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Behram Kurşunoğlu

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Center For Theoretical Studies
University of Miami, Coral Gables, Florida 33124

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UNITARY REPRESENTATIONS OF U(2,2) AND MASSLESS FIELDS*

Behram Kurşunoğlu
Center for Theoretical Studies
University of Miami
Coral Gables, Florida

This paper contains a discussion of unitary irreducible representations of the group U(2,2) in terms of the non-compact algebra of creation and annihilation operators and some applications to massless fields. In particular, the U(2,2) algebra yields discrete values for p_4 (energy), one of its generators. The little group and wave equations of massless fields are also derived from the Lie algebra of U(2,2).

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I. INTRODUCTION

This paper is a contribution to the explosion of group theoretical publications pertaining to elementary particle concepts. The present state of theoretical research on elementary particles seems to indicate that there exist ever increasing possibilities for the so-called "classification" of particles. Recent attempts(1) for the unification of internal and space-time

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1 Proceedings of the First Coral Gables Conference on SYMMETRY PRINCIPLES AT HIGH ENERGY, January 1964, Florida (W.H. Freeman and Company, San Francisco, 1964). See also Phys. Rev. 135, 761 (1964).

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symmetries into a single group theoretical structure aiming at an hypothesis of simultaneous charge, hypercharge, and spin independence of strong interactions (at high energy) have led to further discussions of the subject by others(2). These authors have shown that there are some basic difficulties in the models proposed earlier(1).

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2 W.D. McGlinn, Phys. Rev. Letters 12, 467 (1964)

F. Coester, M. Hamermesh and W.D. McGlinn, Phys. Rev. 135B, 451 (1964)

H. Bacry and J. Nuyts, Physics Letters 12, 2, 156 (1964)

M.E. Mayer, H.S. Schnitzer, E.C.G. Sudarshan, R. Acharya, M.Y. Han, Phys. Rev. 136 B, 888 (1964)

A. Beskow and U. Ottoson, Nuovo Cimento XXXIV. 1, 248 (1964)

In particular, if one adheres to the existing interpretations of the isotopic spin, then spin and isotopic spin assignments to various generators of the group U(3,1) lead to non-commuting operators for the respective observables. Therefore what remains as acceptable is the product of two commuting groups i.e. the cover group is just the direct product of the Poincaré group with an internal symmetry group.

In the light of these investigations the fundamental issue appears to be the possible existence of a non-compact symmetry group containing several commuting "little groups" whose representations can provide enough quantum numbers to fit in all the "free" particles.

One of the subjects which is considered to be closed in the theory of representations of Poincaré group refers to massless states. In view of a great interest in the last few years, in the problem of masses of strongly interacting particles, it may not be a waste of time to further discuss the extreme situation: the massless state of matter in general (very high energies). It is hoped that a further understanding of "masslessness" may be exploited for the study of a more special case, the particles with mass.

We shall, as in the previous paper[1], use the techniques of creation and annihilation operators for the representation of the massless conformal group U(2.2). Our discussions will be confined only to unitary, irreducible representations.

II. REPRESENTATION OF U(2,2)

In order to establish the method we consider a special set of ten Hermitian operators satisfying the commutation relations for the inhomogeneous Lorentz group. These are given by p_μ (four translation operators), and by the relativistic definition of angular momenta[3],

$$R_{\mu\nu} = x_\mu p_\nu - x_\nu p_\mu = i \langle x| F\, M_{\mu\nu} |p\rangle \tag{II.1}$$

where

$$F = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad |x\rangle = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}, \quad |p\rangle = \begin{bmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \end{bmatrix} \tag{II.2}$$

and x_μ, p_ν ($\mu,\nu = 1,2,3,4$) are subject to commutation relations

$$[x_\mu, p_\nu] = -i\hbar g_{\mu\nu}, \; [x_\mu, x_\nu] = [p_\mu, p_\nu] = 0, \tag{II.3}$$

with $g_{\mu\nu}$ being the elements of F. Every Lorentz matrix L satisfies the condition

$$\tilde{L}\, F\, L = F, \tag{II.4}$$

where $\tilde{L}$ is the transposed form of L.

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[3] B. Kurşunoğlu, MODERN QUANTUM THEORY (W.H. Freeman and Company,

San Francisco, 1962). See page 254 eq. (VIII.8.3) also page 50 for the definition of the 4 x 4 matrices $M_{\mu\nu}$ which are generators of rotations and Lorentz transformations. The matrices $M_{\mu\nu}$ constitute a non-unitary representation of the homogeneous group. This book, in this paper, will be referred to as MQT.

The operators x_μ and p_μ under a Lorentz transformation transform according to

$$|x'\rangle = L|x\rangle, \quad |p'\rangle = L|p\rangle \ . \tag{II.5}$$

In a way similar to (II.1) we introduce complex creation and annihilation operators. For example, the Hermitian generators of the homogeneous Lorentz group can be represented by

$$J_{\mu\nu} = \frac{1}{2}\langle a|\beta\sigma_{\mu\nu}|a\rangle \tag{II.6}$$

where

$$\beta = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}, \quad |a\rangle = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix}, \quad \langle a| = [a_1^\dagger, a_2^\dagger, a_3^\dagger, a_4^\dagger], \tag{II.7}$$

and the operators a_α, $a_\rho^\dagger$ $(\alpha, \rho = 1,2,3,4)$ satisfy the commutation relations

$$[a_\alpha, a_\rho^\dagger] = \beta_{\alpha\rho}, \quad [a_\alpha, a_\rho] = [a_\alpha^\dagger, a_\rho^\dagger] = 0 \tag{II.8}$$

with β being taken as the "metric" of the 4-dimensional complex space. We are using a representation of γ's given by

$$\left[\gamma_\mu,\gamma_\nu\right]_+ = -2\, g_{\mu\nu}\, I_4,\ \gamma_4 = i\beta$$

$$\gamma_5 = \gamma_1\,\gamma_2\,\gamma_3\,\gamma_4 \quad ,$$

$$\Lambda_\pm = \frac{1}{2}\,(1 \pm i\,\gamma_5) \quad ,$$

$$\sigma_{\mu\nu} = -\frac{1}{2}\,i\,\left[\gamma_\mu,\ \gamma_\nu\right] \quad ,$$

$$\gamma_5\,\sigma_{\mu\nu} = \frac{1}{2}\,\varepsilon_{\mu\nu\alpha\beta}\,\sigma^{\alpha\beta} \quad ,$$

and

$$g_{11} = g_{22} = g_{33} = -\,g_{44} = 1,\ g_{j4} = g_{4j} = 0,\ g_{k\ell} = 0,\ k \neq \ell,$$

where γ_j ($j = 1,2,3$) are hermitian and γ_4 is anti-hermitian.

The corresponding commutation and anti-commutation relations are

$$\left[\frac{1}{2}\,\sigma_{\mu\nu},\ \frac{1}{2}\,\sigma_{\alpha\beta}\right] = \frac{1}{2}\,i\,(g_{\alpha\nu}\,\sigma_{\mu\beta} + g_{\beta\nu}\,\sigma_{\alpha\mu} - g_{\alpha\mu}\,\sigma_{\nu\beta} - g_{\mu\beta}\,\sigma_{\alpha\nu}), \tag{II.9}$$

$$\frac{1}{2}\left[\sigma_{\mu\nu},\ \sigma_{\alpha\beta}\right]_+ = -\gamma_5\,\varepsilon_{\mu\nu\alpha\beta} + g_{\alpha\mu}\,g_{\beta\nu} - g_{\mu\beta}\,g_{\alpha\nu} \quad , \tag{II.10}$$

$$\left[\sigma_{\mu\nu},\ \gamma_5\right] = 0 \quad , \tag{II.11}$$

$$\frac{1}{2}\left[\sigma_{\mu\nu},\ \gamma_\rho\right] = i\,(g_{\rho\nu}\,\gamma_\mu - g_{\rho\mu}\,\gamma_\nu) \ , \tag{II.12}$$

$$\frac{1}{2}\left[\sigma_{\mu\nu}, \gamma_\rho\right]_+ = -i\, \varepsilon_{\mu\nu\alpha\beta}\, \gamma^\beta \gamma_5 \quad . \tag{II.13}$$

From the isomorphism of the two representations (II.1) and (II.6) it follows that the transformation operator S, corresponding to a Lorentz transformation L, satisfies the condition

$$S^\dagger \beta S = \beta \tag{II.14}$$

in complex four dimensional space. The condition (II.14) is valid only for proper Lorentz transformations. For improper Lorentz transformations the right side of (II.14) should be replaced by $-\beta$. In this paper we shall not be concerned with the latter case[4]. Under a Lorentz transformation of the

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[4] See eq. (VIII.5.55) on page 240, and eqs. (VIII.8.21), (VIII.8.22) on page 257 of MQT. The eqs. (VIII.5.56) and (VIII.5.57) on page 241 of MQT are examples of S-transformations. The operators γ_5, $i\, \gamma_\mu$, $\gamma_5\gamma_\mu$ are also generators of S-transformations.

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generators $J_{\mu\nu}$, the operator column vector $|a\rangle$ transforms according to

$$|a'\rangle = S\, |a\rangle \quad . \tag{II.15}$$

The commutation relations (II.8) are invariant under S-transformations satisfying the condition (II.14).

A special type of S-transformations are gauge transformations of the type exp $(i\phi)$. Furthermore, from (II.14) it follows that the determinant of an S-transformation is defined up to a phase factor. Hence the group of S-transformations can be decomposed according to $U = U_1 \times S_o \times Z$, where U_1 is the one-dimensional unitary group and S_o is the group of S-transformations with determinant + 1. The factor Z is of the form exp $(\frac{1}{2}\, i\, \pi n)$, n = 1,2,..., representing an invariant S-transformation subgroup of fourth order whose members consist of $\pm$ 1 and $\pm$ i. This means that there are four types of vector operators a_α pertaining to the representations of the group U(2,2).

In terms of the operators a_α and $a_\alpha^\dagger$ the hermitian generators of U(2,2), for the positive energies, are given by

$$J_{\mu\nu} = \frac{1}{2} < a|\beta\ \sigma_{\mu\nu}\ |a> \tag{II.16}$$

$$p_\mu^+ = - < a|\gamma_4\ \Lambda_+\ \gamma_\mu\ |a> \tag{II.17}$$

$$p_\mu^- = - < a|\gamma_4\ \Lambda_-\ \gamma_\mu\ |a> \tag{II.18}$$

$$\Gamma = < a|\beta\ \gamma_5\ |a> \tag{II.19}$$

$$\Gamma = < a|\beta\ |a> \quad . \tag{II.20}$$

The 16 Hermitian operators as defined by (II.16) - (II.20) provide an irreducible unitary representation[5] of U(2,2).

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5 B. Kurşunoğlu, Proceedings of Second Coral Gables Conference

on Symmetry Principles at High Energy, Freeman and Co., San Francisco, 1965, page 163.

The commutation rules of U(2,2) are given by

$$\left[J_{\mu\nu},\ J_{\alpha\beta}\right] = i\ (g_{\alpha\nu}\ J_{\mu\beta} + g_{\beta\nu}\ J_{\alpha\mu} - g_{\alpha\mu}\ J_{\nu\beta} - g_{\mu\beta}\ J_{\alpha\nu}), \tag{II.21}$$

$$\left[J_{\mu\nu},\ p^{+}_{\rho}\right] = i\ (g_{\rho\nu}\ p^{+}_{\mu} - g_{\rho\mu}\ p^{+}_{\nu})\ , \tag{II.22}$$

$$\left[p^{+}_{\mu},\ p^{+}_{\nu}\right] = 0\ , \tag{II.23}$$

$$\left[J_{\mu\nu},\ p^{-}_{\rho}\right] = i\ (g_{\rho\nu}\ p^{-}_{\mu} - g_{\rho\mu}\ p^{-}_{\nu})\ , \tag{II.24}$$

$$\left[p^{-}_{\mu},\ p^{-}_{\nu}\right] = 0\ , \tag{II.25}$$

$$\left[p^{+}_{\mu},\ p^{-}_{\nu}\right] = 2i\ (g_{\mu\nu}\ \zeta - \tfrac{1}{2}\ J_{\mu\nu}), \tag{II.26}$$

$$\left[p^{+}_{\mu},\ \zeta\right] = 2\ i\ p^{+}_{\mu}\ , \tag{II.27}$$

$$\left[p^{-}_{\mu},\ \zeta\right] = -2\ i\ p^{-}_{\mu}\ , \tag{II.28}$$

$$\left[J_{\mu\nu},\ \zeta\right] = 0. \tag{II.29}$$

These are satisfied by (II.16) - (II.20).

The operator Γ commutes with all the rest of the generators. From the above commutation rules it is seen that the group U(2,2) contains the Poincaré group as its sub-group. The special representation (II.16) - (II.20) refers to massless case.

An invariant of U(2,2) is given by

$$m_I = \frac{1}{2} (p_\mu^+ p_-^\mu + p_\mu^- p_+^\mu + J_{\mu\nu} J^{\mu\nu}) - r^2 \quad . \qquad \text{(II.30)}$$

The invariants $I_1 = p_\mu p^\mu$, $I_2 = \frac{1}{2} J^{\mu\nu} J_{\mu\nu} p_\rho p^\rho - J_{\mu\rho} J^{\nu\rho} p_\nu p^\mu$

of the sub-group as can easily be shown (via (II.16) - (II.20)) vanish.

Now, from the definition (II.16) of $J_{\mu\nu}$ we obtain

$$J_\ell = J_{1\ell} + J_{2\ell} \quad , (\ell = 1,2,3) \qquad \text{(II.31)}$$

$$J_\ell = \frac{1}{2} \varepsilon_{\ell sk} J_{sk}$$

and

$$J_{1i} = \frac{1}{2} \langle A|\sigma_i|A\rangle \ , \quad J_{2i} = -\frac{1}{2} \langle B|\sigma_i|B\rangle \qquad \text{(II.32)}$$

where σ_i (i=1,2,3) are the usual Pauli matrices and

$$|A\rangle = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} , \qquad |B\rangle = \begin{bmatrix} a_3 \\ a_4 \end{bmatrix} .$$

They satisfy the commutation rules for the commuting angular momenta[6],

$$[J_{1i}, J_{2j}] = 0, \ [J_{1i}, J_{1j}] = i \ \varepsilon_{ij\ell} J_{1\ell}, \ [J_{2i}, J_{2j}] = i \ \varepsilon_{ij\ell} J_{2\ell} \ , \qquad \text{(II.33)}$$

where

$$J^2 = j(j+1), \; J_1^2 = j_1(j_1+1), \; J_2^2 = j_2(j_2+1), \qquad \text{(II.34)}$$

$$j_1 = \frac{1}{2}(a_1^\dagger a_1 + a_2^\dagger a_2), \quad j_2 = \frac{1}{2}(a_3 a_3^\dagger + a_4 a_4^\dagger), \qquad \text{(II.35)}$$

[6] The commutation rules (II.33) are the same as the commutation relations corresponding to the Lie algebra of the 4-dimensional Euclidean group, namely the group O_4.

Hence we see that the space part of $J_{\mu\nu}$ is decomposable into a direct product of two three-dimensional rotation groups. The resultant angular momentum j is associated with angular momenta

$$j = |j_1 - j_2|, \; |j_1 - j_2| + 1, \ldots, \; j_1 + j_2 \qquad \text{(II.36)}$$

where $|j_1 - j_2|$ (=s) is the minimum value of j, it is the spin quantum number of the representation assuming the values 0, $\frac{1}{2}$, 1,... .

From (II.17) and definitions of γ's (see page 235 of MQT) the translation operators p_μ can be written as

$$\begin{aligned} p_1 &= J_{1x} - J_{2x} + \frac{1}{2}(a_1^\dagger a_4 + a_4^\dagger a_1 + a_3^\dagger a_2 + a_2^\dagger a_3) \\ p_2 &= J_{1y} - J_{2y} - \frac{1}{2} i (a_1^\dagger a_4 - a_4^\dagger a_1 + a_3^\dagger a_2 - a_2^\dagger a_3) \\ p_3 &= J_{1z} - J_{2z} + \frac{1}{2}(a_3^\dagger a_1 + a_1^\dagger a_3 - a_4^\dagger a_2 - a_2^\dagger a_4) \end{aligned} \qquad \text{(II.37)}$$

$$p_4 = j_1 + j_2 + 1 + \frac{1}{2}\,(a_3^\dagger a_1 + a_1^\dagger a_3 + a_4^\dagger a_2 + a_2^\dagger a_4)$$

Using these definitions we can construct the helicity operator of massless particles in the form

$$\zeta_o = J\cdot\hat{p} = 1 + \frac{1}{2}\langle a|\beta|a\rangle = \frac{1}{2}\,(a_1^\dagger a_1 + a_2^\dagger a_2 - a_3^\dagger a_3 - a_4^\dagger a_4) + 1 \quad \text{(II.38)}$$

where

$$\hat{p} = \frac{p}{p_4} \quad \text{(II.39)}$$

and where $\frac{1}{2}\langle a|\beta|a\rangle$ commutes with the ten generators of the group and is therefore a group invariant. We shall consider only positive energy representations where the helicity operator ζ_o together with p_4, J^2, and J_3 form a complete commuting set. A set of simultaneous eigenstates of these commuting operators will be designated by $|n,\xi\rangle$. The requirement of non-negativity for j_1 and j_2 assures also positive sign for the energy and the former is obtained only by defining the vacuum state by the conditions

$$\begin{aligned} a_1 |0\rangle &= 0 & \qquad a_3^\dagger |0\rangle &= 0 \\ & & \text{and} & \\ a_2 |0\rangle &= 0 & \qquad a_4^\dagger |0\rangle &= 0 \; . \end{aligned} \quad \text{(II.40)}$$

In complete analogy with Fock representation of harmonic oscillator (see chapter 7 of MQT) we find that occupation number operators are given by

$$N_1 = a_1^\dagger a_1,\ N_2 = a_2^\dagger a_2,\ N_3 = a_3 a_3^\dagger,\ N_4 = a_4 a_4^\dagger \tag{II.41}$$

which satisfy the eigen-value equations

$$N_\alpha |n_\alpha\rangle = n_\alpha |n_\alpha\rangle \quad (\alpha = 1,2,3,4) \tag{II.42}$$

where

$$n_\alpha = 0,1,2,3,\ldots \quad .$$

The normalized eigenstates are defined by

$$\begin{aligned} |n_1\rangle &= \frac{1}{\sqrt{(n_1!)}} (a_1^\dagger)^{n_1} |0\rangle, \quad |n_2\rangle = \frac{1}{\sqrt{(n_2!)}} (a_2^\dagger)^{n_2} |0\rangle, \\ |n_3\rangle &= \frac{1}{\sqrt{(n_3!)}} (a_3)^{n_3} |0\rangle, \quad |n_4\rangle = \frac{1}{\sqrt{(n_4!)}} (a_4)^{n_4} |0\rangle, \end{aligned} \tag{II.43}$$

so that the simultaneous eigen-states $|n,\xi\rangle$ of the complete commuting set p_4, ζ_o, J^2, and J_3 are products of these eigen-states.

From (II.38) it follows that the helicity operator can be expressed in the form

$$J\cdot\hat{p} = \frac{1}{2}(N_1 + N_2 - N_3 - N_4) = \frac{1}{2} N \tag{II.44}$$

and it acts on the state $|n,\xi\rangle$ according to

$$J\cdot\hat{p}\ |n,\xi\rangle = \frac{1}{2} n |n,\xi\rangle$$

where

$$\frac{1}{2} n = \frac{1}{2} (n_1 + n_2 - n_3 - n_4) = j_1 - j_2 = \pm s$$

assumes both positive and negative half odd-integral and integral values including zero. Hence we can write

$$J \cdot \hat{p} \, |n,\xi\rangle = \pm s \, |n,\xi\rangle \quad . \tag{II.45}$$

The eigen-value equation can further be simplified by noting that it is equivalent to

$$J \cdot p \, |n_\tau, \xi\rangle = \tau_3 \, s \, p_4 | \, n_\tau, \xi\rangle \tag{II.46}$$

where

$$|n_\tau, \xi\rangle = \begin{bmatrix} |n_+, \xi\rangle \\ |n_-, \xi\rangle \end{bmatrix}, \qquad \tau_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} ,$$

$$J \cdot p \, |n_+, \xi\rangle = s \, p_4 |n_+, \xi\rangle, \quad |n_+, \xi\rangle = \frac{1}{2} (1 + \tau_3) |n_\tau, \xi\rangle$$

$$J \cdot p \, |n_-, \xi\rangle = -s \, p_4 |n_-, \xi\rangle \quad |n_-, \xi\rangle = \frac{1}{2} (1 + \tau_3) \, n_\tau, \xi\rangle$$

Hence, the most general state is a superposition of two orthogonal states

$$|n_\tau, \xi\rangle = |n_+, \xi\rangle + |n_-, \xi\rangle \tag{II.47}$$

referring either to two different states of polarization or to two different particles. It depends on the reflection symmetries of various spin states whether one has just a different state of polarization or a different particle. Two states of polarizations, whether they refer to identical particles (e.g. zeron $s = 0$, photons $s = 1$) or two different particle states (e.g. $\nu, \bar{\nu}$ with $s = \frac{1}{2}$) as eigen-states of τ_3, span a two-dimensional Hilbert space.

Finally we note from (II.37) and (II.43) that the diagonal element of the operator p_4 with respect to the state $|n,\xi\rangle$ is given by

$$\langle n,\xi|p_4|n,\xi\rangle = \frac{1}{2}(n_1 + n_2 + n_3 + n_4)+1 = \frac{1}{2}n + 1 \tag{II.48}$$

where $n = 0,1,2,\ldots$ so that zero point oscillations are also included in the algebra of $U(2,2)$.

III. WAVE EQUATIONS

As is well-known the group of translations, being an Abelian sub-group of the Poincaré group, has only one-dimensional irreducible, unitary representations. For a translation of states by a real vector $\underline{b}$ the unitary operator is $\exp[-i\, b^\mu p_\mu]$. This group of translations contains also the representations $\exp[-i\, b^\mu p'_\mu]$ provided p'_μ is obtained from p_μ by a proper Lorentz transformation

$$p'_\mu = L^\nu_\mu\, p_\nu \; .$$

An infinitesimal translation of a function of coordinates by an amount $\varepsilon\, b_\mu$ is represented by

$$\exp[i\varepsilon b^\mu p_\mu]\,\psi(x)\,\exp[-i\varepsilon b^\nu p_\mu] = (1+i\varepsilon b^\mu p_\mu)\psi(x)\,(1-i\varepsilon b^\nu p_\nu) =$$

$$\psi(x) + i\varepsilon b^\mu [p_\mu, \psi(x)] \quad .$$

Hence in the limit of $\varepsilon \to o$ we obtain

$$b^\mu \frac{\partial \psi}{\partial x^\mu} = i\, b^\mu [p_\mu,\ \psi(x)]$$

or, since this is valid for all b_μ, we have

$$-i \frac{\partial \psi}{\partial x^\mu} = [p_\mu,\ \psi] \ . \qquad \text{(III.1)}$$

Now consider the eigen-states $|r,t\rangle$ of the complete commuting set q

$$q\,|r,t\rangle = r\,|\,r,\ t\rangle \ . \qquad \text{(III.2)}$$

The translation operator will act according to

$$\exp[-i\varepsilon b^\mu p_\mu]|x\rangle = |x + \varepsilon b\rangle = (1 - i\varepsilon b^\mu p_\mu)|x\rangle \quad .$$

Hence, this being valid for every b, we get

$$p_\mu\ |x\rangle = -\, i\ \frac{\partial}{\partial x^\mu}\ |x\rangle \quad . \qquad \text{(III.3)}$$

A way of obtaining a wave equation may proceed by representing the state $|r,t\rangle$ in a Hilbert space spanned by $|n,\xi\rangle$. Thus writing

$$\langle r,t|n,\xi\rangle = |r,t,s\rangle$$

and regarding it as 2s+1 component wave function we can derive a wave equation.

From (II.45) we obtain

$$\langle r,t|J\cdot p|n,\xi\rangle = \pm\, \varepsilon\ \langle r,t|p_4|n,\xi\rangle$$

or introducing the unit operator

$$\int |r',t\rangle\langle r',t|\ d^3r',$$

using (III.3) and performing the obvious steps we get the wave equations

$$H|r,t,s\rangle = \pm\, i\hbar\, \frac{\partial}{\partial t}\ |r,t,s\rangle \qquad \text{(III.4)}$$

where the Hamiltonian H is given by

$$H = \frac{c}{\hbar s}\, J\cdot p \quad . \qquad \text{(III.5)}$$

For spin $\frac{1}{2}$ particle $(s=\frac{1}{2})$ we have $J = \frac{1}{2}\,\hbar\sigma$. The corresponding wave equations are

$$H|\nu\rangle = i\hbar \frac{\partial}{\partial t} | \nu \rangle \tag{III.6}$$

$$H|\nu\rangle = -i\hbar \frac{\partial}{\partial t} | \nu \rangle \tag{III.7}$$

where

$$H = c\ \sigma\cdot p,\ |\nu\rangle = |r,t,\tfrac{1}{2}\rangle = \text{two component spinor.}$$

If we call $|\nu\rangle$ the neutrino state then the anti-neutrino state can be defined by

$$|\bar{\nu}\rangle = T\ |\nu\rangle \tag{III.8}$$

where $T = i\ \sigma_2\ \bar{C}$ is the time reversal operator for a two-component spinor state and $\bar{C}$ is just complex conjugation operation. The operator T acts on σ_1 according to (see page 221 of MQT)

$$T^{-1}\ \sigma_1\ T = -\ \sigma_1\ . \tag{III.9}$$

Hence the wave equation (III.7) can be written as

$$H|\bar{\nu}\rangle = i\hbar\ \frac{\partial}{\partial t}\ |\bar{\nu}\rangle \tag{III.10}$$

which is of the same form as (III.6) but refers to anti-neutrino. Reflection symmetry here consists of time reversal operation alone, since space parity is not valid in this case.

As a second example we take s=1 with J being represented by $J_i = \hbar K_i$, $(i=1,2,3)$ where K_i are the generators of three-dimensional rotations. Thus (III.4) yields the wave equations

$$H|\eta\rangle = i\hbar \frac{\partial}{\partial t}|\eta\rangle \qquad \text{(III.11)}$$

$$H|\eta\rangle = -i\hbar \frac{\partial}{\partial t}|\eta\rangle \qquad \text{(III.12)}$$

where $|\eta\rangle$ is a three-component complex vector, and $H = cK\cdot p$ is the Hamiltonian of a single photon. Now defining $|p\rangle$ as a three-dimensional column vector in terms of p_i, $(i=1,2,3)$ and operating on H on the left we obtain

$$\langle p|H = 0$$

which is due to H being a 3 x 3 anti-symmetric matrix operator in p's. Hence the equation (III.11) yields

$$\nabla\cdot\eta = 0 \qquad \text{(III.13)}$$

which is the transversality condition of the photon wave (see chapter II of MQT).

The wave equation (III.12) refers to a state of polarization opposite to the one described by (III.11). This can be seen by performing a parity operation on $|\eta\rangle$. Thus if we take

$$|\bar{\eta}\rangle = \bar{C}\,|\eta\rangle \qquad \text{(III.14)}$$

and noting the transformation

$$\tilde{C}\, H\, \tilde{C} = -\, H$$

the wave equation (III.12) becomes

$$H|\bar{\eta}\rangle = i\hbar \frac{\partial}{\partial t} |\bar{\eta}\rangle \qquad \text{(III.15)}$$

which is of the same form as (III.11) but refers to a state of polarization opposite to the one contained in (III.11). The corresponding transversality condition is obtained as $\nabla \cdot \eta = 0$.

A third example is the wave equation for zeron. We first observe that

$$\langle n,\xi|\hat{S}\cdot\hat{p}|n,\xi\rangle = \pm 1, \quad n,\xi|n,\xi\rangle = 1 \quad \text{for every } n,$$

where

$$\hat{S} = \frac{1}{s} J \; .$$

Thus for zero spin we must have $\hat{S} = \pm \hat{p}$. Hence

$$p^2\; |0,\xi\rangle = p_4^2\; |0,\xi\rangle$$

which, using the same methods, yields the scalar wave equation

$$\nabla^2\phi - \frac{1}{c^2} \frac{\partial^2\phi}{\partial t^2} = 0 \; . \qquad \text{(III.16)}$$

IV. THE LITTLE GROUP

The group of Lorentz transformations which leave a null vector invariant is isomorphic to the two-dimensional Euclidean group. This is a known result[7]. However, here we shall derive it in

[7] E.P. Wigner, THEORETICAL PHYSICS. International Atomic Energy Agency, 1963, pp 59-82, Edited by A. Salam.

a direct way.

Under an S-transformation the requirement of invariance of p_μ is contained in the statements

$$p'_\mu = - \langle a|S^\dagger\gamma_4\Lambda_+\gamma_\mu S\ |a\rangle = - \langle a|\gamma_4\Lambda_+\gamma_\mu|a\rangle = p_\mu \ . \qquad \text{(IV.1)}$$

This must hold for every a_α and $a_\alpha^\dagger$, which is possible only if the S-transformations in question commute with p_μ. The operator $J\cdot\hat{p} = \frac{1}{2}N$ is the only non-trivial invariant of the group and therefore a given S-transformation must be a function of $\frac{1}{2}N$ and also must satisfy (II.14). Such an operator is uniquely defined to be:

$$S = e^{\frac{1}{2}iN\theta} \qquad \text{(IV.2)}$$

where θ can be regarded as an angle of rotation in the xy-plane. For an electromagnetic wave θ is the angle of rotation of the electric vector in the plane perpendicular to its momentum.

The result (IV.2) proves the required isomorphism between group of Lorentz transformations which leave a null vector invariant and the two-dimensional Euclidean group. Thus the representation of the little group for massless particles is one-dimensional. The representatives of S are of the form

$$\langle n',\xi | S | n,\xi \rangle = \delta_{nn'} e^{\pm i s \theta} \tag{IV.3}$$

where

$$-s \leqq \frac{1}{2} n \leqq s, \quad -s \leqq \frac{1}{2} n' \leqq s$$

and the dimension of the representation is 2 s + 1.