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HYDRODYNAMIC FORCES AND MOMENTS ACTING ON A SLENDER BODY OF REVOLUTION MOVING UNDER A REGULAR TRAIN OF WAVES

by

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Introduction

The author, not being a Hydrodynamicist, was much impressed by the difficulty experienced in finding procedures for computing or approximating hydrodynamic forces, on bodies, generated by waves. In particular, knowledge was desired concerning the hydrodynamic forces and moments acting on a body of revolution moving under waves. Very few papers were found which attempted these ends. Some of these are given in the selected bibliography. Most of the methods were valid for, or developed for, only very restricted cases such as balls or motion parallel to a mean surface, etc.

Attention finally centered on a paper by Cummins [1]. However, the exposition was found to be lacking, for our purposes, in the following respects:

1) restriction to Potential Flow

2) restriction to slender bodies of revolution

3) restriction to motion parallel to the mean free surface

4) restriction to constant velocity along an axis of the body.

We shall consider these points individually.

The restriction to potential flow is most easily treated. If anyone has done something similar for non-potential flows, we have not learned, as yet, of his work.

No mention need be made here concerning the limitation to treatment of slender bodies of revolution, except to remark that with the mapping techniques, e.g. Miles [2], available, this restriction is not as great as it first appears.

The restriction to motion parallel to the mean surface is, we think, removed by this paper. Even so, the orientation of the body must remain constant. This is not too hard to live with since this is just what would be required for numerical integration. That is, if we are integrating numerically with respect to time, say, then over a small time interval we would hold such things as orientation and velocity constant.

The restriction to constant velocity along an axis of the body is not needed. We shall still take the velocity as constant, but will not restrict it to being along an axis.

The present report, then, can not claim to be anything more than a slight extension of Cummin's results to a situation which may be useful in computer applications.

The organization and notation of this report is essentially that of Cummins. This should make comparison easy.

It is recommended that this report only be read if the reader has available a copy of [1].

Coordinate Systems

It will be necessary to consider two systems of coordinates.

The first system, considered as stationary will be called the earth system. It will be right-handed and orthogonal with its origin at the mean free surface of the fluid. The positive direction of the z-axis is to be upward (out of the fluid) and is assumed parallel to the local gravity vector. The fixed axis will be denoted by x, y, z.

The second set of coordinates is also to be right-handed and orthogonal. This will be called the body system and will be aligned in the body, fixed with respect to the body. The axis will be denoted by x', y', z'.

The Fluid and Its Boundary Conditions

We consider a free surface disturbed by a regular train of waves. As a further assumption we consider a linear, irrotational wave theory for deep water. Such a theory might, for example, be specified by

$$\nabla^2 \varphi = 0$$

$$\frac{p}{\rho} = g(z + \eta) + \varphi_t + \text{constant}$$

$$-\varphi_z \Big|_{z=0} = \eta_t$$

$$\varphi_z = 0 \quad \text{on the bottom}$$

$$\eta = -\frac{1}{g}\varphi_t \text{ at } z = 0 \quad \text{in which } \eta \text{ is the surface elevation.}$$

The linear irrotational wave theory for deep water is satisfied by the velocity potential (earth system)

$$\varphi_w = \frac{hc}{2} e^{\frac{2\pi z}{\lambda}} \cos \psi$$

$$\psi = \frac{2\pi}{\lambda}(x \cos \beta + y \sin \beta - ct + \alpha) \quad \text{in which}$$

λ = wavelength

h = waveheight (crest to trough)

c = celerity

α = phase angle

β = direction of propagation as measured from the x-axis.

The fluid velocities are then given as

$$u = -\varphi_x = \frac{\pi h c}{\lambda} \cos \beta \, e^{\frac{2\pi z}{\lambda}} \sin \psi$$

$$v = -\varphi_y = \frac{\pi h c}{\lambda} \sin \beta \, e^{\frac{2\pi z}{\lambda}} \sin \psi$$

$$w = -\varphi_z = -\frac{\pi h c}{\lambda} e^{\frac{2\pi z}{\lambda}} \cos \psi$$

For a body moving toward the surface at a constant velocity $\underline{V}$ we obtain the following relation between coordinates

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \Theta_{EB} \left[\begin{pmatrix} x \\ y \\ z \end{pmatrix} - \underline{V}t \right] \qquad \text{in which}$$

Θ_{EB} is the matrix of direction cosines relating the systems. Furthermore

$$\begin{pmatrix} u' \\ v' \\ w' \end{pmatrix} = O_{EB} \left[\begin{pmatrix} u \\ v \\ w \end{pmatrix} - \underline{V} \right]$$

In the following we shall treat O_{EB} as a constant; and for convenience we set,

$$O_{EB}^{-1} = O_{BE} = (a_{ij})$$

$$\underline{V} = \begin{pmatrix} u_m \\ v_m \\ w_m \end{pmatrix}$$

$$\underline{P} = \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}$$

$$\gamma = \frac{hc}{2}$$

$$\delta = \frac{2\pi}{\lambda}$$

The potential function, with respect to the moving system, is then

$\varphi'_w = \varphi_w + (O_{BE}\underline{P}) \cdot \underline{V}$ which may be written

$$\varphi'_w = \gamma e^{\delta\left[a_{31}x'+a_{32}y'+a_{33}z'+w_m t\right]} \cos \delta \left\{\left[a_{11}x'+a_{12}y'+a_{13}z'\right.\right.$$

$$\left.\left.+u_m t\right]\cos \beta + \left[a_{21}x'+a_{22}y'+a_{23}z'+v_m t\right]\sin \beta - ct + \alpha\right\}$$

$$+ (O_{BE}\underline{P}) \cdot \underline{V}$$

Hence we find that

(1) $$u' = -(\varphi'_w)_{x'} = -(\varphi_w)_{x'} - \left[(O_{BE}\underline{P}) \cdot \underline{V}\right]_{x'}$$

$$= a_{11}(u-u_m) + a_{21}(v-v_m) + a_{31}(w-w_m)$$

(2) $$v' = -(\varphi'_w)_{y'} = -(\varphi_w)_{y'} - \left[(O_{BE}\underline{P}) \cdot \underline{V}\right]_{y'}$$

$$= a_{12}(u-u_m) + a_{22}(v-v_m) + a_{32}(w-w_m)$$

(3) $$w' = -(\varphi'_w)_{z'} = -(\varphi_w)_{z'} - \left[(O_{BE}\underline{P}) \cdot \underline{V}\right]_{z'}$$

$$= a_{13}(u-u_m) + a_{23}(v-v_m) + a_{33}(w-w_m)$$

Singularities

We shall represent the body in the same manner as in [1]. That is, we set

(4) $\mu = \mu_x i + \mu_y j + \mu_z k$ with

$$\mu_x = -\frac{1}{4} a^2 u'$$

$$\mu_y = -\frac{1}{2} a^2 v'$$

$$\mu_z = -\frac{1}{2} a^2 w' \quad .$$

Hydrodynamic Force

The Force is to be computed via:

(5) $dF = dF_\ell + dF_t$ in which

$$dF_\ell = -4\pi\rho \left(\mu_x \frac{\partial}{\partial x'} + \mu_y \frac{\partial}{\partial y'} + \mu_z \frac{\partial}{\partial z'}\right) \underset{\sim}{q}_w \, dx'$$

$$dF_t = -4\pi\rho \left(\frac{\partial \mu_x}{\partial t} i + \frac{\partial \mu_y}{\partial t} j + \frac{\partial \mu_z}{\partial t} k\right) dx'$$

The notation used is that of Cummins [1], see especially page 5 Eqn [19] - [21].

We have then the relation

$$u = \gamma\delta \cos\beta \, e^{\delta z} \sin\psi$$

$$v = \gamma\delta \sin\beta \, e^{\delta z} \sin\psi$$

$$w = -\gamma\delta \, e^{\delta z} \cos\psi$$

$$z = a_{31}x' + a_{32}y' + a_{33}z' + w_m t$$

$$\frac{\partial z}{\partial t} = w_m$$

$$\psi = \delta\left\{\left[a_{11}x'+a_{12}y'+a_{13}z'+u_m t\right] \cos\beta + \left[a_{21}x'+a_{22}y'+a_{23}z'+v_m t\right] \sin\beta - ct + \alpha\right\}$$

$$\frac{\partial\psi}{\partial t} = \delta\,(u_m \cos\beta + v_m \sin\beta - c)$$

$$u_t = w_m \delta u + \delta \cos\beta\,(u_m \cos\beta + v_m \sin\beta - c)(-w)$$

$$v_t = w_m \delta v + \delta \sin\beta\,(u_m \cos\beta + v_m \sin\beta - c)(-w)$$

$$w_t = w_m \delta w + \delta u_m u + \delta v_m v - \delta \frac{c}{\cos\beta}\, u$$

$$\frac{\partial}{\partial t}\mu_x = -\frac{1}{4} a^2 \frac{\partial}{\partial t}(u') = -\frac{a^2}{4}\frac{d}{dt}\left[a_{11}(u-u_m) + a_{21}(v-v_m) + a_{31}(w-w_m)\right]$$

$$= -\frac{a^2}{4}\left(a_{11}u_t + a_{21}v_t + a_{31}w_t\right)$$

$$\frac{\partial}{\partial t}\mu_y = -\frac{a^2}{2}\frac{\partial}{\partial t}(v') = -\frac{a^2}{2}\frac{d}{dt}\left[a_{12}(u-u_m) + a_{22}(v-v_m) + a_{32}(w-w_m)\right]$$

$$= -\frac{a^2}{2}\left(a_{12}u_t + a_{22}v_t + a_{32}w_t\right)$$

$$\frac{\partial}{\partial t}\mu_z = -\frac{a^2}{2}\frac{\partial}{\partial t}(w') = -\frac{a^2}{2}\frac{d}{dt}\left[a_{13}(u-u_m) + a_{23}(v-v_m) + a_{33}(w-w_m)\right]$$

$$= -\frac{a^2}{2}\left(a_{13}u_t + a_{23}v_t + a_{33}w_t\right).$$

We find, from these, that

$$(6) \quad \frac{\partial}{\partial t}\mu_x = -\frac{\delta a^2}{4}\left\{\left(a_{11}w_m + a_{31}u_m - \frac{ca_{31}}{\cos\beta}\right) u + (a_{21}w_m + a_{31}v_m)\, v + \left[a_{31}w_m - (a_{11}\cos\beta + a_{21}\sin\beta)(u_m \cos\beta + v_m \sin\beta - c)\right] w\right\}$$

(7) $$\frac{\partial}{\partial t}\mu_y = -\frac{\delta a^2}{2}\left\{\left(a_{12}w_m + a_{32}u_m - \frac{ca_{32}}{\cos\beta}\right)u + (a_{22}w_m + a_{32}v_m)\,v + \left[a_{32}w_m - (a_{12}\cos\beta + a_{22}\sin\beta)(u_m\cos\beta + v_m\sin\beta - c)\right]w\right\}$$

(8) $$\frac{\partial}{\partial t}\mu_z = -\frac{\delta a^2}{2}\left\{\left(a_{13}w_m + a_{33}u_m - \frac{ca_{33}}{\cos\beta}\right)u + (a_{23}w_m + a_{33}v_m)\,v + \left[a_{33}w_m - (a_{13}\cos\beta + a_{23}\sin\beta)(u_m\cos\beta + v_m\sin\beta - c)\right]w\right\}.$$

If, furthermore, we consider the relations

$$z_{x'} = a_{31}\;;\quad z_{y'} = a_{32}\;;\quad z_{z'} = a_{33}$$

$$\psi_{x'} = \delta\,(a_{11}\cos\beta + a_{21}\sin\beta)$$

$$\psi_{y'} = \delta\,(a_{12}\cos\beta + a_{22}\sin\beta)$$

$$\psi_{z'} = \delta\,(a_{13}\cos\beta + a_{23}\sin\beta)$$

$$\frac{\partial}{\partial ?}\,e^{\delta z}\cos\psi = z_?\delta\,e^{\delta z}\cos\psi - \psi_?\,e^{\delta z}\sin\psi$$

$$\frac{\partial}{\partial ?}\,e^{\delta z}\sin\psi = z_?\delta\,e^{\delta z}\sin\psi + \psi_?\,e^{\delta z}\cos\psi$$

$$\frac{\partial}{\partial ?}\,u = z_?\delta u - \psi_?\cos\beta\,w$$

$$\frac{\partial}{\partial ?}\,v = z_?\delta v - \psi_?\sin\beta\,w$$

$$\frac{\partial}{\partial ?}\,w = z_?\delta w + \psi_?\gamma\delta\,e^{\delta z}\sin\psi\,,$$

we find

(9) $u'_{x'} = \delta\left\{2a_{11}a_{31}u + 2a_{31}a_{21}v + \left[a_{31}^{2} - (a_{11}\cos\beta + a_{21}\sin\beta)^{2}\right] w\right\}$

(10) $u'_{y'} = \delta\left\{(a_{11}a_{32} + a_{31}a_{12})\, u + (a_{21}a_{32} + a_{31}a_{22})\, v + \left[a_{31}a_{32} - (a_{11}\cos\beta + a_{21}\sin\beta)(a_{12}\cos\beta + a_{22}\sin\beta)\right] w\right\}$

(11) $u'_{z'} = \delta\left\{(a_{11}a_{33} + a_{31}a_{13})\, u + (a_{21}a_{33} + a_{31}a_{23})\, v + \left[a_{31}a_{33} - (a_{11}\cos\beta + a_{21}\sin\beta)(a_{13}\cos\beta + a_{23}\sin\beta)\right] w\right\}$

(12) $v'_{x'} = \delta\left\{(a_{12}a_{31} + a_{32}a_{11})\, u + (a_{22}a_{31} + a_{32}a_{21})\, v + \left[a_{32}a_{31} - (a_{12}\cos\beta + a_{22}\sin\beta)(a_{11}\cos\beta + a_{21}\sin\beta)\right] w\right\}$

(13) $v'_{y'} = \delta\left\{2a_{12}a_{32}\, u + 2a_{22}a_{32}\, v + \left[a_{32}^{2} - (a_{12}\cos\beta + a_{22}\sin\beta)^{2}\right] w\right\}$

(14) $v'_{z'} = \delta\left\{(a_{12}a_{33} + a_{32}a_{13})\, u + (a_{22}a_{33} + a_{32}a_{23})\, v + \left[a_{32}a_{33} - (a_{12}\cos\beta + a_{22}\sin\beta)(a_{13}\cos\beta + a_{23}\sin\beta)\right] w\right\}$

(15) $w'_{x'} = \delta\left\{(a_{13}a_{31} + a_{33}a_{11})\, u + (a_{23}a_{31} + a_{33}a_{21})\, v + \left[a_{33}a_{31} - (a_{13}\cos\beta + a_{23}\sin\beta)(a_{11}\cos\beta + a_{21}\sin\beta)\right] w\right\}$

(16) $w'_{y'} = \delta\left\{(a_{13}a_{32} + a_{33}a_{12})\, u + (a_{23}a_{32} + a_{33}a_{22})\, v + \left[a_{33}a_{32} - (a_{13}\cos\beta + a_{23}\sin\beta)(a_{12}\cos\beta + a_{22}\sin\beta)\right] w\right\}$

(17) $w'_{z'} = \delta\left\{2a_{13}a_{33}\, u + 2a_{23}a_{33}\, v + \left[a_{33}^{2} - (a_{13}\cos\beta + a_{23}\sin\beta)^{2}\right] w\right\}$

We may write

$$dF_{\ell} = -4\pi\rho\left[\mu_x (\underline{q}w)_{x'} + \mu_y (\underline{q}w)_{y'} + \mu_z (\underline{q}w)_{z'}\right]dx', \text{ so that}$$

$$dF_{\ell,x} = -4\pi\rho\left[\mu_x \mu'_{x'} + \mu_y \mu'_{y'} + \mu_z \mu'_{z'}\right]dx'$$

$$= -4\pi\rho\Big\{-\frac{\delta a^2}{4}\left[a_{11}(u-u_m)+a_{21}(v-v_m)+a_{31}(w-w_m)\right]\Big[2a_{11}a_{31}u+2a_{31}a_{21}v$$

$$+\left(a_{31}^2-\left[a_{11}\cos\beta + a_{21}\sin\beta\right]^2\right)w\Big] - \frac{\delta a^2}{2}\Big[a_{12}(u-u_m)+a_{22}(v-v_m)$$

$$+a_{32}(w-w_m)\Big]\Big[(a_{11}a_{32}+a_{31}a_{12})u + (a_{21}a_{32}+a_{31}a_{22})v + \Big(a_{31}a_{32}-\Big[a_{11}\cos\beta$$

$$+a_{21}\sin\beta\Big]\left[a_{12}\cos\beta+a_{22}\sin\beta\right]\Big)w\Big] - \frac{\delta a^2}{2}\Big[a_{13}(u-u_m)+a_{23}(v-v_m)$$

$$+a_{33}(w-w_m)\Big]\Big[(a_{11}a_{33}+a_{31}a_{13})u+(a_{21}a_{33}+a_{31}a_{23})v + \Big(a_{31}a_{33}-\Big[a_{11}\cos\beta$$

$$+a_{21}\sin\beta\Big]\left[a_{13}\cos\beta+a_{23}\sin\beta\right]\Big)w\Big]\Big\}dx'$$

considering terms of first order only (neglect terms containing products of the variables). This may be reduced to

$$dF_{\ell,x} \cdot -4\pi\rho\Big\{\frac{\delta a^2}{4}(a_{11}u_m+a_{21}v_m+a_{31}w_m)\Big[2a_{11}a_{31}u+2a_{31}a_{21}v + \Big(a_{31}^2-\Big[a_{11}\cos\beta$$

$$+a_{21}\sin\beta\Big]^2\Big)w\Big] + \frac{\delta a^2}{2}(a_{12}u_m+a_{22}v_m+a_{32}w_m)\Big[(a_{11}a_{32}+a_{31}a_{12})u + (a_{21}a_{32}$$

$$+a_{31}a_{22})v + \Big(a_{31}a_{32}-\left[a_{11}\cos\beta+a_{21}\sin\beta\right]\left[a_{12}\cos\beta+a_{22}\sin\beta\right]\Big)w\Big]$$

$$+\frac{\delta a^2}{2}(a_{13}u_m+a_{23}v_m+a_{33}w_m)\Big[(a_{11}a_{33}+a_{31}a_{13})u+(a_{21}a_{33}+a_{31}a_{23})v$$

$$+\Big(a_{31}a_{33}-\left[a_{11}\cos\beta+a_{21}\sin\beta\right]\left[a_{13}\cos\beta+a_{23}\sin\beta\right]\Big)w\Big]\Big\}dx'$$

$$
\begin{aligned}
dF_{\ell,x} = -4\pi\rho\left(\frac{\delta a^2}{4}\right)\Bigg\{ & 2\Big[(a_{11}u_m+a_{21}v_m+a_{31}w_m)\,a_{11}a_{31}+(a_{12}u_m+a_{22}v_m+a_{32}w_m)(a_{11}a_{32} \\
& +a_{31}a_{12}) + (a_{13}u_m+a_{23}v_m+a_{33}w_m)(a_{11}a_{33}+a_{21}a_{13})\Big]\,u + 2\Big[(a_{11}u_m+a_{21}v_m \\
& +a_{31}w_m)(a_{31}a_{21}) + (a_{12}u_m+a_{22}v_m+a_{32}w_m)(a_{21}a_{32}+a_{31}a_{22}) + (a_{13}u_m+a_{23}v_m \\
& +a_{33}w_m)(a_{21}a_{33}+a_{31}a_{23})\Big]\,v + \Big[(a_{11}u_m+a_{21}v_m+a_{31}w_m)\Big(a_{31}^2-\big[a_{11}\cos\beta \\
& +a_{21}\sin\beta\big]^2\Big) + 2(a_{12}u_m+a_{22}v_m+a_{32}w_m)\Big(a_{31}a_{32}-\big[a_{11}\cos\beta+a_{21}\sin\beta\big]\big[a_{12}\cos\beta \\
& +a_{22}\sin\beta\big]\Big) + 2(a_{13}u_m+a_{23}v_m+a_{33}w_m)\Big(a_{31}a_{33}-\big[a_{11}\cos\beta+a_{21}\sin\beta\big]\big[a_{13}\cos\beta \\
& +a_{23}\sin\beta\big]\Big)\Big]\,w\Bigg\}\,dx'
\end{aligned}
$$

$$
\begin{aligned}
dF_{t,x} = -4\pi\rho\left(-\frac{\delta a^2}{4}\right)\Bigg\{ & \left(a_{11}w_m+a_{31}u_m-\frac{c\,a_{31}}{\cos\beta}\right)u + (a_{21}w_m+a_{31}v_m)\,v \\
& + \Big[a_{31}w_m-(a_{11}\cos\beta+a_{21}\sin\beta)(u_m\cos\beta+v_m\sin\beta-c)\Big]\,w\Bigg\}\,dx'
\end{aligned}
$$

(18)
$$
\begin{aligned}
dF_x = \pi\rho\delta a^2\Bigg\{ & \Bigg(a_{11}w_m+a_{31}u_m-\frac{c\,a_{31}}{\cos\beta}-2\Big[(a_{11}u_m+a_{21}v_m+a_{31}w_m)a_{11}a_{31}+(a_{12}u_m \\
& +a_{22}v_m+a_{32}w_m)(a_{11}a_{32}+a_{31}a_{12}) + (a_{13}u_m+a_{23}v_m+a_{33}w_m)(a_{11}a_{33}+a_{21}a_{13})\Big]\Bigg)u \\
& + \Bigg(a_{21}w_m+a_{31}v_m-2\Big[(a_{11}u_m+a_{21}v_m+a_{31}w_m)(a_{31}a_{21}) + (a_{12}u_m+a_{22}v_m \\
& +a_{32}w_m)(a_{21}a_{32}+a_{31}a_{22}) + (a_{13}u_m+a_{23}v_m+a_{33}w_m)(a_{21}a_{33}+a_{31}a_{23})\Big]\Bigg)\,v \\
& - \Bigg((a_{11}\cos\beta+a_{21}\sin\beta)(w_m\cos\beta+v_m\sin\beta-c) - \Big[(a_{11}u_m+a_{21}v_m+a_{31}w_m)\Big(a_{31}^2 \\
& -\big[a_{11}\cos\beta+a_{21}\sin\beta\big]^2\Big) + 2(a_{12}u_m+a_{22}v_m+a_{32}w_m)\Big(a_{31}a_{32}-\big[a_{11}\cos\beta \\
& +a_{21}\sin\beta\big]\big[a_{12}\cos\beta+a_{22}\sin\beta\big]\Big) + 2(a_{13}u_m+a_{23}v_m+a_{33}w_m)\Big(a_{31}a_{33} \\
& -\big[a_{11}\cos\beta+a_{21}\sin\beta\big]\big[a_{13}\cos\beta+a_{23}\sin\beta\big]\Big)\Big]\Bigg)\,w\Bigg\}\,dx'
\end{aligned}
$$

Proceeding similarly we find

$$
\begin{aligned}
(19)\quad dF_y = \pi\rho\delta a^2 \Bigg\{ \Bigg(& 2\Big[a_{12}w_m + a_{32}u_m - \frac{ca_{32}}{\cos\beta}\Big] - \Big[(a_{11}u_m + a_{21}v_m + a_{31}w_m)(a_{12}a_{31} + a_{32}a_{11}) \\
& + (a_{12}u_m + a_{22}v_m + a_{32}w_m)\, 4a_{12}a_{32} + 2(a_{13}u_m + a_{23}v_m + a_{33}w_m)(a_{12}a_{33} + a_{32}a_{13}) \Big] \Bigg) u \\
& + \Bigg(2\Big[a_{22}w_m + a_{32}v_m\Big] - \Big[(a_{11}u_m + a_{21}v_m + a_{31}w_m)(a_{22}a_{31} + a_{32}a_{21}) + (a_{12}u_m + a_{22}v_m \\
& + a_{32}w_m)(4a_{22}a_{32}) + 2(a_{13}u_m + a_{23}v_m + a_{33}w_m)(a_{22}a_{33} + a_{32}a_{23}) \Big] \Bigg) v + \Bigg(2\Big[a_{32}w_m \\
& - (a_{12}\cos\beta + a_{22}\sin\beta)(u_m\cos\beta + v_m\sin\beta - c) \Big] - \Big[(a_{11}u_m + a_{21}v_m + a_{31}w_m)\Big(a_{32}a_{31} \\
& - \big[a_{12}\cos\beta + a_{22}\sin\beta\big]\big[a_{11}\cos\beta + a_{21}\sin\beta\big] \Big) + 2(a_{12}u_m + a_{22}v_m + a_{32}w_m)\Big(a_{32}^2 \\
& - \big[a_{12}\cos\beta + a_{22}\sin\beta\big]^2 \Big) + 2(a_{13}u_m + a_{23}v_m + a_{33}w_m)\Big(a_{32}a_{33} - \big[a_{12}\cos\beta \\
& + a_{22}\sin\beta\big]\big[a_{13}\cos\beta + a_{23}\sin\beta\big] \Big) \Big] \Bigg) w \Bigg\}\, dx'
\end{aligned}
$$

$$
\begin{aligned}
(20)\quad dF_z = \pi\rho\delta a^2 \Bigg\{ \Bigg(& 2\Big[a_{13}w_m + a_{33}u_m - \frac{ca_{33}}{\cos\beta}\Big] - \Big[(a_{11}u_m + a_{21}v_m + a_{31}w_m)(a_{13}a_{31} + a_{33}a_{11}) \\
& + 2(a_{12}u_m + a_{22}v_m + a_{32}w_m)(a_{13}a_{32} + a_{33}a_{12}) + 2(a_{13}u_m + a_{23}v_m + a_{33}w_m)(2a_{13}a_{33}) \Big] \Bigg) u \\
& + \Bigg(2\Big[a_{23}w_m + a_{33}v_m\Big] - \Big[(a_{11}u_m + a_{21}v_m + a_{31}w_m)(a_{23}a_{31} + a_{33}a_{21}) + 2(a_{12}u_m + a_{22}v_m \\
& + a_{32}w_m)(a_{23}a_{32} + a_{33}a_{22}) + 2(a_{13}u_m + a_{23}v_m + a_{33}w_m)(2a_{23}a_{33}) \Big] \Bigg) v + \Bigg(2\Big[a_{33}w_m \\
& - (a_{13}\cos\beta + a_{23}\sin\beta)(u_m\cos\beta + v_m\sin\beta - c) \Big] - \Big[(a_{11}u_m + a_{21}v_m + a_{31}w_m)\Big(a_{33}a_{31} \\
& - \big[a_{13}\cos\beta + a_{23}\sin\beta\big]\big[a_{11}\cos\beta + a_{21}\sin\beta\big] \Big) + (a_{12}u_m + a_{22}v_m + a_{32}w_m)\Big(a_{33}a_{32} \\
& - \big[a_{13}\cos\beta + a_{23}\sin\beta\big]\big[a_{12}\cos\beta + a_{22}\sin\beta\big] + (a_{13}u_m + a_{23}v_m + a_{33}w_m)\Big(a_{33}^2 \\
& - \big[a_{13}\cos\beta + a_{23}\sin\beta\big]^2 \Big) \Big] w \Bigg\}\, dx'
\end{aligned}
$$

Let L_1 be the x-coordinate of the rear of the body, and let L_2 be that of the front. Then

$$F_x = \int_{L_1}^{L_2} dF_x dx'; \quad F_y = \int_{L_1}^{L_2} dF_y dx'; \quad F_x = \int_{L_1}^{L_2} dF_z dx'.$$

Now, in each of dF_x, dF_y, dF_z, the only terms which vary with x' are a, u, v, w. If we set $\pi a^2 = A$ then each of dF_x, dF_y, dF_z is of the form,

$$\rho\delta A\,(c_1 u + c_2 v + c_3 w)\,dx', \text{ in which } c_1, c_2, c_3 \text{ are constant.}$$

Hence, each of F_x, F_y, F_z is of the form

$$\rho\delta \left(c_1 \int_{L_1}^{L_2} Audx' + c_2 \int_{L_1}^{L_2} Avdx' + c_3 \int_{L_1}^{L_2} Awdx' \right).$$

This may be written as

$$\frac{h\pi c\rho\delta}{\lambda} \left[(c_1 \cos\beta + c_2 \sin\beta) \int_{L_1}^{L_2} A\, d^{\delta z} \sin\psi\, dx' - c_3 \int_{L_1}^{L_2} A\, e^{\delta z} \cos\psi\, dx' \right].$$

We may then write

$$
\begin{aligned}
(21) \quad F_x = \frac{\delta\rho\pi hc}{\lambda} \Bigg\{ \Bigg[\Bigg(& a_{11}w_m \cos\beta + a_{31}u_m \cos\beta - ca_{31} - 2\cos\beta \Big[(a_{11}u_m + a_{21}v_m + a_{31}w_m)(a_{11}a_{31}) \\
& + (a_{12}u_m + a_{22}v_m + a_{32}w_m)(a_{11}a_{32} + a_{31}a_{12}) + (a_{13}u_m + a_{23}v_m + a_{33}w_m)(a_{11}a_{33} + a_{21}a_{13}) \Big] \Bigg) \\
& + \Bigg(a_{21}w_m + a_{31}v_m - 2\Big[(a_{11}u_m + a_{21}v_m + a_{31}w_m)(a_{31}a_{21}) + (a_{12}u_m + a_{22}v_m + a_{32}w_m)(a_{21}a_{32} \\
& + a_{31}a_{22}) + (a_{13}u_m + a_{23}v_m + a_{33}w_m)(a_{21}a_{33} + a_{31}a_{23}) \Big] \Bigg) \sin\beta \Bigg] \int_{L_1}^{L_2} A\, e^{\delta z} \sin\psi\, dx' \\
& - \Bigg(\Big(\big[a_{11}\cos\beta + a_{21}\sin\beta \big] \big[w_m \cos\beta + v_m \sin\beta - c \big] - \Big[(a_{11}u_m + a_{21}v_m + a_{31}w_m) \Big(a_{31}^2 \\
& - \big[a_{11}\cos\beta + a_{21}\sin\beta \big]^2 \Big) + 2(a_{12}u_m + a_{22}v_m + a_{32}w_m) \Big(a_{31}a_{32} - \big[a_{11}\cos\beta \\
& + a_{21}\sin\beta \big] \big[a_{12}\cos\beta + a_{22}\sin\beta \big] \Big) + 2(a_{13}u_m + a_{23}v_m + a_{33}w_m) \Big(a_{31}a_{33} - \big[a_{11}\cos\beta \\
& + a_{21}\sin\beta \big] \big[a_{13}\cos\beta + a_{23}\sin\beta \big] \Big) \Big] \Big) \Bigg) \int_{L_1}^{L_2} A\, e^{\delta z} \cos\psi\, dx' \Bigg\}
\end{aligned}
$$

$$
\begin{aligned}
(22)\quad F_y = \frac{\delta\rho\pi hc}{\lambda} \Bigg\{ \Bigg[\Bigg(& 2\big[a_{12}w_m\cos\beta + a_{32}u_m\cos\beta - ca_{32}\big] - \cos\beta\Big[(a_{11}u_m + a_{21}v_m + a_{31}w_m)(a_{12}a_{31} \\
& + a_{32}a_{11}) + (a_{12}u_m + a_{22}v_m + a_{32}w_m)\, 4a_{12}a_{32} + 2(a_{13}u_m + a_{23}v_m + a_{33}w_m)(a_{12}a_{33} + a_{32} + a_{13})\Big]\Bigg) \\
& + \Bigg(2\big[a_{22}w_m + a_{32}v_m\big] - \Big[(a_{11}u_m + a_{21}v_m + a_{31}w_m)(a_{22}a_{31} + a_{32}a_{21}) + (a_{12}u_m + a_{22}v_m \\
& + a_{32}w_m)(4a_{22}a_{32}) + 2(a_{13}u_m + a_{23}v_m + a_{33}w_m)(a_{22}a_{33} + a_{32}a_{23})\Big]\Bigg)\sin\beta\Bigg] \int_{L_1}^{L_2} A\, e^{\delta z}\sin\psi\, dx' \\
& - \Bigg(2\Big[a_{32}w_m - (a_{12}\cos\beta + a_{22}\sin\beta)(u_m\cos\beta + v_m\sin\beta - c)\Big] - \Big[(a_{11}u_m + a_{21}v_m \\
& + a_{31}w_m)\Big(a_{32}a_{31} - \big[a_{12}\cos\beta + a_{22}\sin\beta\big]\big[a_{11}\cos\beta + a_{21}\sin\beta\big]\Big) + 2(a_{12}u_m + a_{22}v_m \\
& + a_{32}w_m)\Big(a_{32}^2 - \big[a_{12}\cos\beta + a_{22}\sin\beta\big]^2\Big) + 2(a_{13}u_m + a_{23}v_m + a_{33}w_m)\Big(a_{32}a_{33} - \big[a_{11}\cos\beta \\
& + a_{22}\sin\beta\big]\big[a_{13}\cos\beta + a_{23}\sin\beta\big]\Big)\Big]\Bigg) \int_{L_1}^{L_2} A\, e^{\delta z}\cos\psi\, dx' \Bigg\}
\end{aligned}
$$

$$
\begin{aligned}
(23)\quad F_z = \frac{\delta\rho\pi hc}{\lambda} \Bigg\{ \Bigg[\Bigg(& 2\big[a_{13}w_m\cos\beta + a_{33}u_m\cos\beta - ca_{33}\big] - \cos\beta\Big[(a_{11}u_m + a_{21}v_m + a_{31}w_m)(a_{13}a_{31} \\
& + a_{33}a_{11}) + 2(a_{12}u_m + a_{22}v_m + a_{32}w_m)(a_{13}a_{32} + a_{33}a_{12}) + 2(a_{13}u_m + a_{23}v_m + a_{33}w_m)(2a_{13}a_{33})\Big]\Bigg) \\
& + \Bigg(2\big[a_{23}w_m + a_{33}v_m\big] - \Big[(a_{11}u_m + a_{21}v_m + a_{31}w_m)(a_{23}a_{31} + a_{33}a_{21}) + 2(a_{12}u_m + a_{22}v_m \\
& + a_{32}w_m)(a_{23}a_{32} + a_{33}a_{22}) + 2(a_{13}u_m + a_{23}v_m + a_{33}w_m)(2a_{23}a_{33})\Big]\Bigg)\sin\beta\Bigg] \int_{L_1}^{L_2} A\, e^{\delta z}\sin\psi\, dx' \\
& - \Bigg(2\Big[a_{33}w_m - (a_{13}\cos\beta + a_{23}\sin\beta)(u_m\cos\beta + v_m\sin\beta - c\Big] - \Big[(a_{11}u_m + a_{21}v_m + a_{31}w_m)\Big(a_{33}a_{31} \\
& - \big[a_{13}\cos\beta + a_{23}\sin\beta\big]\big[a_{11}\cos\beta + a_{21}\sin\beta\big]\Big) + (a_{12}u_m + a_{22}v_m + a_{32}w_m)\Big(a_{33}a_{32} \\
& - \big[a_{13}\cos\beta + a_{23}\sin\beta\big]\big[a_{12}\cos\beta + a_{22}\sin\beta\big]\Big) + (a_{13}u_m + a_{23}v_m + a_{33}w_m)\Big(a_{33}^2 \\
& - \big[a_{13}\cos\beta + a_{23}\sin\beta\big]^2\Big)\Big]\Bigg) \int_{L_1}^{L_2} A\, e^{\delta z}\sin\psi\, dx' \Bigg\}.
\end{aligned}
$$

Of course, we must remember when evaluating the integrals that z, ψ, A are functions of x'.

Hydrodynamic Moment

The analysis as done by Cummins, from Eq. [40] on page 7 to Eq. [59] on page 10 of Ref. [1] is carried over with but little change. For convenience we shall write some of the pertinent formulae.

(24) $$M = M_{\ell} + M_t$$

(25) $$M_{\ell} = \sum_i (r \times F_{\ell}) + 4\pi\rho \sum_i (q_w \times A)_i$$

(26) $$M_t = \rho \frac{d}{dt} \int_S \Phi (r \times n)\, d\sigma$$

To evaluate the surface integral we proceed as in Cummins. Corresponding to his Eq. [48] we have

(27) $$dM_t = \rho a x' \left[\frac{d}{dt} \int_0^{2\pi} (\varphi_w' + \varphi_s)(j \sin\theta - k \cos\theta)\, d\theta \right] dx'$$

Writing

$$\varphi_w' (x', a \cos\theta, a \sin\theta) = \varphi_w'(x', 0, 0) - a(v' \cos\theta + w' \sin\theta)$$

and

$$\varphi_s = -a(v' \cos\theta + w' \sin\theta)$$, as in Cummins,

we obtain

(28) $$dM_t = -2\pi\rho a^2 x' \left[\frac{\partial}{\partial t} (w')\, j - \frac{\partial}{\partial t} (v')\, k \right] dx' .$$

We may write

(29) $$dM_{\ell} = r \times dF_{\ell} + 4\pi\rho\, (q_w \times \mu)\, dx'.$$

This is

(30) $dM_{\ell} = (-x'dF_{\ell,z} + \pi\rho a^2 u'w'dx')j + (x'dF_{\ell,y} - \pi\rho a^2 u'v'dx')k,$ since

$q_w \times u = \frac{1}{L} a^2 u'(w'j - v'k)$ and the terms $z'dF_{\ell,x}$, $-y'dF_{\ell,x}$ do not appear because the legally moment is, in this case, evaluted along the x'-axis.

Then

(31) $$dM = \left(-x'dF_{\ell,z} + \pi\rho a^2 u'w'dx - 2\pi\rho a^2 x' \frac{\partial}{\partial t}(w')\, dx'\right)j$$
$$+ \left(x'dF_{\ell,y} - \pi\rho a^2 u'v'dx' + 2\pi\rho a^2 x' \frac{\partial}{\partial t}(v')\, dx'\right)k$$

We shall write this as

(32) $dM = dM_y j + dM_z k$.

Then we find that

$$dM_y = \pi\rho a^2 \Big\{u'w' - x'(u'w'_{x'} + 2v'w'_{y'} + 2w'w'_{z'} + 2a_{13}u_t + 2a_{23}v_t + 2a_{33}w_t)\Big\} dx'$$

$$dM_z = \pi\rho a^2 \Big\{x'(u'v'_{x'} + 2v'v'_{y'} + 2w'v'_{z'} + 2a_{12}u_t + 2a_{22}v_t + 2a_{23}w_t) - u'v'\Big\} dx'$$

If as a first approximation we take

$$u'w' = -(a_{11}u_m + a_{21}v_m + a_{31}w_m)\, w' - (a_{13}u_m + a_{23}v_m + a_{33}w_m)\, u'$$

$$u'v' = -(a_{11}u_m + a_{21}v_m + a_{31}w_m)\, v' - (a_{12}u_m + a_{22}v_m + a_{32}w_m)\, u'$$

$$u'w'_{x'} = -(a_{11}u_m + a_{21}v_m + a_{31}w_m)\, w'_{x'}$$

$$v'w'_{y'} = -(a_{12}u_m + a_{22}v_m + a_{32}w_m)\, w'_{y'} \quad , \text{etc.},$$

we find that

$$(33)\qquad dM_y = \pi\rho a^2 \Big\{ x'\Big[(a_{11}u_m+a_{21}v_m+a_{31}w_m)w'_{x'}+ 2(a_{12}u_m+a_{22}v_m+a_{32}w_m)\, w'_{y'}$$
$$+ 2(a_{13}u_m+a_{23}v_m+a_{33}w_m)\, w'_{z'} - 2a_{13}u_t - 2a_{23}v_t - 2a_{33}w_t\Big]$$
$$- (a_{13}u_m+a_{23}v_m+a_{33}w_m)\, u' - (a_{11}u_m+a_{21}v_m+a_{31}w_m)\, w\Big\}\, dx'$$

$$(34)\qquad dM_z = \pi\rho a^2 \Big\{ (a_{11}u_m+a_{21}v_m+a_{31}w_m)\, v' + (a_{12}u_m+a_{22}v_m+a_{32}w_m)\, u'$$
$$- x'\Big[(a_{11}u_m+a_{21}v_m+a_{31}w_m)\, v'_{x'} + 2(a_{12}u_m+a_{22}v_m+a_{32}w_m)\, v'_{y'}$$
$$+ 2(a_{13}u_m+a_{23}v_m+a_{33}w_m)\, v'_{z'} - 2a_{12}u_t + 2a_{22}v_t + 2a_{32}w_t\Big\}\, dx' .$$

A straightforward, but messy, substitution for the primed variables will show that each of dM_y, dM_z is of the form

$$\pi\rho a^2 \Big[(c_1 + c_2 x')u + (c_3 + c_4 x')v + (c_5 + c_6 x')\, w \Big]\, dx'$$

in which c_1, c_2, c_3, c_4, c_5, c_6 are constants. On putting $\pi a^2 = A$ and integrating, we find that dMy and dMz have the form

$$\rho\delta_\gamma \Big[(c_1 \cos\beta + c_3 \sin\beta) \int_{L_1}^{L_2} A\, e^{\delta z} \sin\psi\, dx' + (c_2 \cos\beta$$
$$+ c_4 \sin\beta) \int_{L_1}^{L_2} Ax'\, e^{\delta z} \sin\psi\, dx - \int_{L_1}^{L_2} (c_5$$
$$+ c_6\, x')\, A\, e^{\delta z} \cos\psi\, dx' \Big] .$$

The results of the integration are

$$
\begin{aligned}
(35)\quad M_y = \frac{\pi\rho hc}{\lambda} \Big\{ & - \Big[(a_{11}u_m + a_{21}v_m + a_{31}w_m)(a_{13}\cos\beta + a_{23}\sin\beta) + (a_{13}u_m + a_{23}v_m \\
& + a_{33}w_m)(a_{11}\cos\beta + a_{21}\sin\beta) \Big] \int_{L_1}^{L_2} A\, e^{\frac{2\pi z}{\lambda}} \sin\psi \, dx + \frac{2\pi}{\lambda} \Big[(a_{11}u_m + a_{21}v_m \\
& + a_{31}w_m) \big([a_{13}a_{31} + a_{33}a_{11}] \cos\beta + [a_{23}a_{31} + a_{33}a_{21}] \sin\beta \big) + 2(a_{12}u_m + a_{22}v_m \\
& + a_{32}w_m) \big([a_{13}a_{32} + a_{33}a_{12}] \cos\beta + [a_{23}a_{32} + a_{33}a_{22}] \sin\beta \big) + 4(a_{13}u_m + a_{23}v_m \\
& + a_{33}w_m)(a_{13}a_{33}\cos\beta + a_{23}a_{33}\sin\beta) - 2w_m(a_{13}\cos\beta + a_{23}\sin\beta) - 2(a_{33}u_m\cos\beta \\
& - ca_{33} + a_{33}v_m\sin\beta) \Big] \int_{L_1}^{L_2} Ax'\, e^{\frac{2\pi z}{\lambda}} \sin\psi \, dx' + \Big[(a_{11}u_m + a_{21}v_m + a_{31}w_m) a_{33} \\
& + (a_{13}u_m + a_{23}v_m + a_{33}w_m)\, a_{31} \Big] \int_{L_1}^{L_2} A\, e^{\frac{2\pi z}{\lambda}} \cos\psi \, dx' + \frac{2\pi}{\lambda} \Big[(a_{11}u_m + a_{21}v_m \\
& + a_{31}w_m) \big(a_{33}a_{31} - [a_{13}\cos\beta + a_{23}\sin\beta][a_{11}\cos\beta + a_{21}\sin\beta] \big) + 2(a_{12}u_m \\
& + a_{22}v_m + a_{32}w_m) \big(a_{33}a_{32} - [a_{13}\cos\beta + a_{23}\sin\beta][a_{12}\cos\beta + a_{22}\sin\beta] \big) \\
& + 2(a_{13}u_m + a_{23}v_m + a_{33}w_m) \big(a_{33}^2 - [a_{13}\cos\beta + a_{23}\sin\beta]^2 \big) + \\
& + 2a_{13}\cos\beta(u_m\cos\beta + v_m\sin\beta - c) - 2a_{33}w_m + 2a_{23}\sin\beta(u_m\cos\beta \\
& + v_m\sin\beta - c) \Big] \int_{L_1}^{L_2} Ax'\, e^{\frac{2\pi z}{\lambda}} \cos\psi \, dx' \Big\}
\end{aligned}
$$

(36)

$$
\begin{aligned}
M_z = \frac{\pi\rho hc}{\lambda} \Bigg\{ \Bigg[& (a_{11}u_m + a_{21}v_m + a_{31}w_m)(a_{12}\cos\beta + a_{22}\sin\beta) + (a_{21}u_m + a_{22}v_m \\
& + a_{33}w_m)(a_{11}\cos\beta + a_{21}\sin\beta)\Bigg] \int_{L_1}^{L_2} A\, e^{\frac{2\pi z}{\lambda}} \sin\psi \, dx' - \frac{2\pi}{\lambda}\Bigg[(a_{11}u_m \\
& + a_{21}v_m + a_{31}w_m)\Big(\big[a_{12}a_{31} + a_{32}a_{11}\big]\cos\beta + \big[a_{22}a_{31} + a_{32}a_{21}\big]\sin\beta\Big) \\
& + 4(a_{12}u_m + a_{22}v_m + a_{32}w_m)(a_{12}a_{32}\cos\beta + a_{22}a_{32}\sin\beta) + 2(a_{13}u_m + a_{23}v_m \\
& + a_{33}w_m)\Big(\big[a_{12}a_{33} + a_{32}a_{13}\big]\cos\beta + \big[a_{22}a_{33} + a_{32}a_{23}\big]\sin\beta\Big) - 2w_m(a_{12}\cos\beta \\
& + a_{22}\sin\beta) - 2a_{32}(u_m\cos\beta - c) - 2v_m a_{32}\sin\beta\Bigg] \int_{L_1}^{L_2} Ax'\, e^{\frac{2\pi z}{\lambda}} \sin\psi \, dx' \\
& - \Big[(a_{11}u_m + a_{21}v_m + a_{31}w_m)\, a_{32} + (a_{21}u_m + a_{22}v_m + a_{32}w_m)\, a_{31}\Big] \int_{L_1}^{L_2} A\, e^{\frac{2\pi z}{\lambda}} \cos\psi \, dx' \\
& + \frac{2\pi}{\lambda}\Bigg[(a_{11}u_m + a_{21}v_m + a_{31}w_m)\Big(a_{32}a_{31} - \big[a_{12}\cos\beta + a_{22}\sin\beta\big]\big[a_{11}\cos\beta \\
& + a_{21}\sin\beta\big]\Big) + 2(a_{21}u_m + a_{22}v_m + a_{32}w_m)\Big(a_{32}^2 - \big[a_{12}\cos\beta + a_{22}\sin\beta\big]^2\Big) \\
& + 2(a_{31}u_m + a_{32}v_m + a_{33}w_m)\Big(a_{32}a_{33} - \big[a_{12}\cos\beta + a_{22}\sin\beta\big]\big[a_{13}\cos\beta + a_{23}\sin\beta\big]\Big) \\
& + 2(a_{12}\cos\beta + a_{22}\sin\beta)(u_m\cos\beta + v_m\sin\beta - c) - 2a_{32}w_m\Bigg] \int_{L_1}^{L_2} Ax'\, e^{\frac{2\pi z}{\lambda}} \cos\psi \, dx' \Bigg\}
\end{aligned}
$$

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