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KINETIC EQUATIONS FOR PLASMAS

by

Toyoki Koga

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by

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SUMMARY

Because of the time scale assumption necessary for its derivation, the Bogoliubov-Born-Green-Kirkwood-Yvon (BBGKY) hierarchy of equations is not applicable to systems consisting of charged particles, except for special subsystems. A new class of equations governing the evolutions of charged particles is derived from the Liouville equation and is coarsed-grained with respect to time and similar particles. In the zeroth approximation, there are two basic types of inter-particle interactions: One is of the Vlasov type and the other of the Boltzmann type characterizing interactions among nearest neighboring particles. In higher order approximations, mutual perturbations among those basic interactions result in secondary effects; for example, two nearest-neighboring particles exert a force of microscopic order to another particle. Depending on the ratio between the

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number of electrons and the number of ions in a real system, the simulating model varies. The main purpose of the paper is to present schemes of rational treatment, rather than to provide numerical results in detail for a particular system.

I. INTRODUCTION

The Boltzmann collision integral, based on the binary collision assumption and the Stosszahlansatz (the assumption of collision number), diverges in the case of charged particles. As a more radical approach, many authors[1] have attempted to investigate ionized gases by treating the Bogoliubov-Born-Green-Kirkwood-Yvon (BBGKY) hierarchy of equations derived from the Liouville equation. As has been argued often by the present author[2], however, the BBGKY hierarchy is not feasible for consideration as the basis of treating ionized gases in general, because of the time scale assumption necessary for the derivation of the hierarchy; the time scale relation necessary for the derivation differs, except for special cases, from the time scale relation conceivable in a real system of charged particles. The purpose of the present article is to propose schemes of treating the Liouville equation feasible for ionized gases.

In Section II, we shall integrate partly the Liouville equation to derive equations of evolution of distributions of sub-systems. After investigating possible time scales of dynamical processes in real systems in Section III, we shall coarse-grain these equations for sub-systems with respect to time and

similar particles so that they will yield kinetic equations for plasmas in the following sections. Each of the resultant equations has its limited regime of feasibility; the feasibility depends on the nature and the condition of an ionized gas under consideration. Providing numerical results in detail for a particular system is left for future publication.

II. EQUATIONS OF EVOLUTION OF SUB-SYSTEMS

We consider a system consisting of N particles which are represented by material points with masses $m_1, m_2, \ldots, m_N$. The distribution of the system in the 6N-dimensional phase space is denoted by $D^{(N)}$. Function $D^{(N)}$ satisfies the Liouville equation

$$\left(\frac{\partial}{\partial t} + \sum_{i=1}^{N} \frac{p_i}{m_i} \cdot \frac{\partial}{\partial r_i} + \sum_{i=1}^{N} \mathcal{F}_i \cdot \frac{\partial}{\partial p_i} \right) D^{(N)} = 0 \tag{2.1}$$

where p_i is the momentum of particle i, r_i the position vector of the same particle, and

$$\mathcal{F}_i = \mathcal{F}_{io} + \sum_j \mathcal{F}_{ij}$$

$\mathcal{F}_i$ is the total force exerted on particle i, $\mathcal{F}_{io}$ the external force, $\mathcal{F}_{ij}$ the force exerted by particle j. Since we are investigating the evolution of a single system, $D^{(N)}$ is known to be given by

$$D^{(N)} = \prod_{i=1}^{N} \delta \left(X_i - X_i^*(t) \right) \tag{2.2}$$

where X_i denotes the six-dimensional vector p_i, r_i, and $X_i^*(t)$ represents the trajectory of particle i obtained by solving the equations of motion of the N particles[3]. The equations of motion

constitute the characteristic equations of Eq. (2. 1). Of course, $D^{(N)}$ given above satisfies Eq. (2. 1). It is noted that X_i is an independent variable while X_i^* is a function of time. We may define $F^{(s)}$'s by

$$F^{(1)}(X_i;t) = V \int D^{(N)} \prod_{j \neq i} dX_j$$

$$F^{(2)}(X_i X_j;t) = V^2 \int D^{(N)} \prod_{k \neq i,j} dX_k \qquad (2.3)$$

etc.

where V is invariant and denotes the volume of the space within which the system is known to exist throughout the period of time of our investigation. Since the velocity of a particle is finite, V is also finite. It is convenient to define

$$F^{(1)}(i) = F^{(1)}(X_i;t)$$

$$F^{(2)}(ij) = F^{(2)}(X_i X_j;t), \quad \text{if } i < j$$

$$= F^{(2)}(X_j X_i;t), \quad \text{if } i > j$$

$$= 0 \qquad , \text{ if } i = j \qquad (2.3)'$$

etc.

On integration of each term of Eq. (2.1) with respect to the phase-space coordinates of all the particles, except for particle i, we have

$$\left(\frac{\partial}{\partial t}+\frac{\mathbf{p}_i}{m_i}\cdot\frac{\partial}{\partial \mathbf{r}_i}+\mathcal{F}_{io}\cdot\frac{\partial}{\partial \mathbf{p}_i}\right)F^{(1)}(i)$$

$$+\frac{1}{V^{\nu}}\sum_{j\neq\ldots\neq k}\int\left(\mathcal{F}_{ij}+\ldots+\mathcal{F}_{ik}\right)\cdot\frac{\partial}{\partial \mathbf{p}_i}F^{(\nu+1)}(ij\ldots k)\,dX_j\ldots dX_k=0 \qquad (2.4)$$

Here $\sum_{j\neq\ldots\neq k}$ means the summation of similar factors, each of which is formulated with respect to one of the ν-particle subsystems made by grouping the particles, except for particle i, of the system. Grouping of the particles to ν-particle subsystems is possible only when $(N-1)/\nu$ is an integer, and the number of the subsystems is $(N-1)/\nu$. If $(N-1)/\nu$ is not an integer, for example 101.36, we have to admit that one of the 102 subsystems consists of less than ν particles. In general $(N-1) \gg \nu$, and hence the number of the subsystems is large so that we may ignore the peculiarity of one subsystem. Therefore we always treat $(N-1)/\nu$

as if an integer. For i, there are N different choices. When i is given, the number of different ways of dividing the remaining N - 1 particles into ν-particle subsystems is

$$\frac{(N-1)!}{(\nu!)^{(N-1)/\nu}\,[(N-1)\nu]!} \tag{2.5}$$

In the microscopic sense, the implication of Eq. (2.4) does not change by changing the way of grouping.

It is easily shown that

$$\frac{1}{V^{\nu}} \int \mathcal{F}_{ij} \cdot \frac{\partial}{\partial p_i} F^{(\nu+1)}(ij\ldots k)\, dX_j \ldots dX_k = \frac{1}{V} \int \mathcal{F}_{ij} \cdot \frac{\partial}{\partial p_i} F^{(2)}(ij)\, dX_j$$

and hence we may write for Eq. (2.4)

$$\left(\frac{\partial}{\partial t} + \frac{p_i}{m_i} \cdot \frac{\partial}{\partial r_i} + \mathcal{F}_{io} \cdot \frac{\partial}{\partial p_i} \right) F^{(1)}(i) + \frac{1}{V} \sum_j \int \mathcal{F}_{ij} \cdot \frac{\partial}{\partial p_i} F^{(2)}(ij)\, dX_j = 0 \tag{2.6}$$

where particle j represents all the particles of the system except for particle i. As is well-known, Eq. (2.6) is the very equation

which yields, after being coarse-grained, the first equation of the BBGKY hierarchy. There is no difference between Eq. (2.4) and Eq. (2.6), as long as we consider them in the precise and microscopic sense. After being coarse-grained, however, the two resultant equations are different. The difference is due to the fact that the equations of motion of particles in mutual interaction are non-linear, and interactions are more multiple than binary.

For the equation of evolution of $F^{(\nu+1)}$, we obtain, by integrating properly Eq. (2.1)

$$\left[\frac{\partial}{\partial t}\cdot + \frac{p_i}{m_i}\cdot\frac{\partial}{\partial r_i} + \frac{p_j}{m_j}\cdot\frac{\partial}{\partial r_j} + \cdots + \frac{p_k}{m_k}\cdot\frac{\partial}{\partial r_k}\right.$$
$$+ (\tilde{\mathcal{F}}_{io} + \tilde{\mathcal{F}}_{ij} + \cdots + \tilde{\mathcal{F}}_{ik})\cdot\frac{\partial}{\partial p_i}$$
$$+ (\tilde{\mathcal{F}}_{jo} + \tilde{\mathcal{F}}_{ji} + \cdots + \tilde{\mathcal{F}}_{jk})\cdot\frac{\partial}{\partial p_j}$$
$$+ \cdots$$
$$\left. + (\tilde{\mathcal{F}}_{ko} + \tilde{\mathcal{F}}_{ki} + \tilde{\mathcal{F}}_{kj} + \cdots)\cdot\frac{\partial}{\partial p_k}\right] F^{(\nu+1)}(ij\ldots k)$$
$$+ \frac{1}{V^{\nu'}}\sum_{l,\ldots,m}\int\left[(\tilde{\mathcal{F}}_{il} + \cdots + \tilde{\mathcal{F}}_{im})\cdot\frac{\partial}{\partial p_i}\right.$$
$$+ (\tilde{\mathcal{F}}_{jl} + \cdots + \tilde{\mathcal{F}}_{jm})\cdot\frac{\partial}{\partial p_j}$$
$$+ \cdots$$
$$\left. + (\tilde{\mathcal{F}}_{kl} + \cdots + \tilde{\mathcal{F}}_{km})\cdot\frac{\partial}{\partial p_k}\right] F^{(\nu+1+\nu')}(ij\ldots kl\ldots m)$$
$$dX_l\ldots dX_m = 0 \qquad (2.6)$$

In the above, in general, it is not trivial to choose a proper value for ν'. As has been stated with respect to coarse-graining of Eq. (2.4), coarse-graining of Eq. (2.6) with a different value for ν' leads to an equation of different physical implications. We assume, however, that ν is sufficiently large that the effect of additional ν' particles is trivial; this assumption is of a nature similar to that of the assumption of binary collision where $\nu = 1$ and $\nu' = 0$.

Further we may define $f\,dX$ by

$$f(X)\,dX = \frac{1}{V} \sum_{i=1}^{N} \int_{X}^{X+dX} F^{(1)}(X_i)\,dX_i. \tag{2.7}$$

$f\,dX$ gives the number of the particles which exist in the states between X and X + dX.

III. GENERAL SCHEMES OF DERIVATION OF KINETIC EQUATIONS

First of all, we should realize that the limitations of our mathematical ability prevents us from treating precisely more than two body problem .. Therefore our effort must be directed toward converting N-body problems to a combination of many two-body problems in an approximation.

Suppose that particle 1 is the test particle of which the time evolution of our main concern. One of the field particles is named by 2. What is the effect of particle 2 on the test particle? If particle 2 happens to come close to the test particle within a certain distance, they will interact with each other as if they are isolated from the other field particles: their mutual interaction is much stronger than their interactions with the other field particles. But such a strong and distinctive interaction continues to exist only for a short period of time. Particle 2 will soon be remote from the test particle, and particle 2 will be in a strong interaction with its new nearest neighbor, particle 3. Of course, particle 2 does yet continue to exert a force on the test particle. But the behavior of particle 2 is almost independent of the behavior of particle 1 and hence the force exerted by particle 2 on particle 1 as a function of time,

changes according to the nature of the interaction between particle 2 and particle 3; the behavior of particle 2 appears as a matter of fluctuation from the view point of particle 1. In spite of such fluctuations appearing in the behavior of particle 2, the state of particle 2 is steadily localized in a larger scale, because of the finiteness of its velocity. Therefore, the average of the fluctuating force due to particle 2 over a proper time period does not vanish; particle 2 exerts a weak but almost stationary force on particle 1. Such weak forces due to many field particles of similarly localized states culminate to the Vlasov type force exerted on particle 1. **See Fig.3.1.**

According to the above investigation, there are at least three different time scales of the force exerted on particle 1 by particle 2: 1) the time scale of the close interaction between the test particle and particle 2; 2) the time scale of the close interaction between a field particle and particle 2; 3) the time scale of the average fligh of particles 2. If all the particles are of the same single species, time scale 1 and time scale 2 are of the same order. On the other hand, if the system under consideration is composed of particles of more than two species, the time scale of the interaction between the test particle and particle 2 may be different from the time scale of the interaction between particle 2 and a third particle;

the latter may also vary depending on the species of the third particle. Finally the time scale of the third category is almost macroscopic.

A conclusion derived from the above consideration is that, in the simplest case where the system is constituted of particles of a single species, it is necessary to take for ν in Eq. (2.4) at least 2 so that the close interaction between two neighboring field particles is taken into consideration with the same time scale as that of the close interaction between the test particle and a field particle. This caution is essential for coarse graining Eq. (2.4) with respect to time.

Since it has been realized that consideration of the time scales of close interactions between two nearest neighboring particles is essential, we investigate the orders of such time scales with respect to typical particles in the following:

By assuming that the gas under consideration is fully ionized the constituent particles are ions and electrons. It is also assumed that those ions are of a single species. The charge of an electron is denoted by $-e$ the mass by m_e, and the total number of the electrons in the system by N_e. The charge of an ion is denoted by ε, the mass by m_I and the total number of ions by N_I. In general we may assume that

$$-N_e e + N_I \varepsilon = 0 \tag{3.1}$$

$$m_I / m_e \geq 2000 \tag{3.2}$$

We also assume that

$$\frac{\varepsilon}{e} \ll \frac{m_I}{m_e} \tag{3.3}$$

Due to relation (3.2) the velocity of an electron is usually much larger than the velocity of an ion, not only when the gas is in thermal equilibrium, but also when the state is non-uniform: In thermal equilibrium, it may be easily shown that

$$\frac{\text{(the average magnitude of the velocity of electron)}}{\text{(the average magnitude of the velocity of an ion)}} \geq 50 \tag{3.4}$$

If the gas is not in thermal equilibrium, the situation is not as simple. However, if the acceleration of an electron and the acceleration of an ion occur due to the same external electromagnetic field relation (3.4) may be a reasonable expectation, because the acceleration of an ion is $(\varepsilon/e)(m_e/m_I)$ times larger than the acceleration of an electron.

The time scale of the interaction between two nearest neighbroing electrons is of the order of

$$\tau_{ee} = n_e^{-1/3} / \langle c_e \rangle \tag{3.5}$$

where n_e is the number density of electrons and $\langle c_e \rangle$ is the average speed of an electron. As compared with the average

speed of an electron, the average speed of an ion is much smaller; and an ion appears as if it is at rest, while an electron appears to move swiftly. The time scale of the interaction between an electron and its nearest neighboring ion is of the order of

$$\tau_{eI} = n_I^{-\frac{1}{3}} / \langle c_e \rangle \qquad (3.6)$$

where $\langle c_e \rangle$ is the average velocity of an electron and n_I is the number density of ions. Since n_I and n_e are assumed to be of the same order, we have

$$\tau_{eI} \simeq \tau_{ee}$$

On the other hand, the time scale of the interaction between two nearest-neighboring ions is of the order of

$$\tau_{II} = n_I^{-\frac{1}{3}} / \langle c_I \rangle \qquad (3.7)$$

and is much longer than τ_{eI} and / or τ_{ee} due to $\langle c_I \rangle \gg \langle c_e \rangle$

In summation, we have

$$\tau_{II} \gg \tau_{eI} \simeq \tau_{ee} \qquad (3.8)$$

IV. MODEL I

4. 1. General Consideration

The first model is a system constituted of similar particles and a uniform background of electric charge which compensates the total charge of the particles. The present model is simply a model by which the procedure of manipulations typical for our schemes of approach is demonstrated. It is too hasty to assume that the model simulates either the ion system or the electron system in a real plasma. As is discussed in subsection 4. 10, it is necessary to investigate model II as is given in the next section for evaluating more precisely all the effects of possible interactions on the evolution of the ion distribution as well as of the electron distribution.[4] A test particle may experience three main sorts of interaction with the field particles. 1) Strong interaction with its nearest neighbor; 2) Interaction with particles which are not nearest neighbors but are fairly close to it; each of those field particles has its own nearest neighbor and exerts a fluctuating force to the test particle. 3) The force exerted by remote field particles; the force due to each field particle is trival but culminates to a Vlasov type force in the macroscopic sense.

Let us take particle i for the test particle; by taking 2 for ν, Eq. (2.4) yields

$$\left(\frac{\partial}{\partial t} + \frac{p_i}{m} \cdot \frac{\partial}{\partial r_i}\right) F^{(1)}(i) + \frac{1}{V^3} \sum_{j \neq k} \int (\mathcal{F}_{ij} + \mathcal{F}_{ik}) \cdot \frac{\partial}{\partial p_i} F^{(3)}(ijk)\, dX_j dX_k = 0 \tag{4.1}$$

and Eq. (2.6) leads to

$$\left[\frac{\partial}{\partial t} + \frac{p_i}{m} \cdot \frac{\partial}{\partial r_i} + \frac{p_j}{m} \cdot \frac{\partial}{\partial r_j} + \frac{p_k}{m} \cdot \frac{\partial}{\partial r_k} + (\mathcal{F}_{ij} + \mathcal{F}_{ik}) \cdot \frac{\partial}{\partial p_i} + (\mathcal{F}_{ji} + \mathcal{F}_{jk}) \cdot \frac{\partial}{\partial p_j} + (\mathcal{F}_{ki} + \mathcal{F}_{kj}) \cdot \frac{\partial}{\partial p_k}\right] F^{(3)}(ijk) = 0 \tag{4.2}$$

Here

$$F^{(3)}(ijk) = V^3 \int D^{(N)} \pi\, dX_\ell \qquad (\ell \neq i, j, k) \tag{4.3}$$

$$= F^{(3)}(X_i X_j X_k; t), \text{ if } i < j < k$$

$$= F^{(3)}(X_j X_i X_k; t), \text{ if } j < i < k$$

etc.

Equation (4. 1) is a partial differential equation of $F^{(1)}(X_i;t)$ with X_i and t for the independent variables. It is possible, in general, to determine $F^{(1)}(X_i;t)$ along a certain trajectory during the time period from t to t + s. If the trajectory is specified by

$$X_i' = X_i + x_i'(s), \qquad X_i' = p_i', r_i' \tag{4.4}$$

$$x_i'(0) = 0 \qquad p_i'(0) = p_i, r_i'(0) = r_i$$

the function is given, along the trajectory, by

$$F^{(1)}(X_i' ; t + s) \tag{4.5}$$

Considering the above, we coarse-grain Eq. (4. 1) with respect to s from 0 to τ along trajectories determined by

$$\left.\begin{aligned} p_i' &= p_i \qquad \text{(independent of s)} \\ \frac{dr_i'}{dt} &= \frac{p_i}{m} \\ r_i'(0) &= r_i \qquad \text{(trajectory 0)} \end{aligned}\right\} \tag{4.6}$$

From now on, trajectories determined by Eq. (4.6) will be referred to by class 0, and we assume that τ is much shorter than the macroscopic time scale and much longer than the time scale of interaction between two neighboring particles. The coarse-graining of the first two members of Eq. (4.1) leads to

$$\frac{1}{\tau}\int_0^{\tau}\left[\left(\frac{\partial}{\partial s}+\frac{p_i'}{m}\cdot\frac{\partial}{\partial r_i'}\right)F^{(1)}(X_i';\,t+s)\right]ds \qquad \text{along } 0$$

$$=\frac{1}{\tau}\int_0^{\tau}\frac{d}{ds}F^{(1)}(X_i';t+s)ds$$

$$=\left(\frac{\partial}{\partial t}+\frac{p_i}{m}\cdot\frac{\partial}{\partial r_i}\right)<F^{(1)}(i)>_0 \qquad (4.7)$$

where

$$<F^{(1)}(i)>_0 \;=\frac{1}{\tau}\int_0^{\tau}F^{(1)}(X_i';t+s)\,ds$$

(Along trajectories 0) (4.8)

The integral term in Eq. (4.1), by the same coarse-graining, yields

$$\langle J\rangle_0=\frac{1}{V^2}\;\frac{1}{\tau}\int_0^{\tau}\sum_{jk}\left[\int(\tilde{\mathcal{F}}_{ij}'+\tilde{\mathcal{F}}_{ik}')\cdot\frac{\partial}{\partial p_i'}F^{(3)}(X_i'X_jX_k;t+s)dX_jdX_k\right]ds \qquad (4.9)$$

Note that the time average is made after the summation $\sum_{jk}$.

Here X'_i is a function of s according to Eq. (4.6) and Σ_{jk} means that similar factors are to be summed up with respect to (N-1)/2 pairs of particles. See (2.5). It is noted that the factor of a pair cannot be decomposed into two factors, each made with respect to one single field particle, after the coarse-graining with respect to time; the reason is that the interaction among three particles i, j and k is a non-linear phenomenon. Hence the equation of evolution of $< F^{(1)}(i) >_0$ is given

$$\left(\frac{\partial}{\partial t} + \frac{p_i}{m} \cdot \frac{\partial}{\partial r_i}\right) < F^{(1)}(i) >_0 + < \quad >_0 = 0 \tag{4.10}$$

In order to obtain $F^{(3)}$ to be inserted in the above integrand, it is necessary to solve Eq. (4.2), a three body problem. The solution must be given along trajectories 0 determined by Eq. (4.6). In the following however, we obtain particular solutions of Eq. (4.2) integrated along several different classes of trajectories of particle i, j and k by successive approximation methods. Later the differences among the results due to the differences among these classes of trajectories are eliminated by consideration of proper coarse-graining operations with respect to similar particles. For the purpose of providing

$F^{(3)}$ during the time between t and t + s we rewrite Eq. (4.2) as follows:

$$\left[\frac{\partial}{\partial s} + \frac{p'_i}{m}\cdot\frac{\partial}{\partial r'_i} + \frac{p'_j}{m}\cdot\frac{\partial}{\partial r'_j} + \frac{p'_k}{m}\cdot\frac{\partial}{\partial r'_k} + (\tilde{\mathcal{F}}'_{ij} + \tilde{\mathcal{F}}'_{ik})\cdot\frac{\partial}{\partial p'_i}\right.$$

$$\left. + (\tilde{\mathcal{F}}'_{ji} + \tilde{\mathcal{F}}'_{jk})\cdot\frac{\partial}{\partial p'_j} + (\tilde{\mathcal{F}}'_{ki} + \tilde{\mathcal{F}}'_{kj})\cdot\frac{\partial}{\partial p'_k}\right] F^{(3)}(X'_i X'_j X'_k; t+s) = 0 \qquad (4.11)$$

As is stated in Appendix A, $F^{(3)}$ is invariant, if Eq. (4.11) is integrated along a trajectory specified by X'_i, X'_j, X'_k obtained as functions of s by solving Eqs. (A.1). Since solution of Eqs. (A.1) is a three-body problem, we consider combinations of two-body problems which approximate the three-body problem under various circumstances. For each case of approximation, we take a different set of differential equations in place of Eqs. (A.1). For each case, the initial conditions are always given by

$$X'_i = X_i\,,\quad X'_j = X_j\,,\quad X'_k = X_k \qquad (4.12)$$
$$s = 0$$

Each different set of differential equations replaced for Eqs. (A.1) determines a different set of trajectories. Along those trajectories, $F^{(3)}$ is no more invariant; it is a function of s. In each case, $F^{(3)}$ is obtained by solving Eq. (4.11) by a different sucessive approximation method. In order to

introduce proper sets of differential equations in place of Eqs. (A. 1), we define, as is detailed in appendix B, W_{ij} which is the probability that particle i and particle j are their mutual nearest neighbors. Similarly we may define W_{ik} and W_{jk}

$$W_{ij}W_{ik} = 0, \quad \text{if } j \neq k. \tag{4. 13}$$

Then, defined by

$$F(i/j/k) = (1-W_{ij})\,(1-W_{ik})(1-W_{jk})\,F^{(3)}(ijk)$$

$$= (1-W_{ij} - W_{ik} - W_{jk})F^{(3)}(ijk) \tag{4. 14}$$

F(i/j/k) is the joint distribution of particle i, particle j and particle k among which none is probably the nearest neighbor of the others. Similarly we define

$$F(ij/k) = W_{ij} F^{(3)}(ijk)$$

$$F(ik/j) = W_{ik} F^{(3)}(ijk) \qquad (4.15)$$

$$F(i/jk) = W_{jk} F^{(3)}(ijk)$$

Obviously

$$F^{(3)}(ijk) = F(i/j/k) + F(ij/k) + F(ik/j) + F(i/jk) \qquad (4.16)$$

In view of the definition of the W's, those members in the right hand side of Eq. (4.16) cannot be significant all at the same time; If one is significant, the others are trivial.

In solving Eq. (4.11) for obtaining $F^{(3)}(ijk)$ appearing in (4.14) and (4.15), we have four different methods of approximation. For example, for calculating $F^{(3)}(ijk)$ involved in $F(i/j/k)$, we may assume that interactions among the three particles are always weak, and hence, in the zeroth approximation, those three particles are regarded as in free flight. On the other hand, for $F^{(3)}(ijk)$ in $F(ij/k)$, it is essential to assume that the interaction between particle i and particle j is strong, although these two particles interact with particle k only weakly. On obtaining $F^{(3)}(ijk)$ in four different cases respectively by four different methods, the results are presented as follows:

$$\text{I.}\quad F^{(3)}_{(ijk)} = F(i/j/k)^{(0)} + F(i/j/k)^{(1)} + \cdots$$

$$\text{II.}\quad F^{(3)}_{(ijk)} = F(ij/k)^{(0)} + F(ij/k)^{(1)} + \cdots$$

$$\text{II'}\quad F^{(3)}_{(ijk)} = F(ik/j)^{(0)} + F(ik/j)^{(1)} + \cdots \qquad (4.17)$$

$$\text{III.}\quad F^{(3)}_{(ijk)} = F(i/jk)^{(0)} + F(i/jk)^{(1)} + \cdots$$

After obtaining $F^{(3)}_{(ijk)}$ in four different cases, we substitute each of the solutions to its proper place in (4.14) and (4.15). Thus for $F^{(3)}_{(ijk)}$ as presented by (4.6), we obtain

$$F^{(3)}_{(ijk)} = F^{(3)}_{(ijk)}{}^{(0)} + F^{(3)}_{(ijk)}{}^{(1)} + \cdots \qquad (4.18)$$

where

$$F^{(3)}_{(ijk)}{}^{(0)} = (1 - W_{ij} - W_{ik} - W_{jk})\, F(i/j/k)^{(0)} + W_{ij}\, F(ij/k)^{(0)} + W_{ik} F(ik/j)^{(0)} + W_{jk}\, F(i/jk)^{(0)}$$

$$F^{(3)}(ijk)^{(1)} = (1 - W_{ij} - W_{ik} - W_{jk})\, F(i/j/k)^{(1)}$$

$$+ W_{ij} F(ij/k)^{(1)} + W_{ik} F(ik/j)^{(1)} \qquad (4.19)$$

$$+ W_{jk} F(i/jk)^{(1)}$$

etc. (4.20)

In the next step of our treatment, we substitute $F^{(3)}(ijk)$ presented by (4.18) in the integral member in Eq. (4.1)

$$J = \frac{1}{V^2} \sum_{jk} \int\int (\tilde{\mathcal{F}}_{ij} + \tilde{\mathcal{F}}_{ik}) \cdot \frac{\partial}{\partial p_i} F^{(3)}(ijk)\, dX_j\, dX_k$$

$$= J^{(0)} + J^{(1)} + J^{(2)} + \cdots \qquad (4.21)$$

where $J^{(0)}$, for example, is due to $F^{(3)}(ijk)^{(0)}$ given by (4.18):

$$J^{(0)} = \frac{1}{V^2} \sum_{jk} \int\int (\tilde{\mathcal{F}}_{ij} + \tilde{\mathcal{F}}_{ik}) \cdot \frac{\partial}{\partial p_i} F^{(3)}(ijk)^{(0)}\, dX_j dX_k$$

$$= J_1^{(0)} + J_2^{(0)} + J_{2'}^{(0)} + J_3^{(0)} \qquad (4.22)$$

and

$$J_1^{(0)} = \frac{1}{V^2} \sum_{jk} \iint (\tilde{\mathcal{F}}_{ij} + \tilde{\mathcal{F}}_{ik}) \cdot \frac{\partial}{\partial P_i} (1 - W_{ij} - W_{ik} - W_{jk}) F(i/j/k)^{(0)} dX_j dX_k \tag{4.23}$$

$$J_2^{(0)} = \frac{1}{V^2} \sum_{jk} \iint (\tilde{\mathcal{F}}_{ij} + \tilde{\mathcal{F}}_{ik}) \cdot \frac{\partial}{\partial P_i} W_{ij} F(ij/k)^{(0)} dX_j dX_k \tag{4.24}$$

$$J_{2'}^{(0)} = \frac{1}{V^2} \sum_{jk} \iint (\tilde{\mathcal{F}}_{ij} + \tilde{\mathcal{F}}_{ik}) \cdot \frac{\partial}{\partial P_i} W_{ik} F(ik/j)^{(0)} dX_j dX_k \tag{4.25}$$

$$J_3^{(0)} = \frac{1}{V^2} \sum_{jk} \iint (\tilde{\mathcal{F}}_{ij} + \tilde{\mathcal{F}}_{ik}) \cdot \frac{\partial}{\partial P_i} W_{jk} F(i/jk)^{(0)} dX_j dX_k \tag{4.26}$$

Similarly we have

$$J^{(1)} = J_1^{(1)} + J_2^{(1)} + J_{2'}^{(1)} + J_3^{(1)} \tag{4.27}$$

in which

$$J_1^{(1)} = \frac{1}{V^2} \sum_{jk} \int\int (\mathcal{F}_{ij} + \mathcal{F}_{ik}) \cdot \frac{\partial}{\partial P_i} (1 - W_{ij} - W_{ik} - W_{jk}) F(i/j/k)^{(1)} \, dX_j dX_k \tag{4.28}$$

$$J_2^{(1)} = \frac{1}{V^2} \sum_{jk} \int\int (\mathcal{F}_{ij} + \mathcal{F}_{ik}) \cdot \frac{\partial}{\partial P_i} W_{ij} F(ij/k)^{(1)} dX_j dX_k \tag{4.29}$$

$$J_{2'}^{(1)} = \frac{1}{V^2} \sum_{jk} \int\int (\mathcal{F}_{ij} + \mathcal{F}_{ik}) \cdot \frac{\partial}{\partial P_i} W_{ik} F(ik/j)^{(1)} dX_j dX_k \tag{4.30}$$

$$J_3^{(1)} = \frac{1}{V^2} \sum_{jk} \int\int (\mathcal{F}_{ij} + \mathcal{F}_{ik}) \cdot \frac{\partial}{\partial p_i} W_{jk} F(i/jk)^{(1)} dX_j dX_k$$

(4.31)

etc.

For coarse-graining J with respect to time from t to t + τ, we may have more than one method. In (4.9), X'_i changes as a function of s according to (4.6), while X'_j and X'_k are invariant during the time of coarse-graining. If we choose for X'_i, X'_j and X'_k a different class of functions of s, we may obtain a different result of coarse-graining. In the following we obtain four different functions for $F^{(3)}(X'_i\ X'_j\ X'_k; t + s)$ by integrating Eq. (4.11) along four different classes of trajectories. These classes of trajectories are also different from the one determined by Eq. (4.6). The result of coarse graining J, and/or a part of J, along a class of trajectories is different from the result made along another class of trajectories. Later, however, it will be shown that, by a proper coarse-graining operation with respect to similar particles, the differences among the results due to the difference among classes of trajectories along which the coarse-grainings with respect to time are done, are eliminated. In the following we obtain for $F^{(3)}$ four different functions of s, respectively corresponding to four different series expansions given by (4.17).

4.2. Equations for sub-systems.

4.2.1. $F^{(3)}(ijk)$ in $F(i/j/k)$

According to (4.17), we define $F(i'/j'/k';t+s)^{(\mu)}$ for $F(i/j/k\ )^{(\mu)}$ where X_i, X_j, X_k and t are replaced respectively by X'_i, X'_j, X'_k and $t+s$. Here X'_i, X'_j and X'_k are functions of s as will be specified soon. We assume that $F(i'/j'/k';t+s)^{(0)}$ satisfies

$$\left(\frac{\partial}{\partial s} + \frac{p_i'}{m}\cdot\frac{\partial}{\partial r_i'} + \frac{p_j'}{m}\cdot\frac{\partial}{\partial r_j'} + \frac{p_k'}{m}\cdot\frac{\partial}{\partial r_k'}\right) F(i'/j'/k';t+s)^{(0)} = 0 \tag{4.32}$$

and $F(i'/j'/k';t+s)^{(\mu)}$ for $\mu > 0$ satisfies

$$\left(\frac{\partial}{\partial s} + \frac{p_i'}{m}\cdot\frac{\partial}{\partial r_i'} + \frac{p_j'}{m}\cdot\frac{\partial}{\partial r_j'} + \frac{p_k'}{m}\cdot\frac{\partial}{\partial r_k'}\right) F(i'/j'/k';t+s)^{(\mu)}$$

$$= -\left[\left(\mathcal{F}_{ij}' + \mathcal{F}_{ik}'\right)\cdot\frac{\partial}{\partial p_i'} + \left(\mathcal{F}_{ji}' + \mathcal{F}_{jk}'\right)\cdot\frac{\partial}{\partial p_j'}\right.$$

$$\left. + \left(\mathcal{F}_{ki}' + \mathcal{F}_{kj}'\right)\cdot\frac{\partial}{\partial p_k'}\right] F(i'/j'/k';t+s)^{(\mu-1)}$$

$$\mu = 1, 2, 3, \cdots \tag{4.33}$$

where $\mathcal{F}_{ij}'$ is a function of $r_i' - r_j'$ instead of $r_i - r_j$. By taking

$$F(i'/j'/k';t+s)^{(0)} = F(i';t+s)^{(0)} F(j';t+s)^{(0)} F(k';t+s)^{(0)} \tag{4.34}$$

equation (4.32) leads to

$$\left(\frac{\partial}{\partial s} + \frac{p_i'}{m} \cdot \frac{\partial}{\partial r_i'}\right) F(i'; t+s)^{(0)} = 0.$$

$$\left(\frac{\partial}{\partial s} + \frac{p_i}{m} \cdot \frac{\partial}{\partial r_j'}\right) F(j'; t+s)^{(0)} = 0$$

$$\left(\frac{\partial}{\partial s} + \frac{p_k}{m} \cdot \frac{\partial}{\partial r_k'}\right) F(k'; t+s)^{(0)} = 0 \qquad (4.35)$$

On integration of Eqs. (4.3) along trajectories determined by

$$\frac{dr_i'}{ds} = \frac{p_i'}{m} \cdot \frac{dr_j'}{ds} = \frac{p_k'}{m}$$

$$\frac{dr_k'}{ds} = \frac{p_k'}{m} \qquad (4.36)$$

$$p_i' = p_i, \quad p_j' = p_j, \quad p_k' = p_k$$

$$r_i'(0) = r_i, \quad r_j'(0) = r_j, \quad r_k(0) = r_k \qquad (4.37)$$

[trajectory I]

we have

$$F(i' ; t + s)^{(0)} = F(i, t)^{(0)}$$

$$F(j' ; t + s)^{(0)} = F(j ; t)^{(0)} \qquad (4.38)$$

$$F(k' ; t + s)^{(0)} = F(k ; t)^{(0)}$$

The next procedure is to obtain $F(i'/j'/k' ; t + s)^{(1)}$ by solving Eq. (4.33) for $\mu = 1$. On substitution of (4.38) in the right hand side of Eq. (4.33), we have

$$\left(\frac{\partial}{\partial s} + \frac{p_i'}{m} \cdot \frac{\partial}{\partial r_i'} + \frac{p_j'}{m} \cdot \frac{\partial}{\partial r_j'} + \frac{p_k}{\partial r_k'}\right) F(i'/j'/k' ; t + s)^{(1)}$$

$$= -\left[\left(\mathcal{F}_{ij}' + \mathcal{F}_{ik}'\right) \cdot \frac{\partial}{\partial p_i'} + \left(\mathcal{F}_{ji}' + \mathcal{F}_{jk}'\right) \cdot \frac{\partial}{\partial p_j'} \right.$$

$$\left. + \left(\mathcal{F}_{ki}' + \mathcal{F}_{kj}'\right) \cdot \frac{\partial}{\partial p_k'} \right] F(i'/j'/k' ; t + s)^{(0)} \qquad (4.39)$$

Since the right hand side member is already given, we may obtain $F(i'/j'/k' ; t + s)^{(1)}$ along trajectories I determined by Eqs. (4.36) and (4.37). We may repeat the same procedure to obtain $F(i'/j'/k' ; t + s)^{(\mu)}$ for $\mu > 1$. It is obvious that

$$F(i'/j'/k' ; t + s)^{(\mu)} = 0,$$

$$\text{if} \quad \mu \neq 0, \ s = 0 \qquad (4.40)$$

since we assume that

$$F(i/j/k;t)^{(0)} = F^{(1)}(X_i;t)\, F^{(1)}(X_j;t)\, F^{(1)}(X_k;t) \qquad (4.41)$$

4.2.2. $F^{(3)}(ijk)$ in $F(ij/k)$

In this cae, particle i and particle j are in a close interaction. By taking the second expansion of $F^{(3)}(ijk)$ given by (4. 17) and defining $F(i' j' /k' ; t + s)^{(0)}$, etc. in the same way as of $F(i' /j' /k' ; t + s)^{(0)}$, etc. we have

$$\left(\frac{\partial}{\partial s} + \frac{p_{i'}}{m} \cdot \frac{\partial}{\partial r_{i'}} + \frac{p_{j'}}{m} \cdot \frac{\partial}{\partial r_{j'}} + \frac{p_{k'}}{m} \cdot \frac{\partial}{\partial r_{k'}} - \mathcal{F}_{ij'} \cdot \frac{\partial}{\partial p_i} + \mathcal{F}_{ji'} \cdot \frac{\partial}{\partial p_j}\right) F(i' j' /k' ; t + s)^{(0)} = 0 \qquad (4.42)$$

and

$$\left[\frac{\partial}{\partial s} + \frac{p_{i'}}{m} \cdot \frac{\partial}{\partial r_{i'}} + \frac{p_{j'}}{m} \cdot \frac{\partial}{\partial r_{j'}} + \frac{p_{k'}}{m} \cdot \frac{\partial}{\partial r_{k'}} + \mathcal{F}'_{ij} \cdot \frac{\partial}{\partial p_i} + \mathcal{F}'_{ji} \cdot \frac{\partial}{\partial p_j}\right] F(i'j'/k'; t + s)^{(\mu)}$$

$$= -\left[\mathcal{F}'_{ik} \cdot \frac{\partial}{\partial p_{i'}} + \mathcal{F}'_{jk} \cdot \frac{\partial}{\partial p_{j'}}\right.$$

$$+ \left(\mathcal{F}'_{ki} + \mathcal{F}'_{kj}\right) \cdot \frac{\partial}{\partial p_k'}\Big] F(i'j'/k'; t+s)^{(\mu-1)}$$

$$\mu = 1, 2, 3, \ldots \qquad (4.43)$$

Those equations are integrated successively along trajectories II determined by

$$\left.\begin{aligned} \frac{dr_i'}{ds} &= \frac{p_i'}{m}, \quad \frac{dp_i'}{ds} = \mathcal{F}'_{ij} \\ \frac{dr_j'}{ds} &= \frac{p_j'}{m}, \quad \frac{dp_j'}{ds} = \mathcal{F}'_{ji} \\ \frac{dr_k'}{ds} &= \frac{p_k'}{m}, \quad p_k' = p_k \end{aligned}\right\} \qquad (4.44)$$

(trajectories II)

with initial conditions at $s = 0$

$$\left.\begin{aligned} r_i' &= r_i, \quad p_i' = p_i \\ r_j' &= r_j, \quad p_j' = p_j \\ r_k' &= r_k \end{aligned}\right\} \qquad (4.41)$$

We may assume that

$$F(i'j'/k' ; t + s)^{(0)} = F(i'j'; t + s)^{(0)} \; F(k', t + s)^{(0)} \tag{4.46}$$

where $F(i'j' ;t + s)^{(0)}$ and $F(k' ;t + s)^{(0)}$ are solutions of

$$\left(\frac{\partial}{\partial s} + \frac{p_i'}{m} \cdot \frac{\partial}{\partial r_i} + \frac{p_j'}{m} \cdot \frac{\partial}{\partial r_j}\right.$$

$$\left. + \mathcal{F}_{ij}' \cdot \frac{\partial}{\partial p_i} + \mathcal{F}_{ji}' \cdot \frac{\partial}{\partial p_j}\right) F(i'j'; t + s)^{(0)} = 0 \tag{4.47}$$

and

$$\left(\frac{\partial}{\partial s} + \frac{p_k'}{m} \cdot \frac{\partial}{\partial r_k}\right) F(k'; t + s)^{(0)} = 0 \tag{4.48}$$

when integrated along rajectories II determined by (4.40) and (4.45). Since we assume that

$$F(i'j'/k'; t + s)^{(\mu)} = 0$$

$$\mu > 0$$

when

$$s = 0 \tag{4.50}$$

it is easily seen that

$$F(i'j' ; t + s)^{(0)} = F^{(\lambda)}(ij)$$

$$F(k' ;T + s)^{(0)} = F^{(1)}(k) \tag{4.51}$$

when

$$s = 0$$

Solution of Eq. (4.47) is a two body problem; we may imitate the method of Boltzmann setting Stosszahlansatz for the initial condition of a binary collision. Of course, it is to be expected that the solution will become meaningless as the distance between two particles i and j increases. However, at the same time W_{ij} tends to vanish, and hence such meaningless solutions do not affect the final result, F(ij/k).

4.2.3. $F^{(3)}(ijk)$ in F(ik/j)

We may repeat the same treatment in this case as in the case of F(ij/k), simply by changing subscripts. The trajectories which correspond trajectories II are denoted by trajectories II' in this case.

4.2.4. $F^{(3)}(ijk)$ in F(i/jk).

Trajectories along which coarse-graining operations are done are named trajectories III. The treatment is similar to those in the two preceding cases under the condition that the interaction between j and k are strong.

Considering $F^{(3)}(X'_i X'_j X'_k; t + s)$ obtained in four cases in the above, we may coarse - grain $J_1^{(0)}$, $J_2^{(0)}$, $J_{2'}^{(0)}$ and $J_3^{(0)}$, with respect to time, respectively along trajectories

I, II, II' and III.

4.3. Coarse-graining

4.3.1. Coarse-Graining $J_1^{(0)}, J_2^{(0)}, J_2^{(0)}, J_3^{(0)}$ with Respect to Time.

According to (4.22), (4.33) and (4.36), we have

$$\langle J_1^{(0)} \rangle_I = \frac{1}{V^2} \sum_{jk} \frac{1}{\tau} \int_0^\tau [\int\int (\mathcal{F}'_{ij} + \mathcal{F}'_{ik}) \cdot \frac{\partial}{\partial p'_i} (1 - W'_{ij} - W'_{ik} - W'_{jk})$$

$$F(i'; t+s)^{(0)} F(j', t+s)^{(0)} F(k'; t+s)^{(0)} dX'_j dX'_k]$$

$$= \frac{1}{V^2} \sum_{jk} \int\int (1 - W_{ij} - W_{ik} - W_{jk}) (\mathcal{F}_{ij} + \mathcal{F}_{ik}) \; F^{(1)}(j) \; F^{(1)}(k)$$

$$\cdot \frac{\partial}{\partial P_i} F^{(1)}(i) \, dX_j \, dX_k$$

$$= \frac{N-1}{V} \int (1 - W_{ij}) \mathcal{F}_{ij} \; F^{(1)}(j) \, dX_j \cdot \frac{\partial}{\partial P_i} F^{(1)}(i) \qquad (4.52)$$

In the above, it is noted that $\sum_{jk}$ means the summation with respect to a possible set of pairs made of all the particles except for the test particle; hence the number of the pairs is (N-1)/2. By the time average, the symmetry is produced in the distribution function. It is also assumed that

$$\frac{1}{V^2}\sum_{jk}\int\int W_{ik}\,F^{(1)}(j)\,F^{(1)}F^{(1)}(k)\,dX_j\,dX_k$$

is negligible as compared with the other members. $<J_1^{(0)}>_I$ gives the Vlasov effects. With necessary cautions similar to those for (4.52), we obtain

$$<J_2^{(0)}>_{II} = \frac{1}{V}\,\frac{N-1}{2}<\int \mathcal{F}_{ij}\,W_{ij}\cdot\frac{\partial}{\partial p_i}F^{(2)}(ij)\,dX_j>_{II} \tag{4.53}$$

$$<J_2'^{(0)}>_{II'} = \frac{1}{V}\,\frac{N-1}{2}<\int \mathcal{F}_{ik}\,W_{ik}\cdot\frac{\partial}{\partial p_i}F^{(2)}(ik)\,dX_k>_{II}$$

$$+\frac{1}{V^2}\,\frac{N-1}{2}\int \mathcal{F}_{ij}\,F^{(1)}(j)dX_j\cdot\frac{\partial}{\partial p_i}\int F^{(2)}(ik)\,W_{ik}dX_k>_{II} \tag{4.54}$$

It is noted that $\int F^{(1)}(X)dX/V = 1$, and $/\mathcal{F}_{ij}/ > /\mathcal{F}_{ik}/$ in (4.53) and $/\mathcal{F}_{ij}/ < /\mathcal{F}_{ik}/$ in (4.54). Therefore, the second member in each is ignored, and the sum of them leads to

$$<J_2^{(0)}>_{II} + <J_2'^{(0)}>_{II'}$$

$$= \frac{N-1}{V}<\int \mathcal{F}_{ij}\,W_{ij}\cdot\frac{\partial}{\partial p_i}F^{(2)}(ij)\,dX_j>_{II} \tag{4.55}$$

This will be converted to the Boltzmann collision integral by proper treatment. Here the convergency of the result is assured by the existence of W_{ij}. Finally we have

$$< J_3^{(0)} >_{III} = \frac{1}{V^s} \frac{(N-1)}{2} < \int\int (\mathcal{F}_{ij} + \mathcal{F}_{ik}) W_{jk} F^{(2)}(jk) dX_j dX_k >_{III}$$

$$\cdot \frac{\partial}{\partial P_i} < F^{(1)}(i) >_{III} \qquad (4.56)$$

It appears that the effect is similar to $< J_1^{(0)} >$. The difference is that $< J_3^{(0)} >$ involves W_{jk}. Therefore

$$< J_3^{(0)} >_{III} \prec < < J_1^{(0)} >_I \qquad (4.57)$$

Hence we may ignore the effect given by (4.56)

4.3.2. Coarse-Graining $J_1^{(1)}, J_2^{(1)}, J_{2'}^{(1)}, J_3^{(1)}$ with Respect to Time

The $J^{(1)}$'s given by (4.28), (4.29), (4.30) and (4.31) are coarse-grained in the same way as the $J^{(0)}$'s. The trajectories along which those functions are coarse-grained are respectively the same as those along which the $J^{(0)}$'s are coarse-grained.

$< J_1^{(1)} >_I$. There are correlations among three particles i, j and k in this approximation. It is easily seen, however, that those particles are mutually remote. According to (4.39), we obtain

$$F(i/j'/k'; t+s)^{(1)}$$

$$= -\int_0^s \Big[(\mathcal{F}'_{ij} + \mathcal{F}'_{ik}) \cdot \frac{\partial}{\partial p'_i} + (\mathcal{F}'_{ji} + \mathcal{F}'_{jk}) \cdot \frac{\partial}{\partial p'_j}$$

$$+ (\mathcal{F}'_{ki} + \mathcal{F}'_{kj}) \cdot \frac{\partial}{\partial p'_k} \Big] F(i'/j'/k'; t+s)^{(0)} ds \qquad (4.58)$$

Since

$$F(i'/j'/k'; t+s)^{(0)} = F^{(1)}(i;t)\, F^{(1)}(j;t)\, F^{(1)}(k;t)$$

and those trajectories, along which the coarse-graining is done, are free mutually, the summation $\sum_{jk}$ results in cancellation among the effects of those particles. Hence we ignore the present effect.

$\langle J_2^{(1)} \rangle_{II} + \langle J_{2'}^{(1)} \rangle_{II'}$. It is easily shown that integrations of Eq. (4.43) and of the similar equation for $F^{(1)}(i'j'/k'; t+s)$ along trajectories II and II' lead to

$$\langle J_2^{(1)} \rangle_{II} + \langle J_{2'}^{(1)} \rangle_{II'} = -\frac{N-1}{2V^2} \frac{1}{\tau} \int_0^\tau \Big[\iiint \mathcal{F}'_{ik} \cdot \frac{\partial}{\partial p_i'} W_{ij}' \Big[\int_0^s \mathcal{F}'_{ik} \cdot \frac{\partial}{\partial p_i'} F^{(2)}(i'j';t+s) F^{(1)}(k';t+s) ds dX_j' dX_k' \Big] ds$$

$$- \frac{N-1}{2V^2} \frac{1}{\tau} \int_0 \Big[\iiint \mathcal{F}'_{ik} \cdot \frac{\partial}{\partial p_k'} W_{ij}' \int_0^s \mathcal{F}'_{ki} \frac{\partial}{\partial p_k'} F^{(2)}(i'j';t+s)$$

$$F^{(1)}(k';t+s) ds dX_j' dX_k' \Big] ds \qquad (4.59)$$

In the above, the first member in the right hand side gives the diffusion effect in the momentum space of particles i and the second member the friction. The significant feature of those terms is that the correlation period is regulated by W_{ij}, the probability of strong interaction between i and j.

$< J_3^{(1)} >_{III}$. By means of similar method as before, we obtain

$$< J_3^{(1)} >_{III} = - \frac{N-1}{2V^2} \frac{1}{\tau} \int_0^{\tau} \Big[\iint \mathcal{F}'_{ij} \cdot \frac{\partial}{\partial p_i} W'_{jk} \int_0^s \mathcal{F}'_{ij} \cdot \frac{\partial}{\partial p_{i'}} F^{(2)}(j'k'; t+s) F^{(1)}(i'; t+s) ds \, dX'_j dX'_k \Big] ds$$

$$- \frac{N-1}{2V^2} \frac{1}{\tau} \int_0^{\tau} \Big[\iint \mathcal{F}_{ij}' \cdot \frac{\partial}{\partial p_i'} W_{jk}' \int_0^S \mathcal{F}'_{ji} \cdot \frac{\partial}{\partial p_j} F^{(2)}(jk; t+s) F^{(1)}(i'; t+s) dX'_j dX'_k \Big] ds$$

(4.60)

In effect, there is no difference between $< J_2^{(1)} >_{II} + < J_2^{(1)} >_{II'}$ and $< J_3^{(1)} >_{III}$.

In summation of all those effects, we may write

$$\langle J^{(1)} \rangle = -\frac{(N-1)}{V^2}\,\frac{1}{\tau}\int_0^{\tau}\Big[\iint \mathcal{F}_{ik}'\cdot\frac{\partial}{\partial p_i'}\,W_{ij}'$$

$$+\int_0^{s}\mathcal{F}_{ik}'\cdot\frac{\partial}{\partial p_i'}\,F^{(2)}(i'j';\,t+s)\,F^{(1)}(k';\,t+s)\,ds$$

$$dX_j'\,dX_k'\Big]\,ds$$

$$-\frac{(N-1)}{V^2}\,\frac{1}{\tau}\int_0^{\tau}\Big[\iint \mathcal{F}_{ik}'\cdot\frac{\partial}{\partial p_i'}\,W_{ij}'\int_0^{s}\mathcal{F}_{ki}'\cdot\frac{\partial}{\partial p_k'}$$

$$F^{(2)}(i'j';\,t+s)\,F^{(1)}(k';\,t+s)\,ds$$

$$dX_j'\,dX_k'\Big]\,ds \qquad (4.61)$$

where it is noted that we have ignored the difference among trajectories over which those coarse-graining operations are considered. As is shown below, however, the difference is not essential in the result. The effect given by (4.61) is of the Fokker-Planck type.

4.3.3. Distribution functions in generic spaces

So far, various functions constituting J have been coarse-grained along different classes of trajectories. For completing Eq. (4.10), however, J is to be coarse-grained with respect to the particular class of trajectories named 0 as is given by (4.9). It is necessary to show that the differences among those classes of trajectories along which J and/or parts of J have been coarse-grained do not result in any inconsistency in the final result. For this purpose, we compare, for example, the results of coarse-graining $F^{(1)}(X_i;t)$ along different classes of trajectories in the following: According to (2.3),

$$F^{(1)}(X_i;t) = V\int D^{(N)} \prod_{j \neq i}^{N} dX_j$$

$$= V\,\delta(X_i - X_i^*(t)) \qquad (4.62)$$

Here $X_i^*(t)$ is a complicated function of time because of the interaction of particle i with the other N-1 particles.

First, let us coarse-grain $F^{(1)}$ along trajectories determined by (4.6) over a time period from t to t + τ:

$$\langle F_i^{(1)}(X_i;t)\rangle = \frac{1}{\tau}\int_0^{\tau} F^{(1)}(X_i' ; t+s)ds \qquad (4.63)$$

where

$$X_i' = X_i + x_i'(s)$$

Those trajectories, according to (4.6), are independent of the other particles. As is shown in Fig. 4.1, none of those trajectories is similar to $X_i^*(t+s)$ in view of time scale τ. Hence $<F_i^{(1)}(X_i;t)>$ is spread over a comparatively broad domain of X_i. There may be a group of ν_i particles, $i_1, i_2, \ldots$, of which

$$<F_{i1}^{(1)}(X;t)> = <F_{i2}^{(1)}(X;t)> = \cdots$$

These are localized in a narrow domain of X. We define $<F^{(1)}(X;t)>$ by

$$N<F^{(1)}(X;t)> = \sum_{i=1}^{N} <F_i^{(1)}(X;t)>$$

$$= \nu_i' <F_{i'}^{(1)}(X;t)>$$

$$+ \nu_i'' <F_{i''}^{(1)}(X;t)> + \cdots \qquad (4.64)$$

$$\nu_{i'} + \nu_{i''} + \cdots = N \qquad (4.65)$$

Unlike $<F_i^{(1)}(X;t)>$, $<F^{(1)}(X;t)>$ is spread more uniformly over the entire domain of X.

We may also coarse-grain $F_i^{(1)}(X_i;t)$ along another set trajectories

$$X_i'' = X_i + x_i''(s)$$

which, for example, is determined by (4.44).

$$\ll F_i^{(1)}(X_i;t) \gg = \frac{1}{\tau}\int_0^{\tau} F_i^{(1)}(X_i''; t+s)\, dS \tag{4.66}$$

Those trajectories, according to (4.44), are determined by taking into account the nearest neighbors' interactions. Therefore, a trajecotry coincides with $X_i^*(t+s)$ in the time scale τ. Hence $\ll F_i^{(1)}(X_i;t\gg$ is much more localized than $<F_i^{(1)}(X_i;t>$ as is illustrated in Fig. 4.1.

$$<F_i^{(1)}(X_i;t)> \neq \ll F_i^{(1)}(X_i; t) \gg \tag{4.67}$$

Nevertheless, it is obvious that

$$\int <F_i^{(1)}(X_i;t)> dX_i = \int \ll F_i^{(1)}(X_i;t)\gg dX_i \tag{4.68}$$

We may define $\overline{\ll F^{(1)}(X;t)\gg}$ by

$$\overline{\ll F^{(1)}(X;t)\gg} = \frac{1}{N} \sum_i \ll F_i^{(1)}(X;t)\gg \tag{4.69}$$

If N is sufficiently large, the coarse-graining in the sense of (4.64) and (4.69) may result in

$$\overline{< F^{(1)}(X;t) >} = \overline{\ll F^{(1)}(X;t)\gg} \tag{4.70}$$

In the above, we have seen coarse-graining operations in the sense of (4.64) and (4.69) are essential for reconcilliation (4.70) between two classes of time averages. In other words, time averages are not enough for deriving proper kinetic equations; it is necessary to make averages over many particles.

Applying the same consideration as above, we make average of Eq. (4.10) with respect to particles so that $<J>$'s made with respect to different classes of trajectories are equivalent to those made with respect to trajectories 0. By this procedure of coarse-graining over particles, we obtain in effect the same function as f defined by (2.7). (The Boltzmann equation is the very equation which governs the evolution of f of a gas where the collisions are assumed to be binary.) After the present coarse-graining operation with respect to similar particles, the effect given by (4.55) may amount to the Boltzmann collision integral; because of W_{ij} in the integrand, the effective interaction

terminates quickly as the distance between particle i and particle j increases.

4.4. Evaluation of the Interaction Effects.

Because of the W's involved in those interaction integrals, the integrals converge always. As is given in appendix A, we take for W

$$W = \exp\left(-9\pi n r^3/4\right)$$

where n is the local number density and r is the distance between the test particle and a field particle under consideration. The range of space of a close correlation (almost binary) is of the order of

$$R_n = \left(\frac{9\pi n}{4}\right)^{-1/3}$$

and the time scale is

$$\tau = R_n/(\langle p\rangle/m)$$

Here $\langle p\rangle$ is the average magnitude of momentum of a particle

$$\frac{\langle p\rangle}{m} = 0\left[\frac{3kT}{m}\right]^{\frac{1}{2}}$$

If $r > R_n$, then $W \ll 1$. The Boltzmann type interaction effect $\langle J_2^{(0)}\rangle + \langle J_{2'}^{(0)}\rangle$ is of the order of

$$0[\langle J_2^{(0)}\rangle + \langle J_2'^{(0)}\rangle] = \pi (R_n)^2 \frac{\langle p\rangle}{m} n F^{(1)}(i)$$

$$= \pi n R_n^2 \left(\frac{3kT}{m}\right)^{\frac{1}{2}} F^{(1)}(i)$$

The order of the interaction effect $\langle J_2^{(1)}\rangle + \langle J_2'^{(1)}\rangle$ is shown to be

$$0[\langle J_2^{(1)}\rangle + \langle J_{2'}\rangle^{(1)}] = \frac{Rn}{(3kT/m)^{\frac{1}{3}}} n \int_{Rn}^{\infty} \frac{4\pi e^4 r^2}{r^3} dr \frac{1}{p} F^{(1)}(i)$$

$$= \frac{4\pi}{3} \frac{ne^4}{m^2 (3kT/m)^{3/2}} F^{(1)}(i)$$

Hence we obtain

$$\frac{0[\langle J_2^{(1)}\rangle + \langle J_{2'}^{(1)}\rangle]}{0[\langle J_2^{(0)} + \langle J_{2'}^{(0)}\rangle]} = 0\left[\left(\frac{e^2 n^{1/3}}{kt}\right)^2\right] \qquad (4.71)$$

If we take $T = 10^4$, $n = 10^{19}$, we obtain

$$\left(\frac{e^2 n^{1/3}}{kT}\right)^2 \doteqdot 10^{-1}.$$

$\left(\frac{e^2 n^{1/3}}{kT}\right)^2$ decreases further as T increases and n decreases.

$<J_3^{(0)}>$ is the same order as of $<J_2^{(1)}> + <J_{2'}^{(1)}>$.

The ratio given above suggests that the effect or remote interactions may be ignored in ordinary cases where $T > 10^4$ and $n < 10^{19}$. This conclusion appears to be different from those obtained by other authors, based on the original interpretation of the BBGKY hierarchy. It is noted that the present definition of close interaction according to the nearest neighborship is different from the usual definition of close interaction according to the polarization effect of the Debye-Huckel type.

One, who has been familiar with theories based on the BBGKY hierarchy, might ask: Why is the effect of field particles which are not nearest neighbors of the test particle so small by the present theory? The answer is as follows: By a theory based on the BBGKY hierarchy, the test particle and a field particle may interact with an indefinitely long correlation time, in spite of the presence of perturbations by third particles. On the other hand, the behavior of a particle, by the present theory, appears in microscopic order only for a short period of time during its interaction with its nearest neighboring particles. In other words, the time scales of microscopic orders are always

finite. Note that the above statement is feasible, only because laws of interaction forces are not linear with respect to inter-particle distances.

4.5. Validity of the Model

As is investigated in section III, the time scale of a close interaction between two ions is much longer than the the time scale of a close interaction where an electron participates. Therefore the present model appear to be suitable for simulating, in an approximation, evolution of the ion distribution in a real system, if the number of the ions is the same as the number of the electrons (charge of an ion being the same as of a proton). This is because the electrons appear to make an almost uniform background in view of the motion of an ion. But at the same time, we find no definite reason for saying that the effect of electron-ion (close) interactions on the evolution of the ion distribution is negligible: although the average magnitude of the momentum of an electron is much smaller than the average magnitude of the momentum of an electron, (the ratio being about 1/50 with respect to the lightest ion), the frequency of electron-ion collision (close) is

about 50 times higher than the frequency of close ion-ion interactions. If we wish to consider the situation more precisely and reasonably, both ions and electrons must be considered as discrete particles. In view of the above, model II will be investigated in the next section.

V. MODEL II

As stated in subsection 4.10, the validity of model I as simulating a real system is fairly dubious. It may be plausible to think that the model has a meaning only as providing an example of manipulation of reducing the Liouville equation to coarse-grained kinetic equations governing evolutions of subsystems. In this section, we consider a model which may be more realistic than the previous one. The plasma which the present model is intended to simulate is the same as is considered in the last section; the gas is fully ionized and the number of the ions is the same as the number of the electrons.

We consider four particles, two ions and two electrons, for representing the field particles. By this way, it is necessary to treat the evolution of a five-particle distribution function $F^{(5)}(X_i X_j X_k X_l X_m ; t)$ where one of the five particles is the test particle. However, as a simpler approach we shall consider

$$\begin{aligned} &F^{(3)}(o, i, j) \\ &F^{(3)}(o, k, \ell) \\ &F^{(3)}(o, m, n) \end{aligned} \qquad (5.1)$$

where particle o is the test particle, either an ion or an electron, particles i, j, k are electrons which represent the N/3 groups of the field electrons, and particles ξ, η, ζ are ions representing the N/3 groups of the field ions; 2N is the total number of the field particles, N electrons and N ions. The evolution of $F^{(1)}(o)$ of the test particle is obtained by integrating partly the Liouville equation of the N+1 particle system as follows:

$$\left(\frac{\partial}{\partial t} + \frac{p_o}{m_o} \cdot \frac{\partial}{\partial r_o}\right) F^{(1)}(o)$$

$$+ \frac{1}{V^3} \sum_{ijk} \sum_{\xi\eta\zeta} \Big[\iint \left(\mathcal{F}_{oi} + \mathcal{F}_{oj}\right) \cdot \frac{\partial}{\partial p_o} F^{(3)}(o, i, j)\, dX_i dX_j$$

$$+ \iint \left(\mathcal{F}_{ok} + \mathcal{F}_{o\xi}\right) \cdot \frac{\partial}{\partial p_o} F^{(3)}(o, k, \xi)\, dX_k dX_\xi$$

$$+ \iint \left(\mathcal{F}_{o\eta} + \mathcal{F}_{o\zeta}\right) \cdot \frac{\partial}{\partial p_o} F^{(3)}(o, \eta, \zeta)\, dX_\eta dX_\zeta \Big] = 0 \qquad (5.2)$$

After coarse-graining, $\frac{1}{V^3} \sum_{ijk} \sum_{\xi\eta\zeta}$ will be replaced by multiplier 2N/6. Equation (5.2) corresponds to Eq. (4.1) for model I. Corresponding to Eq. (4.2), we have the following three equations:

$$\left[\frac{\partial}{\partial t} + \frac{p_o}{m_o}\cdot\frac{\partial}{\partial r_o} + \frac{p_\alpha}{m_\alpha}\cdot\frac{\partial}{\partial r_\alpha} + \frac{p_\beta}{m_\beta}\cdot\frac{\partial}{\partial r_\beta}\right.$$

$$+ (\mathcal{F}_{o\alpha} + \mathcal{F}_{o\beta})\cdot\frac{\partial}{\partial p_o} + (\mathcal{F}_{\alpha o} + \mathcal{F}_{\alpha\beta})\cdot\frac{\partial}{\partial p_\alpha}$$

$$\left. + (\mathcal{F}_{\beta o} + \mathcal{F}_{\beta\alpha})\cdot\frac{\partial}{\partial p_\beta}\right] F^{(3)}(o, \alpha, \beta) = 0$$

$$(\alpha, \beta) = (i, j), (k, \xi), (\eta, \zeta)$$

(5.3)

where

$m_i = m_j = m_k$ = electronic mass

$m_\xi = m_\eta = m_\zeta$ = ionic mass.

The interaction to the zeroth approximation is given by

$$J^{(0)} = \frac{-1}{V^3}\sum_{ijk}\sum_{\xi\eta\zeta}\left[\int\int(\mathcal{F}_{oi} + \mathcal{F}_{oj})\cdot\frac{\partial}{\partial p_o} F^{(3)}(o, i, j)^{(0)} dX_i dX_j\right.$$

$$+ \int\int(\mathcal{F}_{ok} + \mathcal{F}_{o\xi})\cdot\frac{\partial}{\partial p_o} F^{(3)}(o, k, \xi)^{(0)} dX_k dX_\xi$$

$$\left. + \int\int(\mathcal{F}_{o\eta} + \mathcal{F}_{o\zeta})\cdot\frac{\partial}{\partial p_o} F^{(3)}(o, \eta, \zeta)^{(0)} dX_\eta dX_\zeta\right]$$

$$= J^{(0)}(oee)_1 + J^{(0)}(oee)_2 + J^{(0)}(oee)_2' + J^{(0)}(oee)_3$$

$$+ J^{(0)}(oeI)_1 + J^{(0)}(oeI)_2 + J^{(0)}(oeI)_2' + J^{(0)}(oeI)_3$$

$$+ J^{(0)}(oII)_1 + J^{(0)}(oII)_2 + J^{(0)}(oII)_2' + J^{(0)}(oII)_3 \qquad (5.4)$$

where $J^{(0)}(oeI)_1$, for example, is an effect due to field particles k and ξ, k representing field electrons and ξ representing field ions. Those members in the above may be obtained as analogous to those $J^{(0)}$'s obtained in the last section. In coarse-graining these terms, it is necessary to pay attention to the difference between the time scale of ion-ion interaction and the time scale of those in which electrons are involved. The time scale of ion-ion interaction is denoted by τ_{II} while the time scale of those involving electrons is denoted by τ_e. Then, according to the investigation given in section III, it holds that

$$\tau_{II} \gg \tau_e$$

If the macroscopis time scale τ_m is larger than τ_{II}, we may choose τ so that

$$\tau_m \gg \tau \gg \tau_{II} \gg \tau_e \qquad (5.5)$$

τ is the very time scale with which we may coarse-grain those terms in Eq. (5.4). All the procedures of coarse-graining are the same as demonstrated in section IV.

Taking an ion for test particle o,

$$<J^{(0)}(\mathrm{Iee})_1> + <J^{(0)}(\mathrm{IeI})_1> + <J^{(0)}(\mathrm{III})_1> \tag{5.6}$$

gives the effect of the Vlasov type interaction. The effect of the Boltzmann type interaction among nearest neighboring particles is presented by

$$\begin{aligned} &<J^{(0)}(\mathrm{Iee})_2> + <J^{(0)}(\mathrm{Iee})_2'> \\ &+ <J^{(0)}(\mathrm{IeI})_2> + <J^{(0)}(\mathrm{IeI})_2'> \\ &+ <J^{(0)}(\mathrm{III})_2> + <J^{(0)}(\mathrm{III})_2'> \end{aligned} \tag{5.7}$$

We may also obtain

$$\begin{aligned} &<J^{(1)}(\mathrm{Iee})_2> + <J^{(1)}(\mathrm{Iee})_2'> \\ &+ <J^{(1)}(\mathrm{IeI})_2> + <J^{(1)}(\mathrm{IeI})_2'> \\ &+ <J^{(1)}(\mathrm{III})_2> + <J^{(1)}(\mathrm{III})_2'> \end{aligned} \tag{5.8}$$

for the significant effect in the approximation of the next order. By comparing the effect given by (5. 8) with the effect given by (5. 7) in the same was as we did by (4. 71), we may see that effect (5. 8) is negligible in ordinary cases. The situation is explained as follows: 1) First, let us take an ion for the test particle o. Then, take an electron for the particle nearest neighboring to the test particle o. The effect of the interaction (a part of the zeroth approximation effect) is larger than the effect of the force exerted by two mutually neighboring field particles of which one is an electron. Note that the time scales of two events are the same. Take an ion for the nearest neighboring particle of the test particle o. Then the effect of the interaction is larger than the effect of the force exerted by two mutually neighboring field particles of which both are ions. 2) The situation is similar for the test particle which is an electron.

V. MODEL III

Suppose that a fully ionized gas is composed of N electrons and N' ions of a single species. Here the charge of an ion is denoted by Q, and

$$Q = \nu e, \quad e: \text{ electronic charge}$$

$$N/N' = \nu \tag{4.1}$$

If ν is much larger than unity and we wish to investigate the evolution of the distribution of ions, the models given so far are not feasible: Model I is not feasible because neither ions nor electrons can be represented by a uniform charge distribution in the present case; model II is not feasible because $N >> N'$. Due to the large mass and the large charge of an ion, two nearest neighboring ions which are separated at a distance much larger than the average distance between two nearest neighboring electrons may have a strong mutual correlation of which the time scale is much longer than the time scale of the behavior of an electron interacting with an ion or another electron. See Fig. 6.1. This condition is quite favorable for applying the ordinary BBGKY hierarchy, up to the third equation, to the investigation of the evolution of

the ion distribution. Here the test particle is an ion, the field particles appearing in the first equation of the BBGKY hierarchy are also ions, and those field particles appearing in the integrand of the second equation are electrons. The test ion and a field ion may interact through a space in which many electrons are present and the distribution of those electrons are polarized by the field induced by the two ions. The case has been investigated by many authors[6] as based on the BBGKY hierarchy.

VII. CONCLUDING REMARKS

If the number of ions is the same as the number of electrons in a plasma, there are two basic types of inter-particle interactions in the zeroth approximation: One is of the Vlasov type and the other of the Boltzmann type characterizing interactions among nearest neighboring particles. In higher order approximations, mutual perturbations among those basic interactions result in secondary effects which are not significant under ordinary conditions of high temperature and of low density.

As the ratio between the number of electrons and the number of ions increases in a real system, the simulating model becomes more complex; in cases where the ratio is extremely large, the BBGKY hierarchy is useful for investigating the evolution of the ion distribution.

The presence of any external force field of which the time scale is much larger than the maximum time scale of close inter-particle interactions does not affect the validity of the present approach. If there is an external force field of which the time scale is comparable to or smaller than the time scale of any close inter-particle interactions, the method must be considerably modified[7].

APPENDIX A

Solution of Eq. (4.11)

Equation (4.11) is a partial differential equation where $F^{(3)}(X'_i X'_j X'_k; t+s)$ is unknown and X'_i, X'_j, X'_k and s are independent variables. It is possible to obtain particular solutions by taking arbitrary functions of s for X'_s, X'_j, X'_k. If one takes for X'_i, X'_j and X'_k particularly those which satisfy the characteristic equations of Eq. (4.11)

$$\frac{dr_i'}{ds} = \frac{p_i'}{m} \quad , \quad \frac{dp_i'}{ds} = \mathcal{F}_{ij}' + \mathcal{F}_{ik}'$$

$$\frac{dr_j'}{ds} = \frac{p_j'}{m} \quad , \quad \frac{dp_j'}{ds} = \mathcal{F}_{ji}' + \mathcal{F}_{jk}' \qquad \text{(A. 1)}$$

$$\frac{dr_k'}{ds} = \frac{p_k'}{m} \quad , \quad \frac{dp_k'}{ds} = \mathcal{F}_{ki}' + \mathcal{F}_{kj}'$$

then $F^{(3)}$, the solution of Eq. (4.11), is invariant with respect to s. It is noted that $X'_i(s)$, $X'_j(s)$ and $X'_k(s)$ which satisfy Eqs. (A.1) present a set of trajectories of particles i, j and k which are in mutual interaction. The solution of Eqs. (A.1) is a three-body problem. In order to avoid the difficulty of solving

the three-body problem precisely, one may choose a set of approximate differential equations in place of (A. 1) depending on circumstances, and may integrate Eq. (4. 11) along a set of trajectories given by X'_i , X'_j and X'_k which satisfy the approximate differential equations. In this case, the solution $F^{(3)}$ is not precisely invariant with respect to s.

APPENDIX B

The Probability of Two Particles being their Mutual Nearest Neighbors[8]

If it is definite that n particles exist in a space domain of unit volume, the probability of no particle being present in a domain of volume v, which is smaller than unit volume, is given by

$$w_0(v) = \exp(-nv) \tag{B.1}$$

The proof is as follows: Let us consider a variation δv of v. the probable number of particles in δv is $n\delta v$. The probability of any ν particles appearing, at the same time, in δv is denoted by $w_\nu(\delta v)$. Then, we have

$$\sum_{\nu=1}^{n} \nu\, w_\nu(\delta v) = n\delta v \tag{B.2}$$

Obviously

$$w_0(\delta v) + \sum_{\nu=1}^{n} w_\nu(\delta v) = 1 \tag{B.3}$$

Since we know that

$$\lim_{\delta v \to 0} w_\nu(\delta v) \propto (\delta v)^\nu$$

(B. 2) leads to

$$\lim_{\delta v \to 0} w_1(\delta v) = n\delta v$$

Hence (B . 3) yields

$$w_o(\delta v) + n\delta v = 1 \qquad \text{(B.4)}$$

at the limit $\delta v \to 0$.

On the other hand, we have

$$w_o(v + \delta v) = w_o(v)\, w_o(\delta v) \qquad \text{(B. 5)}$$

By expanding the left hand side of (B. 5), we obtain

$$w_o(v) + \frac{dw_o}{dv}\,\delta v = w_o(v)\,(1 - n\delta v)$$

where (B.4) is taken into account, or we have

$$\frac{dw_o(v)}{w_o(v)} = -n\,dv$$

On integration of the above under the condition $w_o(0) = 1$, we obtain Eq. (B. 1).

Two particles i and j can be their mutual nearest neighbors, if there is no other particle in the space domain enclosed by two spheres, one with its center at r_i and with radius r_{ij} and the other with its center at r_j and with the same radius. See Fig. B. 1. The volume of the domain is $(9\pi/4)r_{ij}^3$. Hence

$$W_{ij} = \exp\left[- \frac{9\pi}{4} r_{ij}^3 \, n\right] \qquad (B. 7)$$

is the probability that particle i and particle j are their mutual nearest neighbors. It is noted that the above consideration is simply a matter of geometry.

APPENDIX C

Distribution Functions Decomposed According to Inter-Particle Distances

The distribution function $F^{(3)}(X_1X_2X_3;t)$ of three particles 1, 2, and 3 is divided into parts as follows:

$$F^{(3)} \equiv F^{(3)}(X_1X_2X_3;t)$$

$$\begin{aligned}
F(1/2/3) &= (1-W_{12})(1-W_{13}(1-W_{23})\,F^{(3)} \\
F(12/3) &= W_{12}(1-W_{13})(1-W_{23})F^{(3)} \\
F(13/2) &= (1-W_{12})W_{13}(1-W_{23})F^{(3)} \\
F(1/23) &= (1-W_{12})(1-W_{13})W_{23}F^{(3)} \\
F(12/13) &= W_{12}W_{13}(1-W_{23})F^{(3)} \\
F(13/23) &= (1-W_{12})W_{13}W_{23}F^{(3)} \\
F(12/13/23) &= W_{12}W_{13}W_{23}F^{(3)}
\end{aligned} \tag{C. 1}$$

where, for example, W_{12} is the probability of particles 1 and 2 being their mutual nearest neighbors. The sum of the right hand side members is shown to be $F^{(3)}$ itself. If it is considered that a particle has only one nearest neighbor at each moment of time, we may have

$$W_{12}W_{23} = W_{23}W_{12} = W_{13}W_{23} = 0 \tag{C. 2}$$

and equations (C. 1) yield

$$F(1/2/3) = (1 - W_{12} - W_{13} - W_{23})F^{(3)}$$

$$F(12/3) = W_{12}F^{(3)}$$

$$F(13/2) = W_{13}F^{(3)} \qquad \text{(C. 3)}$$

$$F(1/23) = W_{23}F^{(3)}$$

and it holds that

$$F(1/2/3) + F(12/3) + F(13/2) + F(1/23) = F^{(3)} \qquad \text{(C. 4)}$$

Those relations given by (C. 3) are illustrated in Fig. C. 1.

If we consider test particle 1 and four field particles 2, 3, 4, 5, the possible relations among them are illustrated as in Fig. C. 2. By taking into consideration relations similar to (C. 2) we obtain

$$F^{(5)} \equiv F^{(5)}(X_1X_2X_3X_4X_5;t)$$

$$F(1/2/3/4/5) = (1-W_{12})(1 - W_{13}) \cdots (1 - W_{45})F^{(5)}$$

$$= \left(1 - \sum_{1<j} W_{ij} + \sum_{i<j<k<\ell} W_{ij}W_{k\ell}\right.$$

$$= (1 - W_{12} - W_{13} - W_{14} - W_{15} - W_{23} - W_{24}$$

$$-W_{25}-W_{34}-W_{35} - W_{45} + W_{12} W_{34} + W_{12} W_{35} + W_{12} W_{45}$$

$$+ W_{23} W_{45}) F^{(5)}$$

$$F(12/3/4/5) = W_{12}(1 - W_{34} - W_{35} - W_{45}) F^{(5)}$$

$$F(12/34/5) = W_{12}W_{34} F^{(5)} \qquad (C.5)$$

etc.

Finally it holds that

$$F(1/2/3/4/5) + F(12/3/4/5) + - - -$$

$$= F^{(5)} \qquad (C.6)$$

FOOTNOTES AND REFERENCES

*Supported by the Office of Naval Research Under Contract Nonr 839(38) for Project Defender, ARPA Order No. 529; presented partly at the meeting of American Physical Society, Washington D. C. April, 1967; preliminary accounts were published in PIBAL Report No. 836 (Polytechnic Institute of Brooklyn, 1965).

1 See, for example, C. M. Tchen, Phys. 114, 394(1959); Rostoker and Rosenbluth, Phys. Fluids, 3, 1(1960).

2. T. Koga, Phys. Fluids, 3, 454(1963); Bull. Am. Phys. Soc. 11, 554(1966); paper presented at the annual meeting of the Division of Plasma Physics, APS, 1966.

3. The reason for giving $D^{(N)}$ by (2. 2) has been discussed in detail in the second paper mentioned in footnote 2. The reader might think that, as long as $D^{(N)}$ is given by (2. 2), there is no need of consideration of Eq. (2. 1) since $D^{(N)}$ given by (2. 2) is a solution of Eq. (2. 1). In spite of its simple appearance, $D^{(N)}$ given by (2. 2) is not a simple function of t. X^*_i becomes known only by solving simultaneously the equations of motion of the N particles; these equations are the characteristic equations of Eq. (2. 1), and the solution is out of the scope of our mathematical ability.

4. We often use the present model for simulating the electron system in a plasma. The simulation is feasible when our interest is in the interaction between a plasma and an external electro-magnetic field of a macroscopic scale. However, the simulation is not feasible if we are interested in effects of inter-particle interactions.

5. An electron-electron interaction, for example, means an interaction of two electrons which are their mutual nearest neighboring particles.

6. See footnote 1.

7. C. Oberman, A. Ron, and J. Dawson, Phys. Fluids, 5, 1514(1962), investigated the effect of an external force of a short time scale, as based on the BBGKY hierarchy; but the assumption of time scales necessary for the hierarchy was not properly considered there.

8. The problem was first discussed by Hertz, according to S. Chandrasekhar, Rev. Mod. Phys. 15, 1 (1943).

CAPTIONS OF FIGURES

Fig. 3. 1. A field particle 2 interacts with the test particle 1 at close distances with time scale τ; after the interaction particle 2 interacts with particle 3, particle 4, and so forth. The time scale of those close interactions is the same as τ. The space domain in which particle 2 is localized over a longer period of time is indicated with shadow. The force exerted on particle 1 by particle 2 is classified as follows: 1) The force during the close interaction particle 2 with particle 1; 2) the force during the close interactions of particle 2 with particles 3, 4, etc; 3) the averaged force during the existence of particle 2 in the shadowed domain

Fig. 4. 1. In A, X_i^* is the precise trajectory of particle i under the influence of all the field particles; X_i' is the trajectory of particle i when the influence of the field particles is ignored; X_i'' is the trajectory of particle i interacting with particle j only. X_i'' is much closer to X_i^*. In B, $F_i^{(1)}$ at t is shown by a δ-function; $<F_i^{(1)}>$ is the result of averaging $F_i^{(1)}$ from t to $t + \tau$, along a set of trajectories of the same class as X_i'; $\ll F_i^{(1)} \gg$ is the result of averaging $F_i^{(1)}$, from t to $t + \tau$, along a set of trajectories of the same class of X_i''.

Fig. 6. 1. Ion 1, with a large charge and a large mass, interact with a similar ion 2. Many electons, depicted with small dots, interfer the ion-ion interaction.

Fig. B. 1 The volume of the space domain closed with two spheres with radius r_{ij} is $\frac{9\pi}{4} r_{ij}^3$.

Fig. C. 1. Particle 1 is the test particle; particle 2 and particle 3 represent the field particles. There are four typical relations among them: a) the threes are mutually remote; b) particle i and particle 2 are their mutual nearest neighbors; c) particle 1 and particle 3 are their mutual nearest neighbors; d) particle 2 and particle 3 are their mutual nearest neighbors.

Fig. C. 2. Particle 1 is the test particle; particles 2, 3, 4, 5, represent the field particles;among them particles 2 and 3 are ions and particles 4 and 5 are electrons. There are 26 relations among them; full and short lines indicate nearest neighborhood and long and dotted lines remote relations.

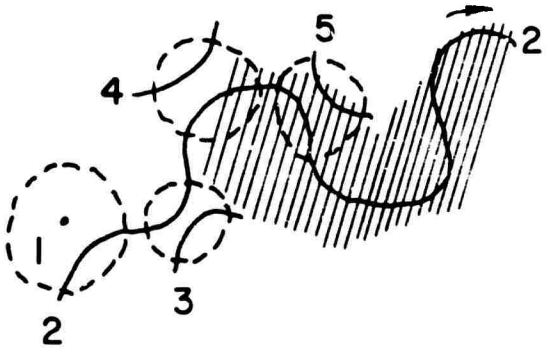

FIG. 3.1

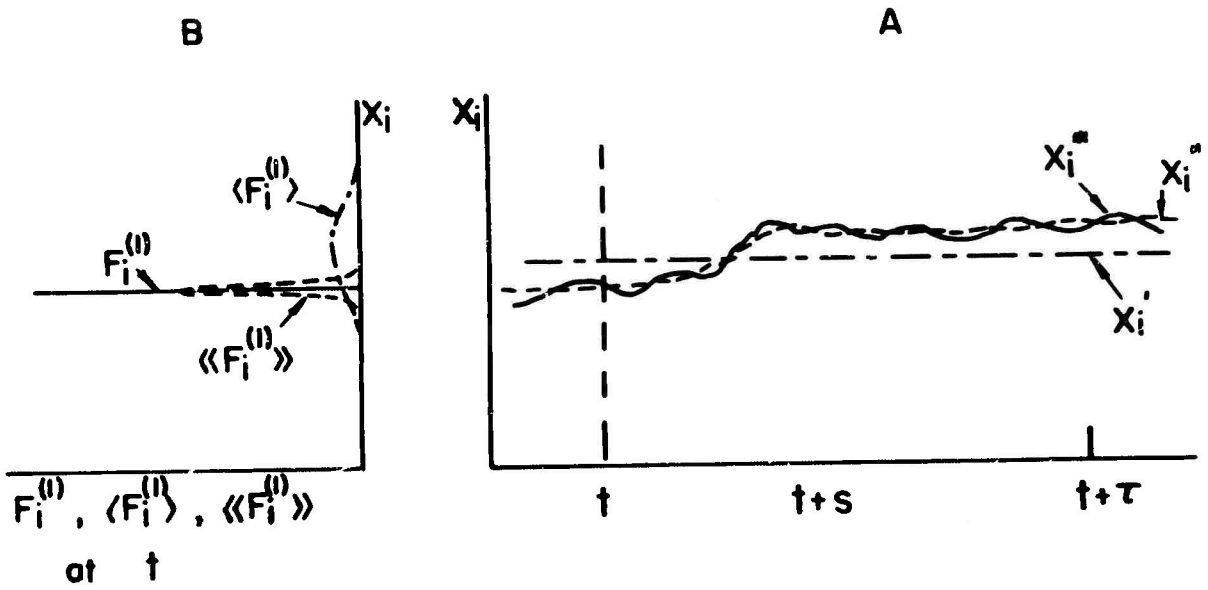
B
A
X_i
$\langle F_i^{(i)} \rangle$
$F_i^{(i)}$
$\langle\langle F_i^{(i)} \rangle\rangle$
$F_i^{(i)}$, $\langle F_i^{(i)} \rangle$, $\langle\langle F_i^{(i)} \rangle\rangle$
at t
X_i
X_i^{**}
X_i^{*}
X_i'
t
t+s
t+τ

FIG. 4.1

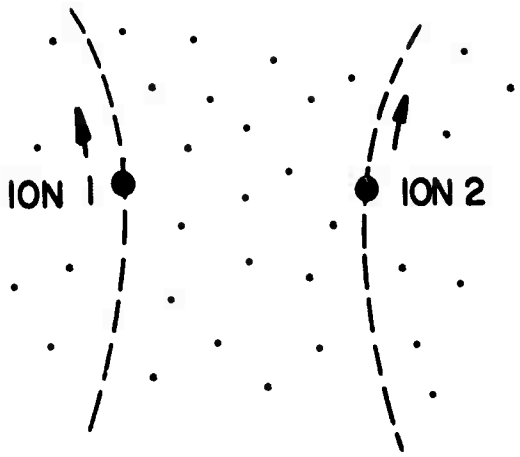

FIG. 6.1

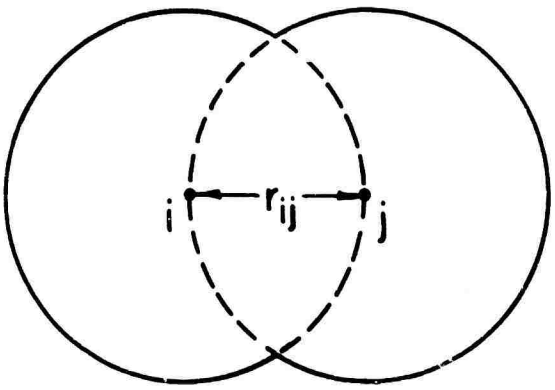

FIG. B.1

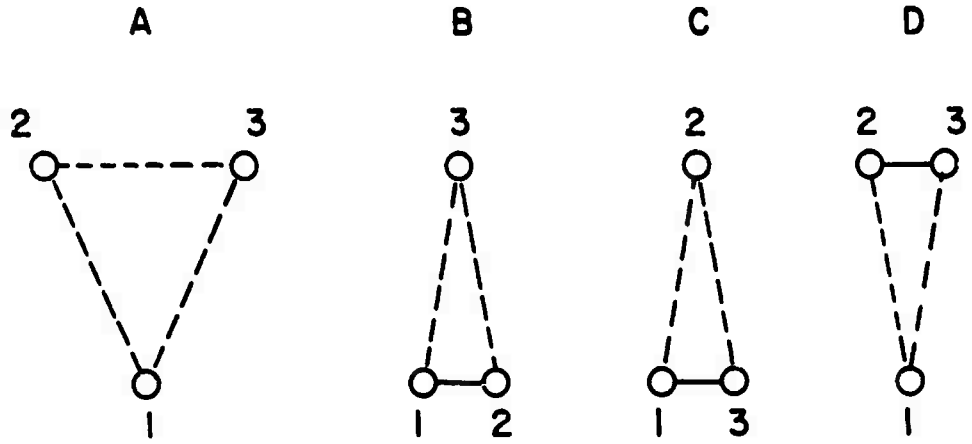

FIG. C. 1

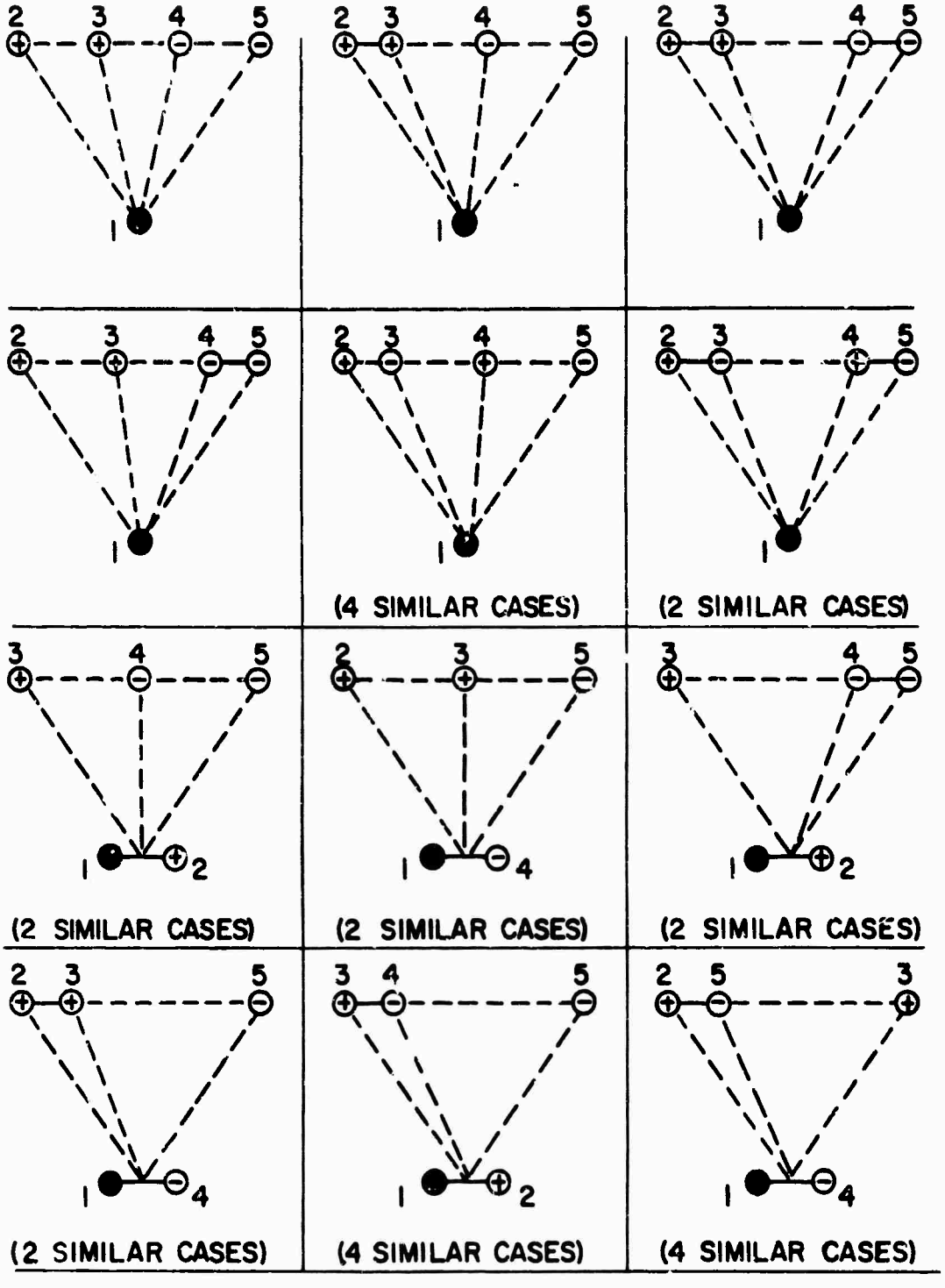
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FIG. C.2

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13. ABSTRACT

Because of the time scale assumption necessary for its derivation, the Bogoliubov-Born-Green-Kirkwood-Yvon (BBGKY) hierarchy of equations is not applicable to systems consisting of charged particles, except for special subsystems. A new class of equations governing the evolutions of charged particles is derived from the Liouville equation and is coarsed-grained with respect to time and similar particles. In the zeroth approximation, there are two basic types of inter-particle interactions: One is of the Vlasov type and the other of the Boltzmann type characterizing interactions among nearest neighboring particles. In higher order approximations, mutual perturbations among those basic interactions result in secondary effects; for example, two nearest-neighboring particles exert a force of microscopic order to another particle. Depending on the ratio between the number of electrons and the number of ions in a real system, the simulating model varies. The main purpose of the paper is to present schemes of rational treatment, rather than to provide numerical results in detail for a particular system.

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