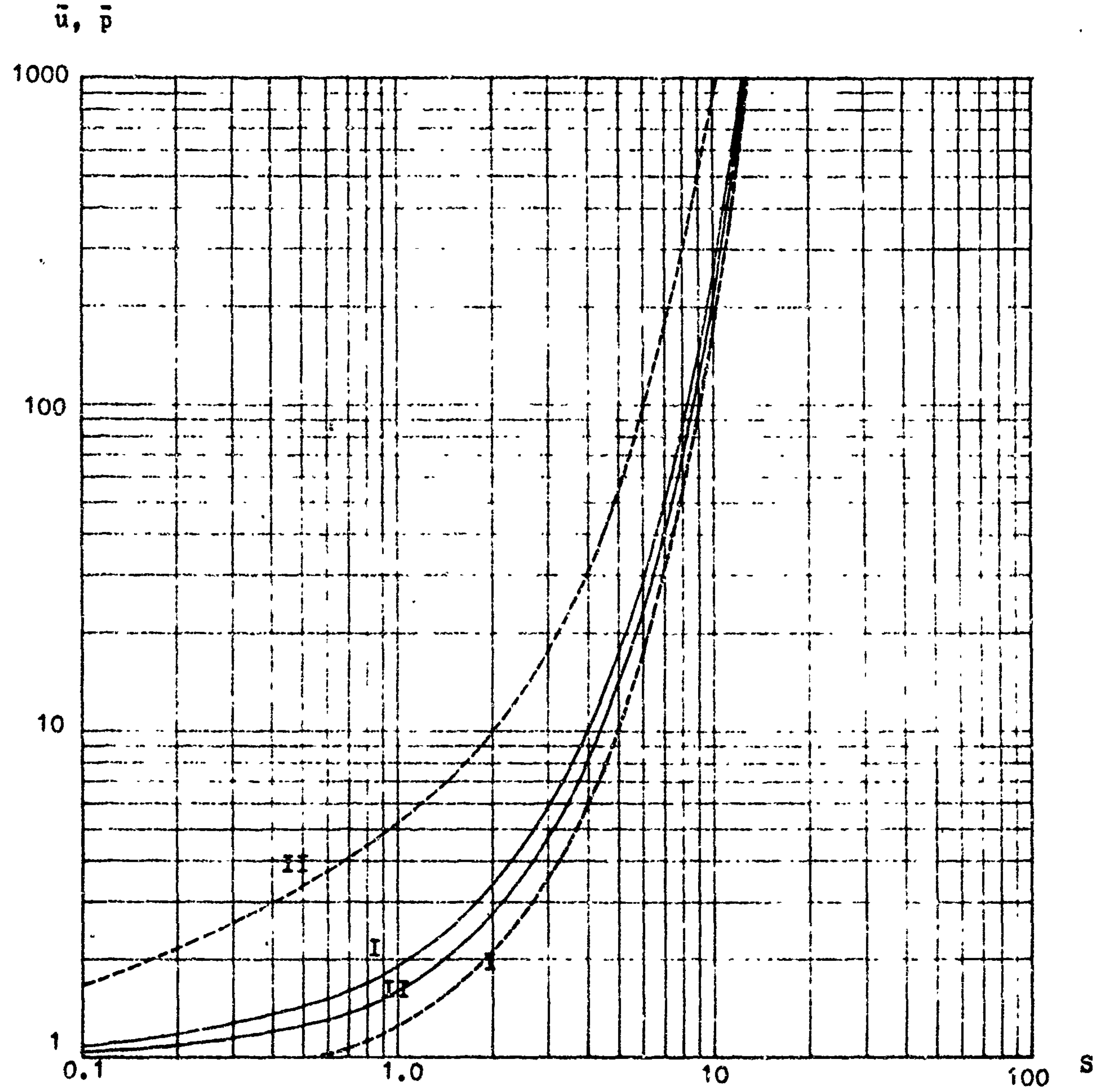

Figure 7

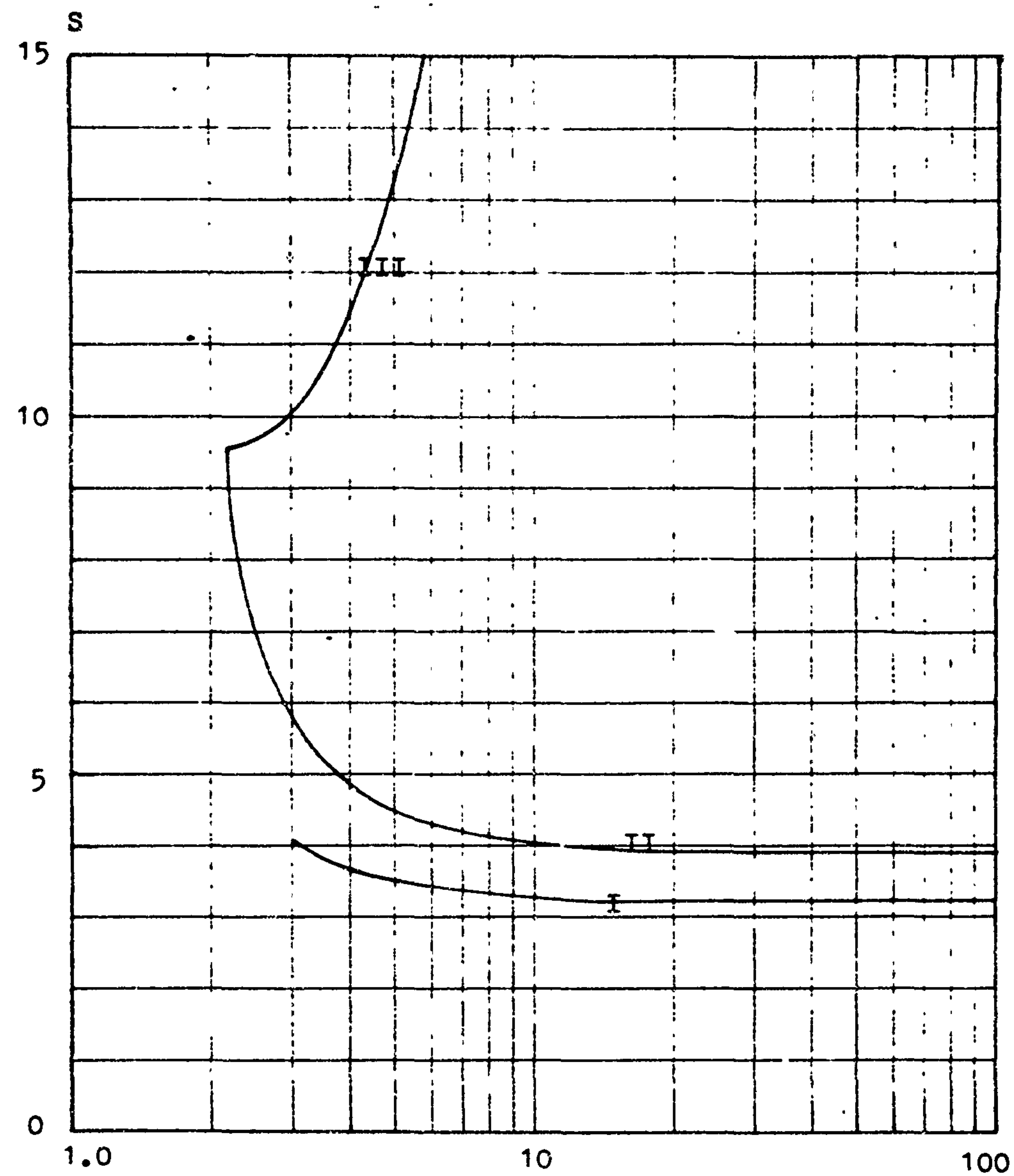

Figure 8

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13. ABSTRACT

The coaxial Hall current accelerator is investigated theoretically taking account of ion slip, variable electrical conductivity and variable magnetic field strength. The operating regimes are outlined and closed analytical solution is given for the axial variation of the flow parameters.

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