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ON THE STATISTICAL PROPERTIES OF TRANSIENT NOISE SIGNALS

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UNITED STATES NAVAL ORDNANCE LABORATORY, WHITE OAK, MARYLAND

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ON THE STATISTICAL PROPERTIES OF TRANSIENT NOISE SIGNALS

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ABSTRACT: A class of random transient signals has been defined as the product of a deterministic envelope waveform of finite integral square and a continuous random process with a well-defined power spectrum and autocorrelation function. The time average autocorrelation function and energy density spectrum of the resulting waveform have been found to be random variables at every value of their arguments. The means and variances of these random variables are derived as functions of the characteristics of the envelope and original noise process. The average autocorrelation function is found to be the product of the autocorrelation functions of envelope and noise, and the average spectrum is given by the convolution of the energy spectrum of the envelope function and the power spectrum of the noise. Examples of the mean and variance calculations are presented for both rectangular and decaying exponential pulses of both broad and narrow band noise. Finally, the implications of these findings for measurement programs and monopulse signal processing are discussed.

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ON THE STATISTICAL PROPERTIES OF TRANSIENT NOISE SIGNALS

A class of random signals has been modeled as the product of a transient, deterministic envelope waveform and a well-behaved continuing random process. The properties of the energy density spectrum and autocorrelation function of such signals are studied and the results related to current problems in signal processing and monopulse detection systems. The work on this project was funded under Task ASW2-21-000-W270-70-00. The report will be of interest to those concerned with statistical communication and detection theory, active sonar systems, signal processing, and noise immunity studies.

The author wishes to acknowledge with thanks the aid of Mr. Ralph Ferguson and Miss Ann Penn of the Computer Applications Division in preparing much of the computer programming underlying these results.

E. F. SCHREITER
Captain, USN
Commander

E. H. Beach
E. H. BEACH
By direction

CONTENTS

ILLUSTRATIONS

ILLUSTRATIONS

TABLE

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Chapter I

INTRODUCTION

In the study of a large class of communication and signal detection systems, one is often faced with the analysis of the effects of interfering noise of a transient, non-continuing nature. Examples of such noise phenomena include the reverberation background of a sonar signal and impulsive interference of the type seen on telephone lines and atmospheric radio links. When formulating a system for the detection of wanted signals in such a background, it is often necessary to know in some detail the frequency distribution of energy in the interference or how such a noise component behaves under correlation processing. The intent of this report is to detail an investigation of certain statistical properties of a class of noise bursts suggested by the above. It is hoped that the results gained here place in somewhat better perspective the problems faced in the analysis and synthesis of processing systems working against non-stationary backgrounds.

The noise signals to be treated here are neither continuing stochastic processes in the usual sense nor deterministic transients amenable to immediate treatment by the Fourier integral. They share the properties of both broad classes but lack the mathematical convenience that arises from the usual assumptions. (It is suggested that these signals, bearing many properties of both transients and random signals, be known as "random transients".) Since a noise burst is defined only for a given epoch, its statistical properties are tied to a given instant of time, and stationarity disappears. Since ensemble averages no longer equal time averages, ergodicity soon evaporates also. On the other hand, such a burst does not have a deterministic Fourier transform, and Fourier integral analysis must be approached with great care. Even so, by carefully defining terms and remaining reasonably aware of the necessity of continually relating the mathematics to the physical situation, it is possible to achieve a consistent and useful interpretation of impulsive noise phenomena.

Eventually, it emerges that the autocorrelation functions and spectra of such noise signals no longer possess deterministic values at every point, but rather become random variables with calculable means and variances. Fortunately, it is possible to show that the means are given by quasi-intuitive expressions similar to those developed in traditional transient or random theory. The variances, in turn, provide an indication of the uncertainty of the spectra and correlation functions for a given value of their arguments. The analysis thus lends a good deal of insight to noise measurement programs and also to the choice of a processing scheme that allows sufficient latitude to encompass the great majority of interfering background noise that may arise.

Chapter II

A MODEL FOR THE GENERATION OF RANDOM TRANSIENTS

The model used to generate the transient noise signals to be treated herein is portrayed in Figure 1. The random transient is taken as the output of a multiplier whose inputs are a zero-mean Gaussian random process $n(t)$ and an "envelope waveform" $e(t)$, both of which are real. The former is assumed to be stationary and ergodic, thus possessing a well-defined power spectrum and autocorrelation function as described in reference (a). The "envelope waveform" is held to be a deterministic transient of finite integral square which is zero for $t < 0$. The output of the multiplier is the product of these functions and evidently equals zero when $e(t) = 0$.

Some justification of this model is provided by noting that many of the processes which yield random-transient-like signals can be approached from a theoretical basis which yields a prediction of some quantity which can loosely be described as the "average level" as a function of time. In sonar applications for example, it is possible to derive theoretical expressions for the acoustic power returned as a function of range from either volume or boundary reverberation. Similarly, in the study of transients caused by impulsive phenomena such as chemical explosions or spark gaps, a theoretical treatment may well provide an expression for some kind of envelope within which the detailed structure of the transient is more or less random. It is certainly visionary to claim that such a multiplicative envelope function can be rigorously defined for the physical processes of the real world. The point is, however, that although the detailed structure of a particular transient may be vastly different from every other, such a quantity as an average "level" or perhaps some analogous statistical measure may well display a uniformity from sample to sample that can be described with the artifice of a multiplicative envelope.

Here, this envelope waveform has been considered a deterministic transient, probably the simplest assumption that might have been made. It is felt that such an approach is consistent with the class of situations described immediately above, but certainly a more elaborately structured model could be envisioned to accomodate a larger class of natural phenomena. As a first step upward from the present assumption, one might randomize the envelope by considering such parameters as length and amplitude to be random variables with appropriate distributions. Another possibility is to model the envelope waveform itself as a segment from a relatively low frequency process so that both functions entering the multiplier are completely random. These

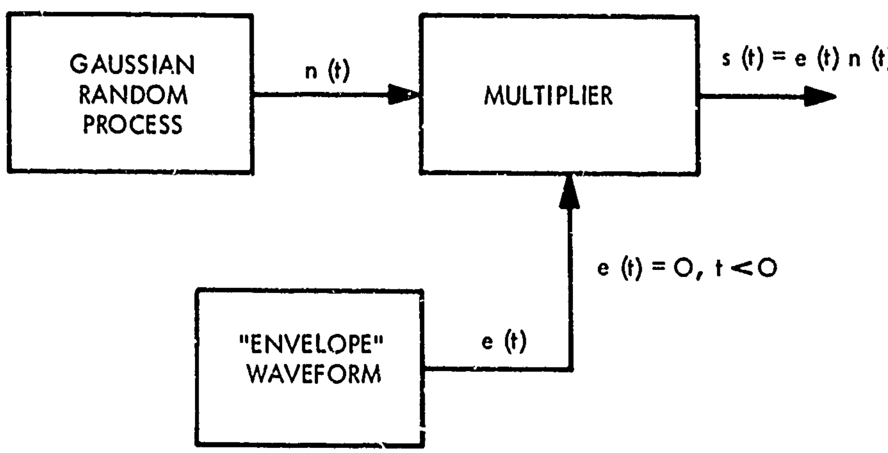

FIGURE 1. IDEALIZED MODEL OF WAVEFORM GENERATION.

approaches are not developed here but remain an interesting avenue for future study. At least at the outset of this investigation it was felt that the simplest model was adequate to deal with the phenomena of immediate interest. We shall now concentrate our attention on the model of Figure 1.

The term "envelope waveform" is actually something of a misnomer here since it does not play the same role as the corresponding concept in, for example, amplitude modulation where e(t) would represent the slowly varying envelope of a more rapid oscillation. Consider the output waveform:

$$s(t) = e(t)\, n(t) \qquad (1)$$

where at every instant of time t, $n(t)$ is a Gaussian random variable with mean zero and variance σ_N^2. At this same instant of time, $s(t)$ is thus also a Gaussian random variable with zero mean and with variance.

$$\sigma_s^2(t) = e^2(t)\,\sigma_N^2 \qquad (2)$$

Alternatively,

$$\sigma_s(t) = |e(t)|\,\sigma_N \qquad (3)$$

and it is seen immediately that the most meaningful interpretation of $e(t)$ is as the time varying factor whose magnitude relates the standard deviations of the input and output. It is also apparent now that $s(t)$ can in no way be considered a stationary random variable since indeed its variance is a function of time. This is hardly surprising since we are now dealing with a transient signal for which concepts of stationarity are irrelevant.

Consider, however, an infinite ensemble of waveform generators on this model, each with its separate independent random process $n(t)$, but with identical envelope waveforms $e(t)$. The ensemble of output transients generated by such an assemblage can be expected to display a large measure of statistical regularity, and it is with this ensemble of all possible transients with identical $e(t)$ that we propose to work.

CHAPTER III

NOISE BURST AUTOCORRELATION FUNCTIONS

For a real deterministic transient such as the envelope waveform e(t), the autocorrelation function is generally defined as

$$\varphi_{ee}(\tau) \equiv \int_{-\infty}^{\infty} e(t)e(t+\tau)dt \qquad (4)$$

such that

$$\varphi_{ee}(0) = \int_{-\infty}^{\infty} e^2(t)dt \qquad (5)$$

where $\varphi_{ee}(0)$ is known as the energy of e(t) in the sense that if e(t) were a voltage or current waveform, this expression gives the total energy dissipated by e(t) in a pure one ohm resistance (reference (a)).

For a real random process such as n(t), extending in time from $-\infty$ to $+\infty$, two autocorrelation functions can be defined. First the ensemble average autocorrelation function:

$$R_n(t_1,t_2) = E\left[n(t_1)n(t_2)\right] \qquad (6)$$

where E[ ] denotes the statistical expectation taken over all members of the ensemble. R_n is, in general, a function of t_1 and t_2, but if the process is stationary, it becomes a function only of τ, the difference between t_1 and t_2:

$$R_n(t_1,t_2) = R_n(\tau) = E\left[n(t)n(t+\tau)\right] \qquad (7)$$

where $\tau = t_1 - t_2$ and t_1 is arbitrary

The <u>time average autocorrelation</u> function is defined* as

$$\psi_{nn}(\tau) \equiv \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} n(t)n(t+\tau)dt \tag{8}$$

with

$$\psi_{nn}(0) = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} n^2(t)dt \tag{9}$$

which is the average <u>power</u> of $n(t)$ in the sense that this expression gives the average power dissipated in a one ohm resistor by $n(t)$.

If the process is ergodic, time and ensemble averaging are equivalent, and in particular,

$$R_n(\tau) = \psi_{nn}(\tau) \tag{10}$$

With this background, we can proceed to a meaningful definition and evaluation of the autocorrelation function of the signal s(t) defined in equation (1) and Figure 1. Consider first the attempt to form an ensemble average autocorrelation function for s(t) as in equation (6):

$$\begin{aligned} R_s(t_1,t_2) &= E\left[s(t_1)s(t_2)\right] \\ &= E\left[e(t_1)e(t_2)\,n(t_1)n(t_2)\right] \end{aligned} \tag{11}$$

Since e(t) is deterministic, we may write

$$R_s(t_1,t_2) = e(t_1)e(t_2)E\left[n(t_1)n(t_2)\right] \tag{12}$$

Because $n(t)$ has been assumed to be ergodic, this becomes

$$R_s(t_1,t_2) = R_s(t_1,\tau+t_1) = e(t_1)e(t_1+\tau)\,\psi_{nn}(\tau) \tag{13}$$

$$\tau = t_2 - t_1$$

Since R_s is a function of both t_1 and τ, s(t) is nonstationary, and the ensemble average autocorrelation function loses much of its interest. Such is generally the case in treating a transient waveform.

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* It may be well to point out that in this report, the letter φ (phi) will be used to denote the autocorrelation functions of <u>energy</u> signals as in equation (4), whereas the letter ψ (psi) will be used for the autocorrelation functions of <u>power</u> signals as in equation (8).

By using, however, the formula of equation (4) which is appropriate for traditional transient analysis, we can write that

$$\varphi_{ss}(\tau) = \int_{-\infty}^{\infty} s(t)s(t+\tau)\,dt$$

$$= \int_{0}^{\infty} e(t)e(t+\tau)n(t)n(t+\tau)\,dt \qquad (14)$$

where the lower limit of the integral becomes zero since $e(t) = 0$ for $t < 0$. Now since $n(t)$ is a random variable for all t, it becomes apparent that the integral of equation (14), if it exists, is also a random variable. The existence of an integral such as that of equation (14) is examined in reference (b) which treats expressions of the form

$$y(\tau) = \int_{a}^{b} h(t,\tau)x(t)\,dt \qquad (15)$$

where only $x(t)$ is a random variable. Now, if

$$\int_{a}^{b} E\left[|h(t,\tau)x(t)|\right] dt = \int_{a}^{b} |h(t)|\, E\left[|x(t)|\right] dt < \infty \qquad (16)$$

then $y(\tau)$ exists for all sample functions of $x(t)$ except for a set of probability zero, and furthermore

$$E\left[y(\tau)\right] = \int_{a}^{b} h(t,\tau)\, E\left[x(t)\right] dt \qquad (17)$$

Looking at equation (14) in this light we see that

$$\int_{0}^{\infty} E\left[|e(t)e(t+\tau)n(t)n(t+\tau)|\right] dt = \int_{0}^{\infty} |e(t)e(t+\tau)|\, E\left[|n(t)|\,|n(t+\tau)|\right] dt \qquad (18)$$

Now $E\left[|n(t)|\,|n(t+\tau)|\right]$ can be interpreted as the autocorrelation function of a full-wave rectified version of n(t), which exists if n(t) is a sufficiently well-behaved stochastic process as we have assumed. But by the basic property of the autocorrelation function that its maximum occurs at the origin for any random or transient function,

$$E\left[|n(t)|\,|n(t+\tau)|\right] \leq E\left[|n(t)|\,|n(t)|\right] = E\left[n^2(t)\right] = \psi_{nn}(0) \qquad (19)$$

Therefore,

$$\int_0^\infty \left|e(t)e(t+\tau)\right| \; E\left[\left|n(t)n(t+\tau)\right|\right] dt \le \psi_{nn}(0) \int_0^\infty |e(t)| \; |e(t+\tau)| \, dt \tag{20}$$

$$\le \psi_{nn}(0) \;\; \varphi_{ee}(0) < \infty$$

since n(t) is assumed to convey finite power and since e(t) is a transient with finite energy.

It has thus been shown that $\varphi_{ss}(\tau)$ exists for all members of the ensemble except for a set of probability zero (whatever that means), and it begins to make sense to speak about the autocorrelation function of s(t) in these terms. Since $\varphi_{ss}(\tau) < \infty$ for all τ,

$$\psi_{ss}(\tau) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} s(t)s(t+\tau)dt = \lim_{T\to\infty} \frac{\varphi_{ss}(\tau)}{2T} \tag{21}$$

$$= 0, \text{ for all } \tau$$

and this form of the time average autocorrelation function, appropriate for continuing random signals, becomes meaningless here. Thus s(t) is an <u>energy</u> signal rather than a <u>power</u> signal and in that sense is more akin to a transient than to a continuing random process.

If one defines the autocorrelation function of s(t) in the form of equation (4), the result is a random variable whose mean and variance at every τ must be related to the correlation functions of e(t) and n(t). Thus if

$$\varphi_{ss}(\tau) \equiv \int_0^\infty s(t)s(t+\tau)dt = \int_0^\infty e(t)e(t+\tau)n(t)n(t+\tau)dt \tag{22}$$

the ensemble average becomes

$$E\left[\varphi_{ss}(\tau)\right] = E\left[\int_0^\infty e(t)e(t+\tau)n(t)n(t+\tau)dt\right] \tag{23}$$

and using equation (17), this is

$$E\left[\varphi_{ss}(\tau)\right] = \int_0^\infty e(t)e(t+\tau) \; E\left[n(t)n(t+\tau)\right]dt \tag{24}$$

Finally, by equations (4), (7), and (10),

$$E\left[\varphi_{ss}(\tau)\right] = \psi_{nn}(\tau) \quad \varphi_{ee}(\tau) \tag{25}$$

It becomes evident, then, that the expected value of the autocorrelation function is, for every τ , equal to the product of the autocorrelation functions of the envelope waveform and the original noise process, and is therefore similar to the result gained in seeking the autocorrelation function of the product of two independent stationary random processes, where the autocorrelation function of the product is equal to the product of the autocorrelation functions.

Some care must be taken in the interpretation of equation (25). What is being claimed is precisely this: Given a sample function from the ensemble generated by the model of Figure 1, we calculate the time average autocorrelation function as in equation (22) - by multiplying the sample function by a shifted replica of itself and integrating the product from 0 to ∞ . If this operation is performed on a number of sample functions and the results averaged for fixed τ, the average will tend to the expression of equation (25) as the number of sample functions grows large.

If $\tau = 0$, equation (25) provides the <u>average total energy</u>:

$$E\left[\varphi_{ss}(0)\right] = \psi_{nn}(0) \quad \varphi_{ee}(0) \tag{26}$$

numerically equal to the product of the energy of e(t) and the average power of n(t) (but bearing, of course, the dimensions of energy). This is to say that if we form the integral

$$\varphi_{ss}(0) = \int_0^\infty s^2(t)dt \tag{27}$$

for a number of sample functions, the average value approaches $E\left[\varphi_{ss}(0)\right]$ as the number of sample functions taken grows large.

Next to be considered is the variance of the random variable $\varphi_{ss}(\tau)$ generated by the correlation process of equation (22). From elementary probability theory it is known that

$$\mathrm{Var}\left[\varphi_{ss}(\tau)\right] = E\left[\varphi^2_{ss}(\tau)\right] - \left\{E\left[\varphi_{ss}(\tau)\right]\right\}^2 \tag{28}$$

or in words, that the variance is equal to the mean square minus the square of the mean. The mean square of $\varphi_{ss}(\tau)$ must thus now be calculated. The square of $\varphi_{ss}(\tau)$ is given by

$$\varphi_{ss}^2(\tau) = \int_0^\infty e(t)e(t+\tau)n(t)n(t+\tau)dt \int_0^\infty e(x)e(x+\tau)n(x)n(x+\tau)dx \qquad (29)$$

$$= \int_0^\infty\int_0^\infty e(t)e(x)e(t+\tau)e(x+\tau)n(t)n(x)n(t+\tau)n(x+\tau)dx\ dt$$

Taking the expectation and interchanging with the integration yields

$$E\left[\varphi_{ss}^2(\tau)\right] = \int_0^\infty\int_0^\infty e(t)e(x)e(t+\tau)e(x+\tau)\ E\left[n(t)n(x)n(t+\tau)n(x+\tau)\right] dx\ dt \qquad (30)$$

At this point, for the first time, we must make explicit use of the fact that n(t) is a Gaussian random process. Indeed, it can be shown (c.f., reference (c)) that if n(t) is a zero mean Gaussian process, then

$$E\left[n(t_1)n(t_2)n(t_3)n(t_4)\right] = E\left[n(t_1)n(t_2)\right] E\left[n(t_3)n(t_4)\right] \qquad (31)$$

$$+ E\left[n(t_1)n(t_3)\right] E\left[n(t_2)n(t_4)\right]$$

$$+ E\left[n(t_1)n(t_4)\right] E\left[n(t_2)n(t_3)\right]$$

Now using ergodicity and equations (6), (7), and (10), we may write that

$$E\left[n(t)n(x)n(t+\tau)n(x+\tau)\right] = \psi_{nn}^2(\tau) + \psi_{nn}^2(t-x) + \psi_{nn}(t-x-\tau)\ \psi_{nn}(t-x+\tau) \qquad (32)$$

Following through,

$$E\left[\varphi_{ss}^2(\tau)\right] = \int_0^\infty \int_0^\infty e(t)e(x)e(t+\tau)e(x+\tau)\,\psi_{nn}^2(\tau)\,dx\,dt$$
$$+\int_0^\infty \int_0^\infty e(t)e(x)e(t+\tau)e(x+\tau)\left[\psi_{nn}^2(t-x) + \psi_{nn}(t-x+\tau)\psi_{nn}(t-x-\tau)\right]dx\,dt \qquad (33)$$

The first integral may immediately be written as

$$\psi_{nn}^2(\tau)\int_0^\infty e(t)e(t+\tau)dt \int_0^\infty e(x)e(x+\tau)dx = \psi_{nn}^2(\tau)\varphi_{ee}^2(\tau) \qquad (34)$$

but by equation (25),

$$\psi_{nn}^2(\tau)\varphi_{ee}^2(\tau) = \left\{E\left[\varphi_{ss}(\tau)\right]\right\}^2 \; ; \qquad (35)$$

and directly, by cancellation in equation (28),

$$\mathrm{Var}\left[\varphi_{ss}(\tau)\right] = \int_0^\infty \int_0^\infty e(t)e(x)e(t+\tau)e(x+\tau)$$
$$\left[\psi_{nn}^2(t-x) + \psi_{nn}(t-x+\tau)\psi_{nn}(t-x-\tau)\right]dt\,dx \qquad (36)$$

The evaluation of this horrendous integral can be somewhat eased by making the change of variables $u = t-x$ and $t = t$, and then eliminating x from the equation ($|\text{Jacobian}| = 1$; first quadrant area of tx plane corresponds to the upper half of the tu plane).

$$\mathrm{Var}\left[\varphi_{ss}(\tau)\right] = \int_0^\infty \int_{-\infty}^\infty e(t)e(t-u)e(t+\tau)e(t+\tau-u)$$
$$\left[\psi_{nn}^2(u) + \psi_{nn}(u+\tau)\psi_{nn}(u-\tau)\right]du\,dt \qquad (37)$$

Reversing the order of integration then yields

$$\mathrm{Var}\left[\varphi_{ss}(\tau)\right] = \int_{-\infty}^\infty \left[\psi_{nn}^2(u) + \psi_{nn}(u+\tau)\,\psi_{nn}(u-\tau)\right]$$
$$\int_0^\infty e(t)e(t-u)e(t+\tau)e(t+\tau-u)dt\,du \qquad (38)$$

Consider now the inner integral, which can be expressed as

$$\int_0^\infty e(t)e(t+\tau)e(t-u)e(t+\tau-u)dt = \int_0^\infty p(t,\tau)p(t-u,\tau)dt \tag{39}$$

where

$$p(t,\tau) \equiv e(t)e(t+\tau), \tag{40}$$

a function defined as e(t) multiplied by a shifted replica of itself. $p(t,\tau)$ possesses an autocorrelation function defined as

$$\varphi_{pp}(u,\tau) = \int_0^\infty p(t,\tau)p(t-u,\tau)dt, \tag{41}$$

identical with the inner integral. The variance may now be written

$$\mathrm{Var}\left[\varphi_{ss}(\tau)\right] = \int_{-\infty}^{\infty}\varphi_{pp}(u,\tau)\left[\psi_{nn}^2(u) + \psi_{nn}(u+\tau)\,\psi_{nn}(u-\tau)\right] du \tag{42}$$

As defined above, $\varphi_{pp}(u,\tau)$ is merely the autocorrelation function of the signal generated by multiplying e(t) by a replica of itself shifted τ seconds. Several examples of this function will be computed in a later section, but some general properties emerge at once. Since $\varphi_{pp}(u,\tau)$ is a bona fide autocorrelation function with u as the independent variable and τ as a parameter, it must be an even function of u and, having no periodicity, assume its maximum value (for given τ) when $u = 0$. Also, since $e(t)e(t+\tau)$ is identical with $e(t)e(t-\tau)$ except for a shift in the time origin, $\varphi_{pp}(u,\tau)$ is an even function of the parameter τ and for a given u obtains its maximum value when $\tau = 0$. Since the integrand factor in square brackets (equation (42)) is also even, the variance may also be written as

$$\mathrm{Var}\left[\varphi_{ss}(\tau)\right] = 2\int_0^\infty \varphi_{pp}(u,\tau)\left[\psi_{nn}^2(u) + \psi_{nn}(u+\tau)\,\psi_{nn}(u-\tau)\right] du \tag{43}$$

The expression for the variance derived here is particularly easy to evaluate on a digital computer using numerical integration techniques, and several examples of this calculation will be presented later. The result is to be interpreted as supplying the variance of the random variable $\varphi_{ss}(\tau)$ as a function of τ. Since by definition

$$\mathrm{Var}\left[\varphi_{ss}(\tau)\right] = E\left\{\left[\varphi_{ss}(\tau) - E\left[\varphi_{ss}(\tau)\right]\right]^2\right\} \tag{44}$$

we have found a measure of how tightly the distribution of $\varphi_{ss}(\tau)$ cleaves to the mean, and hence of the amount of variability to be expected between points of equal τ on the empirical autocorrelation functions of representative samples. Evidently, when $\mathrm{Var}[\varphi_{ss}(\tau)]$ becomes small, we can reasonably expect that the empirical measurement will be close to the expected value. It is of considerable interest to know where the variance is a maximum.

Consider then the expression of equation (43);

$$\mathrm{Var}\left[\varphi_{ss}(\tau)\right] = 2\int_0^\infty \varphi_{pp}(u,\tau)\psi_{nn}^2(u)\,du + 2\int_0^\infty \varphi_{pp}(u,\tau)\psi_{nn}(u+\tau)\psi_{nn}(u-\tau)\,du \tag{45}$$

Since $\varphi_{pp}(u,0) \geq \varphi_{pp}(u,\tau)$ for all u, then

$$\int_0^\infty \varphi_{pp}(u,\tau)\psi_{nn}^2(u)\,du \leq \int_0^\infty \varphi_{pp}(u,0)\psi_{nn}^2(u)\,du \tag{46}$$

and the first integral is obviously a maximum for $\tau = 0$. It is also true that

$$\int_0^\infty \varphi_{pp}(u,\tau)\psi_{nn}(u+\tau)\psi_{nn}(u-\tau)\,du \leq \int_0^\infty \varphi_{pp}(u,0)\psi_{nn}(u+\tau)\psi_{nn}(u-\tau)\,du \tag{47}$$

and it is shown using the Schwarz inequality in Appendix A that if

$\varphi_{pp}(u,0) \geq 0$ for all u, then

$$\int_0^\infty \varphi_{pp}(u,0)\psi_{nn}(u+\tau)\psi_{nn}(u-\tau)\,du \leq \int_0^\infty \varphi_{pp}(u,0)\psi_{nn}^2(u)\,du \tag{48}$$

From equations (47) and (48) then

$$\mathrm{Var}\left[\varphi_{ss}(\tau)\right] \leq \mathrm{Var}\left[\varphi_{ss}(0)\right] \quad \text{for all } \tau \tag{49}$$

subject only to the condition, essentially, that the envelope waveform be everywhere positive. The maximum variance thus occurs at the origin, as one might have expected intuitively, and its value is given by

$$\text{Max}\left\{\text{Var}\left[\varphi_{ss}(\tau)\right]\right\} = \text{Var}\left[\varphi_{ss}(0)\right]$$

$$= 4\int_0^\infty \psi_{nn}^2(u)\int_0^\infty e^2(t)e^2(t-u)dt\ du \tag{50}$$

Since $\varphi_{pp}(u,\tau)$ approaches zero for all u as τ grown large, it should be apparent that

$$\text{Var}\left[\varphi_{ss}(\tau)\right] \rightarrow 0 \quad \text{as} \quad \tau \rightarrow \infty \tag{51}$$

One more general result will now be discussed having to do with the behavior of the autocorrelation variance as the extension in time of the envelope waveform increases. If the statistical properties of the original noise process remain constant while the envelope duration grows larger (in the sense that a rectangular pulse of width T or an exponential pulse of time constant T become "longer" as T increases), we might reasonably expect the variance of the autocorrelation to be decreased since in effect the integration-averaging process of equation (22) is being carried out over a longer and longer period. In other words, by taking a longer and longer "piece" of the input noise, one approaches the operation of equation (8) which yields an expression of zero variance. Actually, since the maximum variance is related to the average total energy (see equation (50)), the variance will increase (absolutely) as the envelope duration (and hence its energy) increases. For this reason it is more instructive to work with a normalized form of the maximum standard deviation. We form, then, the ratio of the standard deviation at $\tau = 0$ to the expected value of $\varphi_{ss}(\tau)$ at $\tau = 0$:

$$R_o \equiv \frac{\sqrt{\text{Var}\left[\varphi_{ss}(0)\right]}}{E\left[\varphi_{ss}(0)\right]} \tag{52}$$

This ratio expresses the standard deviation of $\varphi_{ss}(0)$ as a percentage of the mean of $\varphi_{ss}(0)$ and thus measures the extent that the distribution of $\varphi_{ss}(\tau)$ clusters about the mean in the worst case. From equation (50),

$$R_o = \frac{2\sqrt{\int_0^\infty \varphi_{pp}(u,0)\psi_{nn}^2(u)dt}}{\psi_{nn}(0)\int_0^\infty e^2(t)dt} \tag{53}$$

Consider the following heuristic argument. As the width of the envelope function increases (as $\Psi_{nn}(u)$ remains the same), $\varphi_{pp}(u,0)$ becomes "wider" also, and in most cases can be considered a constant near the origin. If T is some measure of the duration of e(t), then as T increases, a point is reached where the integral in the numerator approaches a constant value (due to the relative narrowness of $\Psi_{nn}(u)$). Meanwhile, the integral of the denominator grows roughly as T, and hence

$$\lim_{T \to \infty} R_o = \lim_{T \to \infty} \frac{k}{T} = 0 \tag{54}$$

This shows that as the duration of the envelope increases, we can expect the autocorrelation function of s(t) to depart percentage-wise less and less from $E\left[\varphi_{ss}(\tau)\right]$ at every point.

We turn next to the spectrum of s(t), where it is found that such convenient limiting behavior does not occur.

Chapter IV

NOISE BURST SPECTRA

To continue this investigation of the properties of noise bursts, it is now appropriate to turn to a consideration of the energy spectrum of signals of this type. As described in reference (a), a deterministic transient can be specified by its Fourier integral:

$$S(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} s(t)\, e^{-j\omega t}\, dt \tag{55}$$

such that

$$s(t) = \int_{-\infty}^{\infty} S(\omega) e^{j\omega t}\, d\omega \tag{56}$$

Now by Parseval's theorem (reference (a), pp 33-39),

$$2\pi \int_{-\infty}^{\infty} S(\omega)\, \overline{S(\omega)}\, d\omega = \int_{-\infty}^{\infty} s^2(t)\, dt \tag{57}$$

where the superscript bar represents complex conjugation. Since the right hand integral is simply the total energy of s(t) as defined in equation (5), the expression

$$\Phi_{ss}(\omega) \equiv 2\pi\, S(\omega)\, \overline{S(\omega)} = 2\pi \left| S(\omega) \right|^2 \tag{58}$$

can be interpreted as an <u>energy density spectrum</u> indicating how the total energy of s(t) is distributed in the frequency domain. This is to say that if s(t) were passed through an ideal low pass filter with a sharp cut-off frequency ω_c, then the total energy dissipated in a pure one ohm resistance following the filter would be

$$E(\omega < \omega_c) = \int_{-\omega_c}^{\omega_c} \Phi_{ss}(\omega) d\omega = 2\int_{0}^{\omega_c} \Phi_{ss}(\omega) d\omega \tag{59}$$

Alternatively, $\Phi_{ss}(\omega)$ can be interpreted as giving the total energy dissipated in the ubiquitous one ohm resistor by a band of frequencies one radian/sec wide centered at the argument radian frequency, ω.

For a signal of the random transient type, the integral of equation (55) can be written, at least formally, as the first step of obtaining the energy density spectrum of the burst. From the same considerations that led to the formation of the autocorrelation integral, it can be shown that the integral of equation (55) exists for all members of the ensemble except for a subset of probability zero. But since every member of the ensemble is different from every other member, every $S(\omega)$ will be different also, and hence there is generated an ensemble of integrals of the type of equation (55) defining $S(\omega)$ as a new random variable. Thus, by equation (58), $\Phi_{ss}(\omega)$ will also be a random variable for every value of ω and will not possess a deterministic value.

The next step, evidently, is to compute the expected value of $\Phi_{ss}(\omega)$. By definition,

$$\Phi_{ss}(\omega) = 2\pi\, S(\omega)\, \overline{S(\omega)} = \frac{1}{2\pi}\int_{-\infty}^{\infty} s(t)e^{-j\omega t}dt \int_{-\infty}^{\infty} s(x)e^{+j\omega x}\,dx$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} s(t)s(x)e^{-j\omega(t-x)}dx\,dt \tag{60}$$

Following through,

$$E\left[\Phi_{ss}(\omega)\right] = \frac{1}{2\pi}\int_{0}^{\infty}\int_{0}^{\infty} e(t)e(x)\, E\left[n(t)n(x)\right]\, e^{-j\omega(t-x)}dx\,dt \tag{61}$$

Because $n(t)$ has been assumed ergodic and stationary,

$$E\left[n(t)n(x)\right] = \Psi_{nn}(t-x) = \Psi_{nn}(u) \tag{62}$$

where $u = t-x$

Thus,

$$E\left[\Phi_{ss}(\omega)\right] = \frac{1}{2\pi}\int_{0}^{\infty}\int_{-\infty}^{\infty} e(t)e(t-u)\,\psi_{nn}(u)e^{-j\omega u}\,du\,dt \tag{63}$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\psi_{nn}(u)e^{-j\omega u}\int_{0}^{\infty} e(t)e(t-u)dt\;du$$

And finally,

$$E\left[\Phi_{ss}(\omega)\right] = \frac{1}{2\pi}\int_{-\infty}\psi_{nn}(u)\;\varphi_{ee}(u)e^{-j\omega u}\,du \tag{64}$$

which can be interpreted as the inverse Fourier transform of $E\left[\varphi_{ss}(u)\right]$ found in equation (25). Actually, by using the Wiener-Khinchin relation, it is possible to relate the expected value of $\Phi_{ss}(\omega)$ to the energy density spectrum of e(t) and the power density spectrum of n(t). Indeed, with $\psi_{nn}(\tau)$ given by equation (8), the power density spectrum of n(t) is given by the Fourier transform

$$\Psi_{nn}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\psi_{nn}(\tau)e^{-j\omega\tau}\,d\tau \tag{65}$$

and the energy density spectrum of e(t) is given by the transform*

$$\Phi_{ee}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\varphi_{ee}(\tau)e^{-j\omega\tau}\,d\tau \tag{66}$$

The inverse transform of $\Phi_{ee}(\tau)$ is given by

$$\varphi_{ee}(\tau) = \int_{-\infty}^{\infty}\Phi_{ee}(\omega)e^{j\omega\tau}\;d\omega \tag{67}$$

and hence equation (64) can be written as

---

*Similar to our convention in Chapter III, the upper case Greek letter Ψ (Psi) will be used to denote power density spectra, whereas the upper case letter Φ (Phi) will be used for energy density spectra.

$$E\left[\Phi_{ss}(\omega)\right] = \frac{1}{2\pi}\int_{-\infty}^{\infty}\psi_{nn}(u)e^{-j\omega u}\int_{-\infty}^{\infty}\Phi_{ee}(\omega)e^{j\sigma u}\,d\sigma\,du \qquad (68)$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\psi_{nn}(u)\;\Phi_{ee}(\sigma)e^{-ju(\omega-\sigma)}\,d\sigma\,du$$

Now setting $\omega' = \omega - \sigma$, we can write that

$$E\left[\Phi_{ss}(\omega)\right] = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\psi_{nn}(u)\;\Phi_{ee}(\omega-\omega')e^{-j\omega' u}\,du\,d\omega' \qquad (69)$$

Using equation (65),

$$E\left[\Phi_{ss}(\omega)\right] = \int_{-\infty}^{\infty}\Psi_{nn}(\omega')\;\Phi_{ee}(\omega-\omega')d\omega' \qquad (70)$$

$$= \Psi_{nn}(\omega) \otimes \Phi_{ee}(\omega)$$

Thus, the expected value of the energy spectrum of the random transient is found to be the convolution of the energy spectrum of the envelope waveform and the power spectrum of the original noise. At least as far as expected values are concerned, it has been shown that one can here apply the well known "folk theorem" that "multiplication in the time domain corresponds to convolution in the frequency domain."

Continuing with this Fourier integral approach, it is possible to investigate the variance of $\Phi_{ss}(\omega)$ at every point. First, by equation (58),

$$\Phi^2_{ss}(\omega) = 4\pi^2\, S^2(\omega)\,\overline{S(\omega)}^{\,2} \qquad (71)$$

Now using equation (55),

$$\Phi^2_{ss}(\omega) = \frac{1}{4\pi^2}\int_0^{\infty}\int_0^{\infty}\int_0^{\infty}\int_0^{\infty} s(t)s(x)s(\mu)s(v)e^{-j\omega(t-x+\mu-v)}dt\,dx\,d\mu\,dv \qquad (72)$$

$$E\left[\Phi^2_{ss}(\omega)\right] = \frac{1}{4\pi^2}\int_0^{\infty}\int_0^{\infty}\int_0^{\infty}\int_0^{\infty} e(t)e(x)e(\mu)e(v)\; E\left[n(t)n(x)n(\mu)n(v)\right]e^{-j\omega(t-x+\mu-v)}dt\,dx\,d\mu\,dv \qquad (73)$$

Since n(t) is assumed to be Gaussian, equation (31) still applies, and this becomes

$$E\left[\Phi_{ss}^{2}(\omega)\right] = \frac{1}{4\pi^2}\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty e(t)e(x)e(\mu)e(v)$$

$$\left[\psi_{nn}(t-x)\psi_{nn}(u-v)+\psi_{nn}(t-u)\psi_{nn}(x-v)+\psi_{nn}(t-v)\psi_{nn}(x-u)\right]$$

$$e^{-j\omega(t-x+\mu-v)}\ dt\ dx\ d\mu\ dv \qquad (74)$$

This is plainly the sum of three integrals, two of which are identical. The first and third are of the form

$$I_1 = I_3 = \frac{1}{4\pi^2}\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty e(t)e(x)e(\mu)e(v)\psi_{nn}(t-x)\psi_{nn}(\mu-v)e^{-j\omega(t-x+\mu-v)}\ dt dx d\mu dv \qquad (75)$$

Using the substitutions $a = t - x$ and $b = \mu - v$, these are equivalent to

$$I_1 = I_3 = \frac{1}{4\pi^2}\int_0^\infty\int_{-\infty}^\infty\int_0^\infty\int_{-\infty}^\infty e(t)e(t-a)e(\mu)e(b-\mu)\psi_{nn}(a)\psi_{nn}(b)e^{-j\omega(a+b)}\ dt da d\mu db$$

$$= \frac{1}{4\pi^2}\int_{-\infty}^\infty \psi_{nn}(a)e^{-j\omega a}\int_0^\infty e(t)e(t-a)dt\ da\int_{-\infty}^\infty\psi_{nn}(b)e^{-j\omega b}\int_0^\infty e(\mu)e(\mu-b)\ d\mu db$$

$$= \frac{1}{4\pi^2}\int_{-\infty}^\infty \psi_{nn}(a)\varphi_{ee}(a)e^{-j\omega a}\ da\int_{-\infty}^\infty\psi_{nn}(b)\varphi_{ee}(b)e^{-j\omega b}\ db$$

$$= \left\{\frac{1}{2\pi}\int_{-\infty}^\infty \psi_{nn}(a)\varphi_{ee}(a)e^{-j\omega a}\ da\right\}^2 = \left\{E\left[\Phi_{ss}(\omega)\right]\right\}^2 \qquad (76)$$

The second integral is

$$I_2 = \frac{1}{4\pi^2}\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty e(t)e(x)e(\mu)e(v)\,\psi_{nn}(t-\mu)\,\psi_{nn}(x-v)e^{-j\omega(t-x+\mu-v)}\,dt\,dx\,d\mu\,dv$$

$$= \frac{1}{4\pi^2}\int_0^\infty\int_0^\infty e(t)e(\mu)\psi_{nn}(t-\mu)e^{-j\omega(t+\mu)}\,dt\,du \int_0^\infty\int_0^\infty e(x)e(v)\psi_{nn}(x-v)e^{+j\omega(x+v)}\,dx\,dv \tag{77}$$

The first integral of this product is identifiable as the complex conjugate of the second, so

$$I_2 = \left|\frac{1}{2\pi}\int_0^\infty\int_0^\infty e(t)e(\mu)\,\psi_{nn}(t-\mu)e^{-j\omega(t+\mu)}\,dt\,d\mu\right|^2 \tag{78}$$

Therefore, from equations (74), (76), and (78),

$$E\left[\Phi_{ss}^2(\omega)\right] = I_1 + I_2 + I_3$$
$$= 2\left\{E\left[\Phi_{ss}(\omega)\right]\right\}^2 + \left|\frac{1}{2\pi}\int_0^\infty\int_0^\infty e(t)e(\mu)\psi_{nn}(t-\mu)e^{-j\omega(t+\mu)}\,dt\,d\mu\right|^2 \tag{79}$$

and by the definition of the variance,

$$\mathrm{Var}\left[\Phi_{ss}(\omega)\right] = \left\{E\left[\Phi_{ss}(\omega)\right]\right\}^2 + \left|\frac{1}{2\pi}\int_0^\infty\int_0^\infty e(t)e(\mu)\psi_{nn}(t-\mu)e^{-j\omega(t+\mu)}\,dt\,d\mu\right|^2$$
$$\mathrm{Var}\left[\Phi_{ss}(\omega)\right] \geq \left\{E\left[\Phi_{ss}(\omega)\right]\right\}^2 \tag{80}$$

We have found, then, that the variance of the energy density spectrum is, for each ω, at least as large as the square of the mean of the energy density spectrum at ω, or alternatively, that the standard deviation of $\Phi_{ss}(\omega)$ is always larger than the mean - regardless of the form and duration of $e(t)$. This somewhat surprising result is a more general statement of the relatively well known fact that the so-called "periodogram" which can be drawn from an empirical record as an estimate of spectral density of a continuing process does not converge in the mean to the true spectral density as the observation period increases without limit. (see reference (b), pp 107-108).

The implication of these findings is the following: If a large number of random transient sample functions are taken and their energy density spectra computed or measured, the arithmetic average of $\Phi_{ss}(\omega)$ will tend toward the expressions of equations (64) or (70) for every ω. Because of the magnitude of the variance, however, it is quite likely that the value of $\Phi_{ss}(\omega)$ for a given sample will be nowhere near the mean. Consequently, at a given value of frequency, there will in general be a wide variation of the energy density spectrum from sample to sample. Unlike the autocorrelation function which becomes percentagewise more precise as the duration of e(t) increases, the empirical spectrum of each sample function remains widely distributed regardless of the form of the envelope.

In Appendix B, an alternative approach to the noise burst spectrum is based on the assumption that the Wiener-Khintchine relation applies to signals of this type. The results obtained are identical to those presented above, and it appears that such an assumption is valid.

Chapter V

EXAMPLES OF NOISE BURST AUTOCORRELATIONS AND SPECTRA

To demonstrate the application of the principles derived above, this section will be devoted to the presentation of several examples of the calculation of noise burst autocorrelation functions and energy spectra. For this purpose, two idealized noise spectra will be treated: the rectangular low-pass broad band spectrum, and the rectangular narrow band spectrum. These will each be modulated by two envelope functions: a rectangular pulse of duration T seconds; and an exponential decay with a time constant of T seconds. This yields four cases:

A. Broad Band Spectrum with Rectangular Envelope
B. Broad Band Spectrum with Exponential Envelope
C. Narrow Band Spectrum with Rectangular Envelope
D. Narrow Band Spectrum with Exponential Envelope

The two noise spectra and their associated autocorrelation functions are shown in Figures 2 and 3. For the rectangular low pass broad band spectrum,

$$\Psi_{nn}(\omega) = N_o, \quad -\omega_2 \leq \omega \leq \omega_2$$
$$= 0, \quad \text{elsewhere} \tag{81}$$

By the Wiener-Khintchine relation,

$$\psi_{nn}(\tau) = \int_{-\infty}^{\infty} \Psi_{nn}(\omega) e^{j\omega\tau}\, d\omega = \int_{-\omega_2}^{\omega_2} N_o\, e^{j\omega\tau}\, d\omega \tag{82}$$

Thus,

$$\psi_{nn}(\tau) = 2N_o\, \omega_2 \frac{\sin \omega_2\tau}{\omega_2\tau} \tag{83}$$

For the rectangular narrow band spectrum,

$$\Psi_{nn}(\omega) = N_o, \quad \omega_1 \leq \omega \leq \omega_2$$
$$= N_o, \quad -\omega_2 \leq \omega \leq -\omega_1 \tag{84}$$
$$= 0, \quad \text{elsewhere}$$

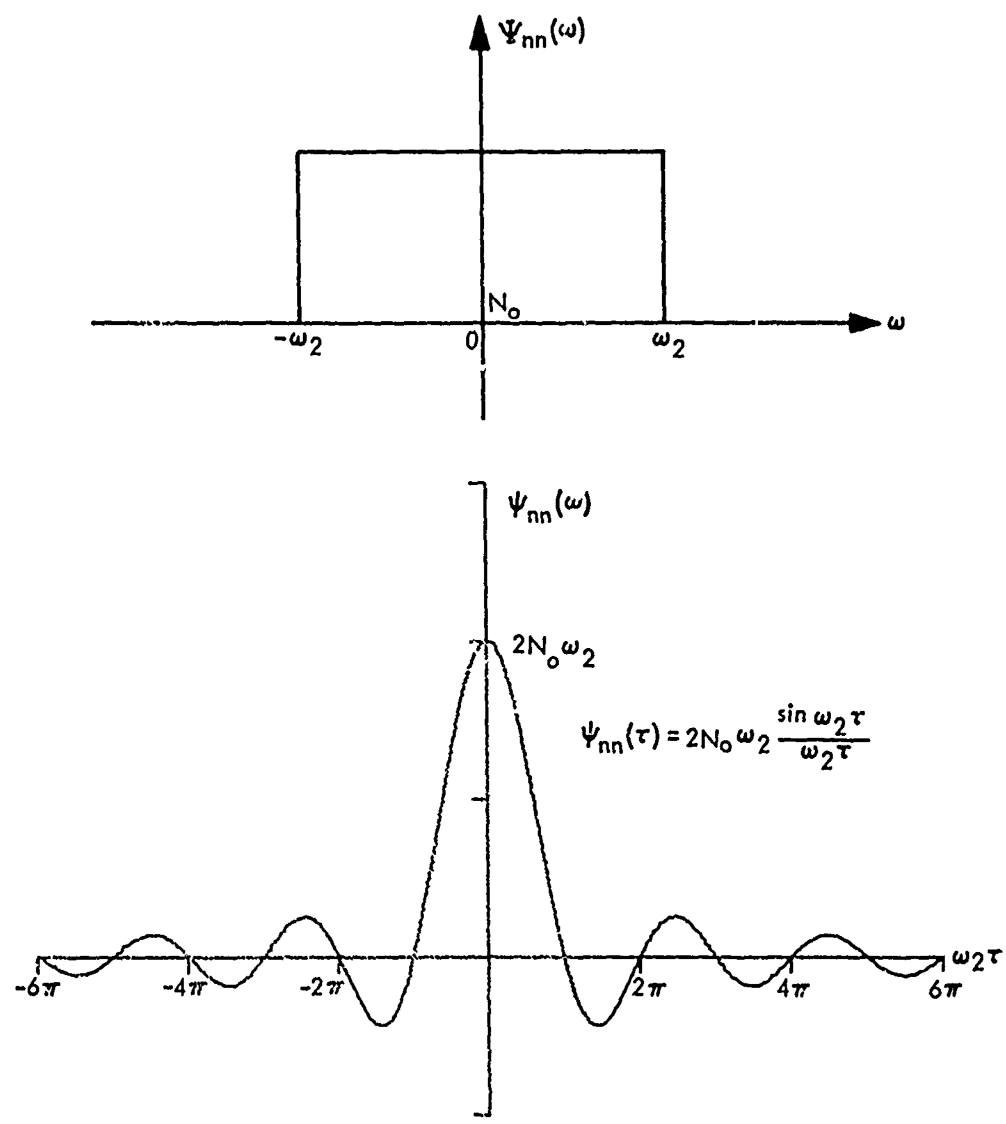


FIGURE 2. RECTANGULAR LOW PASS BROAD BAND NOISE SPECTRUM AND ASSOCIATED AUTOCORRELATION

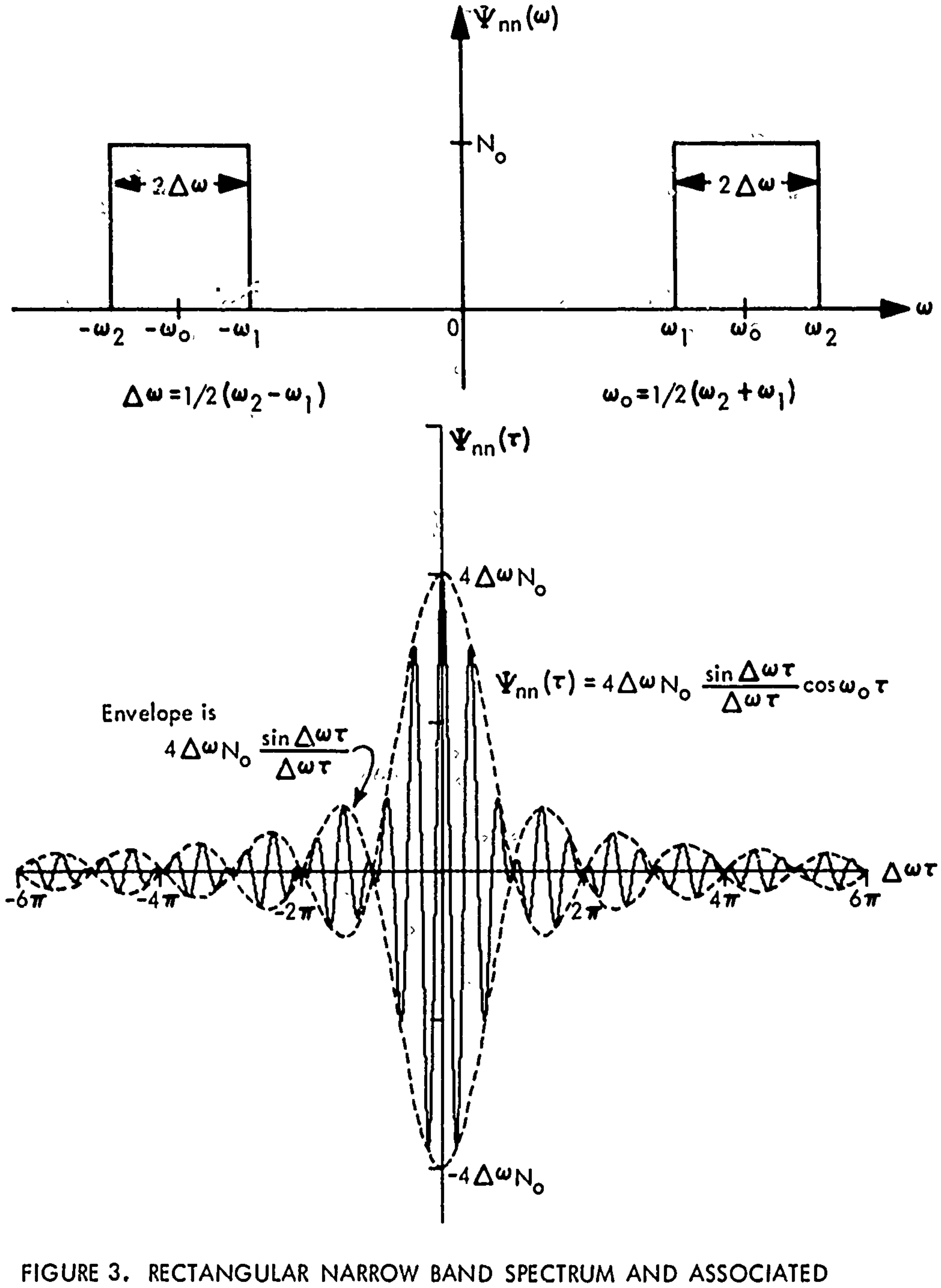


FIGURE 3. RECTANGULAR NARROW BAND SPECTRUM AND ASSOCIATED AUTOCORRELATION.

Thus $$\Psi_{nn}(\tau) = 2\int_{\omega_1}^{\omega_2} N_o \cos \omega\tau \, d\omega \tag{85}$$

$$= 4\Delta\omega \; N_o \frac{\sin \Delta\omega\tau}{\Delta\omega\tau} \cos \omega_o \tau$$

where $$\Delta\omega \equiv \frac{\omega_2 - \omega_1}{2} \quad \text{(interpreted as the half bandwidth)} \tag{86a}$$

$$\omega_o \equiv \frac{\omega_2 + \omega_1}{2} \quad \text{(interpreted as the center frequency)} \tag{86b}$$

Turning now to the first of the envelope functions, consider the rectangular pulse shown in Figure 4.

$$\begin{aligned} e(t) &= 1, \; 0 \le t \le T \\ &= 0, \; \text{elsewhere} \end{aligned} \tag{87}$$

The autocorrelation function is computed from equation (4)

$$\varphi_{ee}(\tau) = \int_{-\infty}^{\infty} e(t)e(t+\tau) \, dt \tag{88}$$

$$= T - |\tau|, \qquad -T \le \tau \le T$$

The energy spectral density can be found as either the Fourier transform of equation (88) or as $2\pi \; |E(\omega)|^2$ where $E(\omega)$ is the Fourier transform of the envelope waveform.

$$\Phi_{ee}(\omega) = \frac{T^2}{2\pi} \; \frac{\sin^2 \omega T/2}{(\omega T/2)^2} \tag{89}$$

One more quantity is needed in the study of rectangular noise bursts, and this is the function $\varphi_{pp}(u,\tau)$ defined by equations (40) and (41). If τ is positive,

$$p(t,\tau) = e(t)e(t+\tau) = 1, \quad 0 \le t \le T-\tau \tag{90}$$

Therefore,

$$\varphi_{pp}(u,\tau) = \int_0^{\infty} p(t,\tau)p(t-u,\tau)dt = \int_0^{T-\tau-u} 1 \, dt, \; \mu \ge 0$$

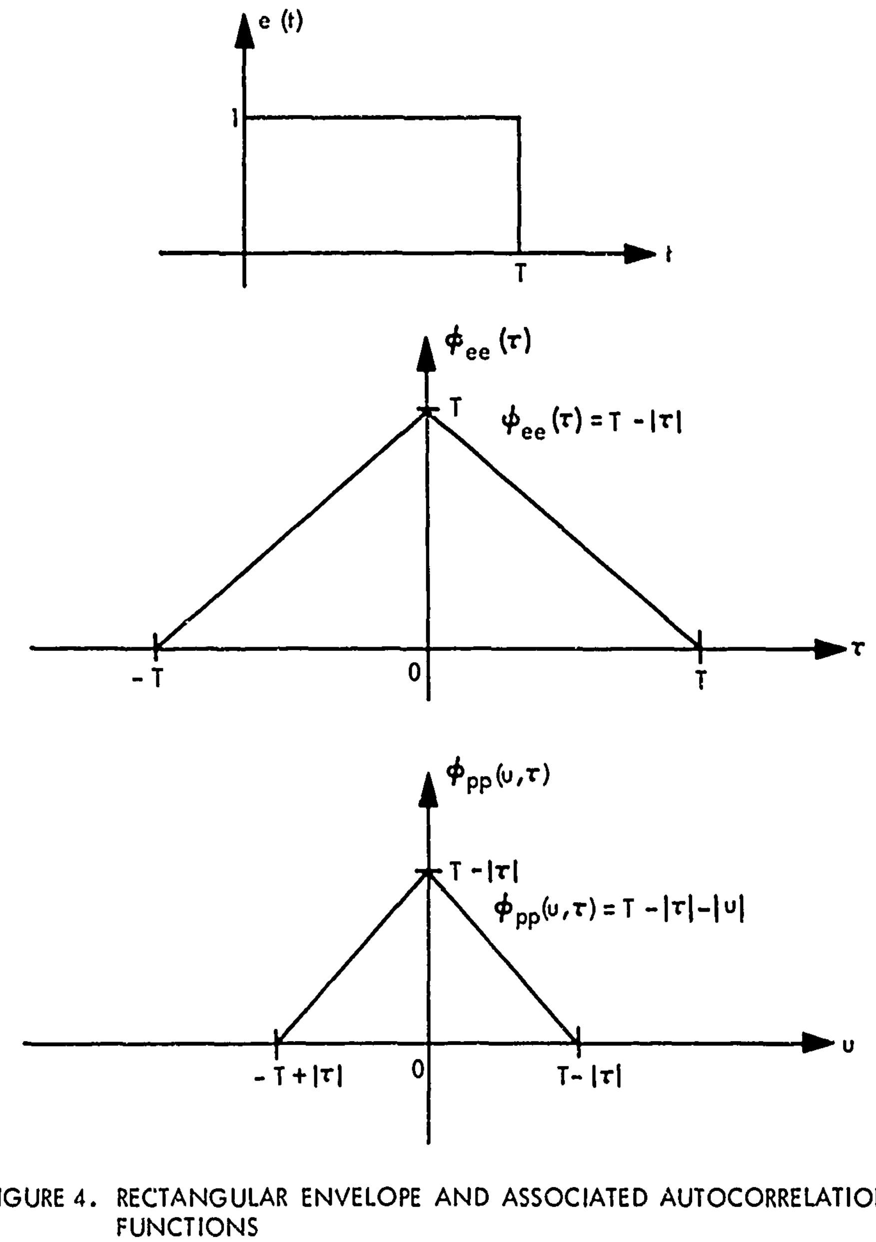


FIGURE 4. RECTANGULAR ENVELOPE AND ASSOCIATED AUTOCORRELATION FUNCTIONS

$$\varphi_{pp}(u,\tau) = T - \tau - u \quad \text{for } \tau \text{ and } u > 0 \qquad 0 \le u \le T - \tau \tag{91}$$

Since $\varphi_{pp}(u,\tau)$ is an even function of both τ and u,

$$\varphi_{pp}(u,\tau) = T - |\tau| - |u| \; , \quad \left[|\tau| + |u|\right] < T \tag{92}$$
$$= 0, \text{ elsewhere}$$

Let the exponential envelope function portrayed in Figure 5 be defined as

$$e(t) = e^{-t/T}, \; 0 \le t \le \infty \tag{93}$$
$$= 0, \text{ elsewhere}$$

Defining the autocorrelation function precisely as in the case of the rectangular pulse, we find

$$\varphi_{ee}(\tau) = \frac{T}{2} e^{-|\tau|/T} \; , \; -\infty < \tau < \infty \tag{94}$$

The energy density spectrum becomes

$$\Phi_{ee}(\omega) = \frac{T^2}{2\pi} \frac{1}{\omega^2 T^2 + 1} \tag{95}$$

and finally,

$$\varphi_{pp}(u,\tau) = \frac{T}{4} e^{-\frac{2}{T}(|u| + |\tau|)} \; , \; -\infty < u,\tau < \infty \tag{96}$$

For convenience, these functions are collected in Table I.

AUTOCORRELATION CALCULATIONS

1. RECTANGULAR PULSE OF BROAD BAND NOISE. By equation (25),

$$E\left[\varphi_{ss}(\tau)\right] = \psi_{nn}(\tau)\, \varphi_{ee}(\tau)$$

and for this case, the autocorrelation functions for the rectangular pulse and the broad band noise spectrum are found in Table I. Using these expressions,

$$E\left[\varphi_{ss}(\tau)\right] = 2N_0 \omega_2 (T - |\tau|) \frac{\sin \omega_2 \tau}{\omega_2 \tau} \; , \; -T \le \tau \le T \tag{97}$$
$$= 0, \text{ elsewhere}$$

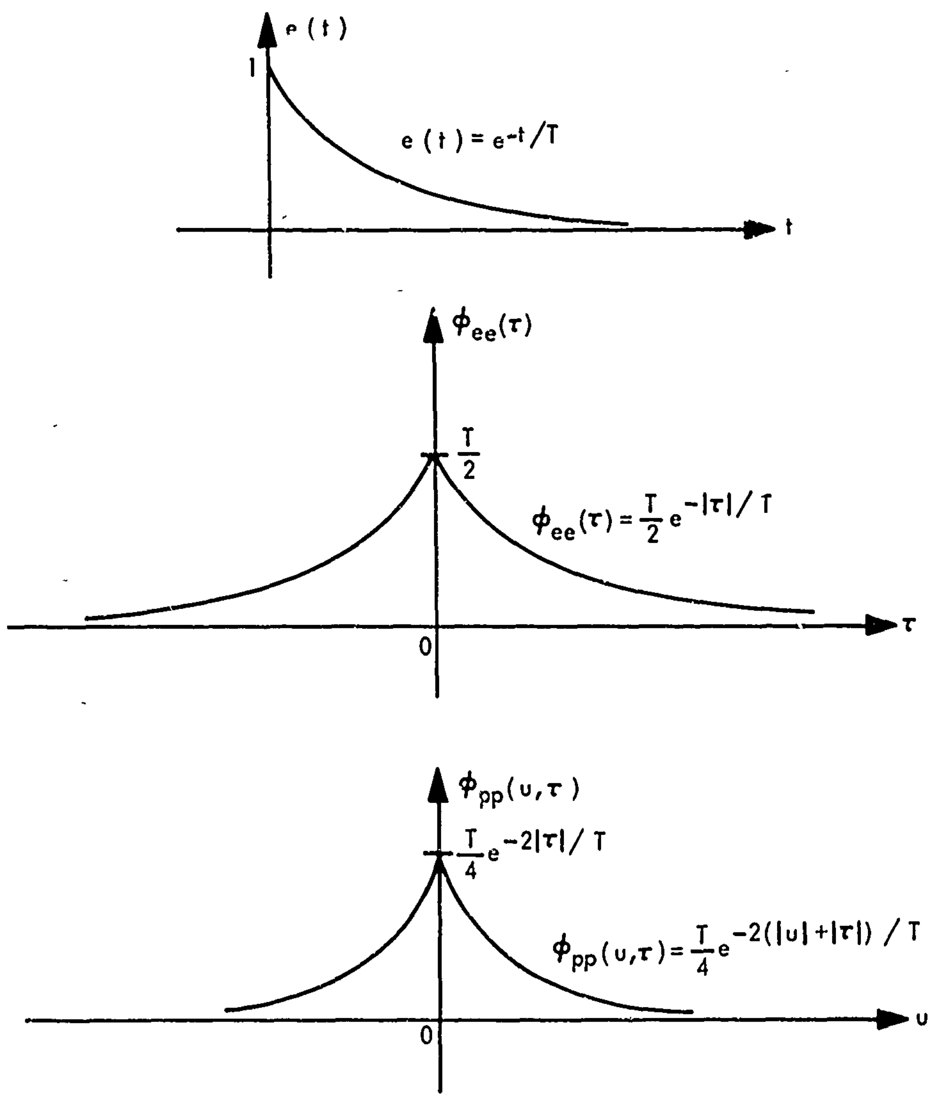


FIGURE 5. EXPONENTIAL ENVELOPE AND ASSOCIATED AUTOCORRELATION FUNCTIONS.

Table I

A. Noise Spectra		
Spectrum Description	Power Density Spectrum $\Psi_{nn}(\omega)$	Autocorrelation Functions $\psi_{nn}(\tau)$
Rectangular Broad Band	$N_o, \ -\omega_2 \le \omega \le \omega_2$ 0, elsewhere	$2 N_o \omega_2 \frac{\sin \omega_2 \tau}{\omega_2 \tau}, \ -\infty < \tau < \infty$
Rectangular Narrow Band	$N_o, \ \omega_1 \le \lvert\omega\rvert \le \omega_2$ 0, elsewhere	$4 \Delta\omega N_o \frac{\sin \Delta\omega\tau}{\Delta\omega\tau} \cos \omega_o \tau, \ -\infty < \tau < \infty$ where $\Delta\omega \equiv (\omega_2 - \omega_1)/2$ $\omega_o \equiv (\omega_2 + \omega_1)/2$

B. Envelope Functions				
Description	Function	Energy Density Spectrum $\Phi_{ee}(\omega)$	Autocorrelation Function $\varphi_{ee}(\tau)$	$\varphi_{pp}(u, \tau)$
Rectangular Pulse	$1, \ 0 \le t \le T$ 0, elsewhere	$\frac{T^2}{2\pi} \frac{\sin^2 \frac{\omega T}{2}}{(\omega T/2)^2}, -\infty < \omega < \infty$	$T - \lvert\tau\rvert, \ -T \le \tau \le T$ 0, elsewhere	$T - \lvert\tau\rvert - \lvert u\rvert, \ [\lvert\tau\rvert + \lvert u\rvert] \le T$ 0, elsewhere
Exponential Pulse	$e^{-t/T}, \ 0 \le t \le \infty$ $0, \ t < 0$	$\frac{T^2}{2\pi} \frac{1}{\omega^2 T^2 + 1}, -\infty < \omega < \infty$	$\frac{T}{2} e^{-\lvert\tau\rvert/T}, -\infty < \tau < \infty$	$\frac{T}{4} e^{-\frac{2}{T}(\lvert u\rvert + \lvert\tau\rvert)}$ $-\infty < u, \tau < \infty$

From equation (43) can be found an expression for the variance of $\varphi_{ss}(\tau)$:

$$\mathrm{Var}\left[\varphi_{ss}(\tau)\right] = 2\int_0^{\infty} \varphi_{pp}(u,\tau)\left[\psi_{nn}^{2}(\mu)+\psi_{nn}(u+\tau)\,\psi_{nn}(u-\tau)\right] du$$

Referring again to Table I,

$$\mathrm{Var}\left[\varphi_{ss}(\tau)\right] = 8N_o^2\omega_2^2 \int_0^{T-|\tau|} (T-\tau-u)\left[\frac{\sin^2\omega_2\,\mu}{\omega_2^2\,\mu^2} + \frac{\sin\omega_2(\tau+u)}{\omega_2(\tau+u)}\,\frac{\sin\omega_2(\tau-u)}{\omega_2(\tau-u)}\right] du \qquad (98)$$

The standard deviation of $\varphi_{ss}(\tau)$ is, of course, the positive square root of this expression.

For more generality of presentation, the expressions of equations (97) and (98) will now be normalized and parametized. We note first that since $\varphi_{ss}(0)$ is equal to the total energy of the noise burst (as defined in equation (27)), $E\left[\varphi_{ss}(0)\right]$ can be described as the average total energy:

$$E_{Tav} = E\left[\varphi_{ss}\;(0)\right] \qquad (99)$$

For the present case,

$$E_{Tav} = 2N_o\;\omega_2 T \qquad (100)$$

which makes sense since it turns out to be equal to the average power per radian times the bandwidth times the duration of the transient. $E[\varphi_{ss}(\tau)]$ will be normalized by dividing by E_{Tav} and $\mathrm{Var}[\varphi_{ss}(\tau)]$ by dividing by $(E_{Tav})^2$. This latter step puts the standard deviation of $\varphi_{ss}(\tau)$ in the same units as the mean. Thus:

$$E_n\left[\varphi_{ss}(\tau)\right] = \left(1-\frac{|\tau|}{T}\right)\frac{\sin\omega_2\tau}{\omega_2\tau},\quad -T\le\tau\le T \qquad (101)$$

$$= 0 \qquad\qquad , \text{ elsewhere}$$

$$\mathrm{Var}_n\left[\varphi_{ss}(\tau)\right] = 2\int_0^{T-|\tau|}\left(\frac{1}{T}-\frac{\tau}{T^2}-\frac{\mu}{T^2}\right)\left[\frac{\sin^2\omega_2\mu}{\omega_2^2\,\mu^2}+\frac{\sin\omega_2(\tau+\mu)}{\omega_2(\tau+\mu)}\,\frac{\sin\omega_2(\tau-\mu)}{\omega_2(\tau-\mu)}\right] du \qquad (102)$$

Now let the following parameters be defined:

(103a)

$p = \frac{\tau}{T}$, i.e. the argument as a fraction of the total pulse length

$q = \frac{T}{\frac{2\pi}{\omega_2}} = \frac{\omega_2 T}{2\pi}$, or the number of periods of the highest noise frequency contained in a pulse length. (103b)

The normalized, parametized equations can now be written as

$$E_p\left[\varphi_{ss}(p)\right] = (1-p)\frac{\sin 2\pi pq}{2\pi pq} \tag{104}$$

$$Var_p\left[\varphi_{ss}(p)\right] = 2\int_0^{1-p}(1-p-x)\left[\frac{\sin^2 2\pi qx}{(2\pi qx)^2} + \frac{\sin 2\pi q(x+p)}{2\pi q\,(x+p)}\,\frac{\sin 2\pi q(x-p)}{2\pi q(x-p)}\right]dx \tag{105}$$

By equation (52), the ratio of the standard deviation at $\tau = 0$ to the expected value of $\varphi_{ss}(\tau)$ at $\tau = 0$ is

$$R_o = \frac{\sqrt{Var\left[\varphi_{ss}(0)\right]}}{E\left[\varphi_{ss}(0)\right]} \tag{106}$$

and it should be recalled that this ratio is the maximum value of the standard deviation expressed as a fraction of the mean. It should be apparent from the foregoing normalization and parameterization that

$$R_o = Ro(q) = 2\sqrt{\int_0^1 (1-x)\frac{\sin^2 2\pi qx}{(2\pi qx)^2}\,dx} \tag{107}$$

The functions of equations (104), (105), and (107) have been programmed for evaluation on the IBM 7090 computer at NOL, the latter two expressions requiring the use of numerical integration subroutines. Computer generated plots of the normalized expected value of $\varphi_{ss}(p)$ are found in Figures 6 and 7 for this case when q = 0.1, 1.0, 10.0, and 100.0. It should be realized that all such autocorrelation functions are even functions of p and disappear for $|p| > 1$. Also note the change of scale for the p axis when q=100 (in Figure 7). Figure 8 shows a typical plot obtained by graphing on the same axes $E_p\left[\varphi_{ss}(p)\right]$ and this mean value plus and minus the standard deviation of $\varphi_{ss}(p)$ as calculated as the square root of equation (105). In some sense, such a graph provides a rough idea of how closely the distribution of $\varphi_{ss}(p)$ cleaves to the mean as a

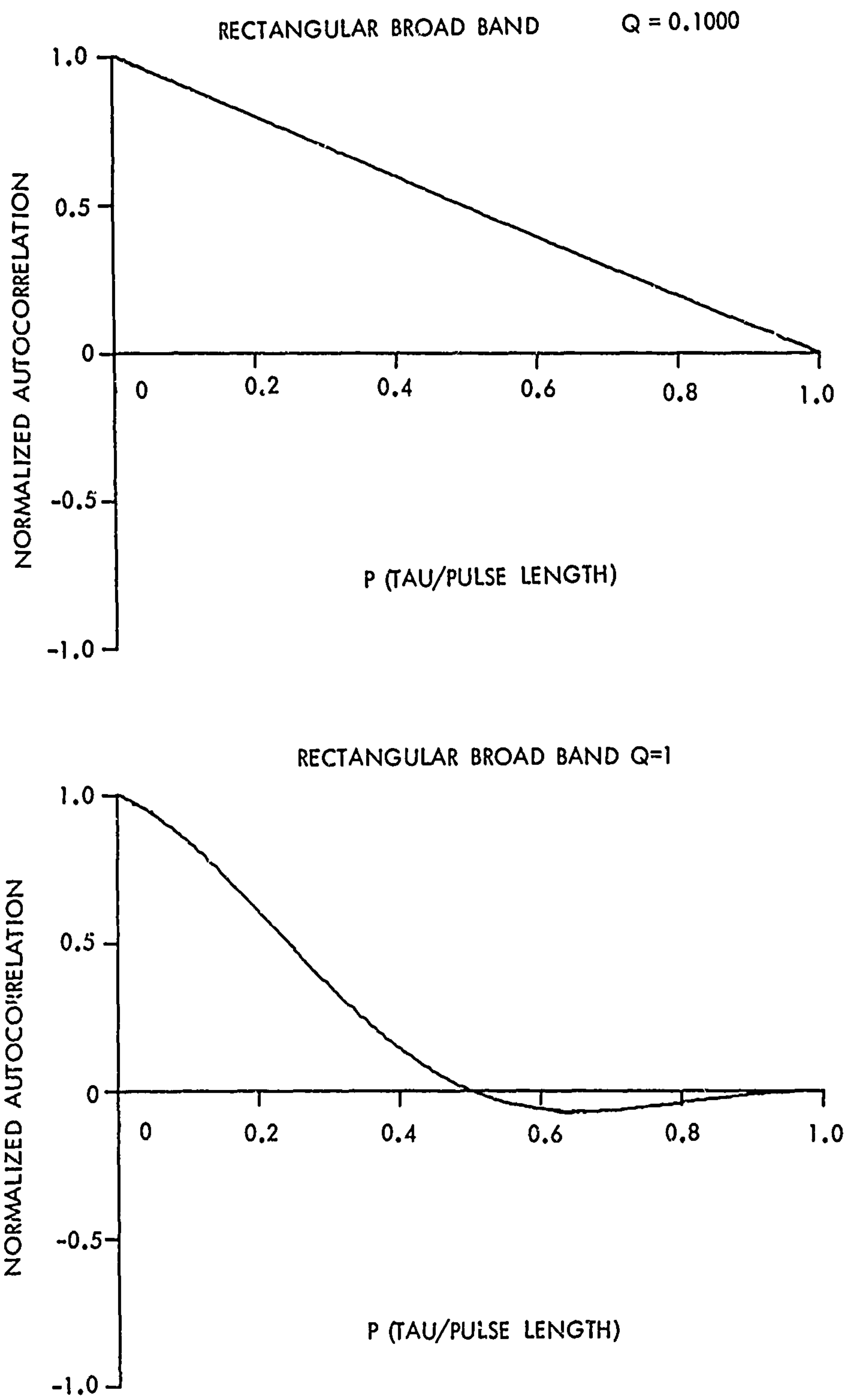
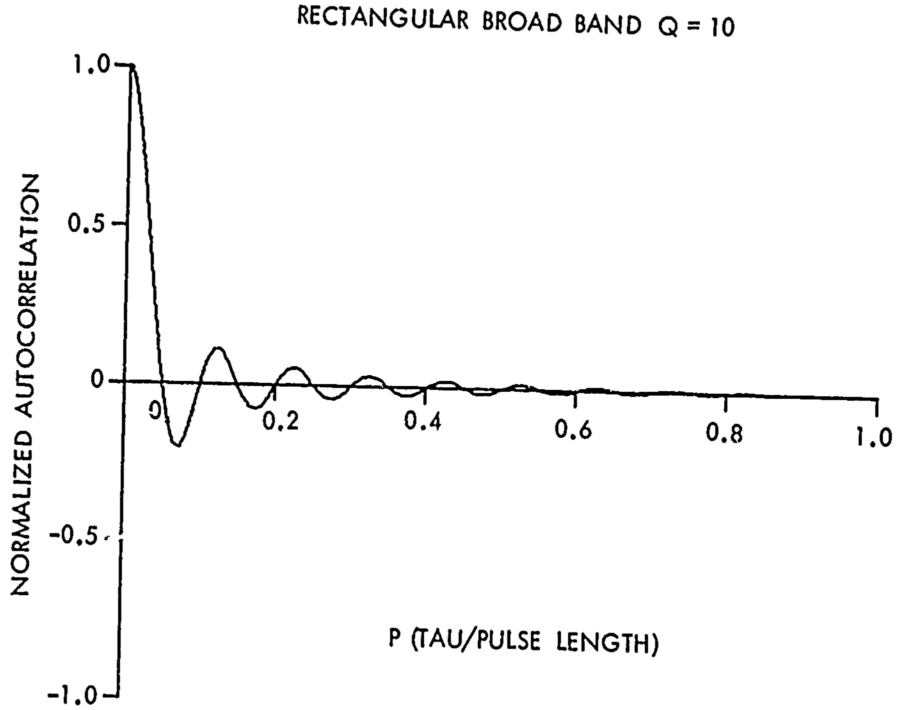


FIGURE 6. RECTANGULAR PULSE OF BROAD BAND NOISE AVERAGE AUTOCORRELATION



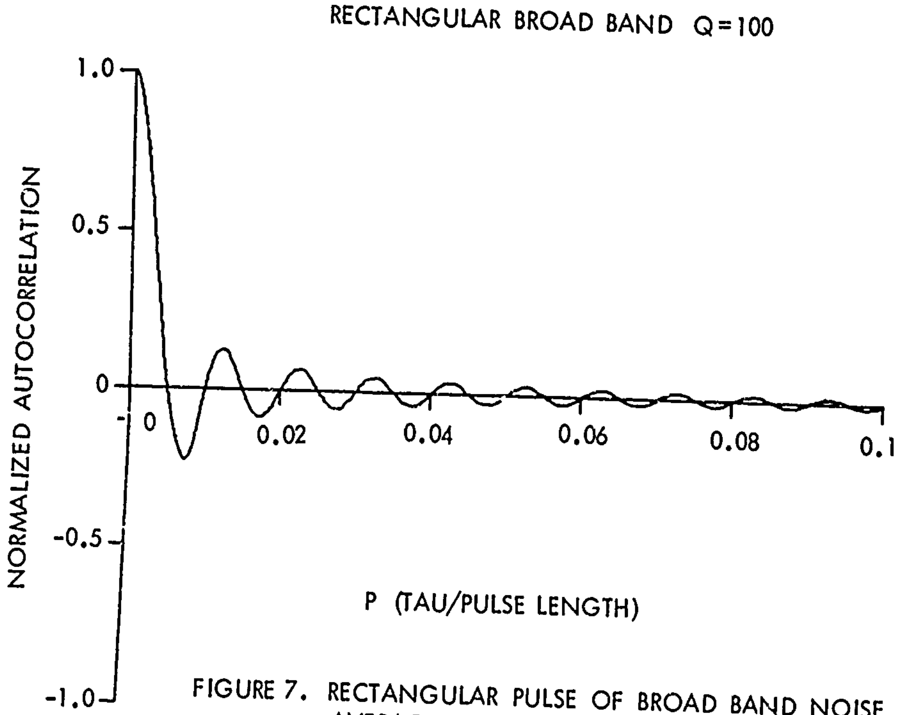


FIGURE 7. RECTANGULAR PULSE OF BROAD BAND NOISE AVERAGE AUTOCORRELATION

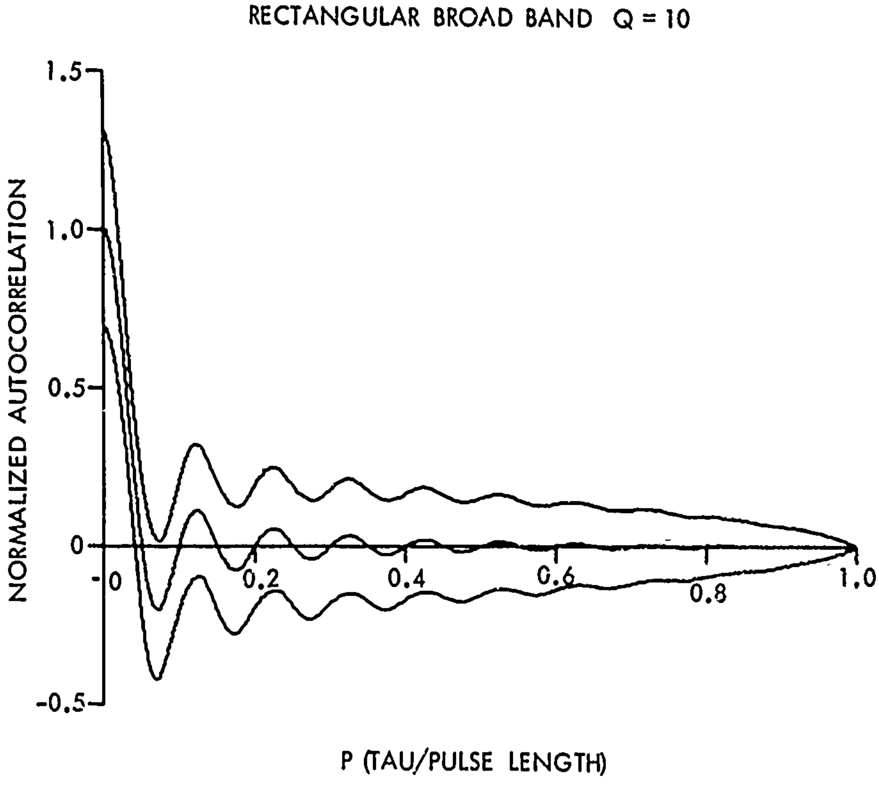


FIGURE 8. RECTANGULAR PULSE OF BROAD BAND NOISE AVERAGE AUTOCORRELATION ± AUTOCORRELATION STANDARD DEVIATION

function of p. Generally, the standard deviation will be largest for p = 0 and then decrease as p increases. This decrease, however, is not always monotonic, and the expression $R_o(q)$ remains perhaps the best single estimate of the tightness of the distribution of $\varphi_{ss}(p)$. A plot of $R_o(q)$ is portrayed in Figure 9 and mirrors the result predicted in equation (54). Since the parameter q is a measure of the length of the envelope pulse in terms of the period of the highest noise frequency, it will be directly proportional to the envelope "duration" discussed in the derivations of equation (54). Thus as q increases, we can expect R_o to decrease sharply, and indeed this is what has been found here. As the pulse length increases relative to the period of the highest noise frequency present, the distribution of $\varphi_{ss}(p)$ clusters closer and closer to the mean for all p, and the value of the mean at p becomes a better and better estimate of the actual value of $\varphi_{ss}(p)$.

2. EXPONENTIAL PULSE OF BROAD BAND NOISE. Again using equations (25) and (43) and the appropriate entries from Table I, it is found immediately that

$$E\left[\varphi_{ss}(\tau)\right] = N_o \omega_2 T e^{-|\tau|/T} \frac{\sin \omega_2 \tau}{\omega_2 \tau} \tag{108}$$

and that

$$\mathrm{Var}\left[\varphi_{ss}(\tau)\right] = 2 N_o^2 \omega_2^2 T e^{-2\tau/T} \int_0^\infty e^{-2u/T}\left[\frac{\sin^2 \omega_2 u}{\omega_2^2 u^2} + \frac{\sin \omega_2(u+\tau)}{\omega_2(u+\tau)}\,\frac{\sin \omega_2(u-\tau)}{\omega_2(u-\tau)}\right] du \tag{109}$$

It follows directly that

$$E_{Tav} = E\left[\varphi_{ss}(0)\right] = N_o \omega_2 T = N_o \omega_2 \int_0^\infty e^2(t)\, dt \tag{110}$$

As before, the functions of equations (108) and (109) will be normalized and parametized. Defining,

$$p = \frac{\tau}{T}$$, i.e., the argument as a fraction of the envelope time constant (111a)

and

$$q = \frac{\omega_2 T}{2\pi}$$, the number of cycles of frequency ω_2 contained in the time constant T (111b)

the functions become

$$E_p\left[\varphi_{ss}(p)\right] = e^{-p} \frac{\sin 2\pi pq}{2\pi pq} \tag{112}$$

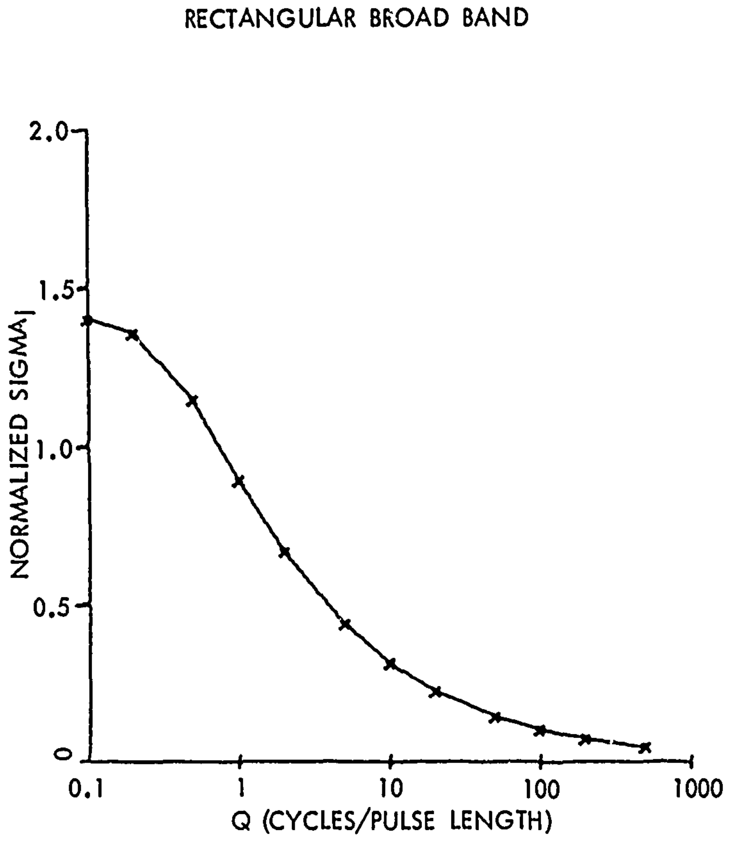


FIGURE 9. RECTANGULAR PULSE OF BROAD BAND NOISE NORMALIZED AUTOCORRELATION STANDARD DEVIATION.

$$\mathrm{Var}_p\left[\varphi_{ss}(p)\right] = 2e^{-2p}\int_0^\infty e^{-2x}\left[\frac{\sin^2 2\pi qx}{(2\pi qx)^2} + \frac{\sin 2\pi q(x+p)}{2\pi q(x+p)}\,\frac{\sin 2\pi q(x-p)}{2\pi q(x-p)}\right]dx \tag{113}$$

and following through,

$$R_o(q) = 2\sqrt{\int_0^\infty e^{-2x}\frac{\sin^2 2\pi qx}{(2\pi qx)^2}\,dx} \tag{114}$$

Equations (112) and (114) have also been programmed on the IBM 7090 computer, and the results appear in Figures 10, 11, and 12. As before, the average autocorrelations are presented for q = 0.1, 1.0, 10., and 100. Note again that as q increases, R_o falls toward zero. As the time constant of the envelope increases, the distribution of $\varphi_{ss}(p)$ closes more tightly about the mean, as one would expect from previous considerations.

3. <u>RECTANGULAR PULSE OF NARROW BAND NOISE</u>. Following exactly the same procedure as before and again referring to Table I, one can write that

$$E\left[\varphi_{ss}(\tau)\right] = 4\,N_o\Delta\omega(T-|\tau|)\,\frac{\sin\Delta\omega\tau}{\Delta\omega\tau}\cos\omega_o\tau \tag{115}$$

$$\mathrm{Var}\left[\varphi_{ss}(\tau)\right] = 32\,N_o^2\Delta\omega^2\int_0^{T-|\tau|}(T-|\tau|-u)\left[\frac{\sin^2\Delta\omega\mu}{(\Delta\omega\mu)^2}\cos^2\omega_o\mu + \frac{\sin\Delta\omega(\mu-\tau)}{(\mu-\tau)}\,\frac{\sin\Delta\omega(\mu+\tau)}{\Delta\omega(\mu+\tau)}\cos\omega_o(\mu-\tau)\cos\omega_o(\mu+\tau)\right]du \tag{116}$$

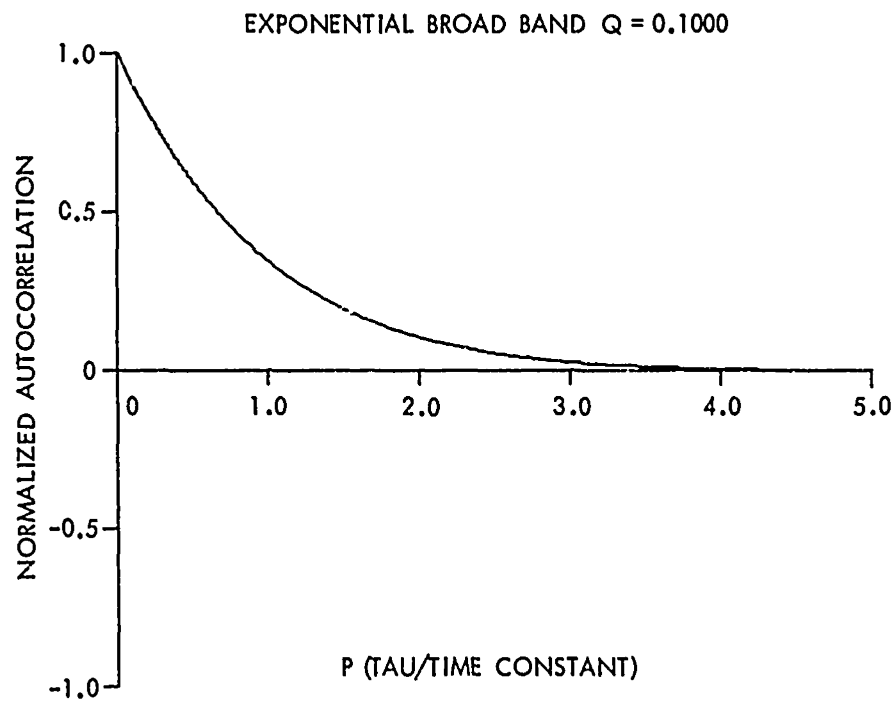
where as before

$$\omega_o = \frac{\omega_1+\omega_2}{2}\ ;\ \Delta\omega = \frac{\omega_2-\omega_1}{2}$$

Now from equation (115),

$$E_{Tav} = 4\,N_o\,\Delta\omega T \tag{117}$$

which is what one would expect intuitively. For this case, the following parameters will be defined:



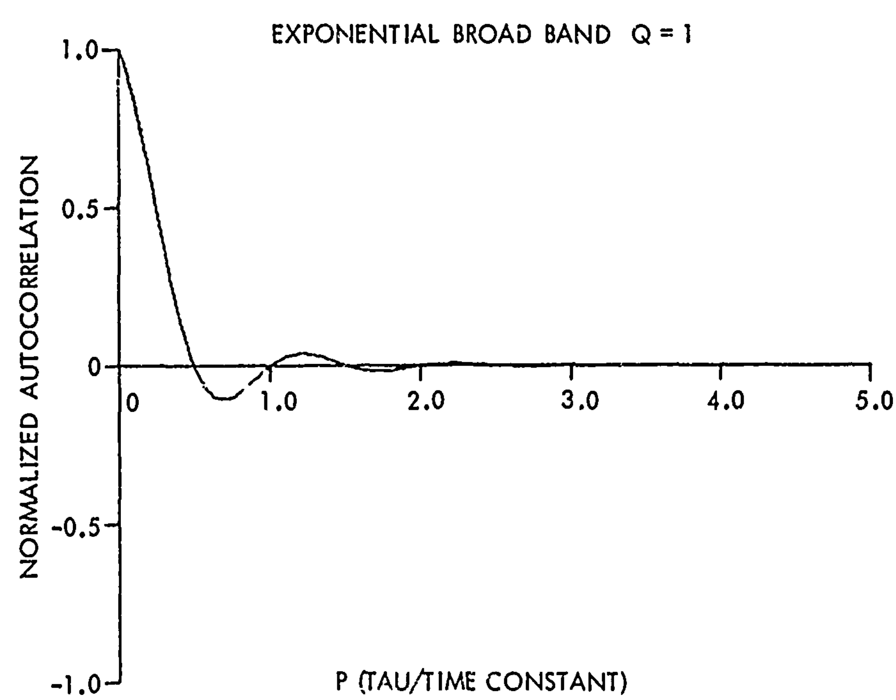


FIGURE 10. EXPONENTIAL PULSE OF BROAD BAND NOISE AVERAGE AUTOCORRELATION

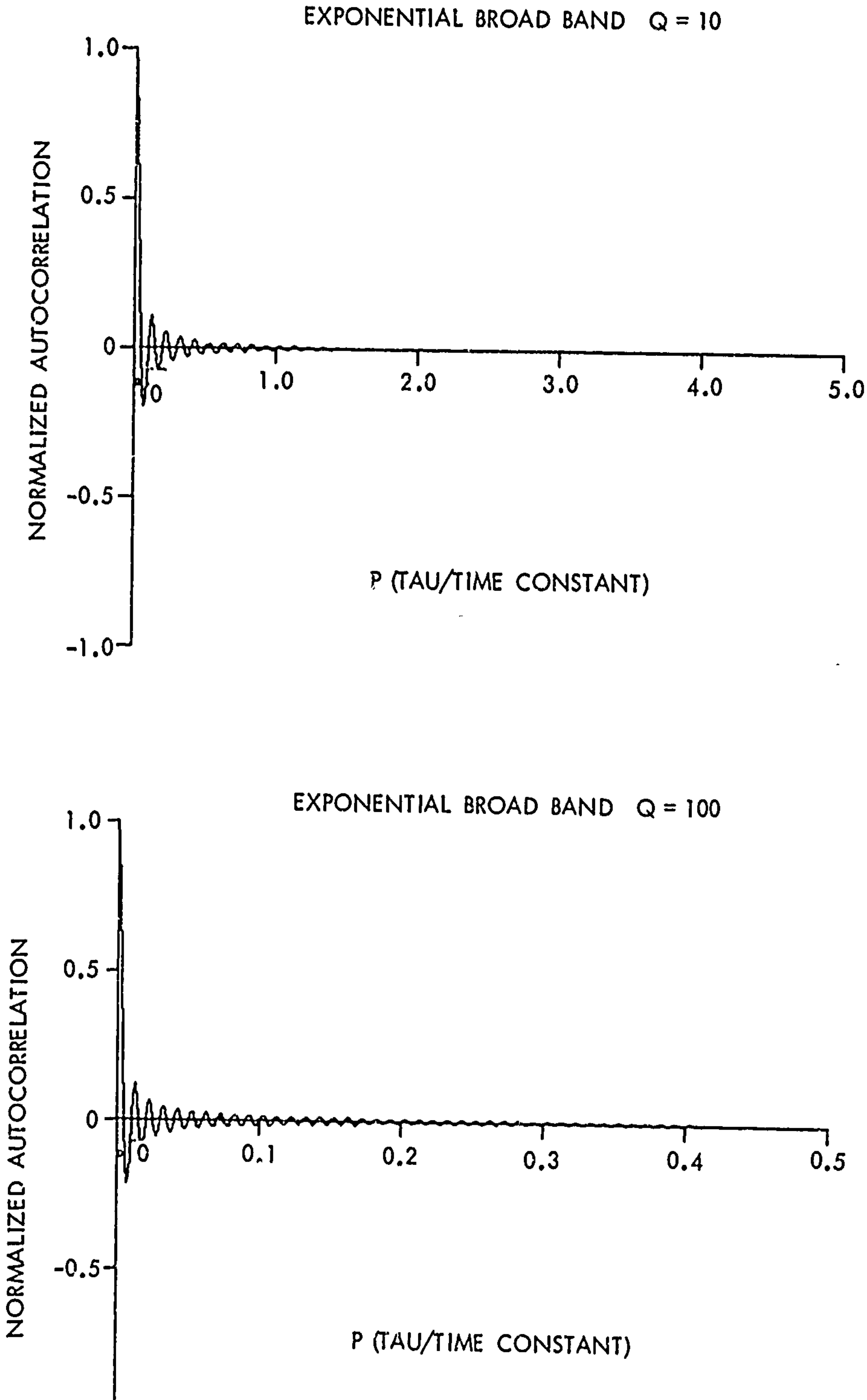


FIGURE 11. EXPONENTIAL PULSE OF BROAD BAND NOISE AVERAGE AUTOCORRELATION

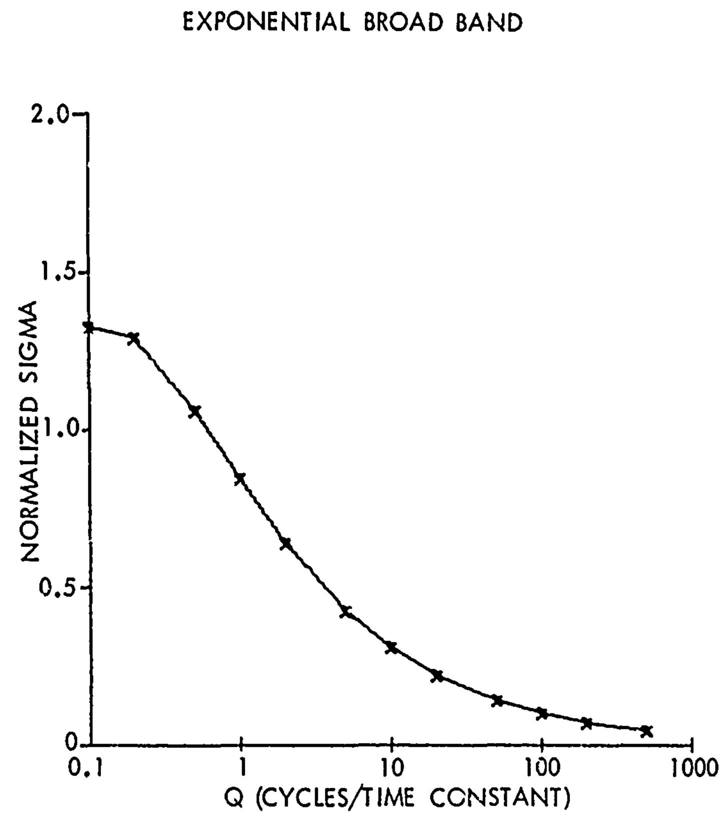


FIGURE 12. EXPONENTIAL PULSE OF BROAD BAND NOISE
NORMALIZED AUTOCORRELATION STANDARD DEVIATION

$p = \frac{\tau}{T}$, the argument as a fraction of the pulse length (118a)

$q = \frac{T}{\frac{2\pi}{\omega_o}} = \frac{\omega_o T}{2\pi}$, the number of cycles of the center frequency found in a pulse length (118b)

$z = \frac{2\Delta\omega}{\omega_o}$, the ratio of bandwidth to center frequency (118c)

Utilizing these expressions, it is found for a rectangular burst of narrow band noise that

$$E_p\left[\varphi_{ss}(p)\right] = (1-p)\,\frac{\sin\pi qzp}{\pi qzp}\,\cos 2\pi pq \tag{119}$$

$$\begin{aligned}\mathrm{Var}_p\left[\varphi_{ss}(p)\right] = 2\int_0^{1-p} (1-p-x)\Bigg[&\frac{\sin^2\pi qzx}{(\pi qzx)^2}\cos^2 2\pi q\,x \\ &+ \frac{\sin\pi qz(x-p)}{\pi qz(x-p)}\,\frac{\sin\pi qz(x+p)}{\pi qz(x+p)}\cos 2\pi q(x-p)\cos 2\pi q(x+p)\Bigg]dx\end{aligned} \tag{120}$$

$$R_o(q,z) = 2\sqrt{\int_0^1 (1-x)\,\frac{\sin^2\pi qzx}{(\pi qzx)^2}\cos^2 2\pi qx\,dx} \tag{121}$$

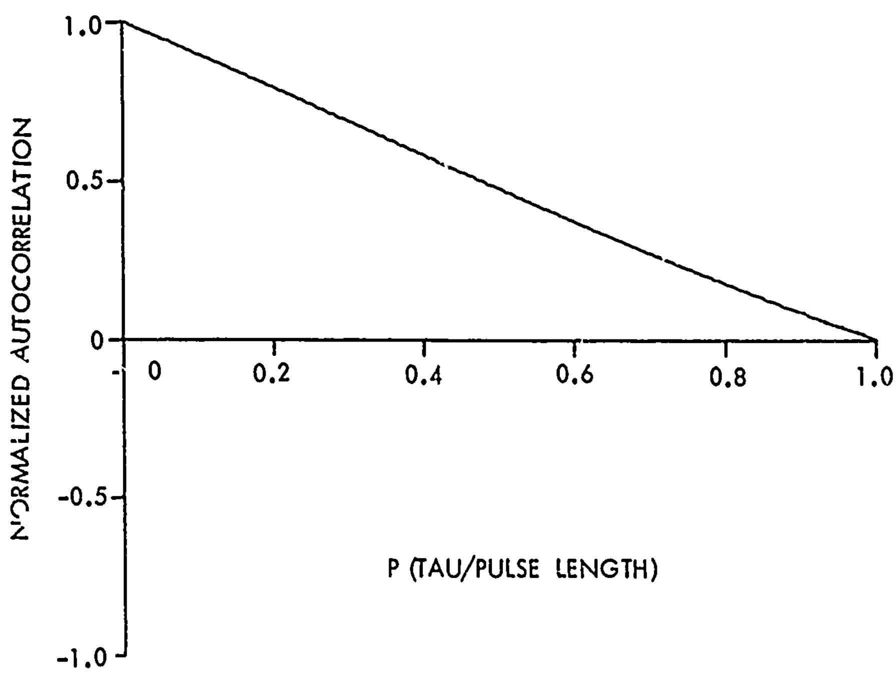
Typical results when equation (119) is evaluated on a digital computer appear in Figures 13 and 14, for which z = 0.1. In Figure 15, $R_o(q,z)$ is plotted as a function of q with z as a parameter. The z-value of 2/3, which may seem like an odd choice at first glance, corresponds to an octave band spectrum, as may easily be verified. Note that $R_o(q,z)$ approaches zero as q increases for all values of z but that the decline is markedly more rapid for the larger values of z, corresponding to wider and wider band-widths.

4. <u>EXPONENTIAL BURST OF NARROW BAND NOISE</u>. Using the proper entries from Table I, one finds immediately that

$$E\left[\varphi_{ss}(\tau)\right] = 2N_o T\,\Delta\omega\, e^{-\frac{\tau}{T}}\,\frac{\sin\Delta\omega\tau}{\Delta\omega\tau}\cos\omega_o\tau \tag{122}$$

where $\Delta\omega$ and ω_o are defined as before

RECTANGULAR NARROW BAND Q = 0.1000



RECTANGULAR NARROW BAND Q = 1

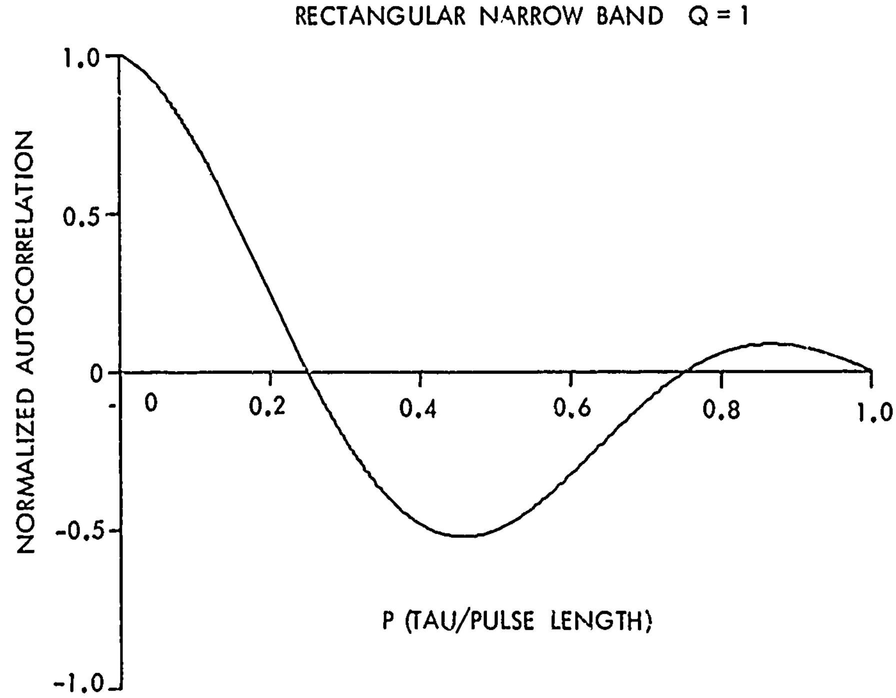
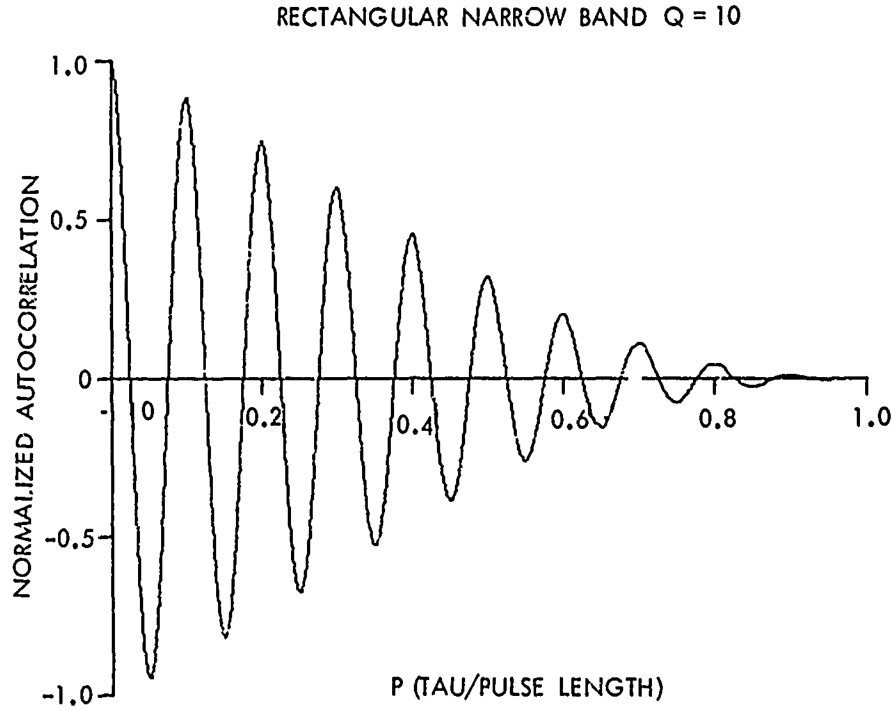


FIGURE 13. RECTANGULAR PULSE OF NARROW BAND NOISE AVERAGE AUTOCORRELATION FOR Z=0.1



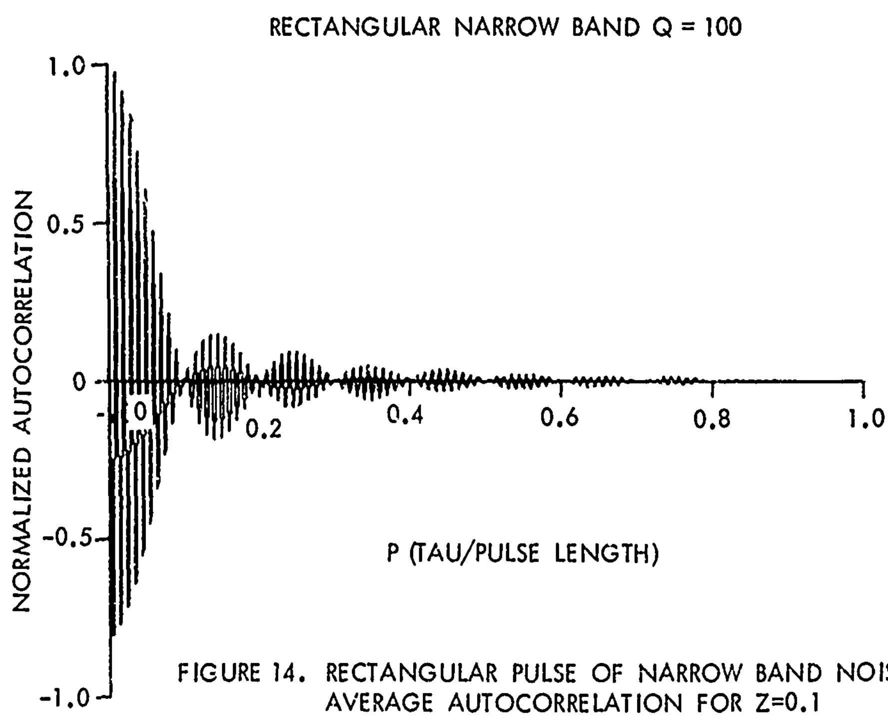


FIGURE 14. RECTANGULAR PULSE OF NARROW BAND NOISE AVERAGE AUTOCORRELATION FOR Z=0.1

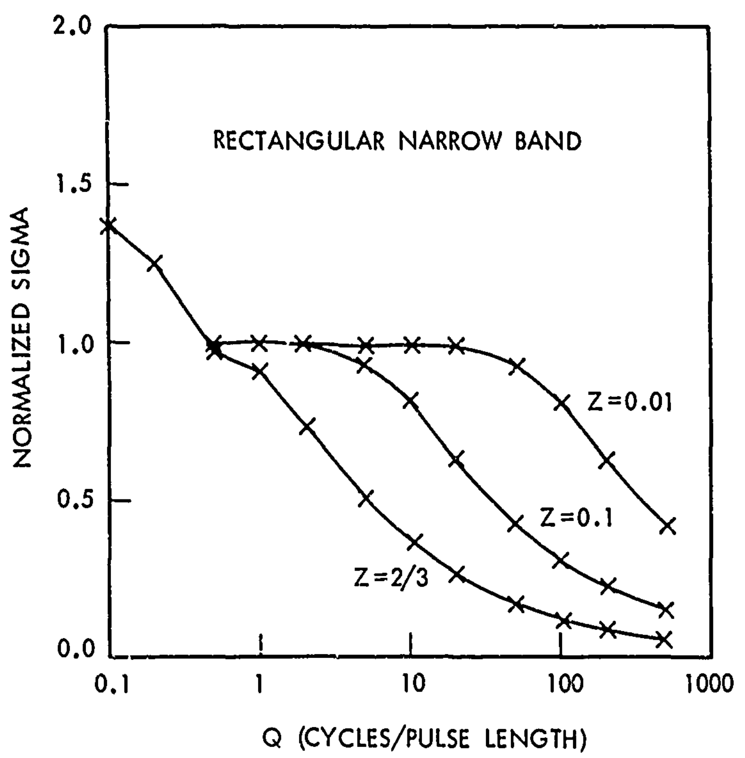


FIGURE 15. RECTANGULAR PULSE OF NARROW BAND NOISE NORMALIZED AUTOCORRELATION STANDARD DEVIATION PARAMETERIZED BY BANDWIDTH.

$$\mathrm{Var}\left[\varphi_{ss}(\tau)\right] = 8N_o^2 T \Delta\omega^2 e^{-2\tau/T} \int_0^\infty e^{-2u/T} \left[\frac{\sin^2 \Delta\omega u}{(\Delta\omega u)^2} \cos^2 \omega_o u + \frac{\sin\Delta\omega(u-\tau)}{\Delta\omega(u-\tau)} \frac{\sin\Delta\omega(u+\tau)}{\Delta\omega(u+\tau)} \cos\omega_o(u-\tau) \cos\omega_o(u+\tau) \right] du \quad (123)$$

$$E_{Tav} = 2N_o T \Delta\omega \quad (124)$$

The parameters for this case are precisely the same as those defined for the rectangular narrow band burst with the exception that T is now to be taken as the envelop time constant. This yields

$$E_p\left[\varphi_{ss}(p)\right] = e^{-p} \frac{\sin \pi qzp}{\pi qzp} \cos 2\pi pq \quad (125)$$

$$\mathrm{Var}_p\left[\varphi_{ss}(p)\right] = 2e^{-2p} \int_0^\infty e^{-2x} \left[\frac{\sin^2 \pi qzx}{(\pi qzx)^2} \cos^2 2\pi qx + \frac{\sin\pi qz(x-p)}{\pi qz(x-p)} \frac{\sin\pi qz(x+p)}{\pi qz(x+p)} \cos 2\pi q(x-p) \cos 2\pi q(x+p) \right] dx \quad (126)$$

$$R_o(q,z) = 2\sqrt{\int_0^\infty e^{-2x} \frac{\sin^2 \pi qzx}{(\pi qzx)^2} \cos^2 2\pi qx \, dx} \quad (127)$$

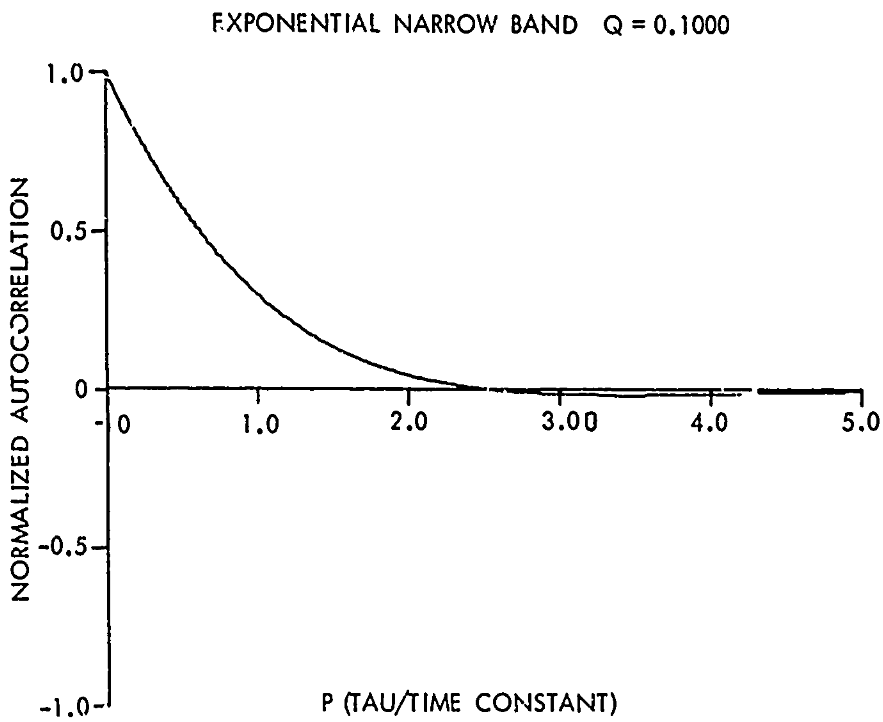
Typical plots of these equations are found in Figures 16, 17, and 18. They tend to bear out the generalizations previously noted for the other cases.

SPECTRUM CALCULATIONS

The expected value of the energy density spectrum of a noise burst for some argument ω has been found to be

$$E\left[\Phi_{ss}(\omega)\right] = \int_{-\infty}^{\infty} \Psi_{nn}(\omega') \Phi_{ee}(\omega-\omega')\, d\omega' = \Psi_{nn}(\omega) \otimes \Phi_{ee}(\omega)$$

by equation (70) and is interpreted as the convolution of the power spectrum of the noise and the energy spectrum of the envelope



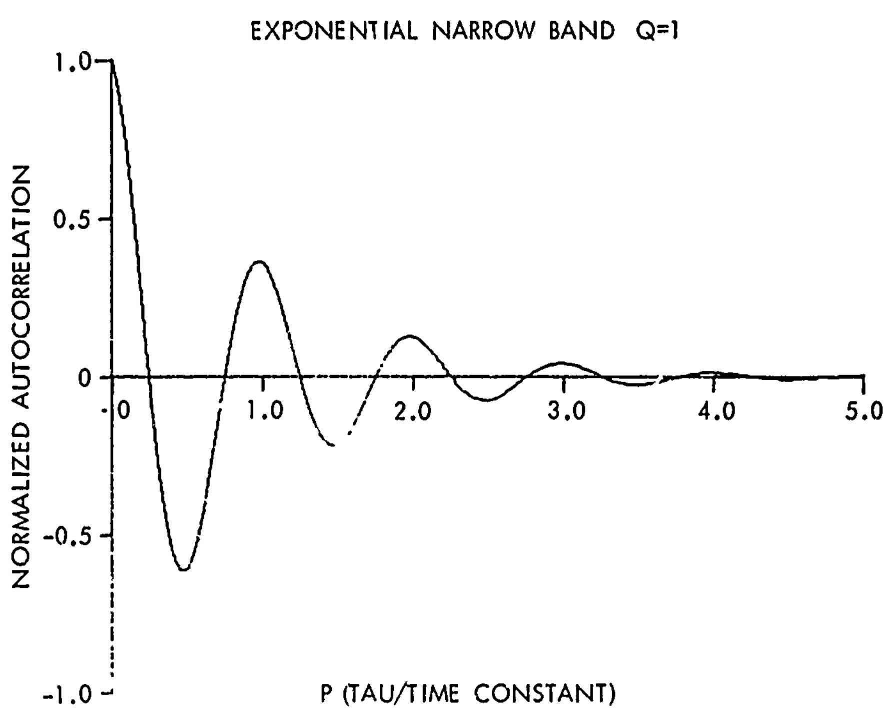


FIGURE 16. EXPONENTIAL PULSE OF NARROW BAND NOISE AVERAGE AUTOCORRELATION FOR Z = 0.1

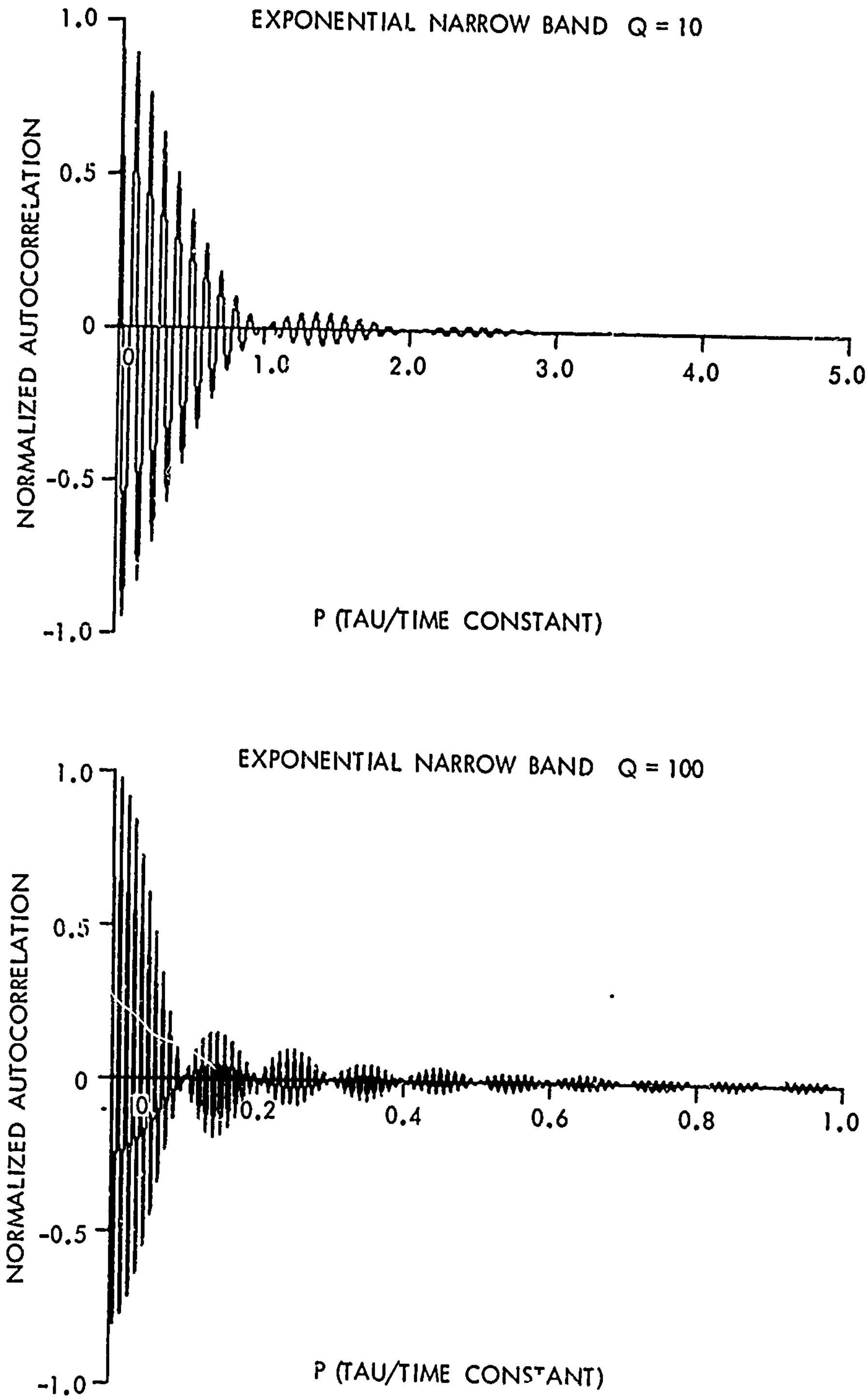


FIGURE 17. EXPONENTIAL PULSE OF NARROW BAND NOISE AVERAGE AUTOCORRELATION FOR Z = 0.1

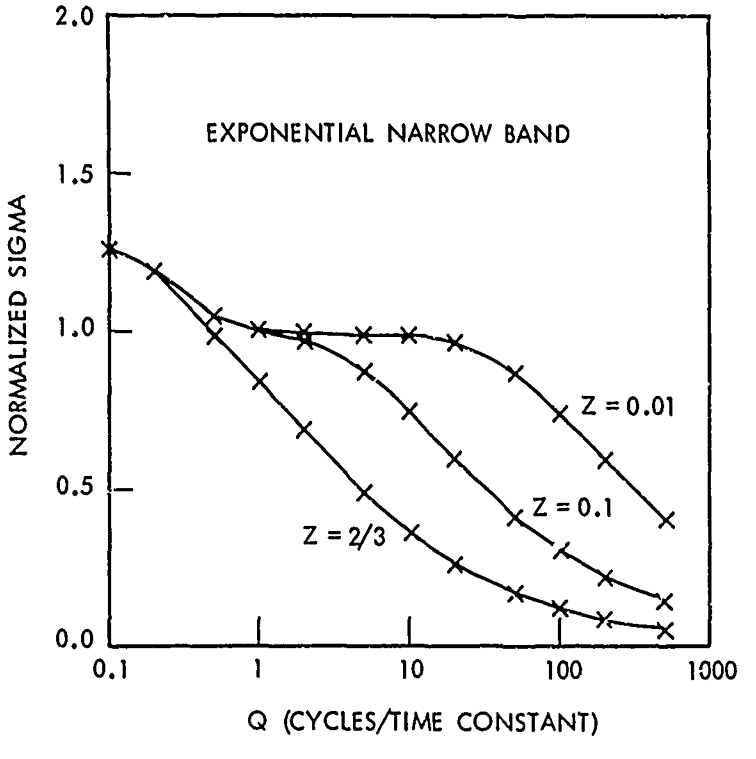


FIGURE 18. EXPONENTIAL PULSE OF NARROW BAND NOISE NORMALIZED AUTOCORRELATION STANDARD DEVIATION PARAMETERIZED BY BANDWIDTH.

waveform. Since both members of the convolution are even functions, the expected value of $\Phi_{ss}(\omega)$ can be written as the cross-correlation of $\Psi_{nn}(\omega)$ and $\Phi_{ee}(\omega)$:

$$E\left[\Phi_{ss}(\omega)\right] = \int_{-\omega}^{\omega} \Psi_{nn}(\omega') \;\; \Phi_{ee}(\omega'-\omega)\; d\omega' \qquad (128)$$

Obtaining the spectral functions of the various envelopes and noises from Table I, it is straightforward, although often tedious to calculate $E\left[\Phi_{ss}(\omega)\right]$ for the examples treated here.

1. RECTANGULAR BURST OF BROAD BAND NOISE. Using formula (128),

$$E\left[\Phi_{ss}(\omega)\right] = \frac{N_o T^2}{2} \int_{-\omega_2 + \omega}^{\omega_2 + \omega} \frac{\sin^2 \omega' \, T/2}{(\omega' \, T/2)^2} \; d\omega' \qquad (129)$$

Now introducing the same parameters defined in the study of the autocorrelation function for this case (equations (103)) and adding another:

$r = \frac{\omega}{\omega_2}$, the argument as a fraction of the highest noise frequency, a normalized and parametized version of equation (130) can be written as

$$E\left[\Phi'_{ss}(r)\right] = \frac{1}{2} \int_{q(r-1)}^{q(r+1)} \frac{\sin^2 \pi x}{(\pi x)^2} \; dx \qquad (131)$$

such that

$$\int_{-\infty}^{\infty} E\left[\Phi'_{ss}(r)\right] dr = 1 \qquad (132)$$

From the normalized expression, one obtains the spectral density for a one radian band at $\omega = r\omega_2$ by writing

$$E\left[\Phi_{ss}(\omega)\right] = \frac{E_{Tav}}{\omega_2} \; E\left[\Phi'_{ss}\left(r = \frac{\omega}{\omega_2}\right)\right] \qquad (133)$$

$$= N_o T \int_{q(r-1)}^{q(r+1)} \frac{\sin^2 \pi x}{(\pi x)^2} \; dx$$

Now integrating equation (131), it emerges that

$$E\left[\Phi'_{ss}(r)\right] = \frac{1}{2\pi^2}\left\{\frac{1-r \sin 2\pi qr \sin 2\pi q - 2 \cos 2\pi qr \cos 2\pi q}{q(r^2-1)}\right\} + \frac{1}{2\pi}\left\{Si\left[2\pi q(r+1)\right] - Si\left[2\pi q(r-1)\right]\right\} \tag{134}$$

where Si is the sine integral function defined as

$$Si(x) \equiv \int_0^x \frac{\sin x}{x}\,dx \tag{135}$$

(See reference (d))

Evidently, if the spectral width of the envelope is narrow compared to that of the original noise, the convolution of equation (128) will just return an approximation to the latter. Such a condition implies q very large (by reciprocal spreading arguments), and in equation (134), the first term will become negligible in comparison with the second. Since for large positive argument, $Si(x) \approx \pi/2$, and for large negative argument, $Si(x) \approx -\pi/2$,

$$\begin{aligned} E\left[\Phi'_{ss}(r)\right] &\approx \frac{1}{2\pi}\left\{Si\left[2\pi q(r+1)\right] - Si\left[2\pi q(r-1)\right]\right\} \\ &\approx 1/2, \quad -1 \le r \le +1 \\ &\approx 0, \text{ elsewhere} \end{aligned} \tag{136}$$

which checks with the intuitive solution.

2. <u>EXPONENTIAL BURST OF BROAD BAND NOISE</u>. Using formula (128),

$$E\left[\Phi_{ss}(\omega)\right] = \frac{N_0 T^2}{2\pi}\int_{-\omega_2+\omega}^{\omega_2+\omega} \frac{d\omega'}{\omega'^2 T^2 + 1} \tag{137}$$

where T is the decay time constant. Normalizing and parametizing as before,

$$E\left[\Phi'_{ss}(r)\right] = \int_{q(r-1)}^{q(r+1)} \frac{dx}{4\pi^2 x^2 + 1} \tag{138}$$

where $r = \omega/\omega_2$.

Integrating,

$$E\left[\Phi'_{ss}(r)\right] = \frac{1}{2\pi} \tan^{-1}\left[\frac{4\pi q}{1 + 4\pi^2 q^2 (r^2-1)}\right] \quad (139)$$

3. RECTANGULAR BURST OF NARROW BAND NOISE. The condition integral can be written as

$$E\left[\Phi_{ss}(\omega)\right] = \frac{N_o T^2}{2\pi}\int_{-\omega_o+\omega-\Delta\omega}^{-\omega_o+\omega+\Delta\omega} \frac{\sin^2 \omega' T/2}{(\omega' T/2)^2}\, d\omega' + \frac{N_o T^2}{2\pi}\int_{\omega_o+\omega-\Delta\omega}^{\omega_o+\omega+\Delta\omega} \frac{\sin^2 \omega' T/2}{(\omega' T/2)^2}\, d\omega' \quad (140)$$

Defining as before,

$$q = \frac{\omega_o T}{2\pi}, \quad z = \frac{2\Delta\omega}{\omega_o}, \quad r = \frac{\omega}{\omega_o},$$

this can be written as

$$E\left[\Phi_{ss}(r)\right] = \frac{1}{2z}\int_{q(r-1-z/2)}^{q(r-1+z/2)} \frac{\sin^2 \pi x}{(\pi x)^2}\, dx + \frac{1}{2z}\int_{q(r+1-z/2)}^{q(r+1+z/2)} \frac{\sin^2 \pi x}{(\pi x)^2}\, dx \quad (141)$$

in the parametized unit energy version. In this form, $\int_{-\infty}^{\infty} E\left[\Phi'_{ss}(r)\right] dr = 1$, and the relationship between $E\left[\Phi_{ss}(\omega)\right]$ and $E\left[\Phi'_{ss}(r)\right]$ can be expressed as

$$E\left[\Phi_{ss}(\omega)\right] = 2\, N_o T z\, E\left[\Phi'_{ss}\left(r = \frac{\omega}{\omega_o}\right)\right]$$

$$= \frac{E_{Tav}}{\omega_o}\, E\left[\Phi'_{ss}\left(r = \frac{\omega}{\omega_o}\right)\right] \quad (142)$$

At any rate, integration yields

$$E\left[\Phi_{ss}^{'}(r)\right] = \frac{1}{8\pi^2}\left[\frac{1}{q\left[(r-1)^2 - z^2/_4\right]} + \frac{1}{q\left[(r+1)^2 - z^2/_4\right]}\right]$$

$$- \frac{1}{4\pi^2 qz}\left\{\frac{2(r-1)\sin 2\pi q(r-1)\ \sin\pi qz + z\cos 2\pi q(r-1)\cos\pi qz}{\left[(r-1)^2 - z^2/_4\right]}\right.$$

$$\left. + \frac{2(r+1)\sin 2\pi q(r+1)\ \sin\pi qz + z\cos 2\pi q(r+1)\ \cos\pi qz}{\left[(r+1)^2 - z^2/_4\right]}\right\}$$

$$+ \frac{1}{2\pi z}\left\{S_i\left[2\pi q(1+r+z/_2)\right] - S_i\left[2\pi q\ (1+r - z/_2)\right]\right\} \tag{143}$$

It can be shown that for q large,

$$E\left[\Phi_{ss}^{'}(r)\right] \approx \frac{1}{2z}\ ,\quad -1 - z/_2 \leq r \leq -1 + z/_2 \tag{144}$$

$$\approx \frac{1}{2z}\ ,\quad 1 - z/_2 \leq r \leq 1 + z/_2$$

$$\approx 0,\ \text{elsewhere}$$

as it should.

4. EXPONENTIAL BURST OF NARROW BAND NOISE. Evidently,

$$E\left[\Phi_{ss}(\omega)\right] = \frac{N_0 T^2}{2\pi}\int_{-\omega_0+\omega-\Delta\omega}^{-\omega_0+\omega+\Delta\omega}\frac{d\omega'}{\omega'^2\ T^2 + 1} + \frac{N_0 T^2}{2\pi}\int_{\omega_0+\omega-\Delta\omega}^{\omega_0+\omega+\Delta\omega}\frac{d\omega'}{\omega'^2\ T^2+1} \tag{145}$$

Proceeding as before,

$$E\left[\Phi'_{ss}(r)\right] = \frac{1}{z}\int_{q(r-1-z/2)}^{q(r-1+z/2)} \frac{dx}{4\pi^2x^2+1} + \frac{1}{z}\int_{q(r+1-z/2)}^{q(r+1+z/2)} \frac{dx}{4\pi^2x^2+1} \tag{146}$$

such that

$$E\left[\Phi_{ss}(\omega)\right] = \frac{E_{Tav}}{\omega_o} E\left[\Phi'_{ss}(r)\right] = N_o\, Tz\, E\left[\Phi'_{ss}(r)\right] \tag{147}$$

Integration of equation (146) yields

$$E\left[\Phi'_{ss}(r)\right] = \frac{1}{2\pi z}\left\{\tan^{-1}\left[\frac{2\pi qz}{1+4\pi^2 q^2\left[(r-1)^2 - z^2/4\right]}\right] + \tan^{-1}\left[\frac{2\pi qz}{1+4\pi^2 q^2\left[(r+1)^2 - z^2/4\right]}\right]\right\} \tag{148}$$

The resulting expressions for $E\left[\Phi_{ss}(\omega)\right]$ in all cases emerge so complicated that in truth they are of but limited value. It is probably best just to keep in mind the relatively simple graphical interpretation of the convolution operation to provide the required insight into the resulting spectra.

This concludes our consideration of illustrative examples.

Chapter VI

CONCLUSIONS AND RECOMMENDATIONS

The foregoing discussions have attempted to formulate a means for obtaining statistical descriptions of the properties of a class of random transients perhaps well described as "noise bursts". In so doing, it has been necessary to thread a careful course between the methodology that applies to deterministic transients on the one hand and that intended for continuing stochastic processes on the other. Attention has been restricted to the time average autocorrelation function and the energy density spectrum for given values of time displacement and radian frequency, respectively.

Given a random transient drawn from the ensemble of all such signals available from the generating mechanism, it is possible, at least formally, to compute the time average autocorrelation function by the familiar process of displacement, multiplication, and integration. It is similarly possible to calculate the energy spectral density of the sample function by either Fourier transformation of the measured autocorrelation function or by a Fourier integral treatment of the function itself. Now since each of the transient sample functions is different, it is hardly surprising to find that each of the measured autocorrelations and spectra will be different also. This implies that these latter functions are, for every value of their arguments, random variables, in the sense that we lack exact a priori knowledge of their values, and therefore cannot predict the autocorrelation function and spectrum of each individual transient with exactitude. Thus, the autocorrelation function and spectrum of a random transient must be described by probability distributions parametized, in a sense, by the arguments of the functions. This investigation has not attempted the derivation of the form of these distributions at each point, but has been restricted to a calculation of the means and variances of the spectra and autocorrelations as functions of the arguments.

If the random transients treated here are modeled as the product of an envelope waveform and a continuing random process, the calculation of the means and variances described above is straightforward. In the case of the resultant autocorrelation function, the mean value at every point is found to be the product of the autocorrelation functions for the envelope and the original noise process. This result agrees with intuition and is similar to that found in seeking the autocorrelation function of the product of two independent random processes. The variance of the resultant autocorrelation function can similarly be expressed in terms of the autocorrelation functions of the original signals, but admits no ready intuitive explanation. The most important characteristics of the variance, however, are first that it achieves its maximum value at the origin (where the mean corresponds to the total average energy), and second that as the suitably defined "duration" of the transient increases with respect to a typical noise period, the standard deviation becomes a smaller and smaller percentage of the mean. This merely reflects the fact that as the transient lengthens, the signal looks more and more like a continuing random process, for which the autocorrelation function at every point is well defined.

Turning to the mean value of the spectrum at a point, a satisfying and intuitive result is found. Since the random transient is formed by the multiplication of two signals in the time domain, one might expect the resulting energy density spectrum to emerge as the convolution of the two corresponding spectra in the frequency domain. When speaking of the mean value at each point, this is found to be the case. The variance of the spectrum has also been treated and expressed in terms of the parameters of the original signals. In contrast to the autocorrelation standard deviation, that of the energy density spectrum is always larger than the mean at a point, regardless of the duration of the transient. Thus, the measured spectrum does not converge in the mean to the value predicted by the convolution as the transient lengthens, and it appears that, at least on the basis of a pointwise comparison, a large spread of measured spectra will always be observed. This result, as was pointed out previously, is the major defect of so-called periodogram analysis, which can be treated as a special case of the problem faced here.

It has been stressed throughout this report that the results derived apply only at specified points on the autocorrelation and spectral functions <u>when no knowledge is assumed</u> about the behavior of the function at other points. In other words, the means and variances derived here stem from unconditional probability distributions for every argument value, in which the behavior at a point is treated in isolation. For this reason, it is risky to attempt to extend the present findings to describe the extent to which the empirical autocorrelations and spectra <u>as a whole</u> are predicted by the calculated mean values. One could envisage, for example, a sample function that yielded an empirical autocorrelation function quite similar in form to the expected value but having one or two pathological points of substantial disagreement. The examination of this sort of effect requires the study of conditional distributions of the autocorrelation functions and spectra, or alternatively, the determination of the joint density functions of their values at two or more arguments. This has not been done here and remains a large and interesting area for future investigation. At present, we must limit ourselves to the consideration of single points and resist the temptation to extend the pointwise conclusions to the autocorrelations and spectra in their entirety.

The problem of power spectral estimation from empirical records is an area that resembles, in many ways, the study of random transients.

In both fields one must work with finite length segments of random processes whose good behavior arises primarily from their extending in time from $-\infty$ to $+\infty$ and this leads to computational difficulties. A good many of the techniques of power spectrum measurement can probably be applied to the present st dy. Blackman and Tukey (reference (f)), for instance, study the problem of joint estimation of neighboring points on an empirical power spectrum and derive expressions for smoothed spectral density estimates which abandon the concept of point estimation in favor of band-wise calculations which display a higher statistical reliability. This appears to be a particularly fruitful approach for the class of problems treated here and may lead to more meaningful prediction of the spectrum and autocorrelation of a sample noise burst.

A closely related area is the derivation of a linear system theory for signals of this type. In effect, this would indicate the results to be expected when random transients are subjected to filtering and other linear operations. It would relate directly to practical problems of measurement, detection, and interference elimination, and may even lead to the development of optimum linear filters in the Wiener sense. These are only a few of the new directions that can be followed in further work on noise-burst-like waveforms.

Despite the limitations set forth above, the present theory has several interesting implications for the processing and measurement of random transient signals. It indicates to some extent, for example, the degree of reliability that can be assumed in basing a measurement program upon a given number of sample functions. The average autocorrelation function for a given displacement or the average spectral density for a given value of ω can be found by averaging a sufficiently large number of empirical calculations. The variance of these averages can in turn be estimated by turning to the theory set out here. Evidently, the longer the transient, the fewer the sample functions required to give reasonably good knowledge of the average autocorrelation at a point. This consideration does not, however, apply to the estimation of the energy spectrum in the form defined here.

More important, though, is the illumination shed on the problem of formulating signal processing systems intended for use with non-stationary backgrounds or in situations, such as explosive echo ranging, where the waveform to be detected is itself a random transient. We have seen that under certain conditions of envelope duration and noise characteristic that it is possible for the spectra and autocorrelations of the individual transients to be rather different at the same value of an argument, even though the expectations are the same from sample to sample. This implies immediately that it may not be advisable to tailor the characteristics of a monopulse processing system too closely to the mean values of autocorrelation and spectrum. A filter painstakingly devised to reproduce or complement the mean spectral density may well do serious violence to the individual transients just because they very well may have spectra which differ significantly from the mean. The same considerations apply to correlation processing. It is hoped that the present theory contributes at least some understanding of this problem and aids in quantifying the latitude that must be allowed for accommodating pulse to pulse differences. Admittedly, a rigorous treatment of this question must await the extensions described above, particularly the derivation of joint and conditional probabilities and a suitable linear system theory. These first considerations, though, should at least alert the researcher to the existence of the problems involved. It is especially hoped that more care will be taken in the measurement and description of such phenomena as sonar reverberation in connection with the study of monopulse detection systems. There is little evidence that these areas have been approached in the past with the rigor they deserve.

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APPENDIX A

ON BOUNDING THE INTEGRAL $\int_{0}^{\infty} \varphi_{pp}(u,o)\, \Psi_{nn}(u+\tau)\, \Psi_{nn}(u-\tau)\, du$

Since the integrand is an even function, we can consider the infinite integral

$$i_1(\tau) = \int_{-\infty}^{\infty} \varphi_{pp}(u,o)\, \Psi_{nn}(u+\tau)\, \Psi_{nn}(u-\tau)\, du \qquad \text{A-1}$$

and write it as follows:

$$i_1(\tau) = \int_{-\infty}^{\infty} \sqrt{\varphi_{pp}(u,o)}\, \Psi_{nn}(u+\tau)\, \sqrt{\varphi_{pp}(u,o)}\, \Psi_{nn}(u-\tau)\, du \qquad \text{A-2}$$

Now using the Schwarz inequality as in reference (e)

$$\left| i_1(\tau) \right|^2 \leq \left[\int_{-\infty}^{\infty} \varphi_{pp}(u,o)\, \Psi_{nn}^2(u+\tau)\, du \right] \left[\int_{-\infty}^{\infty} \varphi_{pp}(u,o)\, \Psi_{nn}^2(u-\tau)\, du \right] \qquad \text{A-3}$$

Since $\varphi_{pp}(u,o)$ is a real even function, the factors on the right are equal and positive. Therefore,

$$i_1(\tau) \leq \int_{-\infty}^{\infty} \varphi_{pp}(u,o)\, \Psi_{nn}^2(u-\tau)\, du \qquad \text{A-4}$$

Consider now the integral on the right of A-4 and denote it $i_2(\tau)$. Thus,

$$i_1(\tau) \leq i_2(\tau) = \int_{-\infty}^{\infty} \varphi_{pp}(u,o)\, \Psi_{nn}^2(u-\tau)\, du \qquad \text{A-5}$$

Since $\Psi_{nn}^2(x)$ is a real even function, $\Psi_{nn}^2(u-\tau) = \Psi_{nn}^2(\tau-u)$ and therefore

$$i_2(\tau) = \int_{-\infty}^{\infty} \varphi_{pp}(u,0)\, \psi_{nn}^2(\tau-u)\,du = \varphi_{pp}(u,0) \circledast \psi_{nn}^2(u) \qquad \text{A-6}$$

where $\circledast$ denotes the operation of convolution. Now, let us enter the frequency domain where the transforms of $i_2(\tau)$, $\varphi_{pp}(u,o)$, and $\psi_{nn}^2(u)$ are respectively $I_2(\omega)$, $P(\omega)$, and $Q(\omega)$. In reference (a) it is pointed out that the Fourier transform of a realizable autocorrelation function must always be positive. Thus $P(\omega)$ is always positive, and since $Q(\omega)$ is the transform of the square of a realizable autocorrelation function, it must be the convolution of a positive transform with itself. Thus $Q(\omega)$ is also everywhere greater than zero. Writing (A-6) in the transform domain yields

$$I_2(\omega) = P(\omega)\, Q(\omega) \qquad \text{A-7}$$

and re-transforming to the time domain gives

$$i_2(\tau) = \int_{-\infty}^{\infty} P(\omega)\, Q(\omega) e^{j\omega\tau}\, d\omega \qquad \text{A-8}$$

Now

$$|i_2(\tau)| \leq \int_{-\infty}^{\infty} |P(\omega)|\, |Q(\omega)|\, d\omega\,, \qquad \text{A-9}$$

but since by A-6, $i_2(\tau)$ must always be positive and since $P(\omega)$ and $Q(\omega)$ are greater than zero, this becomes

$$i_2(\tau) \leq \int_{-\infty}^{\infty} P(\omega)\, Q(\omega)\, d\omega = i_2(0) \qquad \text{A-10}$$

By A-5,

$$i_1(\tau) \leq i_2(\tau) \leq i_2(0) \qquad \text{A-11}$$

and finally

$$\int_{o}^{\infty} \varphi_{pp}(u,0)\, \psi_{nn}(u+\tau)\, \psi_{nn}(u-\tau)\,du \leq \int_{-\infty}^{\infty} \varphi_{pp}(u,0)\, \psi_{nn}^2(u)\, du \qquad \text{A-12}$$

when $\varphi_{pp}(\mu, 0)$ is always positive. Under this restriction, it has been shown that the maximum of the integral of interest occurs when $\tau = 0$.

APPENDIX B

AN ALTERNATIVE APPROACH TO THE NOISE BURST SPECTRA USING THE WIENER-KHINTCHINE RELATION

As applied to transient signals, the Wiener-Khintchine relation states that the autocorrelation function of a signal (as defined in equation (4) of the main body) and its energy spectrum constitute a Fourier transform pair:

$$\varphi_{ss}(\tau) = \int_{-\infty}^{\infty} \Phi_{ss}(\omega) e^{j\omega\tau} d\omega \qquad \text{B-1a}$$

$$\Phi_{ss}(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \varphi_{ss}(\tau) e^{-j\omega\tau} d\tau \qquad \text{B-1b}$$

Thus, it appears that one can compute the energy density spectrum from the second of these equations. From our previous considerations, however, we know that $\Phi_{ss}(\omega)$ is a random variable for all ω and that we thus must be content with computing means and variances.

$$E\left[\Phi_{ss}(\omega)\right] = \frac{1}{2\pi} \int_{-\infty}^{\infty} E\left[\varphi_{ss}(\tau)\right] e^{-j\omega\tau} d\tau \qquad \text{B-2}$$

By equation (23) of the main body,

$$E\left[\varphi_{ss}(\tau)\right] = \psi_{nn}(\tau)\, \varphi_{ee}(\tau)$$

and thus

$$E\left[\Phi_{ss}(\omega)\right] = \frac{1}{2\pi} \int_{-\infty}^{\infty} \psi_{nn}(\tau)\, \varphi_{ee}(\tau) e^{-j\omega\tau} d\tau \qquad \text{B-3}$$

precisely the same expression found in the main body from the Fourier integral approach (equation (64)).

Turning now to a calculation of the variance of $\Phi_{ss}(\omega)$ from this standpoint, we must first compute the mean square value of the spectrum for each ω.

$$\left[\Phi_{ss}(\omega)\right]^2 = \frac{1}{4\pi^2}\left[\int_{-\infty}^{\infty}\varphi_{ss}(\tau)e^{-j\omega\tau}\,d\tau\right]^2$$

$$= \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\varphi_{ss}(\tau)\,\varphi_{ss}(\sigma)e^{-j\omega(\tau+\sigma)}d\tau\,d\sigma \qquad \text{B-4}$$

$$E\left[\Phi_{ss}^2(\omega)\right] = \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}E\left[\varphi_{ss}(\tau)\,\varphi_{ss}(\sigma)\right]e^{-j\omega(\tau+\sigma)}\,d\tau\,d\sigma \qquad \text{B-5}$$

Now by definition,

$$\varphi_{ss}(\tau)\,\varphi_{ss}(\sigma) = \int_0^{\infty}e(t)e(t+\tau)n(t)n(t+\tau)dt\int_0^{\infty}e(x)e(x+\sigma)n(x)n(x+\sigma)dx$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e(t)e(x)e(t+\tau)e(x+\sigma)n(t)n(x)n(t+\tau)n(x+\sigma)dt\,dx \qquad \text{B-6}$$

$$E\left[\varphi_{ss}(\tau)\,\varphi_{ss}(\sigma)\right] = \int_0\int_0 e(t)e(x)e(t+\tau)e(x+\sigma)\,E\left[n(t)n(x)n(t+\tau)n(x+\sigma)\right]dt\,dx \qquad \text{B-7}$$

and combining (B-5) and (B-7), there emerges that

$$E\left[\Phi_{ss}^2(\omega)\right] = \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_0^{\infty}\int_0^{\infty}e(t)e(x)e(t+\tau)e(x+\sigma)\,E\left[n(t)n(x)n(t+\tau)n(x+\sigma)\right]$$

$$e^{-j\omega(\tau+\sigma)}\,dt\,dx\,d\tau\,d\sigma \qquad \text{B-8}$$

Now with the substitution of variables $u = t + \tau$, $v = x + \sigma$,

$$E\left[\Phi_{ss}^2(\omega)\right] = \frac{1}{4\pi^2} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e(t)e(x)e(u)e(v)\ E\left[n(t)n(x)n(u)n(v)\right] e^{-j\omega(u-t+v-x)} dt\ dx\ du\ dv \qquad \text{B-9}$$

which is equivalent to the integral

$$E\left[\Phi_{ss}^2(\omega)\right] = \frac{1}{4\pi^2} \int_{0}^{\infty}\int_{0}^{\infty}\int_{0}^{\infty}\int_{0}^{\infty} e(t)e(x)e(u)e(v)\ E\left[n(t)n(x)n(u)n(v)\right] e^{-j\omega(t-x+u-v)} dt\ dx\ du\ dv \qquad \text{B-10}$$

This expression is identical with that found as equation (73) in approaching the variance from Fourier integral considerations. If both the mean and mean square are identical for the two methods, the variance must be also and hence it appears that the Wiener-Khintchine relation is directly applicable to the expectations of the statistical quantities associated with random transients.

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A class of random transient signals has been defined as the product of a deterministic envelope waveform of finite integral square and a continuous random process with a well-defined power spectrum and autocorrelation function. The time average autocorrelation function and energy density spectrum of the resulting waveform have been found to be random variables at every value of their arguments. The means and variances of these random variables are derived as functions of the characteristics of the envelope and original noise process. The average autocorrelation function is found to be the product of the autocorrelation functions of envelope and noise, and the average spectrum is given by the convolution of the energy spectrum of the envelope function and the power spectrum of the noise. Examples of the mean and variance calculations are presented for both rectangular and decaying exponential pulse of both broad and narrow band noise. Finally, the implications of these findings for measurement programs and monopulse signal processing are discussed.

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