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AN ANALYTICAL FORMULATION FOR A SATELLITE GROUND-TRACE

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SPACE DEFENSE SYSTEMS PROGRAM OFFICE
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FOREWORD

For planning purposes and questions concerning sensor coverage, it is useful to know the analytical ground track of an earth satellite. This report provides methods for determining this ground track.

This technical report has been reviewed and is approved.

THOMAS O. WEAR, Colonel, USAF
Director, Space Defense Systems Program Office
Deputy for Surveillance and Control Systems

ABSTRACT

This report derives the analytical expressions for an earth satellite ground track for a general elliptical, a circular and a circular-synchronous orbit under the assumptions of two-body conditions and in the absence of perturbations.

SECTION I

INTRODUCTION

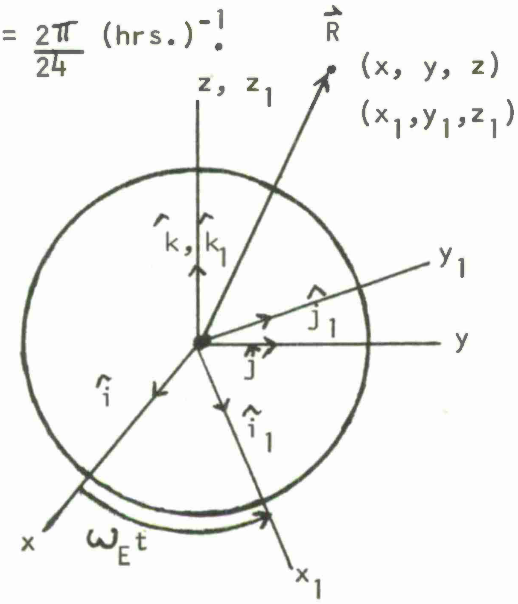
The purpose of this paper is to derive parametric formulas for the ground track of an artificial earth satellite. Certain assumptions are made, namely, that the earth is a sphere and that two-body orbital mechanics hold, i.e., the satellite undergoes no perturbations. After deriving the general expression, two particular cases are considered. These are the case of a circular orbit and the case of a circular orbit of 24-hour period which is inclined to the earth's equator. In these cases, time is eliminated from the expressions and the longitude of the sub-satellite point is expressed as a function of latitude.

SECTION II

GENERAL CASE

Consider a right handed inertial x-y-z coordinate system with origin at the center of the earth, with the x-y plane coincident with the plane of the earth's equator and with the z-axis intersecting the north pole. Finally, let the x-axis point towards the vernal equinox. Also consider a rotating coordinate system, x_1-y_1-z_1, with origin at the center of the earth, the x_1-y_1 plane rotating in the x-y plane and with the z_1-axis coincident with the z-axis. Let $\hat{i}$, $\hat{j}$ and $\hat{k}$ be orthogonal unit vectors in the x-y-z system and $\hat{i}_1$, $\hat{j}_1$ and $\hat{k}_1$ be orthogonal unit vectors in the x_1-y_1-z_1 system. Let $\vec{R}$ be a vector from the center of the earth to the satellite and let the satellite have coordinates (x, y, z) in the x-y-z system and (x_1, y_1, z_1) in the x_1-y_1-z_1 system. Finally, let the period of the rotation of the x_1-y_1 plane be 24 hours so that its angular frequency

$$\omega_E = \frac{2\pi}{24} \text{ (hrs.)}^{-1}.$$



From the above diagram,

$$\vec{R} = x\hat{i} + y\hat{j} + z\hat{k} = x_1\hat{i}_1 + y_1\hat{j}_1 + z_1\hat{k}_1$$

so that

$$x_1 = \hat{i}_1 . \vec{R} = x\hat{i}_1 . \hat{i} + y\hat{i}_1 . \hat{j} + z\hat{i}_1 . \hat{k}.$$

Now, the angle between the x and the x_1 axes will be $\omega_E t$ if the x_1 axis pierces the point of intersection of the prime meridian and the earth's equator and t is the time in mean siderial hours from when these two axes coincide. Therefore,

$$\hat{i}_1 . \hat{i} = \cos\omega_E t \text{ and } \hat{i}_1 . \hat{j} = \sin\omega_E t.$$

Also,

$$\hat{i}_1 . \hat{k} = 0 \text{ since } \hat{k} = \hat{k}_1.$$

Thus,

$$x_1 = x\cos\omega_E t + y\sin\omega_E t, \; y_1 = x\hat{j}_1 . \hat{i} + y\hat{j}_1 . \hat{j} =$$

$$- x\sin\omega_E t + y\cos\omega_E t, \text{ and } z_1 = z.$$

From any book on celestial mechanics,

$$x = aP_x \, (\cos E - e) + bQ_x \sin E$$

$$y = aP_y \, (\cos E - e) + bQ_y \sin E$$

$$z = aP_z \, (\cos E - e) + bQ_z \sin E$$

where a is the semi-major axis of the satellite's orbit, e is the orbit's eccentricity and E is the eccentric anamoly of the satellite at time t. E is given by Kepler's equation

$$E - e\sin E = \frac{2\pi}{P_o} \, (t - T_o)$$

where T_o is the time of perigee passage of the satellite and P_o is the orbital period of the satellite. All times are measured in the same units as t. The semi-minor axis of the satellite orbit

$$b = a\,(1 - e^2)^{\frac{1}{2}}.$$

The direction components P_x, Q_x, P_y, Q_y, P_z and Q_z are defined as follows:

$$P_x = \cos\Omega\cos\omega - \sin\Omega\sin\omega\cos i$$

$$Q_x = -\cos\Omega\sin\omega - \sin\Omega\cos\omega\cos i$$

$$P_y = \sin\Omega\cos\omega + \cos\Omega\sin\omega\cos i$$

$$Q_y = -\sin\Omega\sin\omega + \cos\Omega\cos\omega\cos i$$

$$P_z = \sin\omega\sin i$$

$$Q_z = \cos\omega\sin i$$

where Ω is the right ascension of the ascending node of the orbit relative to the vernal equinox, ω is the argument of perigee of the orbit and i is the inclination of the orbit to the earth's equator.

Substituting,

$$x_1 = \left[a\,(\cos\Omega\cos\omega - \sin\Omega\sin\omega\cos i)\,(\cos E - e) - b \times (\cos\Omega\sin\omega + \sin\Omega\cos\omega\cos i)\sin E\right]\cos\omega_E t + \left[a\,(\sin\Omega\cos\omega + \cos\Omega\sin\omega\cos i)\,(\cos E - e) - b \times (\sin\Omega\sin\omega - \cos\Omega\cos\omega\cos i)\sin E\right]\sin\omega_E t;$$

$$y_1 = \left[a\,(\sin\Omega\cos\omega + \cos\Omega\sin\omega\cos i)\,(\cos E - e) + b \times (\cos\Omega\cos\omega\cos i - \sin\Omega\sin\omega)\sin E\right]\cos\omega_E t - \left[a\,(\cos\Omega\cos\omega - \sin\Omega\sin\omega\cos i)\,(\cos E - e) - b\,(\cos\Omega\sin\omega + \sin\Omega\cos\omega\cos i)\sin E\right]\sin\omega_E t;$$

$$z_1 = a\,(\sin\omega\sin i)\,(\cos E - e) + b\,(\cos\omega\sin i)\sin E.$$

Let λ be the longitude of a point on the earth's surface measured from the prime meridean eastward from 0° to 360°. Let β be the latitude of the same point measured north of the equator from 0° to $+90^{\circ}$ and south of the equator from 0° to -90°. If this point is the sub-satellite point of the ground-trace, then the satellite's location on the celestial sphere will have the same latitude and longitude as the sub-satellite point, since the celestial sphere is concentric with the sphere of the earth. Also, let

$$R = |\vec{R}|.$$

Then,

$$x_1 = R\cos\lambda\cos\beta \ , \ y_1 = R\sin\lambda\cos\beta \ ,$$

and

$$z_1 = R\sin\beta \text{ so that } \cos\lambda\cos\beta = \frac{1}{R}(x\cos\omega_E t + y\sin\omega_E t),$$

$$\sin\lambda\cos\beta = \frac{1}{R}(y\cos\omega_E t - x\sin\omega_E t),$$

and

$$\sin\beta = \frac{z}{R};$$

substituting

$$R = a\,(1 - e\cos E)$$

and using

$$b = a\,(1 - e^2)^{\frac{1}{2}},$$

$$\tan\lambda = \Big\{\Big[(\sin\Omega\cos\omega + \cos\Omega\sin\omega\cos i)(\cos E - e) + (1 - e^2)^{\frac{1}{2}}(\cos\Omega\cos\omega\cos i - \sin\Omega\sin\omega)\sin E\Big] \times$$

$$\cos\omega_E t - \left[(\cos\Omega\cos\omega - \sin\Omega\sin\omega\cos i)\ (\cos E - E) -\right.$$

$$\left.(1 - e^2)^{\frac{1}{2}}\ (\cos\Omega\sin\omega + \sin\Omega\cos\omega\cos i)\ \sin E\right] \sin\omega_E t \Big\} \times$$

$$\Big\{\left[(\cos\Omega\cos\omega - \sin\Omega\sin\omega\cos i)\ (\cos E - e) - (1 - e^2)^{\frac{1}{2}} \times\right.$$

$$\left.(\cos\Omega\sin\omega + \sin\Omega\cos\omega\cos i)\ \sin E\right] \cos\omega_E t +$$

$$\left[(\sin\Omega\cos\omega + \cos\Omega\sin\omega\cos i)\ (\cos E - e) - (1 - e^2)^{\frac{1}{2}} \times\right.$$

$$\left.(\sin\Omega\sin\omega - \cos\Omega\cos\omega\cos i)\ \sin E\right] \sin\omega_E t \Big\}^{-1}.$$

$$\sin\beta = \left[(\sin\omega\sin i)\ (\cos E - e) + (1 - e^2)^{\frac{1}{2}}\ (\cos\omega\sin i) \times\right.$$

$$\left.\sin E\right]\left[1 - e\cos E\right]^{-1}.$$

The quadrants of λ and β are easily determined from the following considerations. β must be between $+90^\circ$ and -90°. If $\sin\beta$ is +, the ground track is in the northern hemisphere, and β is between 0° and 90°. If $\sin\beta$ is -, the ground track is in the southern hemisphere and β is between -90° and 0°. The quadrant for λ is determined in the conventional way be considering the signs of x_1 and y_1; i.e.,

$$\tan\lambda = \frac{y_1}{x_1}.$$

SECTION III

CIRCULAR ORBIT

Two special cases, often useful for various planning purposes, are discussed next. The first of these is the circular orbit. Here, $e = o$. Also, $E = \omega_s t$ where ω_s is the angular frequency of the satellite and

$$\omega_s = \frac{2\pi}{P_o}.$$

Since the argument of perigee is arbitrary for a circular orbit, we may set $\omega = o$, i.e., the line of apsides coincides with the line of nodes. Also, $R = a$ for all t. Substituting these values,

$$\tan\lambda = \left[(\sin\Omega \cos\omega_s t + \cos\Omega \cos i \sin\omega_s t) \cos\omega_E t - (\cos\Omega \cos\omega_s t - \sin\Omega \cos i \sin\omega_s t) \sin\omega_E t \right] \left[(\cos\Omega \times \cos\omega_s t - \sin\Omega \cos i \sin\omega_s t) \cos\omega_E t + (\sin\Omega \cos\omega_s t + \cos\Omega \cos i \sin\omega_s t) \sin\omega_E t \right]^{-1};$$

$$\sin\beta = \sin i \sin\omega_s t.$$

Since from this relation for $\sin\beta$ it follows that

$$t = \frac{1}{\omega_s} \sin^{-1}\left(\frac{\sin\beta}{\sin i}\right),$$

we can write λ explicitly as a function of β:

$$\tan\lambda = \left[\left(\sin\Omega \cos\left\{\sin^{-1}\left(\frac{\sin}{\sin i}\right)\right\} + \cos\Omega \operatorname{ctn} i \sin\beta\right) \cos\left\{\frac{\omega_E}{\omega_s} \sin^{-1}\left(\frac{\sin\beta}{\sin i}\right)\right\}\right)$$

$$- \left(\cos\Omega \cos\left\{ \sin^{-1}\left(\frac{\sin\beta}{\sin i}\right)\right\} - \sin\Omega \, \mathrm{ctn}\, i \sin\beta \right) \sin\left\{ \frac{\omega_E}{\omega_s} \sin^{-1}\left(\frac{\sin\beta}{\sin i}\right)\right\} \Bigg] \times$$

$$\Bigg[\left(\cos\Omega \cos\left\{ \sin^{-1}\left(\frac{\sin\beta}{\sin i}\right)\right\} - \sin\Omega \, \mathrm{ctn}\, i \sin\beta \right) \cos\left\{ \frac{\omega_E}{\omega_s} \sin^{-1}\left(\frac{\sin\beta}{\sin i}\right)\right\}$$

$$+ \left(\sin\Omega \cos\left\{ \sin^{-1}\left(\frac{\sin\beta}{\sin i}\right)\right\} + \cos\Omega \, \mathrm{ctn}\, i \sin\beta \right) \sin\left\{ \frac{\omega_E}{\omega_s} \sin^{-1}\left(\frac{\sin\beta}{\sin i}\right)\right\} \Bigg]^{-1}.$$

It should be noted that for a given β there will be two values of λ per orbit,

one for

$$t = \frac{1}{\omega_s} \sin^{-1}\left(\frac{\sin\beta}{\sin i}\right)$$

and one for

$$t = \frac{1}{\omega_s}\left[\sin^{-1}\left(\frac{\sin\beta}{\sin i}\right) \pm \frac{\pi}{2}\right],$$

the sign of $\pm \frac{\pi}{2}$ corresponding to the sign of $\sin^{-1}\left(\frac{\sin\beta}{\sin i}\right)$.

A further simplification will result if the x-axis is rotated such that it coincides with the ascending node, i.e., such that $\Omega = 0$. Then the longitude of the ascending node relative to the prime meridian, Ω_o, must be used in constructing plots. This simplification is utilized in the next section.

SECTION IV

CIRCULAR SYNCHRONOUS ORBIT

The next interesting special case occurs when the orbit is circular with the period the same as that of the earth, i.e.,

$$\omega_s = \omega_E,$$

and when the direction of motion is the same as the direction of the earth's rotation. Here we may, for convenience, let the x-axis pierce the ascending nodal point of the orbital plane, that is, set $\Omega = 0$. We again get λ as a function of β by the following method:

$$\sin\lambda\cos\beta = \cos i \sin\omega_E t \cos\omega_E t$$

$$- \sin\omega_E t \cos\omega_E t =$$

$$(\cos i - 1) \sin\omega_E t \cos\omega_E t;$$

$$\sin\omega_E t = \frac{\sin\beta}{\sin i} ;$$

therefore, $\sin\lambda\cos\beta =$

$$\pm \frac{\sin\beta(\cos i - 1)}{\sin i} \left(1 - \left[\frac{\sin^2\beta}{\sin^2 i}\right]\right)^{\frac{1}{2}}, \text{ or}$$

$$\sin\lambda = \pm \frac{\tan\beta(\cos i - 1)}{\sin^2 i} (\sin^2 i - \sin^2\beta)^{\frac{1}{2}}.$$

By knowing the inclination of the orbit of such a satellite, we may easily compute its ground-trace and from this deduce the satellite's earth coverage at any instant of time.

It is necessary to reiterate that this ground-trace will not be relative to the vernal equinox but will be relative to the ascending node of the orbit. Also, when $i = 0$, the orbit is synchronous - stationary, i.e., $\beta = \lambda = 0$ for all t.

SECTION X

CONCLUSIONS

In many instances, where only a rough approximation is required, the two-body ground-trace will suffice. The analytical formulas presented will rapidly provide the trace and a particularly interesting case, the circular synchronous orbit, is especially amenable to hand calculation.

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