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Transverse Vibrations of a Shallow Spherical Dome

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Prepared for BALLISTIC SYSTEMS AND SPACE SYSTEMS DIVISIONS
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FOREWORD

This report is published by the Aerospace Corporation, El Segundo, California, under Air Force Contract AF 04(695)-1001. The report was authored by M. H. Lock and H. A. Malcom, of the Applied Mechanics Division and Electronics Division, respectively, El Segundo Technical Operations, and by J. S. Whittier, Aerodynamics and Propulsion Research Laboratory, Laboratories Division, Laboratory Operations.

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ABSTRACT

It is shown that the nondimensional roots of the frequency equation for nonsymmetric transverse vibrations of a clamped-edge shallow spherical dome depend upon a single shell geometric parameter. Numerical values of the frequency parameter are given in tabular and graphical form for an extensive range of shell geometries and mode numbers. A simplified version of the frequency equation is also presented and its accuracy discussed.

CONTENTS

INTRODUCTION

Kalnins [1][1] has treated the nonsymmetric mode vibrations of a clamped-edge shallow spherical shell and has presented frequency data for a limited range of the geometric parameters a/R and a/h, where a, h and R denote the base radius, wall thickness and radius of curvature of the shell, respectively. Kalnins included both transverse and longitudinal (i.e., in-plane) inertial forces in his analysis and compared transverse mode frequencies that were calculated with and without the effect of the longitudinal inertial forces. For the range of parameters treated, he found that these inertial forces produced little change in the transverse mode frequencies.

The purpose of the present Note is to extend the range of calculated frequency data for this type of shell. First it is shown that the frequency equation can be expressed in terms of a single geometric parameter when the longitudinal inertial forces are neglected. The roots of this form of the frequency equation are then determined for an extensive range of mode numbers and shell geometry.

[1] Numbers in brackets designate References.

ANALYSIS

When longitudinal inertial forces are neglected, the equations of motion governing the free vibrations of a shallow spherical shell are [2]

$$D\nabla^2\nabla^2 w + \frac{1}{R}\nabla^2 F + \rho h \frac{\partial^2 w}{\partial t^2} = 0 \qquad (1)$$

$$\nabla^2\nabla^2 F - \frac{Eh}{R}\nabla^2 w = 0 \qquad (2)$$

where ∇^2 is the Laplacian operator, $w(r, \theta, t)$ is the transverse displacement of the shell (see Fig. 1) and F is an Airy stress function. This stress function is related to the transverse displacement $w(r, \theta, t)$ and to the in-plane displacement components $u_\theta(r, \theta, t)$ and $u_r(r, \theta, t)$, reference [3]. The parameters E and ρ are Young's modulus and the mass density of the shell material, respectively. The bending stiffness $D = Eh^3/12(1-\nu^2)$, where ν denotes Poisson's ratio. To determine the vibration frequencies, solutions of the form

$$(w, F) = \begin{Bmatrix} \sin n\theta \\ \cos n\theta \end{Bmatrix} \left[W_n(r), \; F_n(r) \right] e^{i\omega t} \qquad (3)$$

are substituted into the equations of motion. The resulting ordinary differential equations are solved under the condition that the stresses and displacements are finite at the origin. Application of the appropriate boundary conditions results in an equation for the frequencies of the shell.

In the case of a shell with clamped edges, the boundary conditions are

$$w(a,\theta,t) = u_\theta(a,\theta,t) = u_r(a,\theta,t) = 0 \qquad (4a)$$

$$\frac{\partial w}{\partial r}(a,\theta,t) = 0 \qquad (4b)$$

and the resulting frequency equation is[2]

$$\left[N_n - 4n\left(\frac{a}{R}\right)^2\right] I_n(\mu)\, J_{n-1}(\mu) - \left[N_n + 4n\left(\frac{a}{R}\right)^2\right] J_n(\mu)\, I_{n-1}(\mu)$$
$$+ \left(\frac{a}{R}\right)^2 \left[\frac{8n^2}{\mu} J_n(\mu)\, I_n(\mu) + 2\mu J_{n-1}(\mu)\, I_{n-1}(\mu)\right] = 0 \qquad (5)$$

where

$$N_n = \frac{\mu^2}{2(1+\nu)(n+1)}\left[(3-\nu)\,\Omega^2 - 4\left(\frac{a}{R}\right)^2\right]$$

$$\mu^4 = 12(1-\nu^2)\left(\frac{a}{h}\right)^2\left[\Omega^2 - \left(\frac{a}{R}\right)^2\right]$$

$$\Omega^2 = \frac{\rho a^2 \omega^2}{E}$$

The $J_n(\mu)$ and $I_n(\mu)$ are Bessel functions and modified Bessel functions of the first kind; n denotes the number of nodal diameters and ω is the frequency of vibration.

[2] This equation was employed by Kalnins [1] for cases where the longitudinal inertial forces were neglected.

To extend the range of calculated frequency data for this type of shell, we first note that the nondimensional roots $\mu_{n,m}$ (where $m \geq 1$ is the rank of the root) may be obtained as functions of a single geometric parameter rather than in terms of the ratios a/R and a/h. To show this, we introduce the geometric parameter λ used in the buckling theory of such shells. Substitution of

$$\lambda^4 = 12(1-\nu^2)\, a^4/R^2h^2 \tag{6}$$

into equation (5) gives the following frequency equation

$$\left\{\mu^2\left[-1 + \frac{(3-\nu)}{(1+\nu)}\left(\frac{\mu}{\lambda}\right)^4\right] - 8n(n+1)\right\}\mu I_n(\mu)\, J_{n-1}(\mu)$$
$$-\left\{\mu^2\left[-1 + \frac{(3-\nu)}{(1+\nu)}\left(\frac{\mu}{\lambda}\right)^4\right] + 8n(n+1)\right\}\mu I_{n-1}(\mu)\, J_n(\mu)$$
$$+ 16\, n^2(n+1)\, J_n(\mu)\, I_n(\mu) + 4(n+1)\,\mu^2\, J_{n-1}(\mu)\, I_{n-1}(\mu) = 0 \tag{7}$$

For a specified value of n, the roots of this equation depend only upon Poisson's ratio ν and the geometric parameter λ. In order to encompass a wide range of shell configurations, calculations were performed for $\lambda \leq 20$ and $1 \leq n \leq 20$, with $\nu = 0.3$. The resulting roots $\mu_{n,m}$ are presented in Table 1 for values of the rank m up to 3. Some of the tabulated results are presented in graphical form in Fig. 1. This figure is particularly illuminating since it clearly shows that the roots tend to increase linearly with increase in the number of nodal diameters when n is sufficiently large. The figure also indicates that the roots tend to become independent of λ as n becomes larger. This behavior is related to the fact

Table 1. Frequency Parameter $\mu_{n,m}$

n	m \ λ	0	2.5	5.0	7.49	10.0	15.0	20.0
1	1	4.611	4.628	4.865	5.555	6.383	7.006	7.103
	2	7.779	7.801	7.828	7.956	8.389	9.988	10.416
	3	10.958	10.958	10.964	10.989	11.068	11.196	13.416
2	1	5.905	5.912	6.005	6.346	6.964	7.971	8.234
	2	9.197	9.198	9.213	9.279	9.478	10.635	11.520
	3	12.402	12.402	12.406	12.423	12.473	12.896	14.209
3	1	7.144	7.146	7.190	7.364	7.749	8.780	9.279
	2	10.537	10.537	10.546	10.585	10.695	11.399	12.452
	3	13.795	13.795	13.798	13.810	13.843	14.085	14.952
4	1	8.347	8.348	8.371	8.468	8.701	9.549	10.233
	2	11.837	11.837	11.843	11.867	11.935	12.365	13.282
	3	15.150	15.151	15.152	15.160	15.184	15.340	15.878
5	1	9.526	9.527	9.540	9.598	9.743	10.375	11.115
	2	13.107	13.108	13.111	13.128	13.172	13.450	14.141
	3	16.475	16.476	16.477	16.483	16.500	16.608	16.956
6	1	10.687	10.688	10.696	10.732	10.827	11.282	11.971
	2	14.355	14.355	14.358	14.369	14.400	14.589	15.087
	3	17.776	17.777	17.778	17.782	17.795	17.873	18.113
7	1	11.835	11.835	11.841	11.865	11.928	12.256	12.842
	2	15.585	15.585	15.587	15.595	15.617	15.751	16.110
	3	19.058	19.058	19.060	19.062	19.072	19.131	19.304
8	1	12.971	12.971	12.975	12.992	13.036	13.274	13.751
	2	16.799	16.799	16.800	16.806	16.827	16.921	17.186
	3	20.323	20.323	20.324	20.327	20.334	20.379	20.509
9	1	14.098	14.098	14.101	14.113	14.145	14.320	14.701
	2	18.000	18.000	18.000	18.006	18.018	18.092	18.292
	3	21.574	21.574	21.574	21.576	21.582	21.618	21.718
10	1	15.218	15.218	15.220	15.228	15.252	15.383	15.685
	2	19.190	19.191	19.191	19.195	19.204	19.261	19.415
	3	22.812	22.812	22.812	22.814	22.819	22.847	22.926
11	1	16.330	16.330	16.332	16.338	16.356	16.456	16.695
	2	20.371	20.371	20.371	20.375	20.382	20.426	20.547
	3							
12	1	17.437	17.437	17.439	17.444	17.457	17.535	17.725
	2	21.544	21.544	21.544	21.546	21.552	21.588	21.684
	3							
13	1	18.539	18.539	18.540	18.544	18.555	18.616	18.768
14	1	19.637	19.637	19.638	19.641	19.649	19.699	19.821
15	1	20.730	20.730	20.731	20.733	20.740	20.780	20.880
16	1	21.820	21.820	21.820	21.822	21.828	21.860	21.943
17	1	22.906	22.906	22.907	22.908	22.913	22.939	23.008
18	1	23.990	23.990	23.990	23.992	23.995	24.017	24.075
19	1	25.071	25.071	25.071	25.072	25.075	25.093	25.142
20	1	26.149	26.149	26.149	26.150	26.153	26.168	26.209

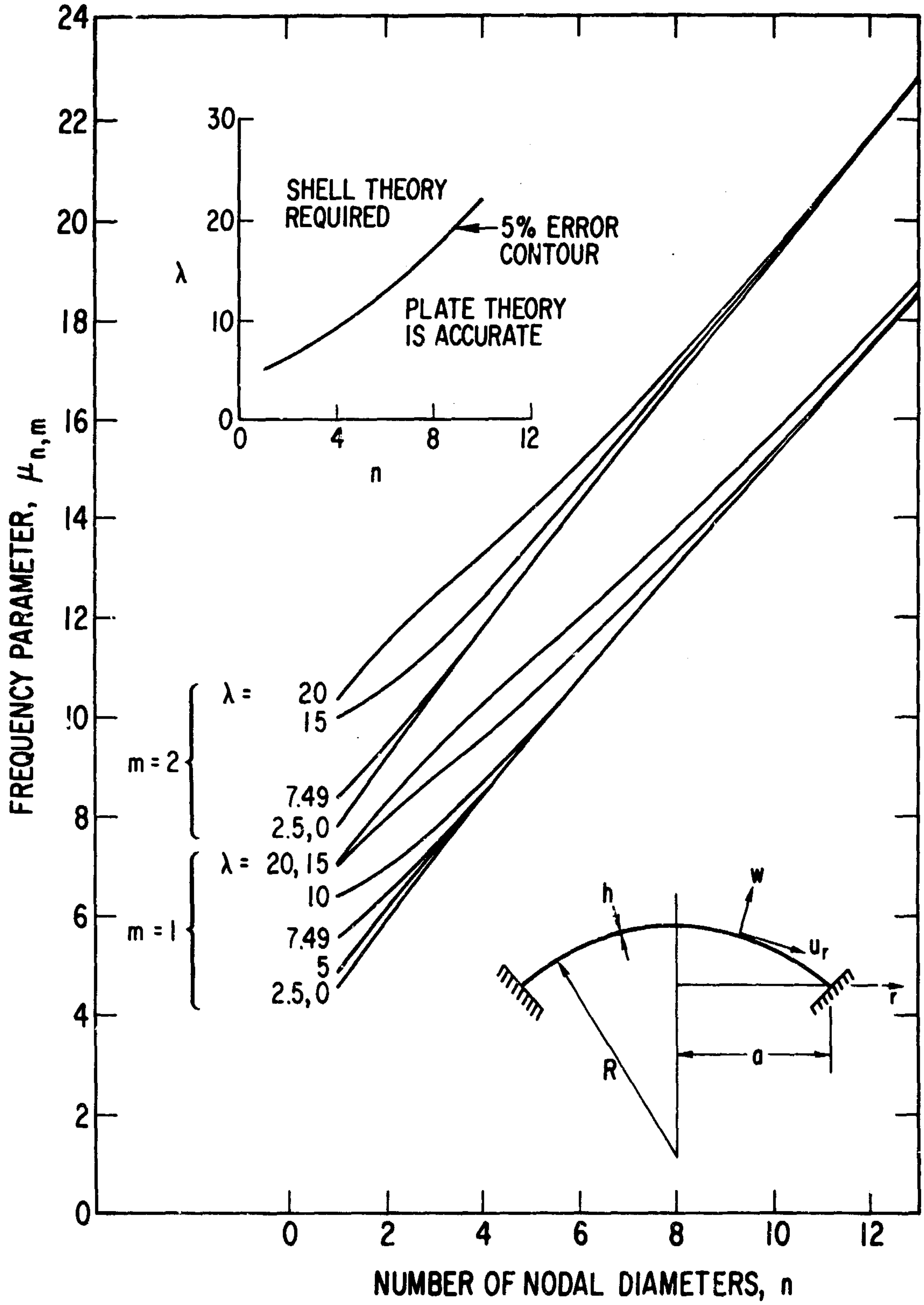

Figure 1. Frequency Parameter $\mu_{n,m}$; $\nu = 0.3$

that the $\mu_{n,m}$ become large as n is increased. When the ratio μ/λ has become sufficiently large, the frequency equation (7) takes the following approximate form

$$I_n(\mu)\, J_{n-1}(\mu) - I_{n-1}(\mu)\, J_n(\mu) = 0 \tag{8}$$

which is independent of the parameter λ. This equation is the frequency equation for the transverse vibrations of a clamped circular plate [4]. The roots of this equation are presented in Table 1 (see the $\lambda = 0$ column) and are also shown in Fig. 1. A comparison of the results from equations (7) and (8) shows that the fundamental roots differ by 5% or less for the region of the λ-n plane indicated in the inset of Fig. 1. The region of agreement appears to be larger for the higher modes ($m = 2, 3$). Thus the roots of the plate equation provide an excellent approximation to the results for the shell over a considerable range.

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