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ANALYTICAL STUDIES OF THE EFFECTS OF IONIZATION ON FLUID FLOWS

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Department of Mechanical Engineering

The Ohio State University
Research Foundation
Columbus, Ohio

Technical Report No. 6

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FOREWORD

This report represents one phase of a general study of the interaction of ions with fluid flow under Contract DA-31-124-ARO-D-246, U. S. Army Research Office-Durham, with Dr. Henry R. Velkoff serving as principal investigator.

The authors wish to thank E. Pejack for his assistance in various phases of the work reported herein.

ABSTRACT

Unusually high pressure drops and flow distortions were observed in a previous experimental program involving laminar flow of a gas in a channel under the action of a corona discharge in a transverse electric field. A hypothesis postulated by Velkoff to explain the phenomenon is extended to the case of laminar boundary layer flow over a flat plate. The problem on hand is found to be analogous to the laminar boundary layer flow in a transverse magnetic field. Three other mechanisms proposed to interpret the above experimental findings are also investigated. The increase in viscosity of a gas because of the ions is not likely and, because of the smallness of ion density, the effect of ion-neutral particle interactions on the flow is believed to be small. One possible mechanism which may explain the phenomenon is the secondary flow resulting from electro-hydrodynamic instability. It is found theoretically that Taylor vortices can be induced in a quiescent fluid between two concentric cylinders under the action of a corona discharge. The Taylor Number of the problem is defined and shown to represent the ratio of the destabilizing electrostatic force to the stabilizing viscous force. It is also found that Goertler vortices can occur in laminar boundary layer over a flat plate provided the applied electric field and the charge density distribution satisfy the condition for instability.

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LIST OF SYMBOLS

(Numbers indicate sections in which the symbol is used)

Symbol	
A	area 1, 4
A	a constant determined by the law of interaction between molecules, 2
a	parameter, $\frac{\ell\lambda}{r_1}$ 4
b	parameter kh, 4
c	parameter, $\alpha\,\delta$, 4
C_f	local coefficient of skin friction, 1
D_i	diffusion coefficient of ion, 4
d	mean distance between neighboring pairs of molecules, 2
$d^{(i)}$	diameter of ions, 3
d^N	diameter of neutral particles, 2, 3
d_{iN}	$\frac{1}{2}(d^{(i)} + d^N)$, 3
E	constant determined by the law of interaction between molecules, 2
E	magnitude of electric field intensity, 3, 4
$\vec{E}$	electric field intensity vector, 4
E_y, E_z, E_r	component of electric field intensity in y; z and radial direction respectively, 1, 4
E_i	i^{th} component of electric field intensity, 3
e	electronic charge, 2
$\mathcal{E}$	constant defined as $E = \mathcal{E}f(\xi)$ or $E = \mathcal{E}f(\eta)$, 4
F_e	electrostatic force, 1
f_0, f_1, f_2	functions of η, 1

Symbol	
f	function of ξ or η defined as $E = \mathcal{E}f$, 4
g	function of ξ or η defined as $\frac{d\bar{\rho}_c}{d\theta} = \mathcal{L}g(\xi)$ or $\frac{d\bar{\rho}_c}{dy} = \mathcal{L}g(\eta)$, 4
h	distance between two parallel plate electrodes, 1, 4
h	channel height, 3
I	current, 1, 4
$\vec{J}$	current density vector, 4
K	ion mobility, 1, 3, 4
k	Boltzman constant, 2
k	wave number, 4
L	characteristic length, 1
$\mathcal{L}$	constant defined as $\frac{d\bar{\rho}_c}{dy} = \mathcal{L}g(\eta)$, or $\frac{d\bar{\rho}_c}{d\theta} = \mathcal{L}g(\xi)$, 4
ℓ	spacing between two concentric cylinders, 4
M_1	reduced mass of neutral molecules, $\frac{m_1}{m_1 + m_2}$, 2
M_2	reduced mass of ions, $\frac{m_2}{m_1 + m_2}$, 2
m	parameter, $\frac{\rho_c}{\rho \underline{K} W_\infty}$, 1
M^*	reduced mass, $\frac{m^{(i)} m^N}{m^{(i)} + m^N}$, 3
$m^{(i)}$	mass of ions, 2, 3
m^N	mass of neutral molecules, 2, 3
n^N	number density of neutral molecules, 2, 3
$n^{(i)}$	number density of ions, 2, 3

LIST OF SYMBOLS (Continued)

Symbol	
n_{12}	$n^N/n^{(i)}$, 2
n_{21}	$n^{(i)}/n^N$, 2
$N\rho_c$	charge number $\frac{\rho_c L^2}{\mu K}$, 1
$N_{\rho_c z}$	$\frac{\rho_c z^2}{\mu K}$, 1
p	pressure, 3, 4
$\bar{p}$	mean pressure, 4
p'	pressure perturbation, 4
p_1	amplitude of p', 4
$\hat{r}$	direction along radius, 4
r^*	dimensionless distance, $\frac{r}{r_1}$, 4
r_1, r_2	radii of inner and outer cylinders,respectively, 4
r_0	mean radius, $\frac{1}{2}(r_1 + r_2)$, 4
T	absolute temperature, 2
T	Taylor number, 4
T_c	critical Taylor number, 4
t	time, 4
t^*	dimensionless time $\frac{t\nu}{h^2}$, 4
U_1	varible defined as $\frac{\ell^2 r_1 K}{\epsilon D_i} u_1$, 4
u	velocity component in x or radial direction, 4
u'	radial component of velocity perturbation, 4
u_1	amplitude of u', 4
u^*	dimensionless radial component of velocity perturbation, $\frac{u' r_1}{\nu}$, 4

LIST OF SYMBOLS (Continued)

Symbol	
V	dimensionless velocity, $\sqrt{Re}\,\frac{v}{w_\infty}$, 1
$\vec{v}$	velocity vector, 4
v	velocity component along y-axis, 1, 4
v'	component of velocity perturbation in y- or θ-direction, 4
v_1	amplitude of v'
v_r	magnitude of relative velocity between colliding particles, 3
$\overline{W}$	dimensionless velocity defined as $\frac{w}{w_\infty}$, 1
$\overline{W}$	dimensionless velocity defined as $\frac{\overline{w}}{w_o}$, 4.5
W_1	variable defined as $\frac{\ell h^2}{D_i}\,w_1$, 4.4
W_1	variable defined as $\left(\frac{w_o \delta}{\nu}\right)^{-1} w_1$, 4.5
w	velocity component along z-axis, 1, 3, 4
w'	component of velocity perturbation along z-axis, 4
w_1	amplitude w', 4
$\overline{w}$	mean velocity component along z-axis, 3, 4
w_∞	upstream velocity, 2
w_o	free stream velocity, 1, 4
x	coordinate direction along flow, 3
Y	dimensionless distance along y-axis, $\frac{y}{L}$, 1
y	coordinate direction normal to flow, 1, 3, 4
Z	dimensionless distance, $\frac{z}{L}$, 1
z	coordinate direction along axis of channel or basic flow, 1, 4

LIST OF SYMBOLS (Continued)

Symbol	
z^*	dimensionless distance, $\frac{z}{r_1}$ or $\frac{z}{h}$, 4
α	$\rho_c/\rho K$, 1
α	wave number appropriate to the direction x, 4
β	amplifiication factor, 4
γ	constant which can be complex, 4
δ	boundary layer thickness, 1, 4
δ_{ij}	Kronecker delta, 4
ϵ	permittivity, 1, 4
ϵ_{ijk}	alternating unit vector, 3, 4
ζ	z-component of vorticity vector, 4
η	dimensionless coordinate, $y \frac{w_\infty}{\nu z}$, 1
η	dimensionless coordinate, y/δ, 4
θ	electrical conductivity, 4
λ	dimensionless wave number, 4
$\vec{\lambda}$	unit vector in the z-direction, 4
μ	viscosity, 1, 3, 4
$[\mu_1]$	first approximation to the viscosity of mixture, 2
$[\mu_1]_1$	first approximation to the viscosity of neutral oxygen, 2
$[\mu_2]_1$	first approximation to the viscosity of ion gas, 2
ν	kinematic viscosity, 1, 4
ν_{iN}	collision frequency for ion-neutral collision, 3
ξ	dimensionless coordinate, $\frac{\hat{r} - r_0}{\ell}$, 4

LIST OF SYMBOLS (Continued)

Symbol	
ρ	density of fluid, 1, 3, 4
ρ_c	charge density, 1, 4
$\bar{\rho_c}$	mean charge density, 4
ρ_c'	charge density perturbation, 4
ρ_{c1}	amplitude of ρ_c' or ρ_c*, 4
ρ_c*	non-dimensional charge density, 4
σ	dimensionless amplification factor, 4.3
σ	parameter $\delta h^2/\nu$, 4.4
σ_D	transport cross section, 3
σ_1	parameter, $\beta\ \delta^2/\nu$, 4
σ_2	parameter, $\beta\ \delta^2/D_i$, 4
τ_s	shear stress at the surface, 1
ψ	stream function, 1
$\vec{\omega}$	vorticity vector, 4

INTRODUCTION AND SUMMARY

In a previous exploratory investigation into the effects of ionization on the gas flow in a channel, interesting results were observed. Complete details of that investigation may be found by consulting reference 1. Here we shall only briefly describe these observations.

Air was passed through a $1\frac{1}{4}$-inch-diameter pipe in which a 0.004-inch-diameter wire was located concentrically. A high voltage applied to the wire gave rise to corona discharge which provided ions in the air stream. Under the action of the field, pressure drops were observed to be doubled, velocity profile distorted, and heat transfer doubled. Similar phenomenon was also observed in a 5/8-inch x 5-inch x 12-foot-long rectangular channel in which ten parallel thin wires were located longitudinally on the center plane of the channel.

In an attempt to explain the nature of the phenomenon, a hypothesis was put forth by Velkoff in Reference 2. According to this hypothesis, an electric field component in the direction opposite to the flow is induced, giving rise to a retarding electrostatic body force. The induced field was found to be

$$E_{ind} = - \frac{u}{K}$$

where u is the flow velocity and K is the ion mobility in the gas under consideration. Application of this hypothesis to channel flows was carried out in great detail in Reference 2. The close agreement of theoretical values predicted by the hypothesis with test data aroused our interest in extending the investigation to external flows.

The first part of this report describes an analysis extending the above hypothesis to the case of laminar boundary layer over a flat plate. An interesting analogy to magnetohydrodynamics was found for the case where the charge density is constant.

In the remaining parts of this report, efforts are directed toward the investigation of a few mechanisms proposed to account for the phenomenon. The second part presents a brief study of the viscosity of a gas mixtrue consisting of ions and neutral particles. Because of insufficient knowledge of an exact expression for the viscosity of ion gas at room temperature, the discussion was based on the data obtained for the constituent gases at high temperature.

In Section III, the flow is considered as a two-fluid model. The interaction between ions and neutral particles is taken into account. For the case where the longitudinal velocity component of ions is neglected, the analysis resulted in an equation of motion similar to the one obtained by Velkoff in Reference 2. For the case with non-vanishing longitudinal velocity component of ions, the solution is rather complicated and the effect of the ions on the flow remains to be determined.

The last part of this report is devoted to the study of electrohydrodynamic instability. It was found that under certain assumptions the quiescent fluid between two concentric cylindrical electrodes under the action of a transverse filed is governed by the same eigenvalue equation as in the case of couette flow between two rotating cylinders. A similar relation exists between the case of fluid at rest between two infinite parallel plates under the action of a transverse electric field and that of classical thermal convection. It has also been shown that Goertler vortices can occur in the boundary layer over a flat plate under the action of a transverse electric field.

I. LAMINAR BOUNDARY LAYER OVER A FLAT PLATE IN A TRANSVERSE ELECTRIC FIELD

1.1 Governing Equations

In this section we study the effect of a transverse electric field on laminar boundary layer over a flat plate. The geometrical configuration of the problem is shown in Fig. 1. Let the yz-plane be taken as the plane of the boundary layer flow with z axis along the plate, and the y axis perpendicular to the plane wall. If ρ denotes the density and μ the viscosity of the fluid, the boundary layer equation for incompressible flow in the absence of an external electric field is[3]

$$v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z} = \frac{\mu}{\rho} \frac{\partial^2 w}{\partial y^2} \tag{1.1}$$

Now suppose the fluid is positively charged through the action of corona discharge or the injection of positive ions from external sources. The charge density in fluid under the action of a transverse electric field (E_y), is ρ_c (charge per unit volume). By the hypothesis postulated by Velkoff, an electric field (E_z) opposing the flow is induced and is given by

$$E_z = - \frac{w}{K} \tag{1.2}$$

Thus the contribution of the transverse electric field to fluid flow is in the form of a retarding electrostatic force. This force is found to be

$$F_e = \rho_c E_z = - \rho_c \frac{w}{K} \tag{1.3}$$

Incorporating this force in the boundary layer equation (1.1), we obtain a modified boundary layer equation for the case under consideration

$$v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z} = \frac{\mu \partial^2 w}{\rho \partial y^2} - \frac{\rho_c w}{\rho k} \tag{1.4}$$

This equation together with the continuity equation

$$\frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

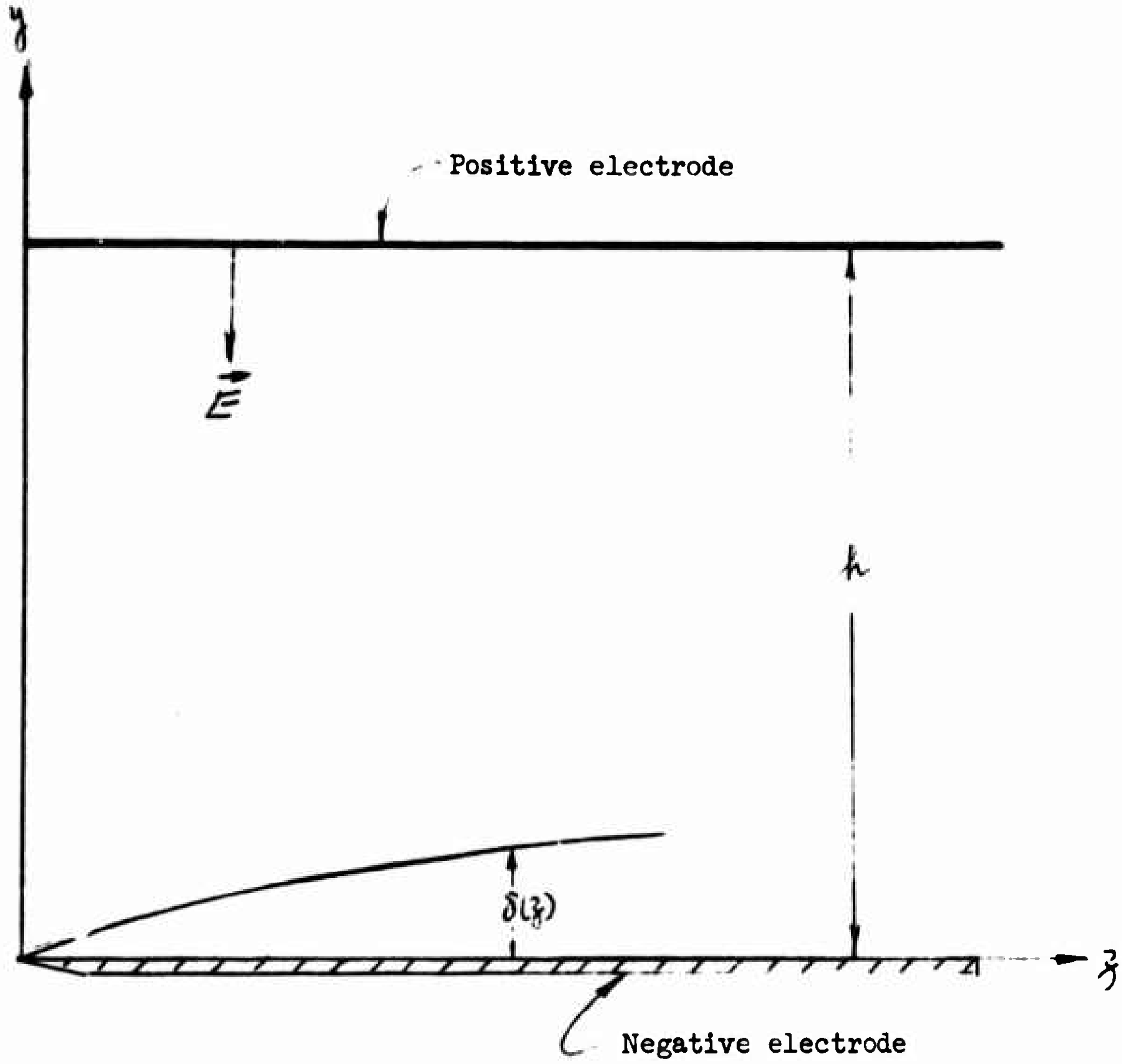

Fig. 1 - Sketch of boundary layer on a flat plate at zero incidence under the action of a transverse electric field

describe the boundary layer flow over a flat plate under the action of a transverse electric field.

1.2 Constant Charge Density

The charge density, ρ_c, is related to the distribution of electric field through the electrostatic equation

$$\nabla \cdot \vec{E} = \frac{\rho_c}{\epsilon}$$

and is, in general, a function of position. This renders Eq. (1.4) difficult to solve. To obtain an idea of what effects the additional term, $-\rho_c/\rho \; w/k$, will have on the boundary layer, let us simplify the problem by assuming that the charge density is constant.* Under this assumption, Eq. (1.4) can be rewritten as

$$v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z} + \alpha w = \nu \frac{\partial^2 w}{\partial y^2} \tag{1.5}$$

where $\alpha = \rho_c/\rho K$ is a constant and $\nu = \mu/\rho$ is the kinematic viscosity of the fluid.

1.2.1 Analogy to MHD

The form of Eq. (1.5) is similar to that obtained for the flow of an electrically conducting fluid over a flat plate in the presence of a transverse magnetic field fixed relative to the plate. The equation, as was derived by Rossow[4] is

$$v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z} + \frac{\theta \mu'^2}{\rho} H_o^2 \, w = \nu \frac{\partial^2 w}{\partial y^2} \tag{1.6}$$

where

θ = electrical conductivity of the fluid

μ' = permeability of the fluid

and

H_o = externally applied constant magnetic field.

By comparison, we see that the constant, ρ_c/K, corresponds to $\theta \mu'^2 H_o^2$ in MHD

*Sec.(1.3) shows that this is a good approximation.

1.2.2 Rossow's solution

Equation (1.6) was first solved by Rossow. He developed an approximate solution. Following Rossow, Eq. (1.5) is solved as follows.

Let us introduce the transformation

$$\eta = y\sqrt{\frac{w_\infty}{\nu z}}$$

and define the stream function as

$$\psi = \sqrt{w_\infty \nu z}\,[f_o + \sqrt{mz}\, f_1 + mz\, f_2 + (mz)^{3/2} f_3 + (mz)^2 f_4 + ---]$$

where w_∞ is the upstream velocity,

$m = \frac{\alpha}{w_\infty} = \frac{\rho c}{\rho K w_\infty}$ and f_o, f_1, f_2, f_3, f_4 --- are functions of η only.

From the definition of stream function, i.e., $w = \partial\psi/\partial y$ and $v = \partial\psi/\partial z$, v, w, $\partial w/\partial y$, $\partial w/\partial z$ and $\partial^2 w/\partial y^2$ can all be expressed in terms of f's and η. This enables Eq. (1.5) to be expressed in powers of mz. Equating the coefficients of equal powers of mz on both sides of the equation, a set of ordinary differential equations for the f's are obtained:

(a) $2f_o''' = - f_o'' f_o$

(b) $2f_1''' = f_o' f_1' - f_o f_1'' - 2f_1 f_o''$

(c) $2f_2''' = 2f_o' f_2' + f_1' f_1' - f_o f_2'' - f_1 f_1'' - 3f_2 f_o'' + 2f_o'$

(d) $2f_3''' = 3f_o' f_3' + 3f_2' f_1' - f_o f_3'' - 2f_1 f_2'' - 3f_2 f_1'' - 4f_3 f_o'' + 2f_1'$

(e) $2f_4''' = 4f_o' f_4' + 3f_1' f_3' + 2f_2' f_2' + f_3' f_1' - f_o f_4''$

$$- 2f_1 f_3'' - 3f_2 f_2'' - 4f_3 f_1'' - 5f_4 f_o'' + 2f_2'$$

The boundary conditions $v = w = 0$ at $y = 0$ and $v = \partial w/\partial y = 0$ at $y = \infty$, when written in terms of f's and η, assume the form

$$f_o = f_1 = f_2 = f_3 = f_4 = --- = 0 \text{ at } \eta = 0$$

$$f_o' = f_1' = f_2' = f_3' = f_4' --- = 0 \text{ at } \eta = 0$$

$$f_o' = 1, \quad f_2' = -1 \quad \text{at } \eta = \infty$$

$$f_1' = f_3' = f_4' = f_5' = --- = 0 \text{ at } \eta = \infty$$

From the boundary conditions and Eqs. (b) and (d), f_1 and f_3 can be taken to be zero throughout the flow field. Equations (c) and (e) then become

$$(c')\quad f_2'''' = f_0'f_2' - \tfrac{1}{2} f_0f_2' - \frac{3}{2} f_0f_0'' + f_0',$$

$$(e')\quad f_4'''' = 2f_0'f_4' + f_2'f_2' - \tfrac{1}{2} f_0f_4'' - \frac{3}{2} f_2f_2'' - \frac{5}{2} f_4f_0'' + f_2'.$$

Equation (a) is the Blasius equation and Eq. (c') was solved numerically using Runge-Kutta method. The solutions were tabulated in Reference 4 and are reproduced in Appendix I.

1.2.3 Interpretation of mz

It should be noted that in the above analysis mz plays the role of a controlling parameter in the flow. To investigate its physical significance, let us non-dimensionalize Eq. (1.5) by introducing the following dimensionless variables

$$Z = \frac{z}{L}\ , \quad Y = \sqrt{Re}\,\frac{y}{L}\ ,$$

$$V = Re\,\frac{v}{w_\infty}\ , \quad W = \frac{w}{w_\infty}\ ,$$

where L is a characteristic length and $Re = w_\infty L/\nu$ is the Reynolds number. In terms of these non-dimensional variables Eq. (1.5) becomes

$$V\frac{\partial W}{\partial Y} + W\frac{\partial W}{\partial Z} = \frac{\partial^2 W}{\partial Y^2} - \frac{\rho_c L}{\rho K w_\infty} W\ . \qquad (1.7)$$

Since all other quantities in the above equation are non-dimensional, the quantity $\rho_c L/\rho K W$ must also be non-dimensional and can be considered as the product of two non-dimensional parameters

$$\frac{\rho_c L}{\rho K w_\infty} = \frac{\rho_c L^2}{\mu K}\cdot\frac{\mu}{\rho w_\infty L}\ .$$

We readily recognize that $\rho w_\infty L/\mu$ is the Reynolds number. It is obvious that $\rho_c L^2/\mu K$ is a new non-dimensional parameter and is called the charge number, N_{ρ_c} (Ref. 2). Now,

$$mz = \frac{\alpha z}{w_\infty} = \frac{\rho_c z}{\rho K w_\infty} = \frac{\rho_c z^2}{K\mu}\cdot\frac{\mu}{\rho w_\infty z}\ .$$

If we let $Rez = \rho w_\infty z/\mu$ and $N_{\rho_c z} = \rho_c z^2/\mu K$, then mz is the product of Rez and $N_{\rho_c z}$. Further analysis[2] indicates that the charge number is physically the ratio of the electrostatic force to the viscous force. Meanwhile, the Reynolds number, Re, is known to represent the ratio of the inertia force to the viscous force. Thus, mz measures the ratio of the electrostatic force to the inertia force.

1.2.4 Velocity profile

To a first approximation, the velocity distribution is given by

$$w = w_\infty (f_o' + mzf_2') \tag{1.8}$$

Divided through by w_o, the free stream velocity, i.e., the velocity at the edge of the boundary layer, on both sides of the equation, Eq. (1.8) becomes

$$\frac{w}{w_o} = \frac{w_\infty}{w_o} (f_o' + mzf_2')$$

In Fig. 2, w/w_o is plotted against η for several values of mz. The figure indicates that increase in mz tends to retard the local flow. From the results of the analysis it is predicted that further increase in mz will decrease the velocity gradient at the wall, $(\partial w/\partial \eta)_{\eta=0}$. When $(\partial w/\partial \eta)_{\eta=0}$ reaches zero, corresponding to mz = 0.372, separation will occur.

If the foregoing theory is applicable to this flow, it may be concluded that increase in electric field will finally cause separation.

1.2.5 Variation of boundary layer thickness

If it is desired to define the boundary layer thickness as that distance for which

$$\frac{w}{w_o} = 0.99,$$

then

$$\frac{\delta}{z} = \frac{\eta_\delta}{Rez},$$

where η_δ is the value of η evaluated at $y = \delta$. In Fig. 3, the ratio δ/z is plotted against Rez. For a given value of z, the boundary layer thickness, δ, increases with increase in m. In other words, the electric field tends to thicken the boundary layer. This can also be seen from Fig. 2.

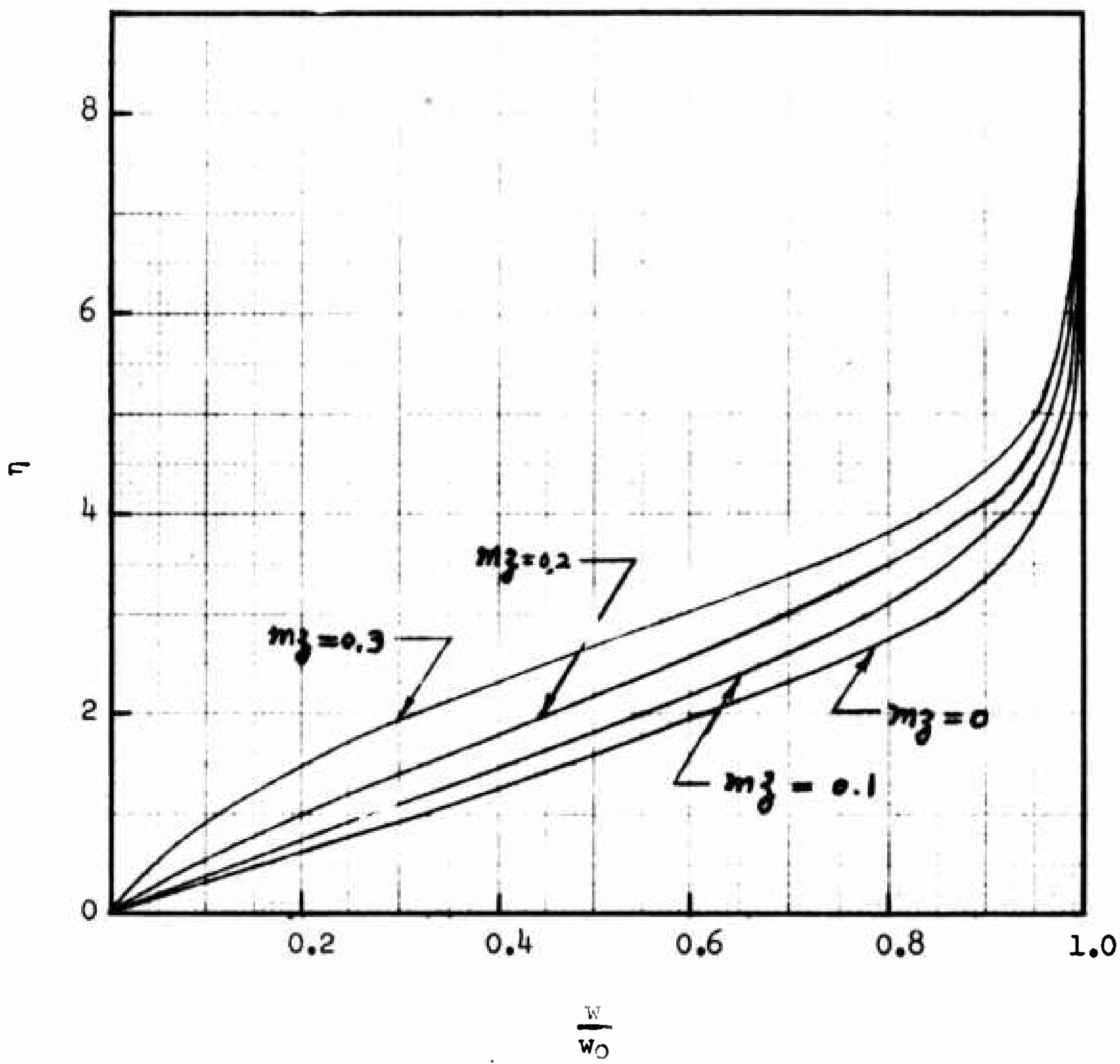

Fig. 2 - Velocity distribution on a flat plate

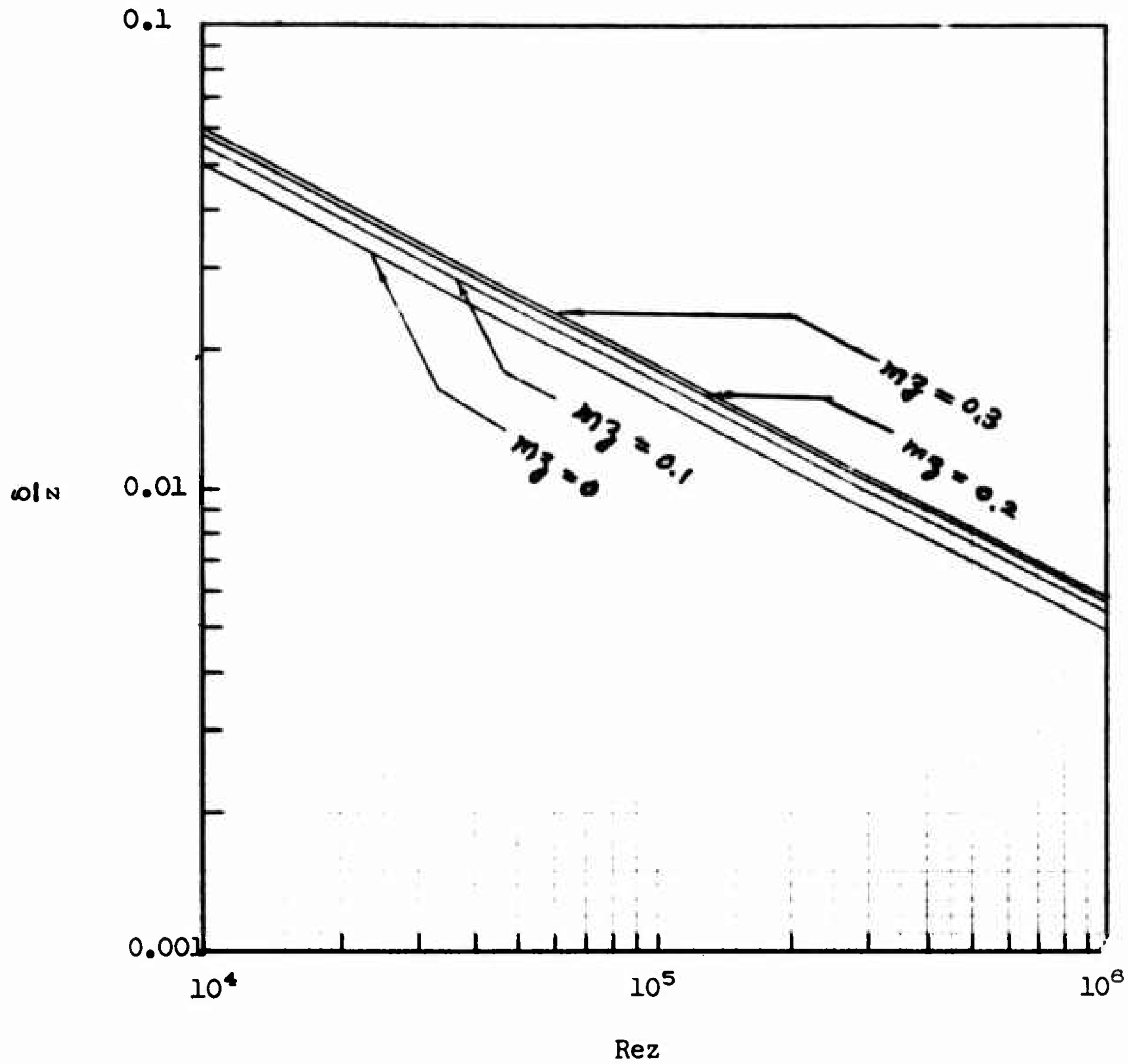

Fig. 3 - oundary layer thickness on a flat plate at zero incidence

1.2.6 Local coefficient of skin friction, C_f,

The local coefficient of skin friction is defined as

$$C_f = \frac{\tau_s}{\frac{1}{2}\rho w_\infty^2}$$

where $\tau_s = \mu(\partial w/\partial y)_s$ is the shear stress on the surface. Using Eq. (1.8) we obtain for C_f the expression

$$C_f = \frac{1}{\sqrt{Rez}}(0.644 - 1.788\ mz + ---) \quad .$$

This equation is plotted in Fig. 4 for several values of mz. In terms of the charge number, C_f can be written as

$$C_f = \frac{0.644}{\sqrt{Rez}} - \frac{1.788}{(Rez)^{3/2}} N_{\rho_c z} + --- \quad .$$

The variation of C_f with the charge number is plotted in Fig. 5. It is seen that for a given Rez, C_f decreases linearly with increasing charge number. Therefore, the application of an electric field reduces the skin friction..

1.3 Variable Charge Density

Consider that ions in the fluid are provided by a corona discharge. For the simple electrode configuration of two parallel plates (a plane of fine wires is necessary to actually get the corona) shown in Fig. 1, the following expression for charge density distribution is obtained from Stuetzer's solution[5], assuming the grounded plate is far away from the region of intense corona,

$$\rho_c = \left[\frac{\epsilon I}{2KA(h-y)}\right]^{1/2} = \left[\frac{\epsilon I}{2KAh}\right]^{1/2}\left(1 - \frac{y}{h}\right)^{-1/2} \quad ,$$

where

I = current,

A = surface area of the plate,

E = permittivity,

h = distance between plates.

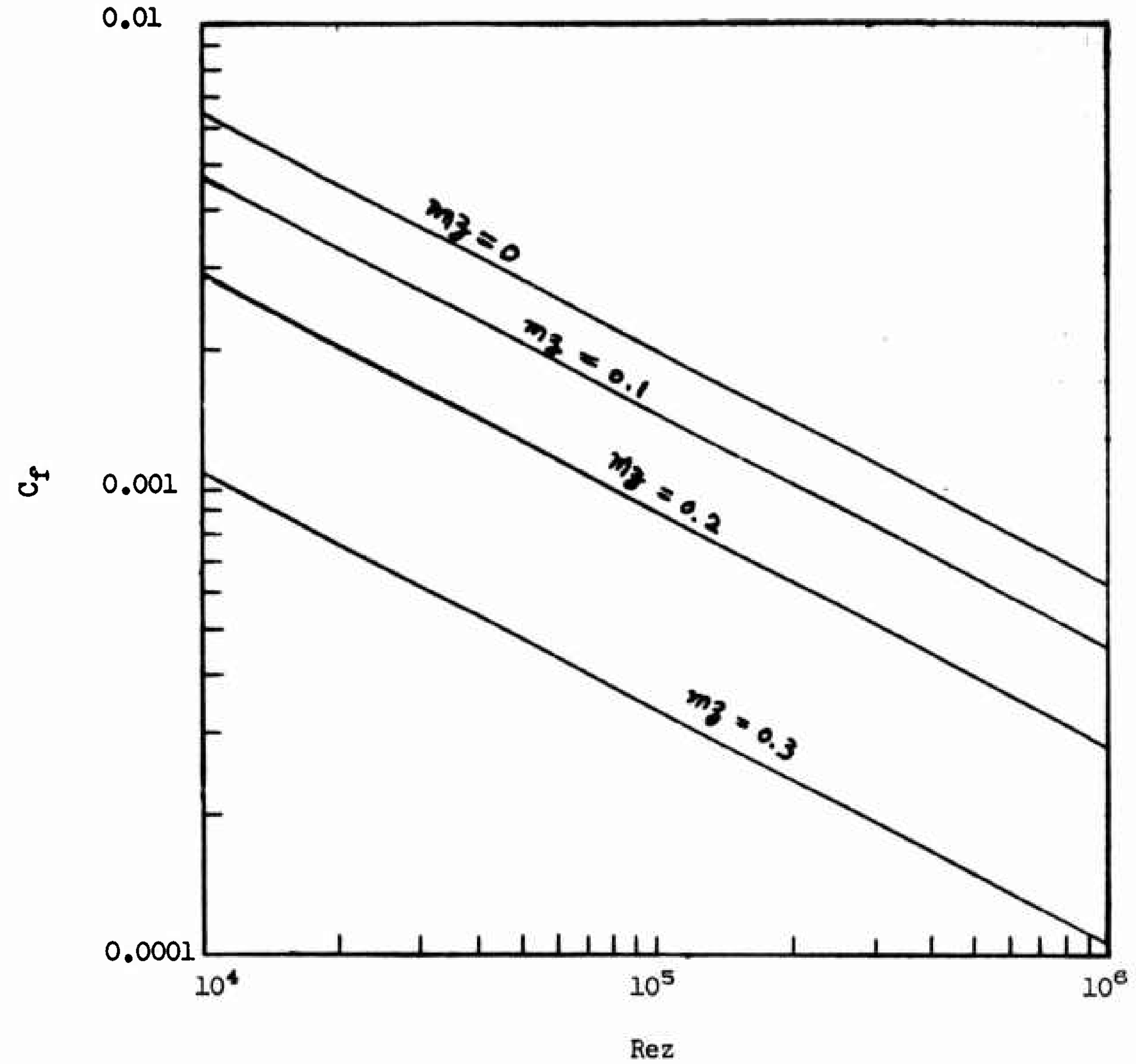

Fig. 4 - Local coefficient of skin friction over a flat plate

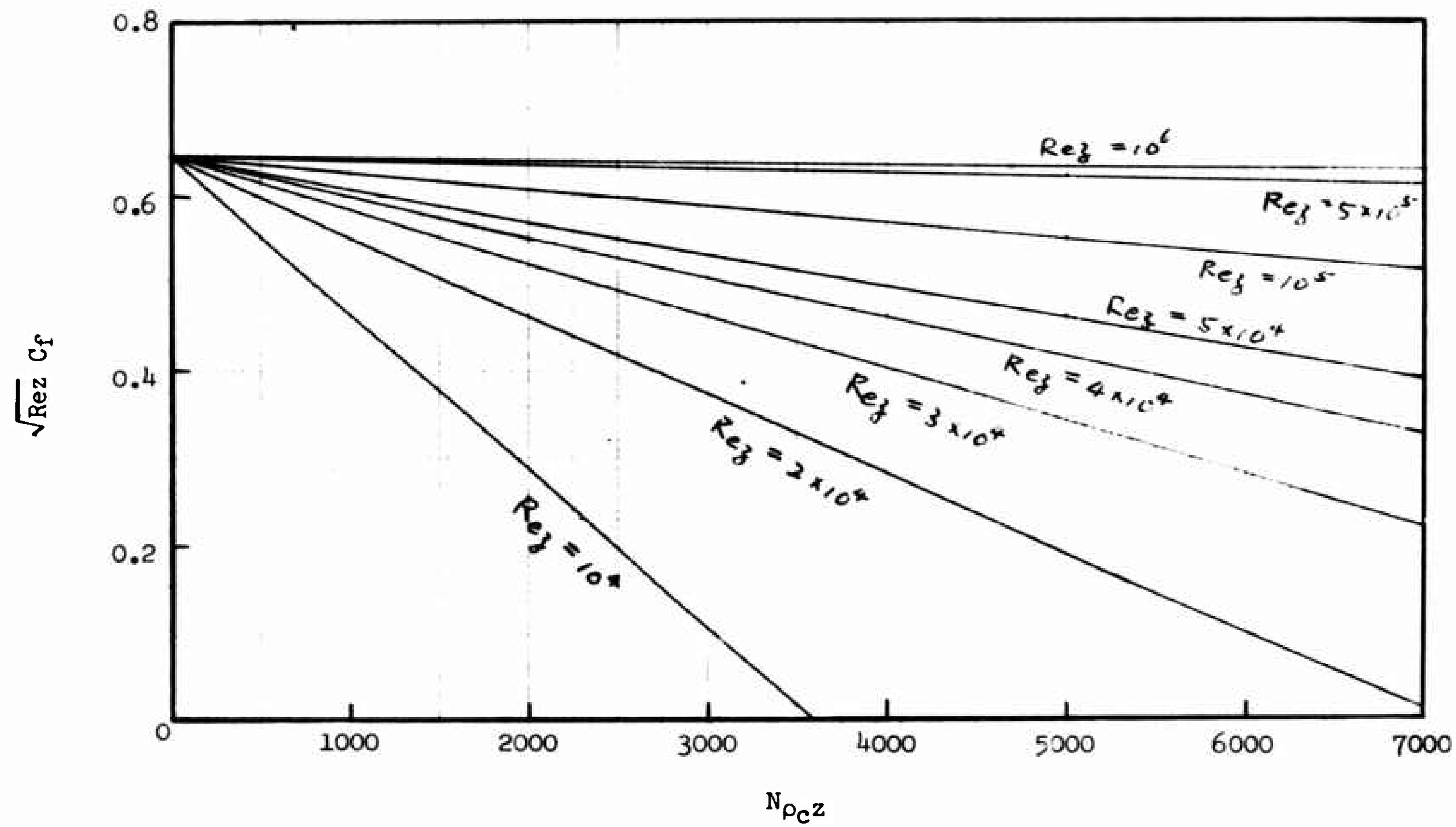

Fig. 5 - $\sqrt{Rez}\ C_f$ versus $N_{\rho_c z}$ on a flat plate

Assume that $y/h << 1$, i.e., $h >> \delta$, then

$$\left[1 - \frac{y}{h}\right]^{1/2} \approx 1 + \frac{1}{2}\frac{y}{h} + \text{---}.$$

Substituting this and ρ_c into Eq. (1.4) and using Rossow's method, we finally obtain

$$2f_o'''' = - f_o''f_o ,$$

$$f_2'''' = f_o'f_2' - \frac{1}{2} f_of_2'' - \frac{3}{2} f_2f_o'' + f_o' ,$$

$$f_4'''' = 2f_of_4' + f_2f_2' - \frac{1}{2} f_of_4'' - \frac{3}{2} f_2f_2'' - \frac{5}{2} f_4f_o''$$

$$+ f_2' + \frac{\eta}{mh \ Rez} f_o .$$

Comparing these equations with Eqs. (a), (c'), and (e"), it is found that under the assumption $y/h << 1$, only f_4 differs from the case of constant charge density. This indicates that for this particular electrode configuration the charge density in the boundary layer may be assumed to be constant.

II. THE EFFECT OF IONIZATION ON VISCOSITY

2.1 Assumptions

In an attempt to determine the effect of ions on the viscosity of oxygen gas, we consider a gas mixture of two components, neutral oxygen molecules and positive, singly ionized oxygen ions. To facilitate the analysis, the following assumptions are made:

(a) The mass of neutral oxygen molecules is approximately equal to that of oxygen ions.

(b) Collisions between neutral molecules as well as those between neutral molecules and ions are of elastic rigid sphere type.

(c) Interactions between ions follow the inverse-square law.

(d) The electric field intensity is small and can be neglected.

2.2 The Viscosity for a Mixture of Neutral Molecules and Ions

The analysis is based on the general expression for the first approximation to the coefficient of viscosity for a mixture of two gases given by Chapman and Cowling:[6]

$$[\mu]_1 = \frac{n_{12}\left(\frac{2}{3} + \frac{M_1}{M_2} A\right) + n_{21}\left(\frac{2}{3} + \frac{M_2}{M_1} A\right) + \frac{E}{2[\mu_1]_1} + \frac{E}{2[\mu_2]_1} + 2\left(\frac{2}{3} - A\right)}{\frac{n_{12}\left(\frac{2}{3} + \frac{M_1}{M_2} A\right)}{[\mu_1]_1} + \frac{n_{21}\left(\frac{2}{3} + \frac{M_2}{M_1} A\right)}{[\mu_2]_1} + \frac{E}{2[\mu_1]_1[\mu_2]_1} + \frac{4A}{3EM_1M_2}} \quad (2.1)$$

where

$[\mu]_1$ = first approximation to the viscosity of mixture,

$[\mu_1]_1$ = first approximation to the viscosity of neutral oxygen,

$[\mu_2]_1$ = first approximation to the viscosity of ions,

n^N = number density of neutral molecules,

$n^{(i)}$ = number density of ions,

$$n_{12} = \frac{n^N}{n^{(i)}} \quad , \quad n_{21} = \frac{n^{(i)}}{n^N} \quad ,$$

m^N = mass of neutral molecules,

$m^{(i)}$ = mass of ions,

$M_1 = \dfrac{m^N}{m^N + m^{(i)}}$ = reduced mass of neutral molecules,

$M_2 = \dfrac{m^{(i)}}{m^N \quad m^{(i)}}$ = reduced mass of ions ,

and A and E are constants determined by the law of interaction between molecules. It follows from assumption (a), that

$$\frac{M_1}{M_2} = \frac{M_2}{M_1} = 1 \quad \text{and} \quad M_1 = M_2 = \tfrac{1}{2} \; .$$

Equation (2.1) then becomes

$$[\mu]_1 = \frac{n_{12}\left(\dfrac{2}{3} + A\right) + n_{21}\left(\dfrac{2}{3} + A\right) + \dfrac{E}{2[\mu_1]_1} + \dfrac{E}{2[\mu_2]_1} + 2\left(\dfrac{2}{3} - A\right)}{\dfrac{n_{12}\left(\dfrac{2}{3} + A\right)}{[\mu_1]_1} + \dfrac{n_{21}\left(\dfrac{2}{3} + A\right)}{[\mu_2]_1} + \dfrac{E}{2[\mu_1]_1[\mu_2]_1} + \dfrac{16A}{3E}} \quad . \tag{2.2}$$

Under assumption (b), it is found that

$$A = \frac{2}{5} \quad \text{and} \quad E = \frac{1}{2}\left(\frac{kmT}{\pi}\right)^{1/2} \frac{1}{(d^N)} \, 2 \quad , \tag{2.3}$$

where

k = Boltzmann constant,

d^N = diameter of neutral molecules ,

and

m = mass of the molecules and T is temperature .

Substitution for A from Eq. (2.3) into Eq. (2.2) gives

$$[\mu]_1 = \frac{\frac{16}{15} n_{12} + \frac{16}{15} n_{21} + \frac{E}{2[\mu_1]_1} + \frac{E}{2[\mu_2]_1} + \frac{8}{15}}{\frac{16}{15} \frac{n_{12}}{[\mu_1]_1} + \frac{16}{15} \frac{n_{21}}{[\mu_2]_1} + \frac{E}{2[\mu_1]_1 [\mu_2]_1} + \frac{32}{15E}} \quad . \qquad (2.4)$$

2.3 Evaluation of $[\mu_1]_1$ and $[\mu_2]_1$

On the basis of the elastic rigid sphere model, the first approximation to the viscosity of neutral oxygen gas is found to be[6]

$$[\mu_1]_1 = \frac{5}{16} \left(\frac{kmT}{\Pi}\right)^{1/2} \frac{1}{(d^N)} 2 \quad . \qquad (2.5)$$

It should be noted that all the above expressions are obtained by taking into consideration binary collisions only. In the evaluation of the viscosity of ions, difficulty arises because of the inverse-square law of electrostatic forces. Since these forces decrease with distance much more slowly than the ordinary forces of interaction, a molecule at a large distance from a given molecule will also be under the electrostatic repulsion of many other molecules. Hence, the distant collision is not binary but multiple. For the analysis based on the binary collision to be applicable, we can only consider the special case in which the temperature is so high and the number density so low that it is sufficient to consider binary collisions. This condition is satisfied at 6000°K. According to Chapman and Cowling[6] the first approximation to the coefficient of viscosity, in this case, is

$$[\mu_2]_1 = \frac{5}{8} \left(\frac{kmT}{\Pi}\right)^{1/2} \left(\frac{2kT}{e^2}\right) \Big/ A_2(2) \quad , \qquad (2.6)$$

with $A_2(2)$ given by

$$A_2(2) = 2 \left\{ \ln \, (1+\nu_{01}{}^2) - \frac{\nu_{01}{}^2}{1 + \nu_{01}} \right\} 2 \quad ,$$

where

$$\nu_{01} = \frac{4dkT}{e^2} \quad ,$$

d = mean distance between neighboring pairs of molecules,

and

e = electronic charge .

2.4 A Numerical Example

To compare the values for $[\mu_1]_1$ and $[\mu_2]_1$, let us evaluate them at $T = 6000°K$. Assume the number density n_2 of ions is about 10^{15} cm^{-3} so that Eq. (2.6) applies. For this case,

$$d = n^{-1/3} \approx 10^{-5} \text{ cm} ,$$

$$\frac{dkT}{e^2} = \frac{10^{-5} \times 1.38 \times 10^{-16} \times 6000}{2.31 \times 10^{-19}} = 35.8 ,$$

$$\nu_{01}{}^2 = (4 \times 35.8)^2 = 2.05 \times 10^4 ,$$

$$A_2(2) = 2 \left\{ \ln (1 + 2.05 \times 10^4) - \frac{2.05 \times 10^4}{1 + 2.05 \times 10^4} \right\} \approx 17 ,$$

and

$$[\mu_2]_1 \approx 7.1 \times 10^{-6} \text{ gm/cm sec.}$$

For neutral oxygen molecules at $T = 6000°K$, Eq. (2.5) gives

$$[\mu_1]_1 \approx 0.9 \times 10^{-3} \text{ gm/cm sec.}$$

$[\mu_1]_1$ and $[\mu_2]_1$ thus obtained may be substituted in Eq. (2.4) to get $[\mu]_1$ for given values of n^N and $n^{(i)}$. However, without going through the calculation, we may draw the conclusion stated in the next section.

2.5 Conclusion and Discussion

From the above calculation, we see that viscosity of ions is about two orders of magnitude smaller than neutral molecules at $T = 6000°K$. This indicates that the viscosity of the mixture is smaller than the original oxygen gas of neutral molecules. This conclusion is justified by the fact that in order for the mixture to have a maximum viscosity, the viscosity of the two pure components must be nearly equal.[7]

As was pointed out previously, Eq. (2.5) is valid only when the condition is such that it is sufficient to consider only binary collisions. The conclusion made above is, strictly speaking, only valid for this specific case. However, unless the ion gas possesses such a peculiar property that its viscosity becomes greater than that of its parent gas at lower temperature, we may conclude from the investigation that the presence of ions intends to make the viscosity of the mixture smaller than the original gas.

A question now arises: Can we draw the same conclusion even in the presence of a strong electric field? So far we do not have sufficient physical reasoning to make the prediction, nor are we able to find related references in the literature. It appears to be an interesting problem deserving further investigation.

III. TWO-FLUID MODEL

This section describes an attempt to consider the flow as consisting of two interacting streams, the neutral air and the ion gas. For channel flow, the neutral gas has only longitudinal velocity, while the latter is assumed to have longitudinal as well as transverse velocity components. As we recall in section I, the electrostatic force was considered to act on the fluid as a whole. It was assumed that there was no transverse velocity component in the flow. However, under the action of an externally applied transverse electric field, ions do possess transverse velocity. Thus, it appears logical to consider the flow as the interaction of two streams of fluid.

3.1 Governing Equations

Instead of including electrostatic force term on the right-hand side of the equation of motion for the entire fluid, we have now to write down a separate set of momentum equations for each species of the fluid, taking into account the change of momentum due to collisions between neutral particles and ions. Such equations can be derived from the Boltzmann equation. The detailed derivation may be found in books on plasma dynamics, Reference 8, for example. Here we give only the final expression. If the external forces consist of an electric field and a magnetic field, the momentum equation for the s^{th} charged species is found to be

$$\frac{\partial v_i^{(s)}}{\partial t} + v_j^{(s)} \frac{\partial v_i^{(s)}}{\partial x_j} + \frac{1}{n^{(s)}m^{(s)}} \frac{\partial \Psi_{ij}^{(s)}}{\partial x_j} - \frac{q^{(s)}}{m^{(s)}} (E_i + \epsilon_{ijk} v_j^{(s)} B_k)$$

$$= \nu_{sn}[v_i^N - v_i^{(s)}] - \frac{v_i^{(s)}}{n^{(s)}} S_{coll}^{(s)} \quad . \qquad (3.1)$$

where

v_i^N = i^{th} velocity component of neutral particles,

$v_i^{(s)}$ = i^{th} velocity component for the s^{th} species,

$n^{(s)}$ = number density of the s^{th} species,

$m^{(s)}$ = mass of particles of the s^{th} species,

$\Psi_{ij}^{(s)}$ = pressure tensor for the s^{th} species
$= n^{(s)}m^{(s)}\langle V_i^{(s)}V_j^{(s)}\rangle$,

$V_i^{(s)}$ = i^{th} peculiar velocity component of the s^{th} species relative to its own mean velocity,

$q^{(s)}$ = charge of the s^{th} type particles,

E_i = i^{th} component of the electric field,

ϵ_{ijk} = the alternating unit vector,

B_k = k^{th} component of the magnetic field,

ν_{sn} = collision frequency for ion-neutral collisions,

S_{coll} = rate at which particles of type s are gained (or lost) because of collisions,

and the $\langle \ \rangle$ indicates mean value.

To apply Eq. (3.1) to the problem under consideration, we make use of the following assumptions:

(a) No external magnetic fields are applied and the induced magnetic field is negligible because of the small current involved in the problem

(b) The fluid consists of two species only, ions and neutral particles. Collisions between ions and neutral particles are elastic so that S_{coll} vanishes. Collision frequency ν_{Ni} is constant.

(c) The neutral gas flows between two infinite parallel plates and the flow is fully developed.

(d) The flow is incompressible

(e) The partial pressure of ions is negligible

(f) The ion gas is inviscid.

(g) The positive electrode is the source of ion gas.

The geometrical configuration of the channel is shown in Fig. 6. If we designate velocity components of the neutral gas in the z- and the y-direction by w^N and v^N, respectively, and the corresponding velocity components of ions by $w^{(i)}$ and $v^{(i)}$, because of the assumptions, $v^N = 0$ and, for neutral gas in the z-direction,

$$\frac{\partial \psi_{ij}}{\partial x_j} = + \frac{\partial p}{\partial z} - \mu \frac{\partial^2 w^N}{\partial y^2} ,$$

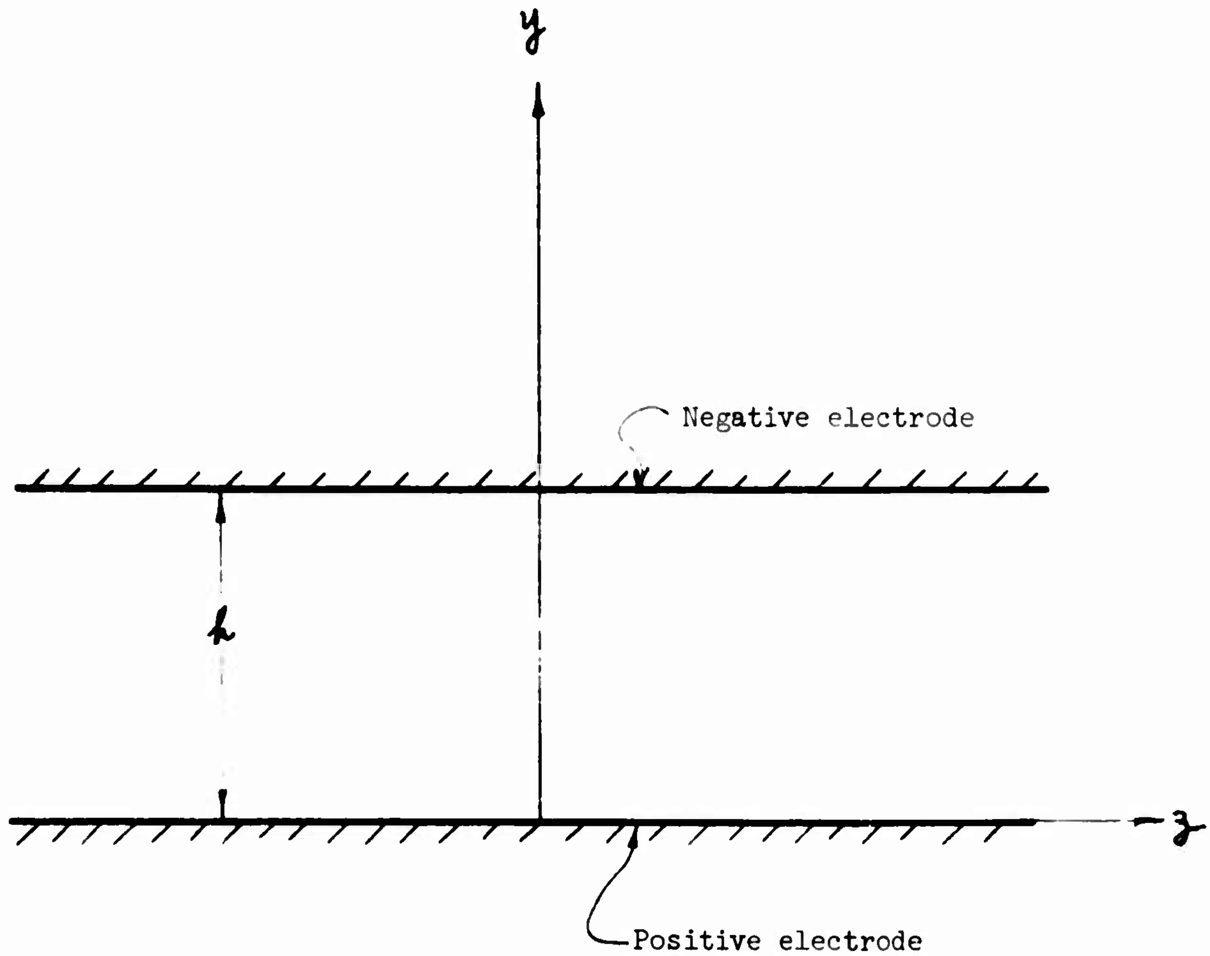

Fig. 6 - Parallel plate flow channel

while in the y-direction

$$\frac{\partial \psi_{ij}}{\partial x_j} = \frac{\partial p}{\partial y} \ .$$

The momentum equations for the neutral gas are

$$\mu \frac{d^2 w^N}{dy^2} = \frac{dp}{dz} - \rho \, \nu_{Ni}(w^{(i)} - w^N) \ , \tag{3.2}$$

$$\frac{\partial p}{\partial y} = \rho \, \nu_{Ni} v^{(i)} \ , \tag{3.3}$$

and those for ions are

$$v^{(i)} \frac{dw^{(i)}}{dy} = \nu_{iN}(w^N - w^{(i)}) \ , \tag{3.4}$$

$$v^{(i)} \frac{dv^{(i)}}{dy} = \frac{q^{(i)}}{m^{(i)}} E - \nu_{iN} v^{(i)} \ , \tag{3.5}$$

where $\rho = m^N n^N$ is mass density of the neutral gas.

The nonlinearity of Eq. (3.5) renders it difficult to solve for $v^{(i)}$. However, the fact that the transverse velocity of ions is the drift velocity of ions in the presence of an electric field, enables us to employ the concept of mobility. The drift velocity of ions is related to the electric field, E, by the relation

$$v^{(i)} = KE \ ,$$

where K is ion mobility.

Equations (3.4) and (3.5) will be solved for the following two cases:

(a) Zero longitudinal velocity component of ions.

(b) Non-vanishing longitudinal velocity component of ions.

3.2 Zero Longitudinal Velocity Component of Ions

In order to obtain an idea of how collisions between ions and neutral particles will affect the flow, another simplifying assumption can be made, namely negligible longitudinal velocity component of ions. With this assumption Eq. (3.2) reduces to the simple form

$$\mu \frac{d^2 w^N}{dy^2} = \frac{dp}{dz} + \rho \nu_{Ni} w^N \ . \qquad (3.6)$$

This equation is similar to the one obtained by Velkoff.[2]

The uncoupling of Eq. (3.2) from Eq. (3.4) greatly simplifies the mathematics. It is readily found that the general solution to Eq. (3.6) is

$$w^N = C_1 \cosh\sqrt{\frac{\rho\nu_{Ni}}{\mu}}\, y + C_2 \sinh\sqrt{\frac{\rho\nu_{Ni}}{\mu}}\, y - \frac{1}{\rho\nu_{Ni}} \frac{dp}{dz} \ . \qquad (3.7)$$

Note that in obtaining this solution, dp/dz is considered constant. This follows from the assumption that the flow is fully developed, i.e., w^N is independent of z. The constants of integration C_1 and C_2 are to be determined by the boundary conditions:

$$w^N(o) = w^N(h) = 0.$$

The solution satisfying these boundary conditions is found to be

$$w^N = \frac{1}{\rho\nu_{Ni}} \frac{dp}{dz} \left[\cosh\sqrt{\frac{\rho\nu_{Ni}}{\mu}}\, y + \left(\frac{1-\cosh\sqrt{\frac{\rho\nu_{Ni}}{\mu}}\, h}{\sinh\sqrt{\frac{\rho\nu_{Ni}}{\mu}}\, h}\right)\sinh\sqrt{\frac{\rho\nu_{Ni}}{\mu}}\, y - 1\right] .$$

Integrating w^N over the channel height, one obtains the mean velocity

$$\bar{w}^N = \frac{1}{h}\int_o^h w^N dy = \frac{1}{h\,\rho\nu_{Ni}} \frac{dp}{dz} \left\{ \sqrt{\frac{\mu}{\rho\nu_{Ni}}} \left[\sinh\sqrt{\frac{\rho\nu_{Ni}}{\mu}}\, h - \frac{\left(1-\cosh\sqrt{\frac{\rho\nu_{Ni}}{\mu}}\, h\right)^2}{\sinh\sqrt{\frac{\rho\nu_{Ni}}{\mu}}\, h} \right] - h \right\} .$$

Or, solving for the pressure gradient yields

$$\frac{dp}{dz} = \frac{\nu_{Ni}\mu Re}{\left\{\sqrt{\frac{\mu}{\rho\nu_{Ni}}}\left[\sinh\sqrt{\frac{\rho\nu_{Ni}}{\mu}}\,h - \frac{\left(1-\cosh\sqrt{\frac{\rho\nu_{Ni}}{\mu}}\,h\right)^2}{\sinh\sqrt{\frac{\rho\nu_{Ni}}{\mu}}\,h}\right] - h\right\}} ,$$

where

$$Re = \frac{\rho h w^N}{\mu} .$$

3.3 Non-Vanishing Longitudinal Velocity Component of Ions

Now let us examine the more complicated case where the longitudinal velocity component of ions is not negligible. Equations (3.2) and (3.4) must be solved simultaneously. If we let

$$\frac{1}{\mu}\frac{dp}{dz} = C_0 \quad , \quad \frac{\rho\nu_{Ni}}{\mu} = C_1 \quad , \quad \frac{\nu_{iN}}{v^{(i)}} = C_2 \quad ,$$

where $v^{(i)}$ is considered constant for simplicity, then Eqs. (3.2) and (3.4) can be rewritten as

$$\left(\frac{d^2}{dy^2} - C_1\right)w^N + C_1 w^{(i)} = C_0 ,$$

$$-C_2 w^N + \left(\frac{d}{dy} + C_2\right)w^{(i)} = 0 .$$

The boundary conditions are

$$w^N = 0 \text{ at } y = h, \; w^{(i)} = w^N = 0 \text{ at } y = 0 .$$

This set of simultaneous equations can be solved by eliminating one of the two variables, or by using Laplace transformation. The detailed calculation is given in Appendix II. Here we give only the final solution.

$$w^N = -\left(\frac{C_0}{C_1} + \frac{C_2^2C_0}{C_1^2} + \frac{C_2}{C_1}B\right) - \frac{C_2C_0}{C_1}y + \frac{1}{\sqrt{C_2^2 + 4C_1}}\left\{\left[2\frac{C_2C_0}{C_1}\right.\right.$$

$$\left.\left. + \frac{C_2^3C_0}{C_1^2} + \left(\frac{C_2^2}{C_1} + 1\right)B\right] + C_5\left(\frac{C_0}{C_1} + \frac{C_2^2C_0}{C_1^2} + \frac{C_2}{C_1}B\right)\right\} e^{C_5y}$$

$$- \frac{1}{\sqrt{C_2^2 + 4C_1}}\left\{\left[2\frac{C_2C_0}{C_1} + \frac{C_2^3C_0}{C_1^2} + \left(\frac{C_2^2}{C_1} + 1\right)B\right]\right.$$

$$\left. + C_6\left(\frac{C_0}{C_1} + \frac{C_2^2C_0}{C_1^2} + \frac{C_2}{C_1}B\right)\right\} e^{C_6y},$$

where

$$C_5 = -\tfrac{1}{2}C_2 + \tfrac{1}{2}\sqrt{C_2^2 + 4C_1},$$

$$C_6 = -\tfrac{1}{2}C_2 - \tfrac{1}{2}\sqrt{C_2^2 + 4C_1},$$

and

$$B = \frac{\sqrt{C_2^2 + 4C_1}\left(\frac{C_0}{C_1} + \frac{C_2^2C_0}{C_1^2} + \frac{C_2C_0h}{C_1}\right)}{\left[C_2(e^{C_5h} - e^{C_6h}) + (C_5e^{C_5h} - C_6e^{C_6h}) - \frac{C_2}{C_1}\sqrt{C_2^2 + 4C_1}\right]}$$

$$- \frac{\left(\frac{2C_2C_0}{C_1} + \frac{C_2^3C_0}{C_1^2}\right)(e^{C_5y} - e^{C_6y}) + \left(\frac{C_0}{C_1} + \frac{C_2^2C_0}{C_1^2}\right)(C_5e^{C_5h} - C_6e^{C_6h})}{\left[C_2(e^{C_5h} - e^{C_6h}) + (C_5e^{C_5h} - C_6e^{C_6h}) - \frac{C_2^2}{C_1}\sqrt{C_2 + 4C_1}\right]}.$$

The mean velocity is

$$\overline{w^N} = -C_0\left(\frac{1}{C_1} + \frac{C_2h}{2C_1} + \frac{C_2}{C_1^2}\right) - \frac{C_2^2}{C_1}B$$

$$- \frac{1}{C_6\sqrt{C_2^2 + 4C_1}}\left\{C_0\left(\frac{2C_2}{C_1} + \frac{C_2^3}{C_1^2}\right) + \left(\frac{C_2^2}{C_1} + 1\right)B + C_0C_6\left(\frac{1}{C_1} + \frac{C_2^2}{C_1^2}\right)\right.$$

$$\left. + \frac{C_2C_6}{C_1}B\right\}(e^{C_6h} - 1) - \frac{1}{C_6\sqrt{C_2^2 + 4C_1}}\left\{C_0\left(\frac{2C_2}{C_1} + \frac{C_2^3}{C_1^2}\right)\right.$$

$$\left. + \left(\frac{C_2^2}{C_1} + 1\right)B + C_0C_6\left(\frac{1}{C_1} + \frac{C_2^2}{C_1^2}\right) + \frac{C_2C_6}{C_1}B\right\}(e^{C_6h} - 1).$$

$$\mu C_0 = \frac{dp}{dx} = \frac{\mu\left[C_5C_6\sqrt{C_2^2 + 4C_1}\left(-\overline{w}^N - \frac{C_2}{C_1}B\right) + \left(\frac{C_6C_2^2}{C_1} + \frac{C_6C_5C_2}{C_1} + 1\right)B(e^{C_5h} - 1) - \left(\frac{C_2^2C}{C_1} + \frac{C_2C_5C_6}{C_1} + 1\right)B(e^{C_6h} - 1)\right]}{\left[C_5C_6\sqrt{C_2^2 + 4C_1}\left(\frac{1}{C_1} + \frac{C_2h}{2C_1} + \frac{C_2^2}{C_1^2}\right) - \left(\frac{2C_2C_6}{C_1} + \frac{C_2^3C_6}{C_1^2} + \frac{C_5C_6}{C_1} + \frac{C_2^2C_5C_6}{C_1^2}\right) \bullet (e^{C_5h} - 1) + \left(\frac{2C_2C_5}{C_1} + \frac{C_2^3C_5}{C_1^2} + \frac{C_5C_6}{C_1} + \frac{C_2^2C_5C_6}{C_1^2}\right)(e^{C_6h} - 1)\right]}$$

3.4 Determination of ν_{iN}

From the form of Eq. (3.8), no explicit relation between the pressure gradient and the electric field exists. It is through the collision frequency for ion-neutral particle collisions, ν_{Ni}, that the electric field manifests its influences on the flow.

Nothing has been said about ν_{Ni} so far. The determination of ν_{Ni} requires a knowledge of the cross section for momentum transfer which must, in general, be found experimentally.

It is shown in the kinetic theory of gases[9] that

$$\nu_{Ni} = \frac{n^{(i)}}{m^N} f_{iN} ,$$

where $n^{(i)}$ is determined by the electric field and f_{iN} is independent of the velocity and the number density of ions or neutral particles. In fact, f_{iN} is given by

$$f_{iN} = \frac{4\pi m^*}{3kT}\left(\frac{m^*}{2\pi kT}\right)^{3/2} \int_0^\infty e^{-m^* v_r^2/2kT} \sigma_D(v_r) v_r^5 dv_r ,$$

where T is the temperature, k is Boltzmann constant,

$$m^* = \frac{m^{(i)} n^N}{m^{(i)} + m^N} \text{ is the reduced mass ,}$$

v_r = magnitude of relative velocity between colliding ions and neutral particles

and

σ_D = transport cross section.

For rigid-elastic sphere model, f_{iN} is found to be

$$f_{iN} = \frac{8}{3} (2m^* \Pi kT)^{1/2} d_{iN}$$

where

$$d_{iN} = \tfrac{1}{2}(d^{(i)} + d^N)$$

and

$d^{(i)}$ = diameter of ions,

d^N = diameter of neutral particles.

The relation between ν_{Ni} and ν_{iN} is given by $m^N n^N \nu_{Ni} = m^{(i)} n^{(i)} \nu_{iN}$.

3.5 Discussion

No numerical data were obtained for the analysis presented in this part, for at this stage of our research, attention was called to the consideration of stability problem. It is generally believed that, for the momentum change due to collisions between ions and neutral molecules to have significant effect on the flow, the number density of ions must be comparable to that of neutral molecules. Unfortunately, under this condition the assumption that the partial pressure of ions is negligible is no longer valid.

IV. ELECTROHYDRODYNAMIC INSTABILITY

As we recall from classical hydrodynamic instability, flows between two concentric rotating cylinders will become unstable because of centrifugal forces. In the cases of flow between concentric cylindrical electrodes with corona discharge, the electric force in the radial direction may have a similar effect on the flow. In fact, it can be shown* that if the components of electric field, E_r, and the charge density gradient, $\partial\rho_c/\partial r$, in the radial direction are such that $E_r \, \partial\rho_c/\partial r < 0$, an unstable situation could result. On the other hand, $E_r \, \partial\rho_c/\partial r > 0$ characterizes the stable equilibrium. The fact that positive corona between coaxial cylinders does satisfy the condition for unstable equilibrium** has stimulated our interest in pursuing investigation along the line of instability.

The question arises as to whether Taylor vortices can be created in quiescent air under the action of corona discharge. Following the classical hydrodynamic instability problem, we propose to make use of the method of small disturbances.

4.1 Governing Equations

As before we consider only incompressible flow. The general equations to be used are listed as follows:

The continuity equation

$$\nabla \cdot \vec{v} = 0 \tag{4.1}$$

The equation of motion

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla)\vec{v} = -\frac{1}{\rho}\nabla p + \nu\nabla^2 \vec{v} + \frac{\rho_c}{\rho}\vec{E} \tag{4.2}$$

Equations from electrostatics

$$\nabla \cdot \vec{E} = \frac{\rho_c}{\epsilon} \tag{4.3}$$

$$\nabla \times \vec{E} = 0 \tag{4.4}$$

---

* See Appendix III.

** See Appendix IV.

Equation for current density

$$\vec{J} = \rho_c(\vec{v} + K\vec{E}) - D_i\nabla\rho_c \tag{4.5}$$

The conservation of charges

$$\frac{\partial\rho_c}{\partial t} + \nabla\cdot\vec{J} = 0 \tag{4.6}$$

where $\vec{v}$ is velocity, D_i diffusion coefficient, $\vec{J}$ current density and the other notations have the same meanings as before.

4.2 Linearized Perturbation Equations

A disturbance of the stationary fluid will give rise to the small velocity (v') and the pressure perturbation (p'). Question now arises as to whether the disturbances will influence the distribution of the electric field intensity.

Two approaches are possible. First, we may consider that the perturbation in the electric field $(\vec{e})$ is not negligible. The total electric field is then given by $\vec{E} + \vec{e}$. The perturbation in charge density is related to $\vec{e}$ through Eqs. (4.3), (4.5), and (4.6). It can be imagined that the electrostatic force term on the right-hand side of Eq. (4.2) will give rise to a rather complicated expression after being expressed in terms of $\vec{E}$ and $\vec{e}$. The formidable mathematics thus introduced urges us to seek an alternative approach.

The second approach to the problem is to consider the perturbation in charge density (ρ_c') and assume the disturbance does not influence the distribution of the electric field. This assumption may be considered to be valid since the perturbation in charge density is supposed to be infinitesimally small and the distortion of the electric field caused by it is thus assumed to be negligible.

With this assumption we obtain the following linearized equations if v', ρ' and ρ_c' are substituted in Eqs. (4.1), (4.2), (4.5), and (4.6) and all quadratic terms in them are neglected:

$$\nabla\cdot\vec{v}' = 0 \ , \tag{4.7}$$

$$\frac{\partial\vec{v}'}{\partial t} = -\frac{1}{\rho}\nabla p' + \nu\nabla^2\vec{v}' + \frac{1}{\rho}\rho_c'\vec{E} \ , \tag{4.8}$$

$$\vec{J}' = \bar{\rho}_c\vec{v}' + K\rho_c'\vec{E} - D_i\nabla\rho_c' \ , \tag{4.9}$$

$$\frac{\partial \rho_c'}{\partial t} + \nabla \cdot \vec{J}' = 0 \cdot \tag{4.10}$$

Here only the case of quiescent fluid is considered. Detailed investigations of these equations will be given in the following sections for two specific cases

(a) Quiescent fluid between two concentric cylinders.

(b) Quiescent fluid between two infinite parallel plates.

4.3 Instability of Quiescent Fluid between Two Concentric Cylinders under the Action of a Transverse Electric Field

4.3.1 Formulation of eigenvalue problem

Let us first consider the viscous fluid at rest between two concentric cylinders of infinite length under the action of a transverse electric field, E_r $(\hat{r})$. Let $\hat{r}$, θ, z be cylindrical polar coordinates and u', v', w' the corresponding components of the perturbation velocity. Because of the geometrical configuration under consideration, we assume that the perturbations are rotationary symmetric, i.e., u', v', w', p', and ρ_c' are independent of θ. Equations (4.7) through (4.10) for this particular case reduce to

$$\frac{\partial u'}{\partial \hat{r}} + \frac{u'}{\hat{r}} + \frac{\partial w'}{\partial z} = 0 \ , \tag{4.11}$$

$$\rho \frac{\partial u'}{\partial t} = - \frac{\partial p'}{\partial \hat{r}} + \mu\left(\nabla^2 u' - \frac{u'}{\hat{r}^2}\right) + \rho_c' E_r \ , \tag{4.12}$$

$$\rho \frac{\partial v'}{\partial t} = \mu\left(\nabla^2 v' - \frac{v'}{\hat{r}^2}\right) \ , \tag{4.13}$$

$$\rho \frac{\partial w'}{\partial t} = - \frac{\partial p'}{\partial z} + \mu \nabla^2 w' \ , \tag{4.14}$$

$$\frac{\partial \rho_c'}{\partial t} + u' \frac{\partial \bar{\rho_c}}{\partial \hat{r}} + K E_r \frac{\partial \rho_c'}{\partial \hat{r}} + \frac{K}{\epsilon} \bar{\rho}_c \rho_c' - D_i \nabla^2 \rho_c' = 0 \ , \tag{4.15}$$

where

$$\bar{\rho_c} = \text{mean charge density} = \epsilon \nabla \cdot \vec{E}$$

and

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\,\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2} \ .$$

From the form of Eq. (4.13), we can conclude that v' will be damped out if it is not initially everywhere zero. Elimination of p' between Eqs. (4.12) and (4.13) yields

$$\rho\,\frac{\partial}{\partial t}\left(\frac{\partial w'}{\partial r} - \frac{\partial u'}{\partial z}\right) = \mu\left(\nabla^2 - \frac{1}{r^2}\right)\left(\frac{\partial w'}{\partial r} - \frac{\partial u'}{\partial z}\right) - E_r\,\frac{\partial \rho_c'}{\partial z} \ . \quad (4.16)$$

We define non-dimensional variables

$$r^* = \frac{r}{r_1} \ , \quad z^* = \frac{z}{r_1} \ , \quad \tau = \frac{\tau\nu}{r_1^2} \ ,$$

$$u^* = \frac{u' r_1}{\nu} \ , \quad w^* = \frac{w' r_1}{\nu} \ , \quad \rho_c{}^* = \frac{\rho_c' K r_1^2}{\nu\epsilon} \ ,$$

where

$$r_1 = \text{radius of inner cylinder}$$

and

$$\epsilon = \text{permittivity} \ .$$

Then, on dropping the stars on r, z, u, w, and ρ_c, we can write Eqs. (4.11), (4.15), and (4.16) as

$$\frac{\partial u}{\partial r} + \frac{u}{r} + \frac{\partial w}{\partial z} = 0 \ , \quad (4.17)$$

$$\frac{\partial \rho_c}{\partial \tau} + \left(\frac{K r_1^3}{\epsilon\nu}\,\frac{\partial \bar{\rho}_c}{\partial r}\right)u + \frac{r_1 K E_r}{\nu}\,\frac{\partial \rho_c}{\partial r} + \frac{r_1^2 K}{\nu\epsilon}\,\bar{\rho}_c\rho_c - \frac{D_i}{\nu}\,\nabla^2\rho_c = 0 \ , \quad (4.18)$$

$$\frac{\partial}{\partial \tau}\left(\frac{\partial w}{\partial r} - \frac{\partial u}{\partial z}\right) = \left(\nabla^2 - \frac{1}{r^2}\right)\left(\frac{\partial w}{\partial r} - \frac{\partial u}{\partial z}\right) - \frac{r_1 \epsilon E_r}{\mu K}\,\frac{\partial \rho_c}{\partial z} \ . \quad (4.19)$$

Following Taylor, we can make the following substitutions:

$$u = u_1(r) \cos \lambda z \, e^{\sigma\tau} \quad ,$$

$$w = w_1(r) \sin \lambda z \, e^{\sigma\tau} \quad ,$$

$$\rho_c = \rho_{c1}(r) \cos \lambda z \, e^{\sigma\tau} \quad ,$$

where λ and σ are non-dimensional wave number for the z-direction and amplification factor respectively. Equations (4.17), (4.18), and (4.19) then become

$$r \frac{du_1}{dr} + u_1 + \lambda r \, w_1 = 0 \quad , \tag{4.20}$$

$$\left(\frac{K r_1^3}{\epsilon D_i} \frac{\partial \overline{\rho_c}}{\partial r}\right) u_1 = \left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{r_1 K E_r}{D_i}\frac{d}{dr} - \frac{r_1^2 K \rho_c}{D_i \epsilon} - \lambda^2 - \frac{\nu\sigma}{D_i}\right)\rho_{c1} \;, \tag{4.21}$$

$$\left(\frac{d^2}{dr^2} + \frac{d}{dr}\frac{1}{r} - \lambda^2 - \sigma\right)\left(\frac{d^2}{dr^2} + \frac{d}{dr}\frac{1}{r} - \lambda^2\right) u_1 = \lambda^2 \left(\frac{r_1 \epsilon E_r}{\mu K}\right)\rho_{c1} \;, \tag{4.22}$$

where Eq. (4.21) is obtained by substituting for w_1 from Eq. (4.20).

Four of the six boundary conditions are that u and w shall be zero at each of the bounding cylinders. The other two should be on ρ_c. Since the perturbation in charge density is supposed to be induced by convection, it may be assumed to vanish at each boundary.

4.3.2 Problem of small spacing

A general discussion of the boundary-value problem presented by Eqs. (4.21) and (4.22) will clearly be a difficult matter. The analysis, however, simplifies considerably when the conduction terms, $r_1 K E_r / D_i \; (d\rho_{c1}/dr)$ and $(r_1^2 K \overline{\rho_c} / D_i \epsilon)\rho_{c1}$, can be disregarded (for a justification of this approximation, see Appendix V) and the distance between the cylinders is very small compared with the mean of their radii, i.e.

$$\ell = r_2 - r_1 \ll \tfrac{1}{2}\,(r_2 = r_1) = r_0 \;. \tag{4.23}$$

We shall, therefore, limit ourselves to this case. When Eq. (4.23) holds, terms of order of ℓ/r_0 can be neglected and the term $1/r$ can be disregarded compared with d/dr. If it is further assumed that the principle of exchange of stabilities is valid, i.e., the situation in neutral stability is given by $\sigma = 0$, we have, for neutral stability, the equations

$$\left(\frac{d^2}{dr^2} - \lambda^2\right)^2 u_1 = \lambda^2\left(\frac{r_1 \epsilon E_r}{\mu K}\right)\rho_{c1} \ , \qquad (4.24)$$

$$\left(\frac{d^2}{dr^2} - \lambda^2\right)\rho_{c1} = \frac{K r_1^3}{\epsilon D_i}\ \frac{d\rho_c}{d\hat{r}}\ u_1 \quad . \qquad (4.25)$$

With $E_r = \mathcal{E} f_1(r)$ and $d\rho_c/dr = \mathcal{L} g_1(r)$, where $\mathcal{E}$ and $\mathcal{L}$ are constants having the dimensions of electric field intensity and charge density gradient, respectively, and, f_1 and g_1 are dimensionless functions assuming positive values when evaluated at a given value of their argument, Eqs. (4.24) and (4.25) can be written

$$\left.\begin{aligned} \left(\frac{d^2}{dr^2} - \lambda^2\right)^2 u_1 &= \lambda^2\ \frac{r_1^3 \epsilon \mathcal{E}}{\mu K}\ f_1 \rho_{c1} \\ \left(\frac{d^2}{dr^2} - \lambda^2\right)\rho_{c1} &= \frac{r_1 K \mathcal{L}}{\epsilon D_i}\ g_1 u_1 \end{aligned}\right\} \qquad (4.26)$$

If we redefine non-dimensional parameters as

$$\xi = \frac{r - r_o}{\ell} \quad , \quad a = \frac{\ell\lambda}{r_1}$$

Eqs. (4.26) can be rewritten as

$$(D^2 - a^2)^2 u_1 = a^2\ \frac{\ell^2}{r_1}\ \frac{\epsilon\mathcal{E}}{\mu K} f(\xi)\rho_{c1}$$

$$(D^2 - a^2)\rho_{c1} = \frac{\ell^2 r_1 K \mathcal{L}}{\epsilon D_i}\ g(\xi) u_1$$

where f and g are obtained from f_1 and g_1 by replacing r by $(\ell/r_1\ \xi + r_o/r_1)$. It is convenient to make the transformation

$$\frac{\ell^2 r_1 K \mathcal{L}}{\epsilon D_i}\ u_1 \longrightarrow U_1$$

the equations then takes the more convenient forms

$$\left.\begin{aligned}(D^2 - a^2)^2U_1 &= - Ta^2f\rho_{c1}\\(D^2 - a^2)\rho_{c1} &= gU_1 \quad ,\end{aligned}\right\} \qquad (4.27)$$

where

$$T = - \frac{\ell^4 \mathcal{E} \mathcal{L}}{\mu D_i} \quad .$$

The boundary conditions can be written

$$U_1 = DU_1 = \rho_{c1} = 0 \text{ at } \xi = \pm \tfrac{1}{2}.$$

Note $DU_1 = 0$ at the boundaries follows from Eq. (4.20).

Further simplification is possible, if we consider the simple case where E_r is constant. For this particular case, $f = 1$, $\mathcal{E} = E_r$, $g = (1 + \ell/r_0\ \xi)^{-2}$ and $\mathcal{L} = - \epsilon E_r/r^2$. Because of Eq. (4.23), g may be replaced by its value evaluated at $\xi = 0$, i.e., $g = 1$. Eliminating U_1 we obtain

$$(D^2 - a^2)^3\rho_{c1} = - a^2T\rho_{c1} \, . \qquad (4.28)$$

The boundary conditions become

$$\rho_{c1} = (D^2 - a^2)\rho_{c1} = D(D^2 - a^2)\rho_{c1} = 0 \text{ at } \xi = \pm \tfrac{1}{2} \quad . \qquad (4.29)$$

4.3.3 The variational principle

The characteristic value problem posed by Eq. (4.28) subject to conditions Eq. (4.29) may also be formulated using variational principle as follows.[10]

Let $P = (D^2 - a^2)\rho_{c1}$, and write Eq. (4.28) as

$$(D^2 - a^2)^2P = - a^2T\rho_{c1} \quad . \qquad (4.30)$$

Now multiply both sides of Eq. (4.30) by P and integrate over the range of ξ. The left-hand side of the equation gives

$$\int_{-1/2}^{1/2} P(D^2 - a^2)^2 P d\xi = \int_{-1/2}^{1/2} PD^4 P d\xi - 2a^2 \int_{-1/2}^{1/2} PD^2 P d\xi + a^4 \int_{-1/2}^{1/2} P^2 d\xi .$$

Since both P and DP vanish at $\xi = \pm \frac{1}{2}$, it readily follows after two integrations by parts that

$$\int_{-1/2}^{1/2} P(D^2 - a^2)^2 P d\xi = \int_{-1/2}^{1/2} [(D^2 - a^2)P]^2 d\xi$$

Turning next to the right-hand side of Eq. (4.30), we have

$$\int_{-1/2}^{1/2} P\rho_{c_1} d\xi = \int_{-1/2}^{1/2} \rho_{c_1} D^2 \rho_{c_1} d\xi - a^2 \int_{-1/2}^{1/2} \rho_{c_1}^{\ 2} d\xi .$$

Using the boundary condition $\rho_c = 0$ at $\xi = \pm \frac{1}{2}$, we find

$$\int_{-1/2}^{1/2} P\rho_{c_1} d\xi = - \int_{-1/2}^{1/2} (D\rho_{c_1})^2 d\xi - a^2 \int_{-1/2}^{1/2} \rho_{c_1}^{\ 2} d\xi .$$

The result of multiplying Eq. (4.30) by P and integrating over ξ is, therefore,

$$T = \frac{\int_{-1/2}^{1/2} [(D^2 - a^2)P]^2 d\xi}{a^2\left[\int_{-1/2}^{1/2} (D\rho_{c_1})^2 d\xi + a^2 \int_{-1/2}^{1/2} \rho_{c_1}^{\ 2} d\xi\right]} . \qquad (4.31)$$

4.3.4 Physical meaning of T

The non-dimensional parameter T corresponds to the Taylor number in flows between two concentric rotating cylinders. It can be shown as follows that the parameter T in the form

$$T = - \frac{\ell^4 \mathcal{E} \mathcal{g}}{\mu D_i}$$

represents the ratio of the destabilizing electrostatic force to the stabilizing viscous force.

Consider a perturbation, of scale $(-\mathcal{g}\ell)$, in the charge density. Associated with this perturbation is a destabilizing electrostatic force per unit volume of scale $(-\mathcal{E}\delta\ell)$. When convection takes place, convection and diffusion of charge density have the same order of magnitude (Eq. (4.25)), this implies that the radial velocity component has a scale of D_i/ℓ. The viscous force opposing the convective motion is thus of scale $\mu D_i/\ell^3$ per unit volume; this is the stabilizing force. The ratio of the destabilizing to the stabilizing force is

$$- \frac{\ell^4 \mathcal{E}\delta}{\mu D_i}$$

which is T.

4.3.5 Interpretation of the stability problem

The problem of the solution to Eq. (4.28) subject to the boundary conditions Eq. (4.29) is an eigenvalue problem leading to a relation between a and T, namely $F(a, T) = 0$. It can be foreseen that instability will be possible if T is greater than some critical value, for then the effect of the destabilizing electrostatic force will outweigh the effect of the stabilizing viscous force. The minimun critical value of T called T_c, is the minimum value of T, as a function of a, given by $F(a, T) = 0$. For $T > T_c$, it can be inferred that a disturbance within the band of unstable wavelength is an amplifying, non-oscillatory flow.

The variational formulation of the problem, Eq. (4.31), facilitates the determination of an approximate value of T_c. If, for given values of a, T is minimized with respect to variation of ρ_{c_1}, the resulting values of T are eigenvalues. In practice, a function of ρ_{c_1} is assumed which contains one arbitrary constant (A) and satisfies the boundary conditions, for given a, the extremals of T, with respect to variation of A, are eigenvalues. The minimum of these eigenvalues for various values of a is the desired critical value of T (T_c). The accepted value of T_c is 1707.8 at a = 3.13.

4.3.6 Critical electric field intensity

From the definition of the non-dimensional parameter T, it is clear that corresponding to the critical value of T (T_c) there is a critical value of the electric field intensity, E_c, for a given fluid. To give an estimation of the magnitude of E_c for air, let us consider again the simple case where E_r is constant. For this case the charge density is found to be

$$\rho_c^- = \epsilon \frac{E_r}{r}$$

and

$$\frac{d\rho_{\overline{c}}}{d\hat{r}} = - \frac{\epsilon E_r}{r^2} \; .$$

If the gradient of the charge density is evaluated at $r = r_0 = \frac{1}{2}(r_1 + r_2)$, we have

$$E_r \frac{d\rho_{\overline{c}}}{d\hat{r}} = \mathcal{E}\mathcal{l} = - \frac{\epsilon E_r{}^2}{r_0^2} < 0$$

and the fluid is unstable. Now,

$$T_c = - \frac{\ell^4}{\mu D_i} \mathcal{E}\mathcal{l} = + \frac{\ell^4}{\mu D_i} \frac{\epsilon E_c^2}{r^2} = 1708 \; .$$

Therefore,

$$E_c = \left(1708 \frac{\mu D_i}{\epsilon}\right)^{1/2} \frac{r_0}{\ell^2} \; .$$

For air at standard condition

$$\mu = 0.000018 \text{ Kg/m sec},$$

$$D_i = 0.028 \times 10^{-4} \text{m/sec for dry air},$$

$$\epsilon = 1.006 \times 8.85 \times 10^{-12} \frac{\text{coulomb}}{\text{volt meter}} \; .$$

If $r_0 = 2.5$ cm and $\ell = 0.5$ cm then $r_0/\ell^2 = 10$ and $E_c = 988$ volt/m. Thus, for this particular case, instability will set in when the electric field reaches the value of 988 volt/m.

4.4 Instability of a Layer of Fluid between Two Infinite, Parallel Plates under the Action of a Transverse Electric Field

4.4.1 Derivation of the perturbation equations

As a second application of Eqs. (4.7) through (4.10), let us investigate the case where a layer of quiescent fluid between two infinite parallel plates is under the action of a transverse electric field, $E(z)$. Let x, y, z be rectangular coordinates with z-axis perpendicular to the two plates. The plates are located at $z = \pm h/2$. To facilitate the derivation of the perturbation equation, we shall use the notation of Cartesian tensor with usual summation convention.[10]

Let u_j' ($j = 1, 2, 3$, with 1, 2, 3 corresponding to x, y, z, respectively) denote the components of the perturbation velocity. Equations (4.7) through (4.10) can then be written as

$$\frac{\partial u_i'}{\partial x_i} = 0 \tag{4.32}$$

$$\frac{\partial u_i'}{\partial t} = -\frac{1}{\rho}\frac{\partial p'}{\partial x_i} + \frac{\rho_c}{\rho} E\,\lambda_i + \nu\nabla^2 u_i' \tag{4.33}$$

$$\frac{\partial \rho_c'}{\partial t} + u_i'\frac{\partial \overline{\rho_c}}{\partial x_i} + K\rho_c'\frac{\partial E}{\partial x_i}\lambda_i + KE\,\lambda_i\frac{\partial \rho_c'}{\partial x_i} - D_i\nabla^2\rho_c' = 0\ , \tag{4.34}$$

where $\vec{\lambda} = (0, 0, 1)$ is a unit vector in the z-direction and

$$\nabla^2 = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}\ .$$

To eliminate the pressure term in Eq. (4.33), apply the operator

$$\mathrm{curl}_k = \epsilon_{ijk}\frac{\partial}{\partial x_j}$$

to the k^{th} component of the equation. Letting

$$\omega_i = \epsilon_{ijk}\frac{\partial u_k'}{\partial x_j}$$

denote the vorticity, we have the equation

$$\frac{\partial \omega_i}{\partial t} = \frac{E}{\rho}\epsilon_{ijk}\frac{\partial \rho_c'}{\partial x_j}\lambda_k + \nu\nabla^2\omega_i\ . \tag{4.35}$$

Taking the curl of this equation once again, we have

$$\frac{\partial}{\partial t}\epsilon_{ijk}\frac{\partial \omega_k}{\partial x_j} = \frac{E}{\rho}\epsilon_{ijk}\epsilon_{k\ell m}\frac{\partial^2 \rho_c'}{\partial x_\ell \partial x_j}\lambda_m + \nu\nabla^2\epsilon_{ijk}\frac{\partial \omega_k}{\partial x_j}\ . \tag{4.36}$$

Making use of the identity

$$\epsilon_{ijk}\epsilon_{k\ell m} = \delta_{i\ell}\delta_{jm} - \delta_{im}\delta_{j\ell} ,$$

we find

$$\epsilon_{ijk} \frac{\partial \omega_k}{\partial x_j} = \epsilon_{ijk}\epsilon_{k\ell m} \frac{\partial^2 u_m}{\partial x_j \partial x_\ell} = \frac{\partial}{\partial x_i}\left(\frac{\partial u_j'}{\partial x_j}\right) - \nabla^2 u_i' = - \nabla^2 u_i' \; .$$

Similarly,

$$\epsilon_{ijk}\epsilon_{k\ell m} \frac{\partial^2 \rho_c'}{\partial x_j \partial x_\ell} \lambda_m = \lambda_i \frac{\partial^2 \rho_c'}{\partial x_j \partial x_i} - \lambda_i \nabla^2 \rho_c' \; .$$

Thus, Eq. (4.36) becomes

$$\frac{\partial}{\partial t} \nabla^2 u_i' = \frac{E}{\rho}\left(\lambda_i \nabla^2 \rho_c' - \lambda_j \frac{\partial^2 \rho_c'}{\partial x_i \partial x_j}\right) + \nu \nabla^4 u_i' \; . \qquad (4.37)$$

Now multiply Eqs. (4.35) and (4.37) by λ_i, we get

$$\frac{\partial \zeta}{\partial t} = \nu \nabla^2 \zeta \qquad (4.38)$$

and

$$\frac{\partial}{\partial t} \nabla^2 w' = \frac{E}{\rho}\left(\frac{\partial^2 \rho_c'}{\partial x^2} + \frac{\partial^2 \rho_c'}{\partial y^2}\right) + \nu \nabla^4 w' \; , \qquad (4.39)$$

where $\zeta = \lambda_j \omega_j$ and $w = \lambda_j u_j'$ are the z-component of the vorticity and the velocity.

If again we neglect the conduction terms, Eq. (4.34) becomes

$$\frac{\partial \rho_c'}{\partial t} = - \frac{\partial \overline{\rho_c}}{\partial z} w' + D_1 \nabla^2 \rho_c' \qquad (4.40)$$

and the required perturbation equations are

$$\left\{\begin{aligned} &\frac{\partial \zeta}{\partial t} = \nu\nabla^2\zeta && (4.38)\\ &\frac{\partial}{\partial t}\nabla^2 w' = \frac{E}{\rho}\left(\frac{\partial^2\rho_c'}{\partial x^2} + \frac{\partial^2\rho_c'}{\partial y^2}\right) + \nu\nabla^4 w' && (4.39)\\ &\frac{\partial\rho_c'}{\partial t} = - \frac{\partial\overline{\rho_c}}{\partial z} w' + D_i\nabla^2\rho_c' \quad . && (4.40)\end{aligned}\right.$$

The fluid is considered to be confined between two conducting plates located at $z = \pm h/2$. The boundary conditions on these plates are, like the previous case

$$\zeta = \rho_c' = w' = \frac{\partial w'}{\partial z} = 0 \text{ at } z = \pm \frac{h}{2} \,.$$

4.4.2 Formulation of eigenvalue problem

Following the analysis in thermal convection, we now ascribe to all quantities describing the perturbation a dependence on x, y, and t of the form

$$\exp[i(k_x x + k_y y) + \gamma t],$$

where $k = \sqrt{k_x^2 + k_y^2}$ is the wave number and γ is a constant (which can be complex). Suppose ρ_c', w', and ζ have the forms

$$\zeta = \zeta_1(z)\exp[i(k_x x + k_y y) + \gamma t] \ ,$$

$$w' = w_1(z)\exp[i(k_x x + k_y y) + \gamma t] \ ,$$

$$\rho_c' = \rho_{c_1}(z)\exp[i(k_x x + k_y y) + \gamma t] \ .$$

For functions with this dependence on x, y, and t,

$$\frac{\partial}{\partial t} = \gamma, \ \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = - k^2, \text{ and } \nabla^2 = \frac{d^2}{dz^2} - k^2,$$

and Eqs. (4.38), (4.39) and (4.40) become

$$\gamma\ \zeta_1 = \nu\left(\frac{d^2}{dz^2} - k^2\right)\zeta_1 \,, \qquad (4.41)$$

$$\gamma\left(\frac{d^2}{dz^2} - k^2\right)w_1 = -\frac{E}{\rho}k^2\rho_c + \nu\left(\frac{d^2}{dz^2} - k^2\right)^2 w_1 \;, \qquad (4.42)$$

$$\gamma\rho_{c_1} = -\frac{\partial\rho_{\overline{c}}}{\partial z}w_1 + D_i\left(\frac{d^2}{dz^2} - k^2\right)\rho_{c_1} \;. \qquad (4.43)$$

In terms of the non-dimensional variables

$$z^* = \frac{z}{h} \;, \quad t^* = \frac{t\nu}{h^2}$$

we obtain from Eqs. (4.42) and (4.43)

$$(D^2 - b^2)(D^2 - b^2 - \sigma)w_1 = \left(\frac{E}{\mu}h^2\right)b^2\rho_{c_1} \;, \qquad (4.44)$$

$$\left(D^2 - b^2\frac{\nu}{D_i}\sigma\right)\rho_{c_1} = \left(\frac{1}{D_i}\frac{\partial\rho_{\overline{c}}}{dz}h^2\right)w_1 \;, \qquad (4.45)$$

where $D = d/dz^*$, $b = k\,h$ and $\sigma = \gamma h^2/\nu$. The associated boundary conditions are

$$\rho_c = 0 \;, \; w_1 = Dw_1 = 0 \text{ at } z^* = \pm\frac{h}{2} \,.$$

With $E = \mathcal{E}f(z^*)$ and $d\rho_{\overline{c}}/dz = \mathcal{L}g(z^*)$ and making the transformation

$$\frac{\mathcal{L}}{D_i}h^2 w_1 \longrightarrow W_1 \;,$$

we obtain

$$\left.\begin{aligned}(D^2 - b^2)(D^2 - b^2 - \sigma)W_1 &= -b^2Rf\rho_{c_1}\\ \left(D^2 - b^2 - \frac{\nu}{D_i}\sigma\right)\rho_{c_1} &= gW_1 \;.\end{aligned}\right\} \qquad (4.46)$$

where $R = -h^4\mathcal{E}\mathcal{L}/\mu D_i$ is equivalent to the Rayleigh number in thermal convection and has the same physical meaning as T in Section 4.3. For marginal state $\sigma = 0$, and Eqs. (4.46) reduce to

$$\left.\begin{aligned}(D^2 - b^2)w_1 &= -b^2Rf\rho_{c_1}\\ (D^2 - b^2)\rho_{c_1} &= gW_1\end{aligned}\right\} \qquad (4.47)$$

4.5 Instability of Laminar Boundary Layer over a Flat Plate under the Action of a Transverse Electric Field

4.5.1 General remarks

It has been shown in Section 1.3 that for the parallel plate electrodes (Fig. 1) the charge density distribution in the boundary layer is approximately

$$\rho_c \approx \left[\frac{\epsilon I}{2KAh}\right]^{\frac{1}{2}}\left(1 + \frac{1}{2}\,\frac{y}{h}\right) .$$

This indicates that $d\rho_c/dy > 0$. But $E_y < 0$, therefore, $E_y\, d\rho_c/dy < 0$ and the flow is unstable. The situation here is analogous to the cases of laminar boundary layers on concave walls and on heated walls. For the last two cases, Goertler demonstrated that the instability occurs in the form of standing, longitudinal cellular vortices which resemble the pattern of Taylor vortices.[11,12] It is believed that this Goertler type of vortices (Fig. 7) may also exist in the problem under consideration. Using the assumptions made in the previous sections, we show below that this is the case.

4.5.2 The equations of instability

Consider a flow past a flat plate at zero incidence; as shown in Fig. 1, the main flow is directed along z, y is normal to the surface, and x is along the surface and perpendicular to both y and z; δ is the boundary layer thickness, and w, v, u are the corresponding velocities.

If all quantities are assumed independent of z, the equations of motion for three-dimensional flow are

$$\frac{\partial u}{\partial t} + u\,\frac{\partial u}{\partial x} + v\,\frac{\partial u}{\partial y} = -\,\frac{\partial p}{\rho \partial x} + \nu\left(\frac{\partial^2 u}{dx^2} + \frac{\partial^2 u}{dy^2}\right) , \tag{4.48}$$

$$\frac{\partial v}{\partial t} + u\,\frac{\partial v}{\partial x} + v\,\frac{\partial v}{\partial y} = -\,\frac{1}{\rho}\,\frac{\partial p}{\partial y} + \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) + \frac{\rho_c}{\rho}\,E_y , \tag{4.49}$$

$$\frac{\partial w}{\partial t} + u\,\frac{\partial w}{\partial x} + v\,\frac{\partial w}{\partial y} = \nu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right) , \tag{4.50}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 . \tag{4.51}$$

And the electrodynamic equations are

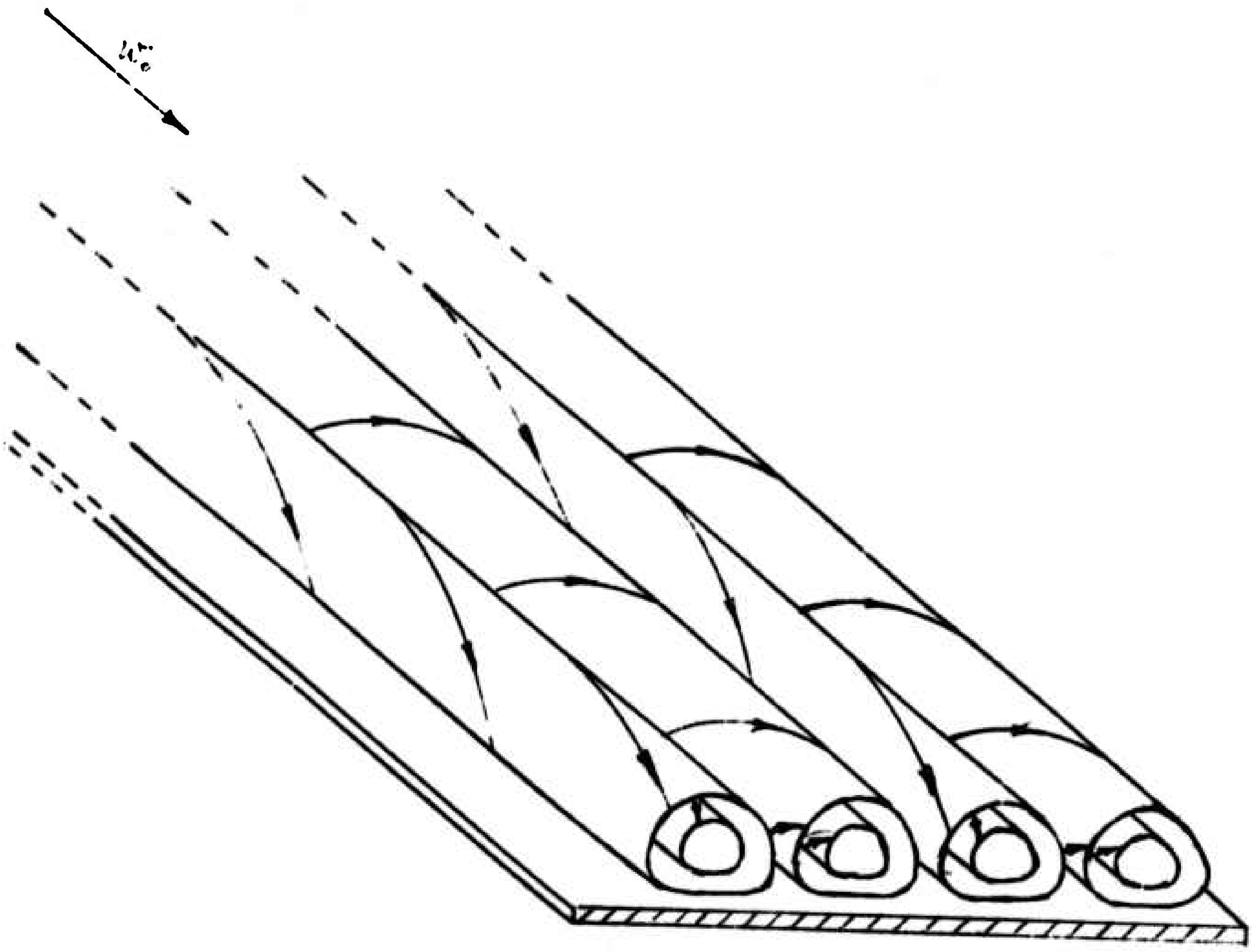

Fig. 7 - Expected Goertler vortices in laminar boundary layer over a flat plate

$$J_x = \rho_c u - D_i \frac{\partial \rho_c}{\partial x} ,$$

$$J_y = \rho_c v + K\rho_c E_y - D_i \frac{\partial \rho_c}{\partial y} ,$$

$$J_z = \rho_c w ,$$

$$\frac{\partial \rho_c}{\partial t} + u \frac{\partial \rho_c}{\partial x} + v \frac{\partial \rho_c}{\partial y} + K\rho_c \frac{\partial E_y}{\partial y} + KE_y \frac{\partial \rho_c}{\partial y} - D_i\left(\frac{\partial^2 \rho_c}{\partial x^2} + \frac{\partial^2 \rho_c}{\partial y^2}\right) = 0 \quad . \quad (4.52)$$

The undisturbed flow $u = v = 0$, $w = \bar{w}(y)$, $p = \bar{p}(y)$, $\rho_c = \bar{\rho}_c(y)$ satisfies the above equations. Assume, now, for the disturbed flow that

$$u = u_1(y) \sin \alpha x \, e^{\beta t} ,$$

$$v = v_1(y) \cos \alpha x \, e^{\beta t} ,$$

$$w = \bar{w}(y) + w_1(y) \cos \alpha x \, e^{\beta t} ,$$

$$p = \bar{p}(y) + p_1(y) \cos \alpha x \, e^{\beta t} ,$$

$$\rho_c = \bar{\rho_c}(y) + \rho_{c1}(y) \cos \alpha x \, e^{\beta t} .$$

Here β is real and denotes the amplification factor, whereas $\lambda' = 2\Pi/\alpha$ represents the wavelength of the disturbance at right angles to the main flow direction. u_1, v_1, w_1 are small quantities whose squares and products can be disregarded. The vortices have the shape shown in Fig. 7, their axes being parallel to the main flow direction.

Assuming that perturbations in electric field intensity and in electrical conductivity, $\theta = K\rho_c$, can be disregarded, we find after introducing Eq. (4.53) in Eqs. (4.48) through (4.52)

$$\beta u_1 - \frac{\alpha}{\rho} p_1 = \nu\left(\frac{d^2 u_1}{dy^2} - \alpha^2 u_1\right) \quad , \qquad (4.54)$$

$$\beta v_1 + \frac{1}{\rho} \frac{dp_1}{dy} = \nu\left(\frac{d^2 v_1}{dy^2} - \alpha^2 v_1\right) + \frac{\rho_{c1}}{\rho} E_y \quad , \qquad (4.55)$$

$$\beta w_1 + v_1 \frac{d\bar{w}}{dy} = \nu\left(\frac{d^2 w_1}{dy^2} - \alpha^2 w_1\right) \quad , \qquad (4.56)$$

$$u_1 = -\frac{1}{\alpha} \frac{dv_1}{dy} \quad , \qquad (4.57)$$

$$\beta\rho_c \quad + v_1 \frac{d\overline{\rho_c}}{dy} = D_1\left(\frac{d^2\rho_{c_1}}{dy^2} - \alpha^2\rho_{c_1}\right) \quad . \qquad (4.58)$$

Eliminating p_1 between Eqs. (4.54) and (4.55) and substituting the value of u_1 from Eq. (4.57), we obtain

$$\nu \frac{d^4v_1}{dy^4} - (\beta + 2\nu\alpha^2) \frac{d^2v_1}{dy^2} + \alpha^2(\beta + \nu\alpha^2)v_1 = \alpha^2 \frac{\rho_{c_1}}{\rho} E_y, \qquad (4.59)$$

$$\nu \frac{d^2w_1}{dy^2} - (\beta + \nu\alpha^2)w_1 = v_1 \frac{d\overline{w}}{dy} \qquad (4.60)$$

$$D_1 \frac{d^2\rho_{c_1}}{dy^2} - (\beta + D_1\alpha^2)\rho_{c_1} = v_1 \frac{d\overline{\rho_c}}{dy} . \qquad (4.61)$$

Transforming the variables, let

$$\eta = \frac{y}{\delta}, \ \overline{W}(\eta) = \frac{\overline{w}}{w_0}, \ c = \alpha\,\delta,$$

$$\sigma_1 = \frac{\beta\delta^2}{\nu}, \ \sigma_2 = \frac{\beta\delta^2}{D_1},$$

$$W_1 = \frac{w_0\delta}{\nu}^{-1} w_1,$$

$$D = \frac{d}{d\eta} ,$$

where w_0 is free stream velocity. We obtain

$$(D^2 - c^2)(D^2 - c^2 - \sigma_1)v_1 = \frac{c^2\delta^2E_y}{\mu} \rho_{c_1} , \qquad (4.62)$$

$$(D^2 - c^2 - \sigma_1)W_1 = v_1D\overline{W} \quad , \qquad (4.63)$$

$$(D^2 - c^2 - \sigma_2)\rho_{c_1} = \frac{\delta^2}{D_1} v_1 \frac{d\overline{\rho_c}}{dy} . \qquad (4.64)$$

Letting $E_y = \mathcal{E}f(\eta)$, $d\overline{\rho_c}/dy = \mathcal{S}g(\eta)$ and making the transformation

$$\frac{\delta^2\mathcal{S}}{D_1}v_1 \longrightarrow V_1$$

we obtain, from Eqs. (4.62) and (4.64)

$$\left.\begin{aligned}(D^2 - c^2)(D^2 - c^2 - \sigma_1)V_1 &= -c^2 T f \rho_{c_1} \\ (D^2 - c^2 - \sigma_2)\rho_{c_1} &= gV_1\end{aligned}\right\} \qquad (4.65)$$

where $T = -\delta^4 \mathcal{E}\alpha g/\rho\nu D_1$ is the Taylor number for the problem. The boundary conditions are

$$V_1 = DV_1 = \rho_{c_1} = 0 \text{ at } \eta = 0 \text{ and } \eta = \infty \qquad (4.66)$$

Equations (4.65), subject to (4.66), define an eigenvalue problem for T, c, and β. Neutral stability is defined by $\beta = 0$.

4.6 Discussion

The theory of hydrodynamic stability discussed in Sections 4.3 and 4.4 are primarily linear and valid for quiescent fluid only. To relate the stability problem to the experimental findings cited previously, it is necessary to take into account the effects of basic flow and finite amplitude disturbance (the non-linear theory). A complete theory about the interaction between the basic flow and the finite amplitude disturbance is as yet far from being completely developed. However, certain general understanding has been reached. Some of these will be discussed in this section.

According to the linear theory a small disturbance will grow exponentially with time. As it amplifies it must eventually reach a size such that the mean transport of momentum by the finite amplitude disturbances is appreciable and such that the associated mean stress (the Reynolds stress[3]) has an appreciable effect on the basic flow. The resulting distortion of the basic flow could strengthen the conversion of energy from the basic flow into the distrubances. Since this energy conversion is the cause of the growth of the disturbance, the rate of growth of the latter is altered. The disturbance is also modified by the generation of harmonics of the fundamental component. Thus, there is a mutual interaction between the basic and disturbance parts of the flow, and a self-distortion of the disturbance.

When the rate of conversion of energy from the basic flow to the disturbance balances the rate of viscous dissipation of kinetic energy by the disturbance, an equilibrium state can be achieved in which the disturbance has a definite finite amplitude and the basic flow exhibits definite deviation from the original laminar flow. This suggests that, in some cases, the effect of instability is to replace the original, laminar motion by another laminar motion, consisting of a basic motion and a superimposed finite disturbance. This may be referred to as the equilibrium flow (or the secondary flow), at a given Taylor number and Reynolds number, appropriate to the given mode of disturbance. If we impose the condition that the rate of discharge for the disturbed motion shall be the same as the original, laminar motion, a larger pressure drop is required because of the work which must be done by the pressure gradient to maintain the disturbance.

CONCLUDING REMARKS

Despite the efforts devoted to the explanation of the phenomenon observed in the exploratory investigation mentioned in the introduction, the problem remains not completely solved. It appears that new determinable parameters must be introduced or that one must understand the transport phenomena of space charges more completely than it is known at present. This was borne out by the fact that in Section 4, Taylor-Goertler type vortices were shown to occur under the assumption of negligible perturbation in electrical conductivity. The neglect of conduction terms in the perturbation equations needs further justification.

Lack of experimental data renders it difficult to verify the analyses presented in Sections 1 and 4. Although no numerical calculations are presented for the analysis in Section 3, it is believed that because of the smallness of the number density of ions, the contribution of ion-neutral particle collisions to the increase in pressure gradient will not be important. The analysis of Section 2 suggests that the pressure gradient increase is not to be attributed to the increase in the viscosity of the fluid because of the presence of ions.

The results of Section 4 are of particular interest. They indicate that theoretically Taylor-Goertler type of instability is possible under certain circumstances. It is recommended that extensive experiments be conducted to verify the occurrence of Taylor-Goertler vortices. If this proves to be successful, the next step will be to work out more rigorous solutions for the eigenvalue problems with various charge density distribution and check the critical Taylor number thus obtained with the experimentally measured values to verify the validity of our assumptions. Because of the highly mathematical complexity and experimental difficulty, this is a rather challenging task.

Our analysis for flow between two concentric cylinders has been restricted to the special case of quiescent fluid and small spacing. Experimentally, space charges are most conveniently provided in a round channel by a corona discharge from a thin wire located concentrically serving as anode. To facilitate comparison of theoretical results with experimental data, it appears that the problem of arbitrary spacings should be also considered. Another problem which concerns us is the modification of the analysis of Section 4.3 to the case when an axial flow is present.

In conclusion, it should be pointed out that electric fields may be used to stabilize laminar boundary layers on concave walls as well as over flat plates, provided it is possible to obtain a charge density distribution such that $E_y \, d\rho_c/dy$ satisfies the condition for stable state, i.e., $E_y \, d\rho_c/dy > 0$.

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APPENDIX I. TABULATED DATA

A. FUNCTION $f_2(\eta)$ -- (after Rossow)

η	f_2	f_2'	f_2''
0	0	0	-0.894
0.2	-0.018	-0.178	-0.888
0.4	-0.071	-0.354	-0.868
0.6	-0.159	-0.525	-0.835
0.8	-0.281	-0.687	-0.789
1.0	-0.433	-0.839	-0.730
1.2	-0.615	-0.978	-0.660
1.4	-0.824	-1.102	-0.579
1.6	-1.055	-1.209	-0.489
1.8	-1.306	-1.298	-0.394
2.0	-1.573	-1.367	-0.296
2.2	-1.852	-1.416	-0.197
2.4	-2.138	-1.446	-0.103
2.6	-2.429	-1.458	-0.017
2.8	-2.720	-1.454	+0.059
3.0	-3.010	-1.435	0.121
3.2	-3.294	-1.406	0.168
3.4	-3.572	-1.369	0.201
3.6	-3.841	-1.327	0.218
3.8	-4.102	-1.283	0.222
4.0	-4.354	-1.239	0.216
4.2	-4.598	-1.197	0.200
4.4	-4.833	-1.159	0.180
4.6	-5.062	-1.125	0.156
4.8	-5.284	-1.097	0.131
5.0	-5.501	-1.073	0.107
5.2	-5.713	-1.054	0.085
5.4	-5.923	-1.039	0.065
5.6	-6.129	-1.028	0.049
5.8	-6.334	-1.019	0.036
6.0	-6.537	-1.013	0.026
6.2	-6.739	-1.009	0.018
6.4	-6.941	-1.006	0.012
6.6	-7.141	-1.004	0.008
6.8	-7.342	-1.002	0.005
7.0	-7.542	-1.001	0.003
7.2	-7.743	-1.001	0.002
7.4	-7.943	-1.001	0.001
7.6	-8.143	-1.0	0.001
7.8	-8.343	-1.0	0
8.0	-8.543	-1.0	0
8.2	-8.743	-1.0	0
8.4	-8.943	-1.0	0
8.6	-9.143	-1.0	0

B. VARIATION OF VELOCITY PROFILES

a) $mz = 0$

η	$f_0 = \frac{w}{w_0} = \frac{w}{w_\infty}$	f_2'
0	0	0
1.0	0.32979	-0.839
2.0	0.62977	-1.367
3.0	0.84605	-1.435
4.0	0.95552	-1.239
5.0	0.99155	-1.073
6.0	0.99898	-1.013
7.0	0.99992	-1.001
8.0	1.00000	-1.0

b) $mz = 0.1$

η	mzf_2'	$\frac{w}{w_\infty}$	$\frac{w}{w_0} = \frac{w}{w_\infty} \cdot \frac{w_\infty}{w_0}$
0	0	0	0
1.0	-0.0839	0.24589	0.27320
2.0	-0.1367	0.49307	0.54785
3.0	-0.1435	0.70245	0.78050
4.0	-0.1239	0.83162	0.92400
5.0	-0.1073	0.88425	0.98250
6.0	-0.1013	0.89768	0.99740
7.0	-0.1001	0.89982	0.99980
8.0	-0.1	0.9	1.0

c) mz = 0.2

η	mzf_2'	$\frac{w}{w_\infty}$	$\frac{w}{w_0}$
0	0	0	0
1.0	-0.1678	0.16199	0.2025
2.0	-0.2734	0.35637	0.4455
3.0	-0.2870	0.55900	0.6990
4.0	-0.2478	0.70772	0.8846
5.0	-0.2146	0.77695	0.9712
6.0	-0.2026	0.79638	0.9955
7.0	-0.2000	0.79972	0.9996
8.0	-0.2	0.8	0.1

d) mz = 0.3

η	mzf_2'	$\frac{w}{w_\infty}$	$\frac{w}{w_0}$
0	0	0	0
1.0	-0.2517	0.07757	0.01108
2.0	-0.4101	0.21967	0.31380
3.0	-0.4305	0.41555	0.59360
4.0	-0.3717	0.58382	0.8340
5.0	-0.3219	0.66965	0.9566
6.0	-0.3039	0.69508	0.9927
7.0	-0.3003	0.69962	0.9994
8.0	-0.3	0.7	1.0

C. BOUNDARY LAYER THICKNESS

mz	η_δ	$\frac{\delta}{z}$
0	4.96	$\frac{4.96}{\sqrt{Rez}}$
0.1	5.5	$\frac{5.5}{\sqrt{Rez}}$
0.2	5.774	$\frac{5.774}{\sqrt{Rez}}$
0.3	5.92	$\frac{5.92}{\sqrt{Rez}}$

D. LOCAL COEFFICIENT OF SKIN FRICTION

mz	0	0.1	0.2	0.3
C_f'	$\frac{0.644}{Rez}$	$\frac{0.4652}{Rez}$	$\frac{0.2864}{Rez}$	$\frac{0.1076}{Rez}$

E. C_f' VERSUS CHARGE NUMBER

Rez	$\sqrt{Rez}\, C_f'$
10^4	$0.644 - 1.788 \times 10^{-4} N_{\rho cz}$
5×10^4	$0.644 - 0.3576 \times 10^{-4} N_{\rho cz}$
10^5	$0.644 - 1.788 \times 10^{-5} N_{\rho cz}$
5×10^5	$0.644 - 0.3576 \times 10^{-5} N_{\rho cz}$
10^6	$0.644 - 1.788 \times 10^{-6} N_{\rho cz}$

APPENDIX II. APPLICATION OF LAPLACE TRANSFORMATION

The simultaneous equations to be solved are

$$\begin{cases} \left(\dfrac{d^2}{dy^2} - C_1\right) w^N + C_1 w^{(i)} = C_o & \text{(II-1)} \\[2ex] - C_2 w^N + \left(\dfrac{d}{dy} + C_2\right) w^{(i)} = 0 \quad . & \text{(II-2)} \end{cases}$$

The boundary conditions are

$$w^N = 0 \text{ at } y = h \quad ; \quad w^{(i)} = w^N = 0 \text{ at } y = 0 \quad .$$

The Laplace transforms of each term of Eqs. (II-1) and (II-2) are listed as follows

$$L\left\{\frac{d^2 w^N}{dy^2}\right\} = S^2 H(S) - S w^N(o) - \left.\frac{dw^N}{dy}\right/_{y=0} = S^2 H(S) - \left.\frac{dw^N}{dy}\right/_{y=0} \quad ,$$

$$L\{w^N\} = H(S) \; ,$$

$$L\{w^{(i)}\} = H^{(i)}(S) \; ,$$

$$L\left\{\frac{dw^{(i)}}{dy}\right\} = SH^{(i)}(S) - w^{(i)}(o) = SH^{(i)}(S) \; .$$

After transformation Eqs. (II-1) and (II-2) become

$$\begin{cases} (S^2 - C_1) H(S) + C_1 H^{(i)}(S) = \dfrac{C_o}{S} + \left.\dfrac{dw^N}{dy}\right/_{y=0} & \text{(II-3)} \\[2ex] - C_2 H(S) + (S + C_2) H^{(i)}(S) = 0 & \text{(II-4)} \end{cases}$$

Elimination of $H^{(i)}(S)$ gives, after some algebraic manipulation and denoting $\left.\dfrac{dw^N}{dy}\right/_{y=0}$ by B

$$H(S) = -\left(\frac{C_0}{C_1} + \frac{C_2^2 C_0}{C_1^2} + \frac{C_2}{C_1} B\right)\frac{1}{S} - \left(\frac{C_2 C_0}{C_1}\right)\frac{1}{S^2}$$

$$+ \left[\frac{C_0 C_2}{C_1} + \frac{C_2^3 C_0}{C_1^2} + \frac{C_0 C_2}{C_1} + \left(\frac{C_2^2}{C_1} + 1\right)B\right]\frac{1}{S^2 + C_2 S - C_1}$$

$$+ \left(\frac{C_0}{C_1} + \frac{C_2^2 C_0}{C_1^2} + \frac{C_2}{C_1} B\right)\frac{S}{S^2 + C_2 S - C_1}$$

where

$$S^2 + C_2 S - C_1 = [S - (-\tfrac{1}{2} C_2 + \tfrac{1}{2}\sqrt{C_2^2 + 4C_1})][S - (-\tfrac{1}{2} C_2 - \tfrac{1}{2}\sqrt{C_2^2 + 4C_1})]$$

Using the inverse transform formula

$$L^{-1}\left\{\frac{1}{S}\right\} = 1 \quad , \quad L^{-1}\left\{\frac{1}{S^2}\right\} = y \quad ,$$

$$L^{-1}\left\{\frac{1}{(S-a)(S-b)}\right\} = \frac{1}{a-b}\,(e^{at} - e^{bt}) \ ,$$

$$L^{-1}\left\{\frac{S}{(S-a)(S-b)}\right\} = \frac{1}{a-b}\,(ae^{at} - be^{bt}) \quad ,$$

we find

$$w^N = -\left(\frac{C_0}{C_1} + \frac{C_2^2 C_0}{C_1^2} + \frac{C_2}{C_1} B\right) - \frac{C_2 C_0}{C_1} y + \left[\left(\frac{C_2^3 C_0}{C_1^2} + \frac{C_2 C_0}{C_1} + \frac{C_2 C_0}{C_1}\right.\right.$$

$$\left.\left. + \frac{C_2^2}{C_1} + 1\, B\right)\right]\left(\frac{1}{\sqrt{C_2^2 + 4C_1}}\right)(e^{C_5 y} - e^{C_6 y}) + \left(\frac{C_0}{C_1} + \frac{C_2^2 C_0}{C_1^2}\right.$$

$$\left. + \frac{C_2}{C_1} B\right)\left(\frac{1}{\sqrt{C_2^2 + 4C_1}}\right)(C_5 e^{C_5 y} - C_6 e^{C_6 y})$$

where

$$C_5 = -\tfrac{1}{2} C_2 + \tfrac{1}{2}\sqrt{C_2^2 + 4C_1} \quad ,$$

$$C_6 = -\tfrac{1}{2} C_2 - \tfrac{1}{2}\sqrt{C_2^2 + 4C_1} \quad .$$

Since $w^N = 0$ at $y = h$, it is found that

$$B = \frac{\sqrt{C_2^2 + 4C_1}\left(\frac{C_0}{C_1} + \frac{C_2^2 C_0}{C_1^2} + \frac{C_2 C_0 h}{C_1}\right)}{\left[(C_2 e^{C_5 h} - C_2 e^{C_6 h}) + (C_5 e^{C_5 h} - C_6 e^{C_6 h}) - \frac{C_2}{C_1}\sqrt{C_2^2 + 4C_1}\right.} \; .$$

$$- \frac{\left(\frac{2C_2 C_0}{C_1} + \frac{C_2^3 C_0}{C_1^2}\right)(e^{C_5 h} - e^{C_6 h}) + \left(\frac{C_0}{C_1} + \frac{C_2^2 C_0}{C_1^2}\right)(C_5 e^{C_5 h} - C_6 e^{C_6 h})}{\left[(C_2 e^{C_5 h} - C_2 e^{C_6 h}) + (C_5 e^{C_5 h} - C_6 e^{C_6 h}) - \frac{C_2}{C_1}\sqrt{C_2^2 + 4C_1}\right]}$$

APPENDIX III. CRITERION FOR THE ELECTROHYDRODYNAMIC STABILITY

Since the fluid is considered at rest, the equation of motions reduces to the simple form

$$\rho_c E_r = \frac{dp}{dr}$$

for the case of concentric cylindrical electrodes. This equation shows that for the fluid to be in equilibrium under the action of an electric field, the electrostatic force must be balanced by the pressure gradient dp/dr. Consider now a fluid element of charge density ρ_{c1} at radius r_1 and suppose that the element is displaced to a radius r_2 which is greater than r_1. Since the charge density of the fluid element remains unchanged, the new electrostatic force on it is $\rho_{c1} E_2$, where E_2 is the electric field intensity at radius r_2. The pressure gradient at radius r_2 supplies an inward force of magnitude $\rho_{c2} E_2$, ρ_{c2} being the original charge density at r_2. If $\rho_{c2} E_2 > \rho_{c1} E_2$, the fluid element will be forced back to its original radius and the fluid is said to be in a stable state. On the other hand, if $\rho_{c_2} E_2 < \rho_{c_1} E_2$, the fluid element will tend to move farther away from its original position and the fluid is in unstable equilibrium.

Now if we replace r_2 by r, r_1 by $r-dr$, E_2 by E, ρ_{c_2} by ρ_c, and ρ_{c_1} by $\rho_c-(d\rho_c/dr)dr$, where dr is taken as a positive quantity, the condition for stable state can be written

$$\rho_c E > \left(\rho_c - \frac{d\rho_c}{dr}\,dr\right)E$$

or

$$-E\,\frac{d\rho_c}{dr}\,dr < 0$$

Thus, for stable state $E\, d\rho_c/dr > 0$. Similarly, the condition for unstable equilibrium becomes $E\, d\rho_c/dr < 0$.

APPENDIX IV. ELECTRIC FIELD AND SPACE CHARGE DISTRIBUTIONS FOR POSITIVE CORONA BETWEEN TWO CONCENTRIC CYLINDERS

Let r_1 be the radius of the inner cylinder (or wire), r_2 the radius of the outer cylinder (or tube) and J_2 the current density at $r = r_2$. The governing equations for the determination of electric field and charge density distributions are

$$\frac{dEr}{dr} + \frac{Er}{r} = \frac{\rho c}{\epsilon} , \tag{IV-1}$$

$$J = \rho_c K E_r = \frac{r_2 J_2}{r} . \tag{IV-2}$$

Combining Eqs. (IV-1) and (IV-2), we obtain

$$\frac{1}{r} \frac{d}{dr} (rE_r) = \frac{r_2 J_2}{\epsilon K r E_r}$$

or

$$\frac{1}{2} \frac{1}{4} \frac{d}{dr} (r^2 E_r{}^2) = \frac{r_2 J_2}{\epsilon K} . \tag{IV-3}$$

The solution of Eq. (IV-3) satisfying the boundary condition $E_r = 0$ at $r = r_1$ is

$$E_r = \left[\frac{r_2 J_2}{\epsilon K} \quad 1 - \frac{r_1^2}{r^2}\right]^{1/2} . \tag{IV-4}$$

From Eq. (IV-2) ρ_c is found to be

$$\rho c = \left[\frac{\epsilon J_2 r_2}{K} \frac{1}{r^2 - r_1^2}\right]^{1/2} , \tag{IV-5}$$

and

$$\frac{d\rho_c}{dr} = - \left[\frac{\epsilon J_2 r_2}{K} \frac{r^2}{(r^2 - r_1^2)^3}\right]^{1/2} .$$

Therefore,

$$E_r \frac{d\rho_c}{dr} = - \frac{J_2 r_2}{K} \frac{1}{(r^2 - r_1^2)} < 0$$

and the fluid is in unstable equilibrium.

APPENDIX V. THE NEGLECT OF CONDUCTION TERMS IN PERTURBATION EQUATIONS

The current density, as given in Section 4.1, is

$$\vec{J} = \rho_c\vec{v} + K\rho_c\vec{E} - D_i\nabla\rho_c \qquad (4.5)$$

By definition, electrical conductivity of a fluid is given by

$$\theta = \rho_c^{+}K^{+} + \rho_c^{-}K^{-}$$

where

θ = electrical conducitivity

ρ_c^{+} = positive charge density

ρ_c^{-} = negative charge density

K^{+} = positive ion mobility

K^{-} = negative ion mobility

For the problem at hand, $\rho_c^{-} = 0$, thus $\theta = \rho_c^{+}K^{+}$ and Eq. (4.5) can be written as

$$\vec{J} = \rho_c\vec{v} + \theta\vec{E} - D_i\nabla\rho_c.$$

Now, if we assume that perturbations in electric field intensity as well as in electrical conductivity are negligible, Eq. (4.9) assumes the form

$$\vec{J} = \rho_c\vec{v} - D_i\nabla\rho_c'$$

and Eq. (4.15) becomes

$$\frac{\partial\rho_c'}{\partial t} + u'\frac{\partial\bar{\rho_c}}{\partial r} - D_i\nabla^2\rho_c' = 0$$

Upon going through the same procedure as in Section 4.3, we finally obtain Eq. (4.29).

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13. ABSTRACT

Unusually high pressure drops and flow distortions were observed in a previous experimental program involving laminar flow of a gas in a channel under the action of a corona discharge in a transverse electric field. A hypothesis postulated by Velkoff to explain the phenomenon is extended to the case of laminar boundary layer flow over a flat plate. The problem on hand is found to be analogous to the laminar boundary layer flow in a transverse magnetic field. Three other mechanisms proposed to interpret the above experimental findings are also investigated. The increase in viscosity of a gas because of the ions is not likely and, because of the smallness of ion density, the effect on ion-neutral particle interactions on the flow is believed to be small. One possible mechanism which may explain the phenomenon is the secondary flow resulting from electro-hydrodynamic instability. It is found theoretically that Taylor vortices can be induced in a quiescent fluid between two concentric cylinders under the action of a corona discharge. The Taylor Number of the problem is defined and shown to represent the ratio of the destabilizing electrostatic force to the stabilizing viscous force. It is also found that Goertler vortices can occur in laminar boundary layer over a flat plate provided the applied electric field and the charge density distribution satisfy the condition for instability.

14. KEY WORDS	LINK A ROLE	LINK A WT	LINK B ROLE	LINK B WT	LINK C ROLE	LINK C WT
Electrohydrodynamic instability						
Taylor vortex						
Goertler vortex						
Taylor number						
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