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TECHNICAL REPORT No. 7 (Rev.)

DIRECT AND INVERSE SOLUTIONS OF GEODESICS

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TABLE OF CONTENTS

			Paragraph	Page
Section	I.	GENERAL		
		Purpose and Scope	1	5
	II.	INTRODUCTION		5
	III.	A RIGOROUS NON-ITERATIVE PROCEDURE FOR RAPID INVERSE SOLUTION OF VERY LONG GEODESICS		6
		Preliminary Modification of Helmert's Iterative Solution	2	6
		Reduction of the Helmert Procedure to Power Series in x	3	9
		Derivation of the Unknown Quantity x .	4	12
		Determination of Geodetic Distance and Azimuths	5	13
		Other Non-Iterative Solutions	6	16
		Numerical Illustration of a Sample Solution (International Spheroid) . .	7	19
		Numerical Coefficients for Other Spheroids	8	21
		Additional Notes on Computational Procedures	9	21
		Antipodal Points	10	23
	IV.	TABULAR AND ELECTRONIC COMPUTER METHOD FOR NON-ITERATIVE SOLUTION OF GEODETIC INVERSE, BASED ON SODANO'S PAPER		
		Extension and Modification of Tabular and Electronic Computer Method for Non-Iterative Solution of Geodetic Inverse for Increased Decimal Accuracy in Short and Long Lines	11	29

TABLE OF CONTENTS (cont.)

			Paragraph	Page
	V.	TABULAR AND ELECTRONIC COMPUTER METHOD FOR SOLUTION OF DIRECT GEODETIC PROBLEM BASED ON SODANO'S PAPER		32
		Extension of Series of the Direct Geodetic Problem for Greater Accuracy	12	36
Appendix	I.	SUMMARY TABLES		
		Part A		39
		Part B		40
Appendix	II.	BIBLIOGRAPHY		41

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ABSTRACT

This Technical Report supersedes TR No. 7 entitled: INVERSE COMPUTATION FOR LONG LINES: A NON-ITERATIVE METHOD BASED ON THE TRUE GEODESIC, which is out of print. It contains the material of the original publication of 1950, and in addition, formulas pertaining to long lines, derived through the years at AMS.

The solutions of the Direct and Inverse Geodetic Problem are presented in forms which are adaptable to desk calculator and to electronic computer.

The maximum errors in the solutions due to the omission of higher order terms have been determined and are presented in tables in the Appendix. These tables will enable the user of the solutions to decide whether the accuracy requirements can be obtained with or without the higher order terms. These higher order terms have been derived and are presented herein.

DIRECT AND INVERSE SOLUTIONS OF GEODESICS

SECTION I. GENERAL

1. Purpose and Scope. The purpose of this report is to present in a single publication the various forms of the Direct and Inverse Solutions of Geodesics which have been solved by the Army Map Service. This report supersedes AMS Technical Report No. 7.

SECTION II. INTRODUCTION

In Section III of this report a procedure is given for a rigorous and rapid non-iterative inverse solution of very long geodesics. This procedure, which is in a convenient form for computation by means of desk calculators, was presented by Mr. Emanuel M. Sodano at the XIth General Assembly of the International Association of Geodesy and Geophysics in Toronto, Canada in 1957. The results represent the gradual extension and accumulated improvements of the original Army Map Service Technical Report No. 7.

This modification contains a more stable formula for azimuths and an alternative formula for very short lines. More general and accurate formulae for both long and short lines are given herein than are contained in Technical Report No. 7. The complete theoretical derivation starting with a rigorous modification of Helmert's(1) classical formulas are given. The final non-iterative formulas have been extended through terms equivalent to the second, fourth and sixth powers of the eccentricity of the spheroid, and therefore, may be shortened according to the required accuracy.

The solution, which requires no special purpose tables, is accurate to at least the tenth decimal place of radians for the azimuths and the arc distance. If the final formulas are shortened to the second and fourth powers of the eccentricity respectively, the results are accurate to seven and nine decimal places of radians respectively, even for distances circumscribing the earth.

In Section IV the formulas for the solution of the Inverse Geodetic Problem have been adapted to electronic computers. These formulas were derived from the basic formulas of Section III. A solution of the Direct Geodetic problem is given in Section V. The formulas are adapted to electronic computers.

SECTION III.

A RIGOROUS NON-ITERATIVE PROCEDURE FOR RAPID INVERSE SOLUTION OF VERY LONG GEODESICS (3)

2. Preliminary Modification of Helmert's Iterative Solution

e = eccentricity of the spheroid $= \sqrt{\dfrac{a_o^2 - b_o^2}{a_o^2}}$

e' = second eccentricity $= \sqrt{\dfrac{a_o^2 - b_o^2}{b_o^2}}$

b_o = semi-minor axis

L = absolute difference of longitude on the spheroid, between the given endpoints of the geodesic.

β_1 and β_2 = parametric (or reduced) latitude of the westward and eastward endpoints, respectively.

The relationship between parametric latitude and geodetic latitude is given by the equation $\tan\beta = \tan B(1-f)$ where f is the spheroidal flattening.

λ = difference of longitude (approximately L) on the reduced sphere, for which a progressively better value is found with each repetition of the following iteration process:

$$\cos \phi_o = \sin\beta_1 \sin\beta_2 + \cos\beta_1 \cos\beta_2 \cos\lambda$$

$$\sin \phi_o = (\text{sign of } \sin\lambda) \sqrt{1 - \cos^2 \phi_o}$$

$$\phi_o = \text{positive radians}$$

$$\sin 2\phi_o = 2 \sin \phi_o \cos \phi_o$$

$$\sin 3\phi_o = 3 \sin \phi_o - 4 \sin^3 \phi_o$$

$$\cos\beta_o = (\cos\beta_1 \cos\beta_2 \sin\lambda) \div \sin \phi_o$$

$$\sin^2\beta_o = 1 - \cos^2\beta_o$$

$$\cos 2\sigma = (2 \sin\beta_1 \sin\beta_2 \div \sin^2 \beta_o) - \cos \phi_o$$

$$\cos 4\sigma = -1 + 2 \cos^2 2\sigma$$

$$\cos 6\sigma = 4 \cos^3 2\sigma - 3 \cos 2\sigma$$

$$A' = \frac{e^2e'}{e'+e} - \frac{e^2e'^2}{16}\sin^2\beta_o + \frac{3e^2e'^4}{128} \sin^4\beta_o$$

$$B' = \frac{e^2e'^2}{16} \sin^2\beta_o - \frac{e^2e'^4}{32} \sin^4 \beta_o$$

$$C' = \frac{e^2e'^4}{256} \sin^4 \beta_o$$

$$\Upsilon = A'\phi_o - B' \sin \phi_o \cos 2\sigma + C' \sin 2\phi_o \cos 4\sigma$$

Next approximation to $\lambda = [(L + \Upsilon \cos\beta_o)$ radians.$]$

After a sufficiently accurate λ is found, and using the set of values from the last iteration, the geodetic distance (S) and azimuths (α) between the endpoints are obtained as follows:

$$A_0 = 1 + \frac{e'^2}{4}\sin^2\beta_0 - \frac{3e'^4}{64}\sin^4\beta_0 + \frac{5e'^6}{256}\sin^6\beta_0$$

$$B_0 = \frac{e'^2}{4}\sin^2\beta_0 - \frac{e'^4}{16}\sin^4\beta_0 + \frac{15e'^6}{512}\sin^6\beta_0$$

$$C_0 = \frac{e'^4}{128}\sin^4\beta_0 - \frac{3e'^6}{512}\sin^6\beta_0$$

$$D_0 = \frac{e'^6}{1536}\sin^6\beta_0$$

$$\left\{ \begin{aligned} S = b_0(A_0\phi_0 &+ B_0 \sin\phi_0 \cos 2\sigma - C_0 \sin 2\phi_0 \cos 4\sigma \\ &+ D_0 \sin 3\phi_0 \cos 6\sigma) \end{aligned} \right.$$

$$\left\{ \begin{aligned} \cot\alpha_{1-2} &= \frac{\tan\beta_2 \cos\beta_1 - \cos\lambda \sin\beta_1}{\sin\lambda} \\ \cot\alpha_{2-1} &= \frac{\sin\beta_2 \cos\lambda - \cos\beta_2 \tan\beta_1}{\sin\lambda} \end{aligned} \right.$$

where α_{1-2} and α_{2-1} range from 0° to 180° and 180° to 360°, respectively, clockwise from north.

3. [illegible] Helmert Procedure to [illegible] Order in x

Let it be assumed that the true value of λ is known (that is, the value that would result from an infinite number of Helmert approximations) and let this true value be represented by the given absolute difference of longitude on the spheroid plus a quantity x which will be determined later.

Thus: $$\lambda = (L + x) .$$

It will be evident, later, that x is a very small positive quantity of the order of e^2, and therefore well suited for setting up a convergent power series in x for each expression contained in the Helmert procedure. For example, from the above assumed equation, the following is derived:

$$\begin{aligned} \cos\lambda &= \cos(L + x) \\ &= \cos L \cos x - \sin L \sin x \\ &= (\cos L)\left(1 - \frac{x^2}{2} + \ldots\right) \\ &\quad - (\sin L)(x - \ldots) \end{aligned}$$

Therefore: $\cos\lambda = (\cos L) - (\sin L)\,x - \frac{1}{2}(\cos L)\,x^2 + \ldots$

There is thus available, at the outset, a series for the true $\cos\lambda$ with which to begin the Helmert solution and develop it in power series in x in its entirety. The process consists of substituting each new series into the succeeding Helmert expressions as required. For convenience, the following additional notation will be used:

$$N = e' \div (e' + e)$$

$$a = \sin\beta_1 \sin\beta_2$$

$$b = \cos\beta_1 \cos\beta_2$$

$$\cos\bar{\phi} = a + b\cos L$$

$$c = b \sin L \csc\bar{\phi}$$

$$m = 1 - c^2$$

$$h = e'^2 m$$

$$P = m \cot \phi - a \csc \phi$$

$$M_1 = (\tan \beta_2 \cos \beta_1 - \cos L \sin \beta_1) \div \sin L$$

$$M_2 = (\sin \beta_2 \cos L - \cos \beta_2 \tan \beta_1) \div \sin L$$

Listed, below, in the same sequence as the corresponding Helmert expressions, is the complete set of series through $\Upsilon \cos \beta_0$. The extent of the powers of x is such as to permit accuracies of the e^4 order in λ, for subsequent application to the distance and azimuths to the same degree of accuracy as the reference Helmert iteration form.

$$\cos \phi_0 = (\cos \phi) - (c \sin \phi)\, x - \tfrac{1}{2}(c^2 \cos \phi + P \sin \phi)\, x^2$$

$$\sin \phi_0 = (\sin \phi) + (c \cos \phi)x - \tfrac{1}{2}(c^2 \sin \phi - P \cos \phi)\, x^2$$

$$\begin{aligned} \phi_0 &= \phi + (\phi_0 - \phi) = \phi + \operatorname{arc\,sin} \left[\sin (\phi_0 - \phi)\right] \\ &= \phi + \operatorname{arc\,sin} \left[\sin \phi_0 \cos \phi - \cos \phi_0 \sin \phi\right] \\ &= \phi + (c)\, x + \tfrac{1}{2}(P)\, x^2 \end{aligned}$$

$$\sin 2\phi_0 = 2 \sin \phi_0 \cos \phi_0$$

$$\sin \lambda = (\sin L) + (\cos L)\, x - [illegible] \sin L)\, x^2$$

$$\cos \beta_0 = (c) + (P)\, x - \tfrac{1}{2}(cm + 3c P \cot \phi)\, x^2$$

$$\sin^2 \beta_0 = m - (2cP)\, x$$

$$\cos 2\sigma = \frac{1}{m} (m \cos \phi - 2P \sin \phi) + \frac{1}{m^2} (cm^2 \sin \phi + 4\, cmP \cos \phi - 4cP^2 \sin \phi)\, x$$

$$\cos 4\sigma = \frac{1}{m^2}\left(m^2 - 2m^2 \sin^2\phi - 8mP\sin\phi\cos\phi + 8P^2\sin^2\phi\right)$$

$$A' = \frac{e^2}{128}\left(128N - 8h + 3h^2\right) + \frac{e^2}{8}\left(e'^2 cP\right)x$$

$$B' = \frac{e^2}{32}\left(2h - h^2\right) - \frac{e^2}{8}\left(e'^2 cP\right)x$$

$$C' = \frac{e^2}{256}\left(h^2\right)$$

$$A'\phi_0 = \frac{e^2}{128}\left(128N\phi - 8h\phi + 3h^2\phi\right) + \frac{e^2}{16}\left(16Nc - hc + 2e'^2cP\phi\right)x$$

$$+ \frac{e^2}{2}\left(NP\right)x^2$$

$$\begin{aligned} -B'\sin\phi_0\cos 2\sigma = \frac{e^2}{128}\Big(&- 8h\sin\phi\cos\phi + 16e'^2 P\sin^2\phi \\ &+ 4h^2\sin\phi\cos\phi - 8e'^2 hP\sin^2\phi\Big) \\ &- \frac{e^2}{16}\left(hc\right)x \end{aligned}$$

$$\begin{aligned} C'\sin 2\phi_0\cos 4\sigma = \frac{e^2}{128}\Big(&h^2\sin\phi\cos\phi - 2h^2\sin^3\phi\cos\phi \\ &- 8e'^2 hP\sin^2\phi\cos^2\phi + 8e'^4P^2\sin^3\phi\cos\phi\Big) \end{aligned}$$

$$\begin{aligned} \Upsilon\cos\beta_0 = \frac{e^2}{128}\Big(&128Nc\phi - 8hc\phi - 8hc\sin\phi\cos\phi + 16e'^2cP\sin^2\phi \\ &+ 3h^2c\phi + 5h^2c\sin\phi\cos\phi - 2h^2c\sin^3\phi\cos\phi \\ &- 8e'^2hcP\sin^2\phi - 8e'^2hcP\sin^2\phi\cos^2\phi \\ &+ 8e'^4cP^2\sin^3\phi\cos\phi\Big) + \frac{e^2}{16}\Big(16Nc^2 + 16NP\phi - 2hc^2 \\ &- hP\phi + 2e'^2c^2P\phi - hP\sin\phi\cos\phi + 2e'^2P^2\sin^2\phi\Big)x \\ &- \frac{e^2}{2}\Big(Ncm\phi - 3NcP + 3NcP\phi\cot\phi\Big)x^2 \end{aligned}$$

4. Derivation of the Unknown Quantity x

Since the substitution into the Helmert iteration began with an algebraic series representing the true λ, the next approximation to λ must of necessity be its equal; that is:

The next approximation to λ = the starting true λ

$$\text{or} \qquad L + \Upsilon \cos \beta_0 = L + x$$

$$\text{and therefore} \qquad \Upsilon \cos \beta_0 = x.$$

By replacing $\Upsilon \cos \beta_0$ with its corresponding power series, the above equation takes the following quadratic form:

$$Q_1 + Q_2 x + Q_3 x^2 = x$$

for which the required solution of x to the proper order is

$$x = Q_1(1 + Q_2 + Q_2^2 + Q_1 Q_3).$$

Finally, substituting for Q_1, Q_2 and Q_3, produces the following end result:

$$\begin{aligned}
x = \frac{e^2 c}{128} \Big[& 128N\phi + 128e^2N^2c^2\phi - 8h\phi - 8h \sin\phi \cos\phi + 128e^2N^2P\phi^2 \\
+ \; & 16e'^2P \sin^2\phi + 128e^4N^3c^4\phi - 24e^2Nhc^2\phi + 3h^2\phi \\
- \; & 8e^2Nhc^2 \sin\phi \cos\phi + 5h^2 \sin\phi \cos\phi - 64e^4N^3c^2\phi^3 \\
- \; & 2h^2 \sin^3\phi \cos\phi + (16e^2e'^2N + 448e^4N^3)\, c^2P\phi^2 \\
- \; & 16e^2NhP\phi^2 + 16e^2e'^2Nc^2P \sin^2\phi - 8e'^2hP \sin^2\phi \\
- \; & 16e^2NhP\phi \sin\phi \cos\phi - 192e^4N^3c^2P\phi^3 \cot\phi \\
- \; & 8e'^2h \sin^2\phi \cos^2\phi + 128e^4N^3P^2\phi^3 \\
+ \; & 32e^2e'^2NP^2\phi \sin^2\phi + 8e'^4P^2 \sin^3\phi \cos\phi \Big]
\end{aligned}$$

The above rigorously developed expression is completely non-iterative, since it requires only the given spheroidal longitude. It therefore permits a direct evaluation of the ultimately true λ (that is, L + x), extended in this case through terms equivalent to the e^2, e^4 and e^6 order consecutively, in accordance to the accuracy that may be desired. Furthermore, it represents the algebraic solution of the hitherto unkown quantity x used in the power series version of each of the intermediate Helmert expressions.

5. Determination of Geodetic Distance and Azimuths

The non-iterative expression that has been developed for **x** suggests at once a numerical solution of distance and azimuths wherein, using the resulting true value of λ, only a single evaluation of Helmert's original formulas is necessary. An illustrative example by such a procedure is given in **paragraph 9.**

On the other hand, instead of reverting to functions of the true λ, the distance and azimuths themselves can be expanded non-iteratively into power series of x with coefficients in terms of the given spheroidal difference of longitude. This is accomplished below, but limited to the e^4 order of accuracy, since this manner of obtaining the distance and azimuths through e^6 would require each component series to one higher power of x than was necessary for λ. Again, the series are developed in the same sequence as the corresponding Helmert expression.

$$A_o = \frac{1}{64}\left(64 + 16h - 3h^2\right) - \tfrac{1}{2}\left(e'^2 cP\right) x$$

$$B_o = \frac{1}{16}\left(4h - h^2\right) - \tfrac{1}{2}\left(e'^2 cP\right) x$$

$$C_o = \frac{h^2}{128}$$

$$A_o \phi_o = \frac{1}{64}\left(64\phi + 16h\phi - 3h^2\phi\right) + \tfrac{1}{4}\left(4c + hc - 2e'^2 cP\phi\right) x$$
$$+ \tfrac{1}{2}\left(P\right) x^2$$

$$B_o \sin\phi_o \cos 2\sigma = \frac{1}{64}\left(16h \sin\phi \cos\phi - 32e'^2 P \sin^2\phi - 4h^2 \sin\phi \cos\phi\right.$$
$$\left. + 8e'^2 hP \sin^2\phi\right) + \tfrac{1}{4}\left(hc\right) x$$

$$-C_o \sin 2\phi_o \cos 4\sigma = \frac{1}{64}\left(-h^2 \sin\phi \cos\phi + 2h^2 \sin^3\phi \cos\phi + 8e'^2 hP \sin^2\phi \cos^2\phi\right.$$
$$\left. - 8e'^4 P^2 \sin^3\phi \cos\phi\right)$$

$$S = \left[\frac{b_o}{64}\left(64\phi + 16h\phi + 16h \sin\phi \cos\phi - 32e'^2 P \sin^2\phi\right.\right.$$
$$- 3h^2\phi - 5h^2 \sin\phi \cos\phi + 2h^2 \sin^3\phi \cos\phi$$
$$+ 8e'^2 hP \sin^2\phi + 8e'^2 hP \sin^2\phi \cos^2\phi$$
$$\left. - 8e'^4 P^2 \sin^3\phi \cos\phi\right) + \frac{b_o}{2}\left(2c + hc - e'^2 cP\phi\right) x$$
$$\left. + \frac{b_o}{2}\left(P\right) x^2\right]$$

$$\cot \alpha_{1-2} = \left[U_1 - \left(\frac{U_2 \cos\beta_1}{\sin L \cos\beta_2}\right) x + \left(\frac{U_1}{2 \sin^2 L} + \frac{U_2 \cos L \cos\beta_1}{2 \sin^2 L \cos\beta_2}\right) x^2\right]$$

$$\cot\alpha_{2-1} = \left[U_2 - \left(\frac{U_1 \cos\beta_2}{\sin L \cos\beta_1}\right)x + \left(\frac{U_2}{2\sin^2 L} + \frac{U_1 \cos L \cos\beta_2}{2\sin^2 L \cos\beta_1}\right) x^2 \right]$$

The x and x^2 for the above formulas of distance and azimuths can be substituted either numerically or algebraically using, in this case, only the first 6 terms of x for accuracies equivalent to the e^4 order. The algebraic substitution gives the following final expressions:

$$\begin{aligned} S = \frac{b_o}{64} \Big[& 64\phi + 64e^2Nc^2\phi + 16h\phi + 16h \sin\phi \cos\phi \\ & - 32e'^2P \sin^2\phi + 64e^4N^2c^4\phi - 3h^2\phi + (32e^2N - 4e^2)hc^2\phi \\ & - 4e^2hc^2 \sin\phi \cos\phi - 5h^2 \sin\phi \cos\phi + 2h^2 \sin^3\phi \cos\phi \\ & + (96e^4N^2 - 32e^2e'^2N)c^2P\phi^2 + 8e^2e'^2c^2P \sin^2\phi \\ & + 8e'^2hP \sin^2\phi + 8e'^2hP \sin^2\phi \cos^2\phi \\ & - 8e'^4P^2 \sin^3\phi \cos\phi \Big] \end{aligned}$$

$$\begin{aligned} \cot\alpha_{1-2} = U_1 & - \frac{e^2Nc\phi U_2 \cos\beta_1}{\sin L \cos\beta_2} - \frac{e^4N^2c^3\phi U_2 \cos\beta_1}{\sin L \cos\beta_2} \\ & + \frac{e^2hc\phi U_2 \cos\beta_1}{16 \sin L \cos\beta_2} + \frac{e^2hcU_2 \sin\phi \cos\phi \cos\beta_1}{16 \sin L \cos\beta_2} \\ & - \frac{e^4N^2cP\phi^2U_2\cos\beta_1}{\sin L \cos\beta_2} - \frac{e^2e'^2cPU_2 \sin^2\phi \cos\beta_1}{8 \sin L \cos\beta_2} \\ & + \frac{e^4N^2c^2\phi^2U_2 \cos L \cos\beta_1}{2\sin^2 L \cos\beta_2} + \frac{e^4N^2c^2\phi^2U_1}{2\sin^2 L} \end{aligned}$$

The corresponding $\cot\alpha_{2-1}$ is obtainable from the above by interchanging U_1 with U_2 and β_1 with β_2.

Thus, progressively, there have been developed three rigorous methods for determining geodetic distance and azimuths non-iteratively: as a function of the true λ, as a power series in x, and culminated by an explicit expression in essentially the given spheroidal latitude and longitude of the endpoints. For shorter lines, or for reduced accuracy on long lines, terms may be still further eliminated according to the next higher powers of e^2, e'^2, h and x, or equivalent combinations thereof.

6. Other Non-Iterative Solutions

The distance and azimuths by the original Helmert method are essentially functions of elements in the following spherical triangle:

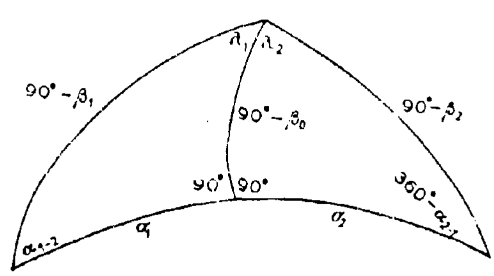

Fig. 1.

where

$$\lambda = \lambda_2 - \lambda_1$$

$$\not{\emptyset}_0 = \sigma_2 - \sigma_1$$

$$2\sigma = \sigma_2 + \sigma_1$$

and λ_1, λ_2, σ_1 and σ_2 are regarded as negative or positive according to whether they are west or east of the perpendicular arc $90° - \beta_0$.

(For this specific configuration, therefore, λ and ϕ_0 actually represent the <u>sum</u> of the absolute components and 2σ the <u>difference</u>.)

Since Helmert's method of successive approximations can only determine λ first, the subsequent solution of the above spherical triangle would always begin with λ and the known β_1 and β_2. The present paper, however, has developed not only a non-iterative expression for λ, but also independent power series for the various elements of this spherical triangle or functions thereof. Therefore the combination of ways to compute quantities leading to the distance and azimuths is increased considerably. In addition, the x for such series can be substituted either numerically or algebraically, in the manner shown for the distance and azimuth series in [illegible].

The above potentiality for increasing the number of non-iterative solutions may be seen from the expressions (1) below, where-in the x and x^2 of the ϕ_0 series are algebraically eliminated.

$$\left\{ \begin{aligned} \phi_0 = \phi &+ \frac{16e^2N - e^2e'^2}{16}(c^2\phi) + \frac{16e^4N^2 + e^2e'^2}{16}(c^4\phi) \\ &+ \frac{e^2e'^2}{16}(c^2\sin\phi\cos\phi) - \frac{e^2e'^2}{16}(c^4\sin\phi\cos\phi) \\ &- \frac{e^2e'^2}{8}(ac^2\sin\phi) + \frac{3e^4N^2}{2}(c^2d\phi^2) - \frac{3e^4N^2}{2}(c^4\phi^2\cot\phi) \end{aligned} \right\} \tag{1}$$

The computed value of ϕ_0 is then combined with β_1 and β_2 to obtain α's, followed by β_0, 2σ, A_0, B_0, C_0 and finally the geodetic distance. When adopting such varied procedures for solving the reference triangle, care should be taken to avoid formulations which lead to a weak determination of required quantities. These difficulties may most likely occur at extremes of latitude, longitude, or azimuth.

The non-iterative series, too, are functions of elements of a spherical triangle, but defined by β_1 and β_2 and the <u>given</u> longitude L. This amounts simply to a substitution of L for λ, which results in a spherical triangle with parts corresponding as follows:

Series	β_1	β_2	L	$\not{\phi}$	c	U_1	U_2
Helmert	β_1	β_2	λ	$\not{\phi}_0$	$\cos\beta_0$	$\cot\alpha_{1-2}$	$\cot\alpha_{2-1}$

Starting with the given β_1, β_2, and L, the values of all quantities used in the non-iterative series may thus be solved trigonometrically in various orders.

It is also to be noted that in the relation $\lambda = (L + x)$, if x is assumed to be zero, L is considered to be equal to λ. Therefore in the various power series in x, the <u>constant term</u> can represent the true value of the series by simply replacing functions of L with λ. This is well illustrated by the A_0 series in x in paragraph [illegible] and its counterpart in paragraph [illegible] to the proper order. The principle can well be incorporated in computation forms, such as the one on the next two pages as applied to H, I, etc.

7. <u>Numerical Illustration of a Sample Solution</u> (International Spheroid)

Given	L = absolute difference of longitude	106°
	B_1 = latitude of westward point	20° N
	B_2 = latitude of eastward point	45° N

$\tan\beta_1 = 0.99663\ 29966\ \tan B_1$	0.36274 47453
$\tan\beta_2 = 0.99663\ 29966\ \tan B_2$	0.99663 29966
$\cos\beta_1 = 1 \div +\sqrt{1+\tan^2\beta_1}$	0.94006 23275
$\cos\beta_2 = 1 \div +\sqrt{1+\tan^2\beta_2}$	0.70829 81969
$\sin\beta_1 = \tan\beta_1 \cos\beta_1$	0.34100 26695
$\sin\beta_2 = \tan\beta_2 \cos\beta_2$	0.70591 33545
$a = \sin\beta_1 \sin\beta_2$	0.24071 83383
$b = \cos\beta_1 \cos\beta_2$	0.66584 44515
$\sin L$	0.96126 16959
$\cos L$	-0.27563 73558
$\cos\phi = a + b\cos L$	0.05718 67343
$\sin\phi = (\text{sign of } \sin L)\sqrt{1-\cos^2\phi}$	0.99836 34996
ϕ = positive radians	1.51357 83766
$A = (b \sin L) \div \sin\phi$	0.64109 99269
$B = A^2$	0.41100 91163
$C = \left[\cos\phi - (\cos\phi)B\right] \div 4.9865\ 20649$	0.00675 47028
$D = -a(0.40108\ 12630)$	-0.09654 76152
$E = -a(0.79948\ 93686)$	-0.19244 45307
$F = (3.9865\ 20649)C$	0.02692 77622
$G = \phi^2 \div \sin\phi$	2.29467 47388

x_{rad}	$= \left\{ A\left[\phi(237.2388918 + B) + \sin\phi(C+D) + C(F+E)\right]\right\} \div 70519.51145$	0.00326 58167
λ	$= L + x$	106°11'13".62305
$\sin\lambda$		0.96035 63902
$\cos\lambda$		-0.27877 51848
$\cos\phi_0$	$= a + b\cos\lambda$	0.05509 74283
$\sin\phi_0$	$= (\text{sign of } \sin\lambda)\sqrt{1 - \cos^2\phi_0}$	0.99848 09830
ϕ_0	= positive radians	1.51567 09835
$\sin 2\phi_0$	$= (\sin\phi_0 \cos\phi_0) \div 0.5$	0.11002 74687
$\cos\beta_0$	$= (b\sin\lambda) \div \sin\phi_0$	0.64042 07839
q	$= 1 - \cos^2\beta_0$	0.58986 12195
$\cos 2\sigma$	$= (2a - q\cos\phi_0) \div q$	0.76108 89231
$\cos 4\sigma$	$= (\cos^2 2\sigma - 0.5) \div 0.5$	0.15851 26977
H	$= 6356911.946 + 10756.165q - 13.650q^2$	6363251.841
I	$= 10756.165q - 18.200q^2$	6338.312
J	$= 2.275q^2$	0.792
$\cot\alpha_{1-2}$	$= \dfrac{(\tan\beta_2\cos\beta_1 - \cos\lambda\sin\beta_1)}{\sin\lambda}$	1.07455 96397
$\cot\alpha_{2-1}$	$= \dfrac{(\sin\beta_2\cos\lambda - \cos\beta_2\tan\beta_1)}{\sin\lambda}$	-0.47245 22891
α_{1-2}	= Clockwise from North, in quad I or II for cot + or -, resp.	42°56'30".03557
α_{2-1}	= Clockwise from North, in quad III or IV for cot + or -, resp.	295°17'18".59865
s meters	$= H\phi_0 + I\sin\phi_0\cos 2\sigma - J\sin 2\phi_0\cos 4\sigma$	9649412.854m.

8. Numerical Coefficients For Other Spheroids

The illustrative solution given in the preceding section contains fixed numerical coefficients which are functions solely of the size and shape of the International spheroid. The algebraic expressions of these coefficients, together with their values, are shown below in the order of appearance in the sample solution. For any other spheroid, these expressions can be quickly re-evaluated once and for all and substituted for the corresponding International values. (Note: e^2N = flattening.)

$+\sqrt{1 - e^2}$	=	0.99663 29966
$(16e^2N^2 + e'^2) \div e'^2$	=	4.9865 20649
$2e'^2 \div (16e^2N^2 + e'^2)$	=	0.40108 12630
$16e^2N^2 \div (16e^2N^2 + e'^2)$	=	0.79945 93686
$16e^2N^2 \div e'^2$	=	3.9865 20649
$(16N - e'^2) \div (16e^2N^2 + e'^2)$	=	237.2388 918
$16 \div e^2(16e^2N^2 + e'^2)$	=	70519.51145
b_o	=	6356911.946
$b_o e'^2 \div 4$	=	10756.165
$3b_o e'^4 \div 64$	=	13.650
$b_o e'^4 \div 16$	=	18.200
$b_o e'^4 \div 128$	=	2.275

9. Additional Notes on Computational Procedures

Although the illustrative solution given in paragraph 7 is primarily intended for accuracy equivalent to the e^4 order, it easily lends itself to any required degree. This is accomplished simply by adding or subtracting appropriate terms of x, H, I, J, and S. The extended terms are given in the latter part of Section III, (paragraphs 4 and 2), respectively. For short lines or reduced accuracy on long lines, x on the International spheroid becomes merely $(A\phi + 297)$ and all terms in q^2 are omitted, with the consequent elimination of many other supporting quantities. Similar savings are realized for other forms of solutions presented herein.

For short lines, the resulting small ϕ is computed more accurately from $\sin\phi$ obtained as follows:

$$\sin\frac{\phi}{2} = +\sqrt{b\,\sin^2\frac{L}{2} + \sin^2\frac{\beta_1 - \beta_2}{2}}$$

$$\cos\frac{\phi}{2} = (\text{sign of } \sin L)\sqrt{0.5(1 + \cos\phi)}$$

$$\sin\phi = \left(\sin\frac{\phi}{2}\cos\frac{\phi}{2}\right) \div 0.5$$

Similarly, $\sin\phi_0$ is obtained as above by replacing ϕ with ϕ_0 and L with λ. In either case, squaring the small sines under the radical increases their significant decimal places.

If the numerator of x is to be cumulated in a ten digit calculator, 9 decimal places should be allotted to ϕ, $\sin\phi$ and G, but only 7 decimals to their multipliers. However, when the value of G is 10 or greater, decrease its decimal places accordingly and increase those of F and E correspondingly. For a smaller calculator, reduce all decimals equally.

Use co-function of $\tan\beta$ or $\cot\alpha$ when their values are too large.

$$\text{Thus} \quad \cot\beta_n = \frac{\cot B_n}{+\sqrt{1-e^2}} \quad \text{and} \quad \tan\alpha = \frac{1}{\cot\alpha}$$

The accuracy of geodetic distances computed through the e^2, e^4 and e^6 order for very long geodesics is within a few meters, centimeters and tenths of millimeters respectively. Azimuths are good to tenths, thousandths, and hundred thousandths of a second. Further improvement of results occurs for shorter lines.

Some of the terms in the sample solution of paragraph 7 have been grouped for ease of computing by desk calculator. For electronic computers, however, the terms are best left in series form, thus being ideally suited to adding or removing them according to accuracy requirements.

10. Antipodal Points

In the various series that have been presented, ϕ represents a spherical arc distance which varies from 0° to 180° and even to 360° according to whether the geodetic line is very short, half around the earth or completely around it. At these specific instances, quantities such as csc ϕ, cot ϕ, and P approach infinity. For the case of the very short lines, this condition is equalized

... the factors ϕ and sin ϕ which ... , however, the series gradually ... becoming larger.

Closer inspection of the various series in x shows, nevertheless, that this condition of divergence never prevails in the constant terms, and for succeeding coefficients it is to no greater degree than the power of the corresponding x. Therefore, here too it could be equalized if x were sufficiently small.

The first equation of ... relates λ as follows:

$$\lambda = (L + x).$$

The true value of λ could have been represented, instead, by:

$$\lambda = (L_n + \chi)$$

where L_n is an arbitrary amount ... nearly equal to λ and therefore χ is correspondingly smaller than x. This new assumption leads to a set of power series in χ ... identical to those in x, except that its coefficients ... function of ... The obvious value to assign to L_n would be the slightly ... result of solving ... in powers of the ...

The relation derived at the beginning of ... will accordingly change from:

$$\Upsilon \cos \beta_0 = x$$

$$(L - L_n) + \Upsilon \cos \beta_0 = \chi$$

where, as noted, the substitution of the $\Upsilon \cos \beta_0$ series given at the end of paragraph 3 will now be in terms of L_n and χ instead of L and x. Solving the above equation for χ (this time through only the e^4 order of accuracy) gives:

$$\chi = \frac{16(L-L_n) + (16e^2Nc\phi - e^2hc\phi - e^2hc \sin\phi \cos\phi + 2e^2e'^2cP \sin^2\phi)_n}{16(1 - e^2Nc^2 - e^2NP\phi)_n}$$

where the subscripts n to the parenthesis indicate that c, ϕ, h, P, etc. are functions of L_n instead of L. This time, the denominator of the expression cannot be algebraically divided into the numerator, because the $e^2NP\phi$ term is relatively large for nearly antipodal lines.

With the above correction χ to an arbitrary but sufficiently accurate value L_n, the true λ of antipodal lines is essentially obtained again non-iteratively, and therefore more rapidly than by numerous individual successive approximations. Thus, also, a previous 4" longitude discrepancy noted by Mr. H. F. Rainsford[2] for a line of about 179°46'18" longitude would be resolved. In this connection, appreciation is expressed to Mr. Rainsford for his interest in the subject which resulted in profitable correspondence.

SECTION IV

TABULAR AND ELECTRONIC COMPUTER METHOD FOR NON-ITERATIVE SOLUTION OF GEODETIC INVERSE, BASED ON SODANO'S PAPER

Due to their series-like nature, the formulas given in this section for distance and azimuth are more adaptable to electronic computer programming than the corresponding closed formulas of Section III.

This method (unlike the one just discussed) does not have the restriction that B_1 and L_1 must be the latitude and longitude, respectively, of the westward point. Here, B_1 and L_1 are the geographic latitude and longitude, respectively, of any point.

The distance equation of this section was derived by making the following substitutions into the distance S equation on page 14.

$$\left.\begin{aligned} f &= e^2 N \\ e'^2 m &= h \text{ (where } e'^2 \text{ was expressed in terms of } f) \\ m &= 1 - c^2 \end{aligned}\right\} \quad (1)$$

$$P = (1 - c^2) \cot \phi - a \csc \phi$$

The expression $(\lambda - L)$ of this method is equivalent to "x" on page [illegible]. The series for $(\lambda - L)$ was derived by making the substitutions (1) into the equation for "x" on page 11. The computation form for this method is as follows:

B_1, L_1 = geographic latitude and longitude, respectively, of any point

B_2, L_2 = geographic latitude and longitude, respectively, of any other point, non-antipodal

Latitudes and longitudes considered (+) north and east, (-) south and west

Required: α, S = azimuths clockwise from north and distance between points, respectively.

$$
\begin{aligned}
\frac{S}{b_o} = & \left[(1 + f + f^2)\,\phi \right] \\
& + a \left[(f + f^2)\sin\phi - \frac{f^2}{2}\,\phi^2 \csc\phi \right] \\
& + m \left[- \frac{(f + f^2)}{2}\phi - \frac{(f + f^2)}{2}\sin\phi\cos\phi + \frac{f^2}{2}\,\phi^2 \cot\phi \right] \\
& + a^2 \left[- \frac{f^2}{2}\sin\phi\cos\phi \right] \\
& + m^2 \left[\left(\frac{f^2}{16}\right)\phi + \frac{f^2}{16}\sin\phi\cos\phi - \frac{f^2}{2}\,\phi^2 \cot\phi - \frac{f^2}{8}\sin\phi\cos^3\phi \right] \\
& + am \left[\left(\frac{f^2}{2}\right)\phi^2 \csc\phi + \frac{f^2}{2}\sin\phi\cos^2\phi \right]
\end{aligned}
$$

$$
\begin{aligned}
\frac{(\lambda - L)}{\phi_c} = & \left[(f+f^2)\phi \right] + a \left[- \left(\frac{f^2}{2}\right)\sin\phi - f^2\phi^2 \csc\phi \right] \\
& + m \left[- \frac{5f^2}{4}\phi + \frac{f^2}{4}\sin\phi\cos\phi + f^2\phi^2 \cot\phi \right]
\end{aligned}
$$

where: a_0, b_0 = semi-major and semi-minor axes, respectively, of spheroid

f = spheroidal flattening = $\left(1 - \frac{b_0}{a_0}\right)$

ρ = number of seconds in one radian = 206,264.80625

$L = (L_2-L_1)$ or $(L_2-L_1) + \left[\text{sign opposite of } (L_2-L_1)\right] 360°$.

[Use whichever L has an absolute value < or > 180°; according to whether the shorter or longer geodetic arc is required. However, for meridional arcs ($|L|$ = 0° or 180° or 360°) use either L but consider it as (+) for the shorter and (-) for the longer arc.]

$\tan\beta = \tan B\,(1-f)$ when $|B| \leq 45°$ or $\cot\beta = \frac{\cot B}{1-f}$ when $|B| > 45°$

$a = \sin\beta_1 \sin\beta_2$

$b = \cos\beta_1 \cos\beta_2$

$\cos\sigma = a + b\cos L$

$\sin\sigma = \pm\sqrt{(\sin L\cos\beta_2)^2 + (\sin\beta_2\cos\beta_1 - \sin\beta_1\cos\beta_2\cos L)^2}$

[The sign of $\sin\sigma$ is (+) or (-) according to whether the shorter or the longer arc is required. The quantity under the radical and its root must be computed by floating decimal to obtain $\sin\sigma$ to full accuracy for short lines.]

σ = positive radians (obtain reference angle from $\sin\sigma$ or $\cos\sigma$ whichever has smaller absolute value.)

$c = (b\sin L) \div \sin\sigma$

$m = 1 - c^2$

$$\cot \alpha_{1-2} = (\tan\beta_2 \cos\beta_1 - \cos\lambda \sin\beta_1) \div \sin\lambda$$

$$\cot \alpha_{2-1} = (\cos\lambda \sin\beta_2 - \tan\beta_1 \cos\beta_2) \div \sin\lambda$$

Omit for meridional arcs

If $\left|\cot \alpha\right| > 1$, divide result into 1 to obtain $\tan\alpha$ instead.

		Quadrant of α_{1-2}		Quadrant of α_{2-1}
Sign L	Sign of tan or (cot) α_{1-2}		Sign of tan or (cot) α_{2-1}	
+	+	I	+	III
	-	II	-	IV
-	+	III	+	I
	-	IV	-	II

For meridional arcs, enter the above table with the sign of the numerator of cot α, and reference angle 0°.

11. Extension and Modification of Tabular & Electronic Computer Method for Non-Iterative Solution of Geodetic Inverse for Increased Decimal Accuracy in Short and Long Lines

Additional accuracy in the distance S may be obtained by adding the f^3 terms to the $\frac{S}{F}$ series given above.
The series then becomes:

$$
\begin{aligned}
\frac{S}{b_o} = {} & (1+f+f^2+f^3)\,\phi \\
& + a\left[(f+f^2+f^3)\sin\phi + \left(-\tfrac{1}{2}f^2-f^3\right)\phi^2\csc\phi + \tfrac{1}{2}f^3\phi^3\csc\phi\cot\phi\right] \\
& + m\Big[\left(-\tfrac{1}{2}f-\tfrac{1}{2}f^2-\tfrac{1}{2}f^3\right)\phi + \left(-\tfrac{1}{2}f-\tfrac{1}{2}f^2-\tfrac{1}{2}f^3\right)\sin\phi\cos\phi \\
& \qquad + \left(\tfrac{1}{2}f^2+f^3\right)\phi^2\cot\phi - \frac{1}{6}f^3\,\phi^3 - \tfrac{1}{2}f^3\phi^3\cot^2\phi\Big] \\
& + a^2\left[\left(-\tfrac{1}{2}f^2-f^3\right)\sin\phi\cos\phi + \tfrac{1}{2}f^3\phi^3\csc^2\phi + \tfrac{1}{2}f^3\phi\right] \\
& + m^2\Big[\left(+\frac{1}{16}f^2+\frac{1}{8}f^3\right)\phi + \left(\frac{1}{16}f^2+\frac{1}{8}f^3\right)\sin\phi\cos\phi \\
& \qquad + \left(-\tfrac{1}{2}f^2-\frac{7}{4}f^3\right)\phi^2\cot\phi + \left(-\frac{1}{8}f^2-\tfrac{1}{4}f^3\right)\sin\phi\cos^3\phi \\
& \qquad + \tfrac{1}{4}f^3\phi\cos^2\phi + \frac{1}{3}f^3\phi^3 + \frac{3}{2}f^3\phi^3\cot^2\phi\Big] \\
& + am\Big[\left(\tfrac{1}{2}f^2+\frac{7}{4}f^3\right)\phi^2\csc\phi + \left(\tfrac{1}{2}f^2+f^3\right)\sin\phi\cos^2\phi \\
& \qquad - \frac{3}{4}f^3\phi\cos\phi - 2f^3\phi^3\csc\phi\cot\phi\Big] \\
& + a^2m\left[-\tfrac{1}{2}f^3\phi - \tfrac{1}{2}f^3\sin\phi\cos\phi - \tfrac{1}{2}f^3\phi^3\csc^2\phi + f^3\sin^3\phi\cos\phi\right] \\
& + am^2\Big[-\frac{3}{4}f^3\phi^2\csc\phi + \tfrac{1}{2}f^3\sin\phi\cos^2\phi + \frac{3}{4}f^3\phi\cos\phi \\
& \qquad + \frac{3}{2}f^3\phi^3\csc\phi\cot\phi - \tfrac{1}{2}f^3\sin\phi + \tfrac{1}{2}f^3\sin^5\phi\Big] \\
& + m^3\Big[-\frac{1}{32}f^3\phi + \frac{3}{4}f^3\phi^2\cot\phi - \frac{1}{32}f^3\sin\phi\cos\phi \\
& \qquad + \frac{1}{16}f^3\sin\phi\cos^3\phi - \tfrac{1}{4}f^3\phi\cos^2\phi - \frac{1}{6}f^3\phi^3 - f^3\phi^3\cot^2\phi \\
& \qquad + \frac{1}{12}f^3\sin^3\phi\cos^3\phi\Big] \\
& + a^3\left[\tfrac{1}{2}f^3\sin\phi - \frac{2}{3}f^3\sin^3\phi\right]
\end{aligned}
$$

The maximum values for the f^3 term of $\frac{S}{b_o}$ have been found for various lengths of arc ϕ and are recorded in Appendix I, Part A.

Similarly, the accuracy of the $\frac{(\lambda - L)}{\rho c}$ series may be extended by adding the f^3 term. The series then becomes:

$$\begin{aligned}\frac{\lambda - L}{\rho c} = & \left[(f+f^2+f^3)\,\phi\right] + a\left[(-\tfrac{1}{2}f^2-f^3)\sin\phi + (-f^2-4f^3)\phi^2\csc\phi\right. \\ & \left. + \tfrac{3}{2}f^3\phi^3\csc\phi\cot\phi\right] + m\left[(-\tfrac{5}{4}f^2-3f^3)\phi + (\tfrac{1}{4}f^2+\tfrac{1}{2}f^3)\sin\phi\cos\phi\right. \\ & \left. + (f^2+4f^3)\phi^2\cot\phi - \tfrac{1}{2}f^3\phi^3 - \tfrac{3}{2}f^3\phi^3\cot^2\phi\right] \\ & + m^2\left[\tfrac{31}{16}f^3\phi - \tfrac{7}{16}f^3\sin\phi\cos\phi + \tfrac{1}{2}f^3\phi^3 - \tfrac{1}{8}f^3\sin^3\phi\cos\phi\right. \\ & \left. - \tfrac{9}{2}f^3\phi^2\cot\phi + \tfrac{1}{2}f^3\phi\cos^2\phi + \tfrac{5}{2}f^3\phi^3\cot^2\phi\right] \\ & + am\left[\tfrac{9}{2}f^3\phi^2\csc\phi - \tfrac{3}{2}f^3\phi\cos\phi - \tfrac{7}{2}f^3\phi^3\csc\phi\cot\phi - \frac{f^3}{2}\sin\phi\cos^2\phi\right. \\ & \left. + f^3\sin\phi\right] + a^2\left[f^3\phi + \tfrac{1}{2}f^3\sin\phi\cos\phi + f^3\phi^3\csc^2\phi\right]\end{aligned}$$

The f^3 term of the above series has been maximized and this value is shown in Appendix I, Part A. The error in the azimuth α which would result from the omission of the $f3$ term of $\frac{(\lambda - L)}{\rho c}$ has been recorded in Appendix I, Part A.

In the case of short geodetic lines (lines shorter than 180 miles) when the values of ψ, λ-L, etc. of the series above become small, it is necessary to use floating point notation in order to insure greater decimal accuracy.

The alternative formulas for $\sin\psi$, $\cot\alpha_{1-2}$ and $\cot\alpha_{2-1}$, which are given below are recommended for short lines. (They may also be used for long lines).

$$\sin\psi = \pm\sqrt{(\sin L\cos\beta_2)^2 + \left[\sin(\beta_2-\beta_1) + 2\cos\beta_2\sin\beta_1\sin^2\frac{L}{2}\right]^2}$$

$$\cot\alpha_{1-2} = \frac{\left[\sin(\beta_2-\beta_1) + \cos\beta_2\sin\beta_1(1-\cos\lambda)\right]}{\cos\beta_2\sin\lambda}$$

$$\cot\alpha_{2-1} = \frac{\left[\sin(\beta_2-\beta_1) - \cos\beta_1\sin\beta_2(1-\cos\lambda)\right]}{\cos\beta_1\sin\lambda}$$

where $$\beta_2-\beta_1 = B_2-B_1 + \left\{n(\sin 2B_1-\sin 2B_2) - \frac{n^2}{2}(\sin 4B_1-\sin 4B_2) + \frac{n^3}{3}(\sin 6B_1-\sin 6B_2)\right\}$$

and $$n = \frac{1-\sqrt{1-e^2}}{1+\sqrt{1-e^2}}$$

(B_1 and B_2 as previously defined are the geographic latitudes of points 1 and 2, respectively). See the quadrant criterion on page 28.

SECTION V.

TABULAR AND ELECTRONIC COMPUTER METHOD FOR SOLUTION OF DIRECT GEODETIC PROBLEM, BASED ON [illegible] TABLE

The formulas given below for the solution of the direct geodetic problem are intended primarily for electronic computer programming. However, they may also be used for computation by means of desk calculators.

The computation form for this method is as follows:

Given: B_1, L_1 = geographic latitude and longitude, respectively of any point 1.

α_{1-2}, S = azimuth clockwise from north, and distance, respectively, to any other point 2.

Required: B_2, L_2 and α_{2-1}. (Latitudes and longitudes considered (+) north and east, (-) south and west).

$$\begin{aligned}
\phi_0 = &\left[\phi_s\right] \\
&+ a_1\left[-\frac{e'^2}{2}\sin\phi_s\right] \\
&+ m_1\left[-\frac{e'^2}{4}\phi_s + \frac{e'^2}{4}\sin\phi_s\cos\phi_s\right] \\
&+ a_1^2\left[\frac{5e'^4}{8}\sin\phi_s\cos\phi_s\right] \\
&+ m_1^2\left[\frac{11e'^4}{64}\phi_s - \frac{13e'^4}{64}\sin\phi_s\cos\phi_s - \frac{e'^4}{8}\phi_s\cos^2\phi_s \right. \\
&\qquad \left. + \frac{5e'^4}{32}\sin\phi_s\cos^3\phi_s\right] \\
&+ a_1 m_1\left[\frac{3e'^4}{8}\sin\phi_s + \frac{e'^4}{4}\phi_s\cos\phi_s - \frac{5e'^4}{8}\sin\phi_s\cos^2\phi_s\right]
\end{aligned}$$

$$\begin{aligned}
\frac{L-\lambda}{\rho\cos\beta_0} = &\left[-f\phi_s\right] + a_1\left[\frac{3f^2}{2}\sin\phi_s\right] \\
&+ m_1\left[\frac{3f^2}{4}\phi_s - \frac{3f^2}{4}\sin\phi_s\cos\phi_s\right]
\end{aligned}$$

where:

a_o, b_o = semi-major and semi-minor axes, respectively, of spheroid

f = spheroidal flattening = $\left(1 - \frac{b_o}{a_o}\right)$

e'^2 = second eccentricity squared = $(a_o^2 - b_o^2) \div b_o^2$

ρ = number of seconds in one radians = 206,264.80625

ϕ_s = $(S \div b_o)$ radians

$\tan \beta = (\tan B)\,(1-f)$ when $|B| \leq 45^\circ$ or $\cot \beta = \frac{(\cot B)}{(1-f)}$ when $|B| > 45^\circ$.

$\cos\beta_o = \cos\beta_1 \sin\alpha_{1-2}$

$g = \cos\beta_1 \cos\alpha_{1-2}$

$a_1 = (1 + \frac{e'^2}{2} \sin^2\beta_1)\,(\sin^2\beta_1 \cos\phi_s + g \sin\beta_1 \sin\phi_s)$

$m_1 = (1 + \frac{e'^2}{2} \sin^2\beta_1)\,(1 - \cos^2\beta_o)$

$$\sin\beta_2 = \sin\beta_1 \cos\phi_o + g \sin\phi_o$$

$$\cos\beta_2 = + \sqrt{(\cos\beta_o)^2 + (\sin\beta_1 \sin\phi_o - g \cos\phi_o)^2}$$

[The quantity under the radical and its root must be computed by floating point notation to obtain $\cos\beta_2$ to full accuracy at large absolute latitude.]

$\tan\beta_2 = \frac{\sin\beta_2}{\cos\beta_2}$ or $\cot\beta_2 = \frac{\cos\beta_2}{\sin\beta_2}$, whichever has the smaller absolute value.

Obtain tan (or cot) of B_2 from its relation to tan (or cot) of β_2.

Obtain B_2, which ranges from -90° through $+90^\circ$ and takes the sign of its tan (or cot).

$$\cot\alpha_{2-1} = (\cos\alpha_{1-2} \cos\phi_o - \tan\beta_1 \sin\phi_o) \div \sin\alpha_{1-2}$$

When $|\cot\alpha_{2-1}| > 1$, divide result into 1 to obtain $\tan\alpha_{2-1}$ instead.

(Omit these last two lines for meridional arcs)

	Quadrant of α_{2-1}
If $(0° \leq \alpha_{1-2} \leq 180°)$	.... and cot (or tan) of α_{2-1} is (+) or (-), α_{2-1} is in Quad. III or IV, respectively.
If $(180° < \alpha_{1-2} < 360°)$	.... and cot (or tan) of α_{2-1} is (+) or (-), α_{2-1} is in Quad. I or II, respectively.

[For meridional arcs, enter the above table with the sign of the numerator of $\cot\alpha_{2-1}$, and reference angle 0°.]

$$\cot\lambda = (\cot\phi_0 \cos\beta_1 - \cos\alpha_{1-2} \sin\beta_1) \div \sin\alpha_{1-2}$$

When $|\cot\lambda| > 1$, divide result into 1 to obtain $\tan\lambda$ instead.

(Omit these last two lines for meridional arcs)

	Quadrant and Sign of λ	
	When ϕ_0 is in Quad. I or II (180° included)	When ϕ_0 is in Quad. III or IV (180° excluded)
and $(0° \leq \alpha_{1-2} \leq 180°)$	... then if cot (or tan) of λ is (+) or (-) λ is in Quad. I or II, respectively.	... then if cot (or tan) of λ is (+) or (-) λ is in Quad. III or IV, respectively.
and $(180° < \alpha_{1-2} < 360°)$	... then if cot (or tan) of λ is (+) or (-) the assoc. angle is in Quad. III or IV, respectively, and λ is obtained by subtracting 360°	... then if cot (or tan) of λ is (+) or (-) the assoc. angle is in Quad. I or II, respectively, and λ is obtained by subtracting 360°

[For meridional arcs, enter the above table with the sign of the numerator of $\cot\lambda$, and reference angle 0°.]

$$L_2 = L_1 + L$$

[If $|L_2| > 180^\circ$, modify L_2 by adding or subtracting 360° according to whether it is initially negative or positive.]

12. Extension of Series of the Direct Geodetic Problem For Greater Accuracy

The e'^6 term of the preceding $\bar{\phi}_o$ series has been derived and numerically maximized in order to determine the error in the $\bar{\phi}_o$ series which would result from an omission of the e'^6 term. This maximum value is given in Appendix I, Part A.

Likewise, the $\left(\dfrac{L-\lambda}{\rho\cos\phi_o}\right)$ series has been extended to include the f^3 term. A maximum numerical value is given in Appendix I, [illegible].

The errors which the omission of the e'^6 term of $\bar{\phi}_o$ and the f^3 term of $\left(\dfrac{L-\lambda}{\rho\cos\phi_o}\right)$ could finally produce in B_2, L_2 and α_{2-1} are also shown in Appendix I, Part A.

The e'^6 term of the $\bar{\phi}_o$ series is as follows:

$$\left[e'^6 \text{ term of } \bar{\phi}_o\right] = a_1^3\left[-\frac{29}{24}e'^6 \sin\bar{\phi}_s\cos^2\bar{\phi}_s + \frac{5}{24}e'^6 \sin\bar{\phi}_s\right]$$

$$+\ a_1^2 m_1\left[\frac{5}{32}e'^6\bar{\phi}_s - \frac{5}{8}e'^6\bar{\phi}_s\cos^2\bar{\phi}_s - \frac{43}{32}e'^6\sin\bar{\phi}_s\cos\bar{\phi}_s + \frac{29}{16}e'^6\sin\bar{\phi}_s\cos^3\bar{\phi}_s\right]$$

$$
\begin{aligned}
&+ a_1 m_1^2 \Big[-\frac{39}{64} e'^6 \phi_s \cos\phi_s + \frac{5}{8} e'^6 \phi_s \cos^3\phi_s + \frac{e'^6}{32} \phi_s^2 \sin\phi_s \\
&\qquad - \frac{e'^6}{32} \phi_s^2 \cos^2\phi_s \csc\phi_s + \frac{79}{64} e'^6 \sin\phi_s \cos^2\phi_s \\
&\qquad - \frac{29}{32} e'^6 \sin\phi_s \cos^4\phi_s - \frac{5}{16} e'^6 \sin\phi_s \Big] \\
&+ m_1^3 \Big[-\frac{35}{256} e'^6 \phi_s + \frac{35}{128} e'^6 \phi_s \cos^2\phi_s - \frac{5}{32} e'^6 \phi_s \cos^4\phi_s \\
&\qquad - \frac{1}{32} e'^6 \phi_s^2 \sin\phi_s \cos\phi_s + \frac{45}{256} e'^6 \sin\phi_s \cos\phi_s \\
&\qquad - \frac{59}{192} e'^6 \sin\phi_s \cos^3\phi_s + \frac{29}{192} e'^6 \sin\phi_s \cos^5\phi_s \Big] \\
&+ a_1^2 t_1 \Big[\frac{3}{16} e'^6 \sin\phi_s \cos\phi_s \Big] \\
&+ a_1 t_1^2 \Big[-\frac{e'^6}{8} \sin\phi_s \Big] \\
&+ a_1 m_1 t_1 \Big[\frac{3}{16} e'^6 \phi_s \cos\phi_s - \frac{3}{16} e'^6 \sin\phi_s \cos^2\phi_s \Big] \\
&+ m_1 t_1^2 \Big[-\frac{e'^6}{16} \phi_s + \frac{e'^6}{16} \sin\phi_s \cos\phi_s \Big] \\
&+ m_1^2 t_1 \Big[\frac{e'^6}{128} \phi_s - \frac{3}{32} e'^6 \phi_s \cos^2\phi_s + \frac{e'^6}{32} \phi_s^2 \cos\phi_s \csc\phi_s \\
&\qquad + \frac{1}{128} e'^6 \sin\phi_s \cos\phi_s + \frac{3}{64} e'^6 \sin\phi_s \cos^3\phi_s \Big]
\end{aligned}
$$

In the above equation $t_1 = \sin^2\beta_1$ and all other quantities are the same as defined on page 33.

As $\bar{\phi}_s$ approaches 180°, each of the two terms containing $\csc\bar{\phi}_s$ in the series above approaches infinity. However, they may be combined into a single finite term as follows:

$$a_1 m_1^2 \left(-\frac{e'^6}{32} \bar{\phi}_s^{\,2} \cos^2\bar{\phi}_s \csc\bar{\phi}_s\right) + m_1^2 t_1 \left(\frac{e'^6}{32} \bar{\phi}_s^{\,2} \cos\bar{\phi}_s \csc\bar{\phi}_s\right) =$$

$$m_1^2 \gamma \left(\frac{e'^6}{32} \bar{\phi}_s^{\,2} \cos\bar{\phi}_s\right), \text{ where } \gamma = \left(\sin^2\beta_1 \sin\bar{\phi}_s - g \sin\beta_1 \cos\bar{\phi}_s\right).$$

The series for $\left(\frac{L-\lambda}{\rho\cos\beta_0}\right)$ extended through the f^3 term is as follows:

$$\begin{aligned}
\frac{(L-\lambda)}{\rho\cos\beta_0} = \left[-f\bar{\phi}_s\right] &+ a_1\left[\left(\frac{3}{2}f^2 + 2f^3\right)\sin\bar{\phi}_s\right] \\
&+ m_1\left[\left(\frac{3}{4}f^2 + f^3\right)\bar{\phi}_s + \left(-\frac{3}{4}f^2 - f^3\right)\sin\bar{\phi}_s\cos\bar{\phi}_s\right] \\
&+ a_1^2\left[-4f^3\sin\bar{\phi}_s\cos\bar{\phi}_s\right] \\
&+ a_1 m_1\Big[-\frac{5}{2}f^3\sin\bar{\phi}_s - \frac{3}{2}f^3\bar{\phi}_s\cos\bar{\phi}_s \\
&\qquad + 4f^3\sin\bar{\phi}_s\cos^2\bar{\phi}_s\Big] \\
&+ m_1^2\Big[-\frac{9}{8}f^3\bar{\phi}_s + \frac{3}{4}f^3\bar{\phi}_s\cos^2\bar{\phi}_s \\
&\qquad - f^3\sin\bar{\phi}_s\cos^3\bar{\phi}_s + \frac{11}{8}f^3\sin\bar{\phi}_s\cos\bar{\phi}_s\Big]
\end{aligned}$$

APPENDIX I.

PART A

SUMMARY TABLE OF ERRORS IN DISTANCE & AZIMUTH IN TABULAR AND ELECTRONIC COMPUTER METHOD FOR NON-ITERATIVE SOLUTION OF GEODETIC INVERSE

ϕ	Error in S (meters)	Error in $\frac{\lambda - L}{\rho_c}$	Error in α_{1-2}	Error in α_{2-1}
350°	.9	″.003	″.006	″.003
170°	.5	″.002	″.001	″.001
80°	.2	″.001	″.001	″.0002
0°12'12"	.001	″.0003	″.005	″.005 *

The errors above result from the omission of the f^3 terms from the $\frac{S}{b_o}$ and $\frac{\lambda - L}{\rho_c}$ series.

Computations were done for high latitudes and are based on the International Spheroid.

* The magnitude of the error in α for lines of approximately 15 miles clearly indicates the necessity of the f^3 term in the $\frac{\lambda - L}{\rho_c}$ series for accuracy of three decimal places of seconds of arc.

PART B

SUMMARY TABLE OF ERRORS IN POSITION & AZIMUTH IN TABLUAR AND ELECTRONIC
COMPUTER [illegible] FOR NON-ITERATIVE SOLUTION OF DIRECT GEODETIC PROBLEM

ϕ_s	α_{1-2}	Error in ϕ_c series	Error in $\frac{1-\lambda}{\rho\cos\beta_0}$ series	Error in β_2	Error In L_2	Error in α_{2-1}
350°	5°	″.2	″.04	″.2	″.02	″.02
350°	45°	″.2	″.05	″.2	″.2	″.2
350°	90°	″.2	″.04	″.1	″.5	″.5
170°	5°	″.1	″.02	″.1	″.2	″.01
170°	90°	″.1	″.02	″.1	″.2	″.3
80°	5°	″.02	″.002	″.02	″.0001	″.0004
80°	90°	″.02	″.002	″.02	″.001	″.004

The errors above result from the omission of the e^{16} term from the ϕ_c series and the f^3 term of $\frac{1-\lambda}{\rho\cos\beta}$ series. Computations were done for high latitudes and are based on the International Spheroid.

APPENDIX II. BIBLIOGRAPHY

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