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INEQUALITIES AND TOLERANCE LIMITS FOR s-ORDERED DISTRIBUTIONS

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ABSTRACT

Define $F \underset{s}{<} G$ $(F \underset{r}{<} G)$ if F and G have the same median, say the origin and $G^{-1}F(x)$ is concave-convex about the origin $(G^{-1}F(x)/x$ is increasing (decreasing) in x positive (negative)). Conservative tolerance limits are derived for distributions which are s-ordered with respect to the Laplace distribution. These are especially reasonable for mensuration data. In addition, many inequalities concerning combinations of order statistics are obtained. These results are useful in robustness studies of tolerance limits, estimates and statistical tests derived for specified distributions such as the normal distribution. Some examples are given.

TOLERANCE LIMITS AND INEQUALITIES FOR s-ORDERED DISTRIBUTIONS

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I. INTRODUCTION

Motivated by classical but somewhat unsatisfactory measures of skewness and kurtosis, van Zwet (1964) introduced two partial orderings on the space of distribution functions. Barlow and Proschan (1966a) investigated the properties of linear combinations of order statistics from distributions ordered in the sense of van Zwet but corresponding to positive random variables. Our objective is to extend the results of Barlow and Proschan (1966a,1966b) to distributions s-ordered in the sense of van Zwet but not restricted to the positive axis. If we were to confine attention exclusively to symmetric distributions, this would be a relatively straightforward task. However, we extend the s-ordering definition of van Zwet to include a wider class of possibly skewed distributions. We obtain tolerance limits which are conservative for a wide class of distributions. This class of distributions is especially reasonable for measurement type data. In addition, we obtain many results concerning linear combinations of order statistics from s-ordered distributions. These results should be useful in robustness studies of estimates and statistical tests derived for specified distributions such as the normal distribution.

The basis of van Zwet's ordering between distribution functions, and hence between random variables, is that one random variable can be expressed as a convex or concave-convex transformation of another random variable. We adopt van Zwet's notation for c-ordering: $F \underset{c}{<} G$ if and only if $G^{-1}F$ is convex on the support of F ; and a more general definition of s-ordering: $F \underset{s}{<} G$ if and only if $F(m) = G(m) = \frac{1}{2}$ and $G^{-1}F$ is concave-convex about m , on the support of F . If F and G are symmetric, then our definition of s-ordering coincides with that of van Zwet. For convenience, we shall assume that the median of F and G is the origin.

One expects that if $F \underset{c}{<} G$ then G is more skewed to the left than F . If skewness is measured by the standardized odd central moments, then we would expect (assuming for convenience that $EX = EY = 0$, and that X (Y) has distribution F (G))

$$\text{(1.1)} \qquad \frac{EX^{2k+1}}{(EX^2)^{k+\frac{1}{2}}} \leq \frac{EY^{2k+1}}{(EY^2)^{k+\frac{1}{2}}} \qquad k = 1,2, \ldots$$

Van Zwet proves this result and also that if i and n tend to infinity and $\lim \frac{i}{n} = r$, $0 < r < 1$, then $F \underset{c}{<} G$ if (1.1) holds asymptotically for some fixed k , all $0 < r < 1$, and order statistics $X_{i,n}$ and $Y_{i,n}$. If F and G are symmetric, then $F \underset{s}{<} G$ would seem to imply that G has heavier tails or more "kurtosis" than F . Van Zwet proves this when the even standardized central moments are taken as the measure of "kurtosis".

Barlow and Proschan (1966a,b) have derived tolerance limits for the distributions which are c-ordered with respect to certain distribution functions, as well as developing many interesting inequalities for linear combinations of the order statistics from c-ordered distributions. They have particularly exploited the properties of distribution functions which are convex with respect to the exponential distribution, and have also introduced and developed for positive random variables, the properties of the weaker star-shaped ordering; i.e., $\frac{G^{-1}F(x)}{x}$ is increasing in x for x on the support of F and $F(0) = 0$.

It is the object of this thesis to develop some of the statistical properties of distribution functions related by both s-ordering and starshaped ordering. We will be particularly interested in a natural class of distribution functions--viz., those which are s-ordered with respect to the Laplace distribution (also called the bilateral exponential distribution).

Examples of such s-ordered distributions are:

$$\text{U-shaped} \underset{s}{<} \text{uniform} \underset{s}{<} \text{normal} \underset{s}{<} \text{logistic} \underset{s}{<} \text{Laplace} \underset{s}{<} \text{Cauchy} \quad .$$

(cf. Van Zwet (1964), pp. 70-71, 72-73.)

There are numerous reasons why we are interested in studying the properties of distributions related by s-ordering. The most apparent one is that very often we do not know the exact distribution but because of physical considerations we can make certain deductions about the properties of the distribution, and hence do not want the disadvantage of a distribution-free approach. For instance if we measure the length of an object, it is plausible that the probability of obtaining an error in the range $(|x|, |x| + \delta x)$, given that the error is at least $|x|$ is increasing in $|x|$. The class of distributions having this property are s-ordered with respect to the Laplace distribution. Another example would be the commonly occurring situation where the normal distribution is assumed, but we suspect that this is not true and that in fact the tails are heavier or lighter than the tails of the normal. We then wish to know if the normal assumption is conservative or not. In short, we want to know how robust the normal distribution is against s-ordered alternatives.

In Chapter II, we develop some new inequalities for concave-convex functions ϕ , when ϕ satisfies a skewness condition. These inequalities are used in Chapter III to construct tolerance limits for distributions s-ordered with respect to the Laplace distribution (these distributions we call SIFR). We further investigate symmetric SIFR distributions by obtaining sharp bounds on F symmetric and SIFR when given only the mean and the variance of

F . This is used to construct a confidence bound on the variance of F . Also in Chapter III, the robustness of tolerance limits from the normal distribution against s-ordered alternatives is investigated. Results are obtained for small sample sizes.

In Chapter IV, we are interested in inequalities for the order statistics and their expectation when related by the weaker r-ordering. We say that $F \underset{r}{<} G$ if $F(0) = G(0) = \frac{1}{2}$ and $\frac{G^{-1}F(x)}{x}$ is increasing (decreasing) in x positive (negative). Bounds on the expectation of the i^{th} order statistic from F are given in terms of the expectation of the order statistics from G when $F \underset{r}{<} G$ and F and G are symmetric. Also, an inequality relating a linear combination of the expectations of the order statistics from F and G is given for $F \underset{s}{<} G$, G symmetric about the origin and the direction of the skew of F known.

If $F \underset{r}{<} G$ and F and G are symmetric, then we prove that not only are the standardized even central moments of G greater than that of F but so are the usual sample estimates of the standardized even central moments in a stochastic sense.

Preliminaries

We adopt the following definitions:

(i) $F \underset{c}{<} G$ if and only if $G^{-1}F$ is convex on the support of F .

(ii) $F \underset{s}{<} G$ if and only if $F(0) = G(0) = \frac{1}{2}$ and $G^{-1}F$ is concave-convex, about the origin, on the support of F .

(iii) $F \underset{r}{<} G$ if and only if $F(0) = G(0) = \frac{1}{2}$ and $\frac{G^{-1}F(x)}{x}$ is increasing (decreasing) for x positive (negative) on the support of F .

(iv) F is SIFR(SDFR) if and only if $F \underset{s}{<} (\underset{s}{>})\ G$, where G is the Laplace distribution; i.e., $G'(x) = \frac{1}{2} e^{-|x|}$, $-\infty < x < \infty$.

(v) $X \overset{st}{\leq} Y$ if and only if $P(X \leq a) \geq P(Y \leq a)$, $-\infty < a < \infty$.

We assume throughout that F and G are continuous with median at the origin. Note that $F \underset{s}{<} G$ implies $F \underset{r}{<} G$.

Define $F_{(i)}$ to be the distribution of the i-th order statistic from F and F_α to be the distribution of $|X|^\alpha$. Since $G^{-1}_{(i)}F_{(i)} = G^{-1}F$ we see that

$$\text{(a)} \quad F \underset{s}{<} G \text{ implies } F_{(i)} \underset{s}{<} G_{(i)}$$

$$\text{(b)} \quad F \underset{r}{<} G \text{ implies } F_{(i)} \underset{r}{<} G_{(i)} \ .$$

If furthermore, F and G are symmetric about the origin, then for $\alpha \geq 1$

$$\text{(c)} \quad F \underset{s}{<} G \text{ implies } F_\alpha \underset{c}{<} G_\alpha$$

$$\text{(d)} \quad F \underset{r}{<} G \text{ implies } F_\alpha \underset{r}{<} G_\alpha$$

on the positive axis.

If G is symmetric about the origin and $\mathcal{b} = \{G(\theta x) \mid \theta > 0\}$ then a sufficient statistic for $\mathcal{b}$ based on a *complete* random sample $\underline{Y} = (Y_1, Y_2, \ldots, Y_n)$ is given by $(|Y_1|, |Y_2|, \ldots, |Y_n|)$. Suppose we are interested in studying the robustness of statistics derived under the assumption that the observations are distributed according to G when in fact they are distributed according to F where $F \underset{s}{<} G$. Since by (c), $F \underset{s}{<} G$ implies $F_1 \underset{c}{<} G_1$, the results of Barlow and Proschan (1966a) apply to linear combinations of the sufficient statistics for G .

Throughout we let $X_1 \leq X_2 \leq \ldots \leq X_n$ $(Y_1 \leq Y_2 \leq \ldots \leq Y_n)$ be an ordered sample from F (G) , and we observe that $Y \overset{st}{=} G^{-1}F(X)$ where $\overset{st}{=}$ denotes stochastic equality.

II. INEQUALITIES FOR CONCAVE-CONVEX FUNCTIONS

Some inequalities on concave-convex functions, interesting in their own right and also required later on, will now be developed. We will need a slight extension of a theorem by Hardy, Littlewood and Pólya (1929). In this connection, see also Barlow, Marshall and Proschan (1967) and Karlin and Novikoff (1963) p. 1252.

We say that ϕ is concave-convex about the origin and defined on $[-a,b]$ if ϕ is concave on $[-a,0]$ and convex on $[0,b]$.

Theorem 2.1

Let μ be a signed measure on $[-a,b]$, $0 \le a$, $b \le \infty$, then

$$\int_{-a}^{b} \phi(x)d\mu(x) \ge 0 \tag{2.1}$$

for all ϕ concave-convex about the origin, continuous at the origin, and defined on $[-a,b]$ if and only if

$$\int_{-a}^{b} xd\mu(x) = 0 \ , \int_{-a}^{b} d\mu(x) = 0 \tag{2.2}$$

and

$$\int_{-a}^{-z'} (x + z')d\mu(x) + \int_{z}^{b} (x - z)d\mu(x) \ge 0 \tag{2.3}$$

for all

$$z \in [0,b]$$
$$-z' \in [-a,0]$$

Proof

Suppose first that inequality (2.1) is satisfied for all ϕ concave-convex about the origin, continuous at the origin and defined on $[-a,b]$. Now (2.2) follows if $\phi(x) = (\pm)1$ or $\phi(x) = (\pm)x$, and (2.3) follows if $\phi(x)$ is a "double angle function," i.e., a function of the form

$$\phi(x) = \begin{cases} x + z' & x \leq -z' \\ 0 & -z' < x < z \\ x - z & x \geq z \end{cases} .$$

Next, suppose that (2.2) and (2.3) hold. Since ϕ is concave-convex about the origin and continuous there, there exists α , β such that

$$\phi(x) - \alpha x - \beta \geq (\leq)\ 0 \quad \text{for} \quad x \geq (\leq)\ 0 \ .$$

Hence we may assume that

$$\phi(x) \geq (\leq)\ 0 \quad \text{for} \quad x \geq (\leq)\ 0 \ .$$

Consider now a sequence of functions $\phi_n(x)$ such that each ϕ_n is the sum of a finite number of positive multiples of "double angle functions" and $\phi_n(x)$ is increasing in n for $x \geq 0$ and decreasing in n for $x \leq 0$. Since ϕ is concave-convex we can construct a sequence of functions $\phi_n(x) \geq (\leq)\ 0$ for $x \geq (\leq)\ 0$ such that $\phi_n(x)$ converges upward (downward) to ϕ for $x \geq 0$ $(x < 0)$. By construction

$$\int_a^b \phi_n(x)\,d\mu(x) \geq 0 \ ,$$

for each ϕ_n and (2.1) follows from the Lebesgue monotone convergence theorem. ||

Remark

If ϕ in Theorem 2.1 is not only concave-convex but also satisfies $\phi(x) \leq -\phi(-x)$, then we see that the inequality (2.3) need only hold for all z , z' satisfying $z \geq z'$.

We will now prove a lemma which although fairly elementary is nevertheless quite useful.

Lemma 2.2

$$\sum_{i=1}^{n} a_i\gamma_i + \sum_{j=1}^{m} b_j\delta_j \geq x$$

for all

$$1 \geq \gamma_1 \geq \gamma_2 \geq \cdots\cdot \geq \gamma_n \geq 0$$

$$1 \geq \delta_1 \geq \delta_2 \geq \cdots\cdot \geq \delta_m \geq 0$$

if and only if

$$\sum_1^j a_i + \sum_1^k b_i \geq x \qquad \text{for } 0 \leq j \leq n \text{ and } 0 \leq k \leq m$$

and we define $\sum_1^0 = 0$.

Proof

Let

$$u_o = 1 - \sum_1^n u_i \quad , \quad u_j = \gamma_j - \gamma_{j+1} \quad , \quad u_n = \gamma_n$$

$$\nu_o = 1 - \sum_1^m \nu_i \quad , \quad \nu_k = \delta_k - \delta_{k+1} \quad , \quad \nu_m = \delta_m$$

and

$$A_j = \sum_1^j a_i \ , \ B_k = \sum_1^k b_i \ .$$

The lemma can now be reformulated as

$$\sum_0^n A_j u_j + \sum_0^m B_k \nu_k \geq x \tag{2.4}$$

for all

$$\begin{aligned} &0 \leq u_j \leq 1 \quad , \quad 0 \leq j \leq n \quad , \quad \text{and} \ \sum_0^n u_j = 1 \\ &0 \leq \nu_k \leq 1 \quad , \quad 0 \leq k \leq m \quad , \quad \text{and} \ \sum_0^m \nu_k = 1 \end{aligned} \tag{2.5}$$

if and only if

$$A_j + B_k \geq x \qquad \text{for} \ \begin{array}{l} 0 \leq j \leq n \\ 0 \leq k \leq m \end{array} \ . \tag{2.6}$$

The proof is straightforward. If (2.4) is true for all u_j , ν_k satisfying (2.5), then clearly the inequality (2.6) must be satisfied. If (2.6) is true, then

$$\sum_{k=0}^m \nu_k \sum_{j=0}^n u_j(A_j + B_k) \geq \sum_{k=0}^m \nu_k \sum_{j=0}^n u_j x = x \ ,$$

which completes the proof. ||

We would like to determine conditions on $a_1, a_2, \ldots, a_n$ such that

$$\phi\left(\sum_1^n a_i x_i\right) \leq \sum_1^n a_i \phi(x_i)$$

holds for all $x_1 \leq x_2 \leq \ldots \leq x_n$ and all ϕ concave-convex. However, it is possible to construct an example to show that no such inequality holds for all ϕ concave-convex, all ordered x's and $a_i \neq 0$, $1 \le i \le n$. If we assume that ϕ is concave-convex about the origin and $\phi(0) = 0$, then a simple additional condition on ϕ which admits a solution to our problem is $\phi(x) \le -\phi(-x)$ for $x \ge 0$.

In proving the next theorem it is necessary only to consider ϕ such that $\phi(0) = 0$, and ϕ is concave-convex about the origin. This can be seen by making the transformation $\phi^*(x) = \phi(x+c) - \phi(c)$, if ϕ is concave-convex about $x = c$. To simplify notation we will define $A_j = \sum_1^j a_i$ and $\bar{A}_j = \sum_j^n a_i$.

Theorem 2.3

If ϕ is concave-convex about the origin and defined on $(-\infty, b)$, $\phi(x) \le -\phi(-x)$ for $x \ge 0$, $\phi(0) = 0$ and ϕ is continuous at the origin then

$$\phi\left(\sum_1^n a_i x_i\right) \le \sum_1^n a_i \phi(x_i) \tag{2.7}$$

for all $x_1 \le x_2 \le \dots \le x_k \le 0 \le x_{k+1} \le \dots \le x_n \le b$ if and only if (2.8) or (2.9) is satisfied, where

$$\left.\begin{aligned} 0 \le -A_i + \bar{A}_j \le 1 \\ A_i \le 0 \\ \bar{A}_j \le 0 \end{aligned}\right\} \text{for } 1 \le i \le k < j \le n , \tag{2.8}$$

$$\left.\begin{aligned} \bar{A}_q = A_p = 0 \\ 1 \le A_{r'} \le A_{r'+1} \le \dots \le A_k \\ \bar{A}_j \le -1 \end{aligned}\right\} \text{for } 0 \le p < r' \le i \le k < j \le r < q \le n , \tag{2.9}$$

and $r'(1 \le r' \le k+1)$, $r(k \le r \le n)$ are fixed.

<u>Proof</u>

It will be convenient to assume $x_k = 0$ with $a_k = 0$. Clearly we can always do this with no loss of generality.

We use Theorem 2.1 and adopt the measure

$$\mu(x) = \begin{cases} -1 & , x = \sum_1^n a_i x_i \\ a_i & , x = x_i \\ 1 - \sum_1^n a_i & , x = 0 \\ 0 & , \text{elsewhere} \end{cases} .$$

Clearly then

$$\int_{-\infty}^{\infty} d\mu(x) = \int_{-\infty}^{\infty} x d\mu(x) = 0$$

and hence from Theorem 2.1 we see that

$$\phi\left(\sum a_i x_i\right) - \sum a_i \phi(x_i) \leq 0$$

if and only if

$$(2.10) \qquad -\int_{-\infty}^{-z'} (x + z')d\mu(x) - \int_{+z}^{\infty} (x - z)d\mu(x) \leq 0$$

$$\forall \; z \in [z',\infty]$$
$$-z' \in [-\infty,0] .$$

In what follows we shall assume that

$$x_s < -z' \leq x_{s+1} \leq 0$$

and

$$0 \leq x_{\ell-1} \leq z < x_\ell \ .$$

If no x_s exists, add an extra term $x_o < -z'$ and $a_o = 0$. Similarly add $x_{n+1} > z$ if $x_n \leq z$ and let $a_{n+1} = 0$.

Case (1)

Assume that $-z' \leq \sum_0^{n+1} a_i x_i \leq z$. Now the left hand side of (2.10) equals

$$(2.11)\qquad \begin{aligned} &-\sum_0^{s-1} A_i(x_i - x_{i+1}) - A_s(x_s + z') - \bar{A}_\ell(x_\ell - z) - \\ &\qquad -\sum_{\ell+1}^{n+1} \bar{A}_i(x_i - x_{i-1}) \ . \end{aligned}$$

Since $x_i - x_{i+1} \leq 0$, $x_i - x_{i-1} \geq 0$ and since these differences can be arbitrarily small for suitable choice of the x values, we see that the necessary and sufficient conditions for (2.11) to be nonpositive are

$$(2.12)\qquad \begin{aligned} A_i &\leq 0 \qquad 1 \leq i \leq k \\ \bar{A}_j &\geq 0 \qquad k < j \leq n \end{aligned}$$

If (2.12) holds then $\sum a_i x_i \geq 0$.

Case (2)

Assume $z \leq \sum_1^n a_i x_i$, and recall that $z \geq z' \geq 0$. The left hand side of (2.10) can be written as

$$\sum_{0}^{k-1} A_i(x_i - x_{i+1}) + \sum_{k+1}^{n+1} \bar{A}_i(x_i - x_{i-1}) - z$$

$$- A_s(x_s + z') - \sum_{0}^{s-1} A_i(x_i - x_{i+1})$$

$$- \bar{A}_\ell(x_\ell - z) - \sum_{\ell+1}^{n+1} \bar{A}_i(x_i - x_{i-1})$$

$$(2.13) \qquad = z\left[\left(\frac{-z'}{z} A_s + \bar{A}_\ell - 1\right) + \frac{x_{s+1}}{z}(-A_s + A_{s+1}) + \dots + \frac{x_{\ell-1}}{z}(-\bar{A}_\ell + \bar{A}_{\ell-1}) + \frac{x_{\ell-2}}{z}(-\bar{A}_{\ell-1} + \bar{A}_{\ell-2}) + \dots\right] .$$

Now from Lemma 2.2 and identifying $x = \frac{-z'}{z} A_s + \bar{A}_\ell - 1$, we see that the necessary and sufficient conditions for (2.13) to be nonpositive are

$$(2.14) \qquad \left.\begin{aligned} -\frac{z'}{z} A_s + \bar{A}_\ell - 1 &\le 0 \\ A_s\left(1 - \frac{z'}{z}\right) + \bar{A}_g - 1 - A_h &\le 0 \end{aligned}\right\} \text{for} \quad s < h \le k < g \le \ell \ .$$

Since (2.14) must hold for all z , z' such that

$$z' \le z \le \sum a_i x_i \ ,$$

then from (2.12) and (2.14),

$$-\frac{z'}{z} A_s + \bar{A}_\ell - 1 \le -A_s + \bar{A}_\ell - 1 \ ,$$

$$A_s\left(1 - \frac{z'}{z}\right) + \bar{A}_g - 1 - A_h \le \bar{A}_g - 1 - A_h \ ,$$

and we have that (2.14) holds if and only if $-A_h + \bar{A}_g \le 1$ for $0 \le h \le k$, $k < g \le n$.

Thus, we have established the necessity and sufficiency of conditions (2.8) for the case $\sum_{j=1}^{n} a_i x_i \ge -z'$.

Case (3)

Assume that $\sum a_i x_i \leq -z'$. Note that if $-z' \leq x_1$ and $z \geq x_n$ then the left hand side of (2.10) is nonpositive. The left hand side of (2.10) can be written as

$$z\left[\left(\bar{A}_\ell - \frac{z'}{z} A_s + \frac{z'}{z}\right) + \frac{x_{s+1}}{z}(-A_s + A_{s+1}) + \ldots + \frac{x_{\ell-1}}{z}(-\bar{A}_\ell + \bar{A}_{\ell-1}) + \ldots\right] \tag{2.15}$$

By Lemma 2.2, the necessary and sufficient conditions for (2.15) to be nonpositive are that

$$\left.\begin{aligned} \bar{A}_\ell - \frac{z'}{z} A_s + \frac{z'}{z} &\leq 0 \\ \frac{z'}{z}(1-A_s) + A_s - A_h + \bar{A}_g &\leq 0 \end{aligned}\right\} \text{ for } s < h \leq k < g < \ell .$$

By noting that if $-z' \leq x_1$ then $A_s = A_0 = 0$, and if $z \geq x_n$ then $\bar{A}_\ell = \bar{A}_{n+1} = 0$, and that the above conditions must hold for all $z \geq z'$, we see that the necessary and sufficient conditions for (2.7) to be true in the case that $\sum a_i x_i \leq -z'$, are (2.9). Now from (2.9) we see that $\sum a_i x_i \leq x_1$ and hence if $\sum a_i x_i \geq -z'$ then the left hand side of (2.10) is zero. ||

Remarks

1. We see that the only concave-convex functions which admit a solution of (2.7) with all weights $a_1, a_2, \ldots, a_n$ nonzero are those which can be generated from a double angle function with $\frac{z'}{z}$ bounded above and below.

2. If in Theorem 2.3, we reverse the skewness condition on ϕ and set $\phi(x) \geq -\phi(-x)$ for $x \geq 0$, then we can see from the proof of the theorem

that (2.8) is replaced by

$$\left.\begin{array}{r} 0 \le \bar{A}_j \le 1 \\ A_i = 0 \end{array}\right\} \text{ for } 1 \le i \le k < j \le n \ ,$$

and for (2.9) we need $r' = k+1$.

By considering $\phi^*(x) = -\phi(-x)$ we get the following corollary.

Corollary 2.4

If ϕ concave-convex about the origin and defined on $(-a,\infty)$, $\phi(0) = 0$, ϕ continuous at the origin and $\phi(x) \ge -\phi(-x)$ for all $x \ge 0$, then

$$\phi\left(\sum_1^n a_i x_i\right) \ge \sum_1^n a_i \phi(x_i)$$

for all $-a \le x_1 \le x_2 \le \dots \le x_k \le 0 \le x_{k+1} \le \dots \le x_n$ if and only if (2.17) or (2.18) is satisfied.

$$(2.17) \qquad \left.\begin{array}{r} A_i \ge 0 \\ \bar{A}_j \le 0 \\ 0 \le A_i - \bar{A}_j \le 1 \end{array}\right\} \text{ for } 1 \le i \le k < j \le n \ ,$$

$$(2.18) \qquad \left.\begin{array}{c} \bar{A}_q = A_p = 0 \\ 1 \le \bar{A}_r \le \bar{A}_{r+1} \le \dots \le \bar{A}_{k+1} \\ A_i \le -1 \end{array}\right\} \text{ for } 0 \le p < r' \le i \le k < j \le r < q \le n \ ,$$

and $r'(1 \le r' \le k+1)$, $r(k \le r \le n)$ are fixed.

By noting that if ϕ is concave-convex then $-\phi$ is convex-concave, we can obtain similar inequalities for convex-concave functions. As they follow trivially, we shall not include them here.

In the special case that ϕ is concave-convex and antisymmetric (i.e., $\phi(x) = -\phi(-x)$), we can set $z = z'$ in the proof of Theorem 2.3 and obtain the necessary and sufficient conditions on $a_1, a_2, \ldots, a_n$ such that (2.7) is true for all ϕ concave-convex and antisymmetric. However for this special case the result also follows directly from Lemmata 4.1 and 4.3 of Barlow and Proschan (1966).

Theorem 2.5

If ϕ is concave-convex and antisymmetric about the origin and defined on $(-\infty,\infty)$ then

$$\phi\left(\sum_1^n a_i x_i\right) \le \sum_1^n a_i \phi(x_i)$$

for all $x_1 \le \ldots \le x_k \le 0 \le \ldots \le x_n$, if and only if (2.8) or (2.19) is satisfied.

$$(2.8)\qquad \left.\begin{array}{r} 0 \le -A_i + \bar{A}_j \le 1 \\ A_i \le 0 \\ \bar{A}_j \ge 0 \end{array}\right\} \text{ for } 1 \le i \le k < j \le n$$

$$(2.19)\qquad \left.\begin{array}{r} A_i \ge 1 \\ \bar{A}_j \le -1 \\ A_p = \bar{A}_q = 0 \end{array}\right\} \text{ for } 0 \le p < r' \le i \le k < j \le r < q \le n .$$

Proof

The proof follows from Barlow and Proschan (1966) with the observation that

$$\sum_{1}^{n} a_i\phi(x_i) = \sum_{1}^{k} - a_i\phi(-x_i) + \sum_{k+1}^{n} a_i\phi(x_i) \quad ,$$

and hence the right hand side only involves ϕ convex on $[0,\infty)$. ||

Theorem 2.6

If ϕ is concave-convex and antisymmetric about the origin and defined on $(-\infty,\infty)$ then

$$\phi\left(\sum_{i=1}^{n} a_i x_i\right) \geq \sum_{i=1}^{n} a_i\phi(x_i)$$

for all $x_1 \leq x_2 \leq \ldots \leq x_k \leq 0 \leq \ldots \leq x_n$ if and only if either (2.20) or (2.21) is satisfied.

$$(2.20) \qquad \left.\begin{array}{r} A_i \geq 0 \\ \bar{A}_j \leq 0 \\ 0 \leq A_i - \bar{A}_j \leq 1 \end{array}\right\} \text{for} \quad 1 \leq i \leq k < j \leq n$$

$$(2.21) \qquad \left.\begin{array}{r} A_i \leq -1 \\ \bar{A}_j \geq 1 \\ A_p = \bar{A}_q = 0 \end{array}\right\} \text{for} \quad 0 \leq p < r' \leq i \leq k < j \leq r < q \leq n \quad .$$

III. TOLERANCE LIMITS AND CONFIDENCE BOUNDS

It will be convenient to let X (Y) have distribution F (G). We assume that G is symmetric about the origin and strictly increasing on its support and that both F and G are continuous. Let $X_1 \leq X_2 \leq \ldots \leq X_n$ $(Y_1 \leq Y_2 \leq \ldots \leq Y_n)$ denote an ordered sample from F (G). We say that a random variable X is stochastically greater than a random variable Y, denoted by $X \overset{st}{\geq} Y$, if $P(X \geq x) \geq P(Y \geq x)$ for all x.

We construct tolerance limits by using the relevant inequalities of Chapter II to give stochastic comparisons between the order statistics from F and G when $F \underset{s}{<} G$. Consider for example Theorem 2.3. The weights $a_1, a_2, \ldots, a_n$ in inequality (2.7) which must be selected to satisfy conditions (2.8) or (2.9), are dependent on the value of the index k defined by $X_1 \leq \ldots \leq X_k \leq 0 \leq X_{k+1} \leq \ldots \leq X_n$, which is therefore a random variable. Let $\underline{X} = (X_1, X_2, \ldots, X_n)$ denote a random vector of ordered observations. Thus when making a stochastic comparison using inequality (2.7), the weights must be chosen as a function of $\underline{X}$, i.e., $(a_1(\underline{X}), \ldots, a_n(\underline{X})) \equiv [\underline{a}(\underline{X})]$. We require that these weights satisfy conditions (2.8) or (2.9) for every possible outcome, $\underline{X}$; and then say that $[\underline{a}]$ satisfies (2.8) or (2.9).

If we let $\phi(x) = G^{-1}F(x)$ in Theorem 2.3, then the condition $G^{-1}F(x) \leq -G^{-1}F(-x)$, $x \geq 0$, will be satisfied, for example, if $G(0) = F(0) = \frac{1}{2}$, G is symmetric about the origin and $1 - F(x) \geq F(-x)$ for $x \geq 0$.

Theorem 3.1

If $F \underset{s}{<} G$, G symmetric about the origin, $F(0) = G(0) = \frac{1}{2}$, $1 - F(x) \geq F(-x)$ for $x \geq 0$ and $[\underline{a}]$ satisfies (2.8) or (2.9), then

$$G\left(\sum_1^n a_i(\underline{Y})Y_i\right) \overset{st}{\geq} F\left(\sum_1^n a_i(\underline{X})X_i\right) .$$

Proof

From Theorem 2.3 we have that if $[\underline{a}]$ satisfies (2.8) or (2.9), then

$$G^{-1}F\left(\sum_1^n a_i(\underline{X})X_i\right) \leq \sum_1^n a_i(\underline{X})G^{-1}F(X_i) \overset{st}{=} \sum_1^n a_i(\underline{Y})Y_i .$$

The stochastic equality follows from the fact that $G^{-1}F$ preserves order with respect to the origin, and $G^{-1}F(X_1)$, $G^{-1}F(X_2)$, ..., $G^{-1}F(X_n)$ are jointly distributed as the order statistics from G . ||

Corollary 3.2

If $F \underset{s}{<} G$, G symmetric about the origin $F(0) = G(0) = \frac{1}{2}$, $1 - F(x) \leq F(-x)$ for $x \geq 0$ and $[\underline{a}]$ satisfies (2.17) or (2.18), then

$$G\left(\sum_1^n a_i(\underline{Y})Y_i\right) \overset{st}{\leq} F\left(\sum_1^n a_i(\underline{X})X_i\right) .$$

If we let F be symmetric about the origin, we can similarly prove the following:

Theorem 3.3

If $F \underset{s}{<} G$, F and G symmetric about the origin and $[\underline{a}]$ satisfies either (2.8) or (2.19), then

$$G\left(\sum_1^n a_i(\underline{Y})Y_i\right) \overset{st}{\geq} F\left(\sum_1^n a_i(\underline{X})X_i\right) .$$

<u>Corollary 3.4</u>

If $F \underset{s}{<} G$, F and G symmetric about the origin, and $[\underline{a}]$ satisfies either (2.20) or (2.21), then

$$G\left(\sum_1^n a_i(\underline{Y})Y_i\right) \underset{-}{\overset{st}{<}} F\left(\sum_1^n a_i(\underline{X})X_i\right) .$$

3.1 Tolerance Limits for SIFR and SDFR Distributions

Barlow and Proschan (1966b) have considered tolerance limits for the class of distributions which are c-ordered with respect to the exponential. These distributions possess an increasing failure rate, and so arise naturally when wear-out is present. The two sided analogue of the increasing failure rate class is the class of distributions which are s-ordered with respect to the Laplace distribution G, i.e., $G'(x) = \frac{1}{2} e^{-|x|}$. We will call the distribution F SIFR(SDFR) when $F \underset{s}{\le} G$, $(F \underset{s}{\ge} G)$ and G is the Laplace. We note that this implies that $F(0) = G(0) = \frac{1}{2}$. The distribution function F with density f, $F(0) = \frac{1}{2}$ is SIFR(SDFR) if $\frac{f(x)}{1 - F(x)}$ is increasing (decreasing) in x for $x \ge 0$ and $\frac{f(x)}{F(x)}$ is decreasing (increasing) in x for $x \le 0$. Note that F need not be symmetric and that we chose the median at the origin only for convenience. Recall that the normal distribution is SIFR and the Cauchy distribution is SDFR.

The SIFR class can arise naturally when we are considering problems such as the distribution of the error of some measurement, for here we would expect that the probability of an error in the range $(|x|, |x| + \delta x)$, given the error is greater than $|x|$, would be increasing in $|x|$. We will develop tolerance limits for F SIFR and F SDFR in the one sided case when F is skewed and in the two sided case when F is symmetric.

If we have a *complete* sample and F is symmetric about the origin, then it is an easy matter to construct conservative two sided tolerance limits for F. We need only assume $F \underset{r}{\le} G$ where $G'(x) = \frac{1}{2\theta} e^{-|x|/\theta}$. Then $(|Y|_1 \le |Y|_2 \le \dots \le |Y|_n)$ is a sufficient statistic for θ and tolerance limits for G may be constructed from these. Since F_a, the distribution of $|X|$ is star shaped with respect to G_a, the distribution of $|Y|$ (which is the exponential distribution) i.e., $\frac{G_a^{-1} F_a(x)}{x}$ is increasing in

x , we can apply Theorem 3.1 of Barlow and Proschan (1966b). Let

$$\hat{\theta}_{r,n}(\underline{X}) = \sum_{i=1}^{r} (n - i+1)r^{-1} (|X|_i - |X|_{i-1})$$

and

$$B_{\alpha,q,r} = \frac{-2r \log(1-q)}{\chi_\alpha^2(2r)}$$

and

$$C^{**}_{\alpha,q,r} = \max (B_{\alpha,q,r}, r(n - r+1)^{-1}) \quad .$$

Then

$$P_F\left\{\int_{-C^{**}_{\alpha,q,r}\hat{\theta}_{r,n}(\underline{X})}^{C^{**}_{\alpha,q,r}\hat{\theta}_{r,n}(\underline{X})} dF(x) \geq q\right\} \geq 1-\alpha \quad .$$

However, if we have a censored sample and/or F is not symmetric this inequality is *not* valid.

We will need to develop some properties of the Laplace distribution. Given an ordered sample $X_1 \leq X_2 \leq \dots \leq X_k \leq 0 \leq X_{k+1} \leq \dots \leq X_n$, we define the statistic $\bar{\theta}_{r',r,n}(\underline{X})$ by

$$\bar{\theta}_{r',r,n}(\underline{X}) = \sum_{r'}^{k-1} - i(X_i - X_{i+1}) + \sum_{k+2}^{r} (n - i+1)(X_i - X_{i-1}) \quad .$$

If $X_1 > 0$ let $k = 1$ and if $X_n < 0$ let $k = n-1$.

We can easily verify that if $X_1 \le X_2 \le \dots \le X_k \le 0 \le X_{k+1} \le \dots \le X_n$ are distributed as the n order statistics from distribution G where $G'(x) = \frac{1}{2\theta} e^{-|x|/\theta}$, $\theta > 0$ (the Laplace distribution with scale parameter θ), then the maximum likelihood estimate of the scale parameter θ, given censorship at r' and r where $r' \le k$, $r > k$ is

$$\hat{\theta} = \frac{-X_{r'}(r'-1) + \sum_{r'}^{r} |X_i| + X_r(n-r)}{r-r'+1} .$$

Hence we see that if r and r' are such that $\lim \frac{r'}{n}$ and $\lim \frac{r}{n}$ exist finitely then $\frac{\bar{\theta}_{r',r,n}(\underline{X})}{r-r'+1}$ is asymptotically the maximum likelihood estimate for the scale parameter when X has the Laplace distribution with scale parameter θ.

Lemma 3.5

If $Y \sim G$ where $G'(y) = \frac{1}{2} e^{-|y|}$ then

$$2\bar{\theta}_{r',r,n}(\underline{Y}) \sim \chi^2(2(r-r'-1)) .$$

(N.B. $\sim$ denotes "is distributed as".)

Proof

If $Y \sim G$, the Laplace distributed function, then

$$P(Y \le x + y \mid Y \ge y \ge 0) = 1 - e^{-x}$$

which is independent of y. Now if Y_h is the h^{th} order statistic from G we have

$$P(Y_{h+1}-Y_h \ge x \mid Y_h = y \ge 0) = P(Y_{h+1} \ge x+y \mid Y_h = y \ge 0) = e^{-(n-h)x}$$

and since this is independent of y

$$P(Y_{h+1} - Y_h \geq x \mid Y_h \geq 0) = e^{-(n-h)x} ,$$

or the random variable

$$(n-h)(Y_{h+1} - Y_h)$$

given that $Y_h \geq 0$ is distributed as the unit exponential. Now from symmetry we have

$$P(Y_{n-h}-Y_{n-h+1} \geq x \mid Y_{n-h+1} \leq 0) = P(Y_{h+1}-Y_h \geq x \mid Y_h \geq 0) = e^{-(n-h)x}$$

and hence the conditional distribution of $i(Y_{i+1} - Y_i)$ given $Y_{i+1} \leq 0$ is exponential with mean unity, which completes the proof. ||

Let $H_{\alpha,q}(r',r) = \dfrac{-2 \log(1-q)}{\chi^2_\alpha 2(r-r'-1)}$, and

$$C^*_{\alpha,q}(r',r) = \begin{cases} H_{\alpha,q}(r',r) & \text{if } \chi^2_\alpha 2(r-r'-1) \leq -2m \log(1-q) \\ \\ \dfrac{1}{m} & \text{if } \chi^2_\alpha 2(r-r'-1) > -2m \log(1-q) \end{cases}$$

where $m = \min(r',n-r+1)$.

We first give the tolerance limit on the tail probability, $1 - F$, for values to the left of the median.

<u>Theorem 3.6</u>

If F is SIFR, $F(0) = \frac{1}{2}$, $1 - F(x) \geq F(-x)$ for $x \geq 0$, and $0 \leq q \leq \frac{1}{2}$ then

$$P_F\left\{1 - F\left[-C^*_{\alpha,1-2q}(r',r)\bar{\theta}_{r',r,n}(\underline{X})\right] \geq 1-q\right\} \geq 1-\alpha \quad .$$

Proof

Let G be the Laplace distribution, i.e., $G'(x) = \frac{1}{2} e^{-|x|}$, then from Theorem 3.1,

$$F\left[-H_{\alpha,1-2q}(r',r)\bar{\theta}_{r',r,n}(\underline{X})\right] \overset{st}{\leq} G\left[-H_{\alpha,1-2q}(r',r)\bar{\theta}_{r',r,n}(\underline{Y})\right]$$

for $mH_{\alpha,1-2q}(r',r) \geq 1$. It follows in this case that since

$$P_G\left\{ G\left[-H_{\alpha,1-2q}(r',r)\bar{\theta}_{r',r,n}(\underline{Y})\right] \leq q\right\} = 1-\alpha$$

we have

$$P_F\left\{F\left[-H_{\alpha,1-2q}(r',r)\bar{\theta}_{r',r,n}(\underline{X})\right] \leq q\right\} \geq 1-\alpha \quad .$$

Now if $mH_{\alpha,1-2q}(r',r) < 1$, we note that

$$F\left(-\frac{1}{m}\bar{\theta}_{r',r,n}(\underline{X})\right) \overset{st}{\leq} G\left(-\frac{1}{m}\bar{\theta}_{r',r,n}(\underline{Y})\right) \leq G\left(-H_{\alpha,1-2q}(r',r)\bar{\theta}_{r',r,n}(\underline{Y})\right)$$

which proves the theorem. ||

Clearly Theorem 3.6 implies a tolerance limit on the distribution F for values to the right of the median. However, we shall not present this result.

We note that the tolerance limit in Theorem 3.6 achieves $1-\alpha$ confidence for the Laplace distribution, (i.e., is sharp) only when $\chi^2_\alpha[2(r-r'-1)] \leq -2m \log 2q$. The following corollary gives bounds on q and α such that this condition is satisfied.

Corollary 3.7

If F is SIFR, $F(0) = \frac{1}{2}$, $1 - F(x) \geq F(-x)$ for $x \geq 0$, $1-\alpha \geq 1-e^{-1}$

and $q \leq \frac{1}{2} e^{-\frac{r-r'-1}{m}}$ then

$$P_F\left\{1 - F\left[-H_{\alpha,1-2q}(r',r)\bar{\theta}_{r',r,n}(\underline{X})\right] \geq 1-q\right\} \geq 1-\alpha \quad .$$

<u>Proof</u>

From Theorem 3.6 it suffices to show that $\chi^2_\alpha 2(r-r'-1) \leq -2m \log 2q$. Let K denote the chi-square distribution with $2(r-r'-1)$ degrees of freedom. Then since $\log K(x)$ is concave, $K(2(r-r'-1)) \geq e^{-1}$ by Jensen's inequality. Therefore

$$\frac{\chi^2_\alpha(2(r-r'-1))}{2(r-r'-1)} \leq 1 \quad .$$

Now

$$q \leq \tfrac{1}{2} e^{-\frac{r-r'-1}{m}}$$

implies

$$\frac{-2m \log(2q)}{r-r'-1} \geq 1$$

and the result follows. ||

If F is SIFR and symmetric about the origin we have from Theorem 3.6

<u>Corollary 3.8</u>

If F is SIFR and symmetric about the origin, then

$$P_F\left\{\int_{-C^*_{\alpha,q}(r',r)\bar{\theta}_{r',r,n}(\underline{X})}^{C^*_{\alpha,q}(r',r)\bar{\theta}_{r',r,n}(\underline{X})} dF(x) \geq q\right\} \geq 1-\alpha \quad .$$

In a similar way to Corollary 3.7 we obtain

Corollary 3.9

If F is SIFR, symmetric about the origin, $1-\alpha \geq 1-e^{-1}$ and $q \geq 1-e^{-\frac{r-r'-1}{m}}$, then

$$P_F\left\{\int_{-H_{\alpha,q}(r',r)\bar{\theta}_{r',r,n}(\underline{X})}^{H_{\alpha,q}(r',r)\bar{\theta}_{r',r,n}(\underline{X})} dF(x) \geq q\right\} \geq 1-\alpha \quad .$$

It can be seen from Corollary 3.9 that we generally have to truncate the sample in order to have a sharp tolerance limit. For example, if we are computing the tolerance limits between which 95% of the symmetric SIFR distribution F lies with 99% confidence when $n = 12$, we will need to set $r' = 2$ and $r = 11$ in order that the limits be sharp.

We now give the tolerance limit on the tail probability, $1 - F$, for values to the right of the median. We let

$$C_{\alpha,q}(r',r) = \begin{cases} H_{\alpha,q}(r',r) & \text{if } \chi^2_\alpha 2(r-r'-1) \geq -2(n-2)\log(1-q) \\ \\ \dfrac{1}{n-2} & \text{if } \chi^2_\alpha 2(r-r'-1) < -2(n-2)\log(1-q) \quad . \end{cases}$$

Theorem 3.10

If F is SIFR, $F(0) = \frac{1}{2}$, $1 - F(x) \geq F(-x)$ for $x \geq 0$ and $0 \leq 1-q \leq \frac{1}{2}$, then

$$P_F\left\{1 - F\left[C_{1-\alpha,2q-1}(r',r)\bar{\theta}_{r',r,n}(\underline{X})\right] \geq 1-q\right\} \geq 1-\alpha \quad .$$

Proof

Let G be the Laplace distribution, then from Theorem 3.1

$$F\left[H_{1-\alpha,2q-1}(r',r)\bar{\theta}_{r',r,n}(\underline{X})\right] \overset{st}{\leq} G\left[H_{1-\alpha,2q-1}(r',r)\bar{\theta}_{r',r,n}(\underline{Y})\right]$$

for $(n-2)H_{1-\alpha,2q-1}(r',r) \leq 1$ and $\frac{1}{2} \leq q \leq 1$. It follows that since

$$P_G\left\{G\left[H_{1-\alpha,2q-1}(r',r)\bar{\theta}_{r',r,n}(\underline{Y})\right] \leq q\right\} = 1-\alpha \quad ,$$

we have proved the theorem in the range

$$(n-2)H_{1-\alpha,2q-1}(r',r) \leq 1 \quad ,$$

that is

$$\chi^2_{1-\alpha}2(r-r'-1) \geq -2(n-2)\log 2(1-q) \ .$$

Now suppose that $(n-2)H_{1-\alpha,2q-1}(r',r) > 1$. By noting that

$$F\left[\frac{1}{n-2}\bar{\theta}_{r',r,n}(\underline{X})\right] \overset{st}{\leq} G\left[\frac{1}{n-2}\bar{\theta}_{r',r,n}(\underline{Y})\right]$$
$$\leq G\left[H_{1-\alpha,2q-1}(r',r)\bar{\theta}_{r',r,n}(\underline{Y})\right] \quad ,$$

the proof follows. ||

We note that the tolerance limit is sharp only when $\chi^2_{1-\alpha}2(r-r'-1) \geq -2(n-2)\log 2(1-q)$. The following corollary gives bounds on q and α such that this condition is satisfied.

Corollary 3.11

If F is SIFR, $F(0) = \frac{1}{2}$, $1 - F(x) \geq F(-x)$ for $x \geq 0$, $1-\alpha \geq 1-e^{-1}$, and $1-q \geq \frac{1}{2} e^{-\frac{r-r'-1}{n-2}}$, then

$$P_F \left\{1 - F\left[H_{1-\alpha,2q-1}(r',r)\bar{\theta}_{r',r,n}(\underline{X})\right] \geq 1-q\right\} \geq 1-\alpha \quad .$$

Proof

By Theorem 3.10 we have only to show that

$$\chi^2_{1-\alpha}2(r-r'-1) \geq -2(n-2)\ \log 2(1-q) \quad .$$

Let K denote the chi-squared distribution with $2(r-r'-1)$ degrees of freedom. Since K is IFR, we have from Barlow and Proschan (1965) that

$$K[2(r-r'-1)] \leq 1-e^{-1} \quad .$$

Now

$$K\left[\chi^2_{1-\alpha}2(r-r'-1)\right] = 1-\alpha$$

therefore

$$\frac{\chi^2_{1-\alpha}2(r-r'-1)}{2(r-r'-1)} \geq 1 \quad .$$

If $1-q \geq \frac{1}{2} e^{-\frac{r'-r-1}{n-2}}$, we have proved the result. ||

In noting that $F(0) = \frac{1}{2}$ by assumption, we see from Corollaries 3.7 and 3.11 that we can obtain sharp tolerance limits on $1 - F$ for most q, and α values of interest.

If F is SIFR and symmetric about the origin, we have from Theorem 3.10,

Corollary 3.12

If F is SIFR and symmetric about the origin, then

$$P_F\left\{\int_{-C_{1-\alpha,q}(r',r)\bar{\theta}_{r',r,n}(\underline{X})}^{C_{1-\alpha,q}(r',r)\bar{\theta}_{r',r,n}(\underline{X})} dF(x) \le q\right\} \ge 1-\alpha \quad .$$

Constructing a proof similar to Corollary 3.11, we obtain Corollary 3.13 for the symmetric case.

Corollary 3.13

If F is SIFR, symmetric about the origin, $1-\alpha \ge 1-e^{-1}$ and $q \le 1-e^{-\frac{r-r'-1}{n-2}}$ then

$$P_F\left\{\int_{-H_{1-\alpha,q}(r',r)\bar{\theta}_{r',r,n}(\underline{X})}^{H_{1-\alpha,q}(r',r)\bar{\theta}_{r',r,n}(\underline{X})} dF(x) \le q\right\} \ge 1-\alpha \quad .$$

We see from Corollary 3.13 that the tolerance limit in Corollary 3.12 is sharp for most q and α values of interest.

Tolerance limits for SDFR distributions although probably not as useful as those for SIFR distributions are nevertheless quite interesting. Again, we give one sided tolerance limits when F is skewed and two sided limits when F is symmetric.

Theorem 3.14

If F is SDFR, $F(0) = \frac{1}{2}$, $1 - F(x) \ge F(-x)$ for $x \ge 0$, $0 \le q \le \frac{1}{2}$ and $(n-2)H_{\alpha,1-2q}(r',r) \le 1$ then

$$P_F\left\{1 - F\left[-H_{\alpha,1-2q}(r',r)\bar{\theta}_{r',r,n}(\underline{X})\right] \ge 1-q\right\} \ge 1-\alpha \quad .$$

Proof

Let G be the Laplace distribution, then by Corollary 3.2

$$F\left[-H_{\alpha,1-2q}(r',r)\bar{\theta}_{r',r,n}(\underline{X})\right] \overset{st}{\leq} G\left[-H_{\alpha,1-2q}(r',r)\bar{\theta}_{r',r,n}(\underline{Y})\right]$$

for $(n-2)H_{\alpha,1-2q}(r',r) \leq 1$. By noting that

$$P_G\left\{G\left[-H_{\alpha,1-2q}(r',r)\bar{\theta}_{r',r,n}(\underline{Y})\right] \leq q\right\} = 1-\alpha$$

we have proved the theorem. ||

In a similar way to Corollary 3.7 we see that if $1-\alpha \geq 1-e^{-1}$, then for the above tolerance limit to exist

$$q \geq \tfrac{1}{2}\exp\left\{-\frac{\chi^2_{\alpha}2(r-r'-1)}{2(n-2)}\right\} \geq \tfrac{1}{2}\exp\left\{-\frac{r-r'-1}{n-2}\right\} \geq \tfrac{1}{2}e^{-1} .$$

Thus we see that we cannot achieve high coverage with this tolerance limit. Now in the case that F is symmetric we can obtain two sided tolerance limits. From Theorem 3.14 we obtain immediately

Theorem 3.15

If F is SDFR, symmetric about the origin, and $(n-2)H_{\alpha,q}(r',r) \leq 1$, then

$$P_F\left\{\int_{-H_{\alpha,q}(r',r)\bar{\theta}_{r',r,n}(\underline{X})}^{H_{\alpha,q}(r',r)\bar{\theta}_{r',r,n}(\underline{X})} dF(x) \geq q\right\} \geq 1-\alpha .$$

Again in a similar way to Corollary 3.7, we see that if $1-\alpha \geq 1-e^{-1}$, then for the above tolerance limit to exist it is necessary that

$$q \le 1 - \exp\left(-\frac{\chi^2_{\alpha}2(r-r'-1)}{2(n-2)}\right) \le 1 - \exp\left(-\frac{r-r'-1}{n-2}\right) \le 1-e^{-1}$$

and so high coverage is not possible.

Theorem 3.16

If F is SDFR, $F(0) = \frac{1}{2}$, $1 - F(x) \ge F(-x)$ for $x \ge 0$, $\frac{1}{2} \le q \le 1$, and $mH_{1-\alpha,2q-1}(r',r) \ge 1$, then

$$P_F\left\{1 - F\left[H_{1-\alpha,2q-1}(r',r)\bar{\theta}_{r',r,n}(\underline{X})\right] \ge 1-q\right\} \ge 1-\alpha \ .$$

Proof

Let G be the Laplace distribution, then by Corollary 3.2,

$$F\left[H_{1-\alpha,2q-1}(r',r)\bar{\theta}_{r',r,n}(\underline{X})\right] \overset{st}{\le} G\left[-H_{1-\alpha,2q-1}(r',r)\bar{\theta}_{r',r,n}(\underline{Y})\right]$$

for $mH_{1-\alpha,2q-1}(r',r) \ge 1$. By noting that

$$P_G\left\{G\left[H_{1-\alpha,2q-1}(r',r)\bar{\theta}_{r',r,n}(\underline{Y})\right] \le q\right\} = 1-\alpha$$

we have proved the theorem. ||

If $1-\alpha \ge 1-e^{-1}$, then by similar reasoning to that used in Corollary 3.11, we see that for the above tolerance limit to exist

$$1-q \le \tfrac{1}{2} \exp\left(-\frac{\chi^2_{1-\alpha}2(r-r'-1)}{2m}\right) \le \tfrac{1}{2} \exp\left(-\frac{r-r'-1}{m}\right) .$$

It will thus be necessary to truncate the sample rather severely in order that the tolerance limit on the tail probability exists.

If F is SDFR and symmetric about the origin, then from Theorem 3.16 we have,

Corollary 3.17

If F is SDFR, symmetric about the origin and $mH_{1-\alpha,q}(r',r) \geq 1$, then

$$P_F\left\{\int_{-H_{1-\alpha,q}(r',r)\bar{\theta}_{r',r,n}(\underline{X})}^{+H_{1-\alpha,q}(r',r)\bar{\theta}_{r',r,n}(\underline{X})} dF(x) \leq q\right\} \geq 1-\alpha \quad .$$

In the same way as for Theorem 3.16, the sample will have to be truncated rather severely in order that the tolerance limit exists.

3.2 Tolerance Limits Based on the Standard Deviation

Frequently sampling plans are based on the assumption that the underlying distribution is normal, which gives rise to tolerance limits of the form $\bar{x} \pm \lambda_1 s$ where $\bar{x}$ and s are the sample mean and standard deviation respectively. If the mean μ is known, which we will assume, the tolerance limits are of the form $\mu \pm \lambda_2 s$. Since we are generally not sure that the assumption of a normal distribution is correct, it is interesting to see how robust the normal tolerance limits are against s-ordered alternatives. The following theorem throws some light on this.

Theorem 3.18

If $F \underset{s}{<} G$ and F, G are symmetric about the origin, then

$$P_F\left(\int_{-\lambda s_X}^{\lambda s_X} dF(x) \geq a\right) \leq (\geq)\; P_G\left(\int_{-\lambda s_Y}^{\lambda s_Y} dG(x) \geq a\right)$$

for all $a \in [0,1]$ and $\lambda \leq 1$, $(\lambda \geq \sqrt{n})$, where

$$s_X = \sqrt{\frac{\sum_1^n X_i^2}{n}}\;, \quad s_Y = \sqrt{\frac{\sum_1^n Y_i^2}{n}}\;.$$

Proof

Define G_2 by

$$G_2(x) = P(Y^2 \leq x)$$

and similarly for F_2. We see that $G_2^{-1}F_2(y^2) = [G^{-1}F(y)]^2$ and by differentiation we find that $G_2^{-1}F_2(x)$ is convex if $G^{-1}F(x)$ is convex for $x \geq 0$. From Lemma 3.1 of Barlow and Proschan (1966a)

$$G_2^{-1}F_2\left(\frac{\lambda^2 \sum_1^n X_i^2}{n}\right) \leq (\geq) \frac{\lambda^2}{n} \sum_1^n G_2^{-1}F_2(X_i^2)$$

for $\lambda^2 \leq 1$ $(\lambda^2 \geq n)$. Since $G_2^{-1}F_2(X_i^2) \overset{st}{=} Y_i^2$

$$F_2(\lambda^2 s_X^2) \overset{st}{\leq} (\overset{st}{\geq})\ G_2(\lambda^2 s_Y^2) \quad .$$

The theorem now follows from the observation that $F_2(x) = F(\sqrt{x}) - F(-\sqrt{x})$. ||

Applying the strong law of large numbers we obtain

Corollary 3.19

If $F \underset{s}{<} G$ and F , G are symmetric about the origin, then

$$\int_{-\lambda\sigma_X}^{\lambda\sigma_X} dF(x) \leq \int_{-\lambda\sigma_Y}^{\lambda\sigma_Y} dG(x) \qquad \text{for } \lambda \leq 1$$

and σ_X^2 is the variance of X .

This is the symmetric analogue of $F(\theta_1) \leq G(\theta_2)$, when $F \underset{c}{<} G$ and θ_1 , θ_2 are the means of F and G respectively. [See Barlow and Marshall (1964), Theorem 7.1.]

As an example of Theorem 3.18 we let G be the normal distribution with known mean but unknown variance. If we consider the tolerance limits $\mu \pm \lambda s$ for G which contain 99% of the population with 99% probability we see that if the number of observations from G is n then $\lambda \geq \sqrt{n}$ for $n \leq 18$. Hence we can assert that if $F \underset{s}{<} G$, F symmetric and G the normal and both with known means, then for a sample size less than 18, the 99% tolerance limit with 99% confidence based on the normal is conservative for F .

3.3 Confidence Bound on the Variance of a Symmetric SIFR Distribution

In order to calculate a confidence bound on the variance of a symmetric SIFR distribution we need to develop bounds on the distribution in terms of the mean, which we will assume to be zero, and the variance. We note that two symmetric SIFR distributions with the same mean and variance must cross at least three times. Bounds will only be given to the right of the mean, bounds to the left following by the symmetry of F.

Lemma 3.20

If F is SIFR and symmetric about the origin with variance σ^2, then

$$F(x) \geq \begin{cases} \frac{1}{2} & 0 \leq x \leq \sigma \\ 1 - \frac{1}{2} e^{-bx} & x > \sigma \end{cases}$$

where b is a function of x and σ^2 and is given by the solution to

$$\sigma^2 = 2 \frac{1}{b^2} - 2e^{-xb} \left(\frac{x^2}{2} + \frac{x}{b} + \frac{1}{b^2} \right)$$

and the bound is sharp.

Proof

The bound for $0 \leq x \leq \sigma$ is obvious and is attained by the symmetric distribution with mass $\frac{1}{2}$ at $-\sigma$ and $+\sigma$. For $x \geq \sigma$ we consider the distribution $G_T(x)$ which is symmetric about the origin and for positive arguments is given by

$$G_T(x) = \begin{cases} 1 - \frac{1}{2} e^{-bx} & 0 \leq x \leq T \\ 1 & x \geq T \end{cases},$$

where b is determined by

$$2\int_0^\infty x^2 dG_T(x) \doteq \sigma^2 \quad .$$

We note that since F must cross G at least once in $(0,\infty)$, and $G^{-1}F(x)$ is convex for $x \geq 0$ that $F(x) \geq G_{T=x}(x)$, $x \geq \sigma$. ||

Lemma 3.21

If F is SIFR and symmetric about the origin with variance σ^2, then

$$F(x) \leq \begin{cases} \sup\limits_{0 \leq \Delta \leq \sigma} e^{-a(x-\Delta)} & 0 \leq x \leq \sigma \\ 1 & x > \sigma \quad , \end{cases}$$

where a is given by the solution to

$$a^2\left(\frac{\Delta^2 - \sigma^2}{2}\right) + a\Delta + 1 = 0 \quad .$$

Proof

Consider the distribution $G_\Delta(x)$, symmetric about the origin, and which for positive arguments is given by

$$G_\Delta(x) = \begin{cases} \frac{1}{2} & 0 \leq x \leq \Delta \\ 1 - \frac{1}{2}e^{-a(x-\Delta)} & x \geq \Delta \quad , \end{cases}$$

and where a is given by

$$2\int_0^\infty x^2 dG_\Delta(x) = \sigma^2$$

and Δ lies in the range $0 \le \Delta \le \sigma$. Since $G_\Delta(x)$ is SIFR with mean zero and variance σ^2 , it must cross F at least once in the interval $(0,\infty)$. Let G_Δ cross F from above at $x = u(\Delta)$. If $u(\Delta)$ does not exist set $u(\Delta) = \infty$. Let us consider the following two cases:

Case (i) $u(0) \ge \sigma$

Since $G^{-1}F(x)$ is convex for $x \ge 0$, we see that

$$F(x) \le G_0(x) \qquad 0 \le x \le \sigma \ .$$

Case (ii) $u(0) < \sigma$

Since $G^{-1}F(x)$ is convex for $x \ge 0$, we see that $u(\Delta) \uparrow \Delta$, $u(\sigma) = \infty$ and $u(\Delta)$ is continuous in Δ on the positive support of F . Therefore

$$F(x) \le \sup_{0\le\Delta\le\sigma} e^{-a(u-\Delta)} \qquad 0 \le x \le \sigma \ .$$

The bound for $x > \sigma$ is obvious and is achieved by the distribution with mass ½ at $x = -\sigma$ and $x = +\sigma$. ||

We are now in a position to derive a lower confidence bound on the variance of a symmetric SIFR distribution.

Theorem 3.22

If F is SIFR and symmetric about the origin, then

$$P\left[\sigma^2 \geq 8\left(\frac{\bar{\theta}_{r',r,n}(\underline{X})}{\chi^2_{1-\alpha}2(r-r'-1)}\right)^2 - 2\bar{\theta}^2_{r',r,n}(\underline{X})\exp\left(-\frac{\chi^2_{1-\alpha}2(r-r'-1)}{2(n-2)}\right)\right.$$
$$\left(\tfrac{1}{2}(n-2)^{-2} + 2\left([n-2]\left[\chi^2_{1-\alpha}2(r-r'-1)\right]\right)^{-1} + \right.$$
$$\left.\left. + 4\left[\chi^2_{1-\alpha}2(r-r'-1)\right]^{-2}\right)\right] \geq 1-\alpha \ .$$

<u>Proof</u>

From Lemma 3.20 we have the bound

$$F(t;\sigma) \geq h(t;\sigma) = \begin{cases} \tfrac{1}{2} & t \leq \sigma \\ 1 - \tfrac{1}{2}e^{-bt} & t > \sigma \ , \end{cases}$$

and b is given by the solution to (3.1). If G is the Laplace distribution, then since F is SIFR

$$G\left[\frac{1}{n-2}\,\bar{\theta}_{r',r,n}(\underline{Y})\right] \overset{st}{\geq} F\left[\frac{1}{n-2}\,\bar{\theta}_{r',r,n}(\underline{X})\right] \geq h\left[\frac{1}{n-2}\,\bar{\theta}_{r',r,n}(\underline{X})\ ;\ \sigma\right] .$$

Since

$$P_G\left[G\left(\frac{1}{n-2}\,\bar{\theta}_{r',r,n}(\underline{Y})\right) \leq 1 - \tfrac{1}{2}e^{-\frac{\chi^2_{1-\alpha}2(r-r'-1)}{2(n-2)}}\right] = 1-\sigma$$

we have

$$P_F\left[h\left(\frac{1}{n-2}\,\bar{\theta}_{r',r,n}(\underline{X})\ ;\ \sigma\right) \leq 1 - \tfrac{1}{2}e^{-\frac{\chi^2_{1-\alpha}2(r-r'-1)}{2(n-2)}}\right]$$
$$= P_F\left[\sigma^2 \geq b^{-1}\left(\frac{\chi^2_{1-\alpha}2(r-r'-1)}{2\bar{\theta}_{r',r,n}(\underline{X})}\right)\right] \geq 1-\alpha$$

since $\sigma \nmid b$. Hence we have the result. ||

IV. PROPERTIES OF THE ORDER STATISTICS

Marshall, Olkin and Proschan (1965) and Barlow and Proschan (1966a) have developed inequalities for order statistics arising from distributions F and G in the case that $G^{-1}F$ is starshaped on the support of F and $G(0^-) = F(0^-) = 0$. Van Zwet (1964) has extensively treated inequalities for the expectations of order statistics arising from c-ordered and symmetric s-ordered distributions. These inequalities are not only interesting but are useful in developing bounds and giving insight into the nature of convex and concave-convex transformations. We shall develop inequalities for the order statistics and for power combinations of the random variables in the case of two symmetric r-ordered distributions, except for one inequality where we require the stronger s-ordering. The inequalities reflect the fact that an antisymmetric starshaped transformation of a random variable shifts mass to the tails.

Barlow and Proschan (1966) showed that if $F(0) = G(0) = 0$ and $\frac{G^{-1}F(x)}{x}$ is nondecreasing in $x \geq 0$, then the ratio of order statistics $EY_{i,n}/EX_{i,n}$ is also increasing in $i = 1,2, \ldots, n$ for all n; i.e., r-ordering on the positive axis is preserved by the expected values of the order statistics. Van Zwet (1966) showed that if $F \underset{c}{<} G$ then $[EY_{i+1,n} - EY_{i,n}]/[EX_{i+1,n} - EX_{i,n}]$ is nondecreasing in $i = 1,2, \ldots, n$ for all n. He also proved that if $F \underset{s}{<} G$, F and G symmetric about the origin, then the expected values of the order statistics are similarly s-ordered. Independently of van Zwet we proved a related result, namely that for r-ordered symmetric distributions the expected values of the order statistics preserve the ordering.

We shall need the concept of total positivity. A function $K(x,y)$ of two real variables $x \in X$, $y \in Y$, where X and Y are ordered sets, is said to be totally positive of order r (TP_r) if for all $1 \leq m \leq r$,

$x_1 \le x_2 \le \dots \le x_m$ and $y_1 \le y_2 \le \dots \le y_m$, where each $x_i \in X$, $y_i \in Y$ we have the determinantal inequalities

$$\left| K(x_i, y_j) \right|_{i,j=1}^{m} \ge 0 \ .$$

For further treatment of TP functions and their properties, see Karlin (1964).

It is convenient to define

$$K(i,n,x) = \frac{n!}{(i-1)!(n-i)!} F^{i-1}(x) \ (1 - F(x))^{n-i} \ .$$

We will need the following well-known result:

Lemma 4.1

$K(i,n,x)$ is TP_∞ in $i = 1,2, \dots$ and $-\infty < x < \infty$.

Proof

Since

$$\left(\left(\frac{F(x_i)}{\bar{F}(x_i)} \right)^{\alpha_k} \right) \quad (x_1 \le x_2 \le \dots \le x_n \ ; \ \alpha_1 < \alpha_2 < \dots < \alpha_n)$$

is a generalized Vandermonde matrix, we know (see Gantmacher (1959), p. 118) that it is totally positive. The lemma follows. ||

Theorem 4.2

Let $F \underset{r}{<} G$, F and G symmetric about the origin, then

(i) $$\frac{EX_{i,n}}{EY_{i,n}} \downarrow \ i \ge \left[\frac{n}{2}\right] + 1 \ , \ \uparrow \ i < \left[\frac{n}{2}\right] + 1$$

(ii) $$\frac{EX_{n-i+1,n}}{EY_{n-i+1,n}} = \frac{EX_{i,n}}{EY_{i,n}} \uparrow n \geq i \quad .$$

<u>**Proof**</u>

Let

$$h(i) = \int_{-\infty}^{\infty} (x - cG^{-1}F(x))K(i,n,x)dF(x)$$

$$= EY_{i,n}\left(\frac{EX_{i,n}}{EY_{i,n}} - c\right) \quad .$$

Since F is symmetric, we have that

$$h(i) = -h(n-i+1) \quad \text{for} \quad i \geq \left[\frac{n}{2}\right] + 1 \quad ;$$

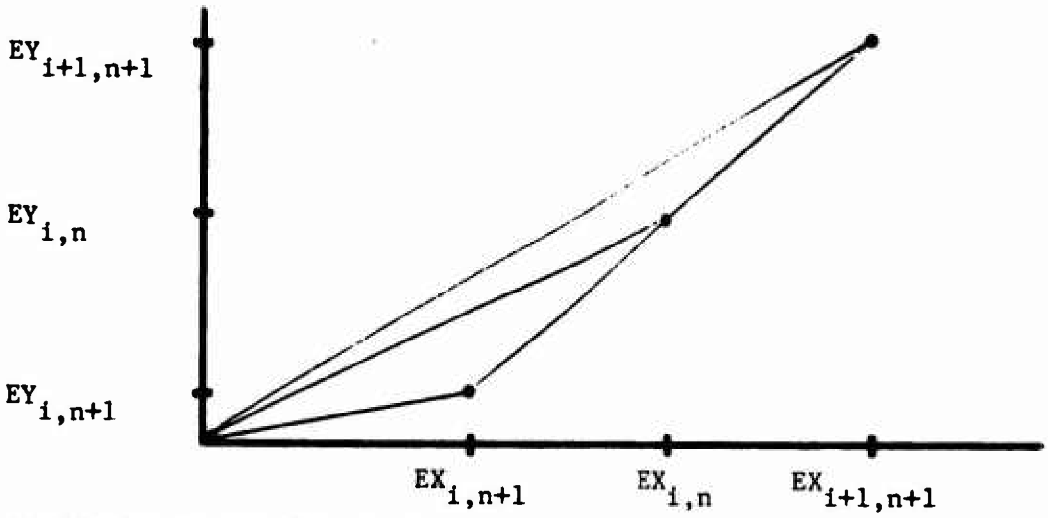
i.e., $h(i)$ is antisymmetric about $\frac{n+1}{2}$. Since $K(i,n,x)$ is TP_∞ for $i = 1,2, \ldots ; -\infty < x < \infty$, and $(x - cG^{-1}F(x))$ changes sign at most three times for $c \geq 0$, we have by the variation diminishing property of TP_∞ functions that $h(i)$ must change sign at most three times. If $h(i)$ does change sign three times then the order of the signs must be the same as for $(x - cG^{-1}F(x))$; viz, + - + - . Since $h(i)$ is antisymmetric about $\frac{n+1}{2}$, we see that $\frac{EX_{i,n}}{EY_{i,n}} \downarrow i \geq \left[\frac{n}{2}\right] + 1$, proving (i).

(ii) may be most readily proved by a geometrical argument.

By conditioning on the $(n+1)$st observation, we see that

$$EX_{i,n} = \left(1 - \frac{i}{n+1}\right)EX_{i,n+1} + \frac{i}{n+1}\, EX_{i+1,n+1} \quad ,$$

and hence for $n < 2i$ we obtain the following diagram from which we observe $\frac{EY_{i,n}}{EX_{i,n}} \downarrow n$ for $n < 2i$.



Similarly we can show that

$$\frac{EY_{i,n}}{EX_{i,n}} \downarrow n \quad \text{for} \quad n \geq 2i \ . \ ||$$

We are now in a position to obtain bounds on $EX_{i,n}$. Note that

$$\frac{EX_{i,i}}{EY_{i,i}} \leq \frac{EX_{i,n}}{EY_{i,n}} \leq \frac{EX_{\left\lceil \frac{n}{2} \right\rceil + 1,n}}{EY_{\left\lceil \frac{n}{2} \right\rceil + 1,n}} \ ,$$

for $i \geq \left\lceil \frac{n}{2} \right\rceil + 1$, and where $\left\lceil \frac{n}{2} \right\rceil$ is the smallest integer larger than or equal to $\frac{n}{2}$. The first inequality follows from Theorem 4.2 (ii), and the second from Theorem 4.2 (i). If

$$\theta = \int_{-\infty}^{\infty} |x| \, dF(x) \ ,$$

then

$$\frac{\theta EY_{i,n}}{EY_{i,i}} \leq EX_{i,n} \leq \frac{\theta EY_{i,n}}{EY_{\left\lceil \frac{n}{2} \right\rceil + 1,n}}$$

for $i \geq \left\lceil \frac{n}{2} \right\rceil + 1$. Note that no nontrivial upper bound exists when n is odd.

Similary for $i \leq \left[\frac{n}{2}\right]$

$$\frac{-\theta EY_{i,n}}{EY_{\left[\frac{n}{2}\right],n}} \leq EX_{i,n} \leq \frac{-\theta EY_{i,n}}{EY_{1,n-i+1}} .$$

Using Theorem 4.2, inequalities for linear combinations of order strtistics from r-ordered distributions may be derived in much the same way as Barlow and Proschan (1966a) do for star-shaped ordering (i.e., $F(0) = G(0) = 0$ and $G^{-1}F$ starshaped on the support of F). As these inequalities parallel those of Bariow and Proschan (1966a) we shall omit them.

Van Zwet has obtained necessary and sufficient conditions on $a_1, a_2, \ldots, a_n$ such that

$$F\left(\sum_{i=1}^{n} a_i EX_{i,n}\right) \leq G\left(\sum_{i=1}^{n} a_i EY_{i,n}\right)$$

for $F \underset{c}{<} G$.* We derive sufficient conditions on $a_1, a_2, \ldots, a_n$ such that the above inequality is true when $F \underset{s}{<} G$. We strongly believe that these conditions are also necessary but have not been able to show this.

Theorem 4.3

If $F \underset{s}{<} G$, G symmetric, $F(0) = G(0) = \frac{1}{2}$ and $1 - F(x) \overset{(\leq)}{\geq} F(-x)$, $x \geq 0$ then

(4.1) $$F\left(\sum_{1}^{n} a_i EX_{i,n}\right) \leq G\left(\sum_{1}^{n} a_i EY_{i,n}\right)$$

when

*Personal communication.

$$(4.2) \qquad 0 \le \sum_{j=0}^{n-1} \binom{n}{j} \bar{A}_{j+1}\left[y^j(1-y)^{n-j} + y^{n-j}(1-y)^j\right] - \bar{A}_1 \le 1$$

and

$$(4.3) \qquad \sum_{j=0}^{n-1} \binom{n}{j} \bar{A}_{j+1}\, y^{n-j}(1-y)^j - \bar{A}_1 \overset{(\le)}{\ge} 0$$

for all $\frac{1}{2} \le y \le 1$.

Proof

We prove (4.1) for the unbracketted inequalities first.

Let

$$h(x) = \sum_{1}^{n} a_i \frac{n!}{(i-1)!(n-i)!} G^{i-1}(x)(1-G(x))^{n-i}G'(x) \quad .$$

Now by letting $\phi = -F^{-1}G$ the theorem can be re-expressed as follows. For ϕ concave-convex about the origin and $\phi(x) \le -\phi(-x)$ for $x \ge 0$ and G is symmetric about the origin then

$$(4.4) \qquad \phi\left(\int_{-\infty}^{\infty} xh(x)dx\right) \le \int_{-\infty}^{\infty} \phi(x)h(x)dx$$

if (4.2) and (4.3) are satisfied for all $\frac{1}{2} \le y \le 1$. By an argument similar to that used in Theorem 2.1 we can see that (4.1) is true if and only if (4.4) holds for all $x_1 \ge x_0 \ge 0$, when ϕ is a double angle function: i.e.,

$$\phi(x) = \begin{cases} x + x_0 & \text{for } x \le -x_0 \\ 0 & \text{for } -x_0 < x < x_1 \\ x - x_1 & \text{for } x \ge x_1 \end{cases}$$

For ϕ a double angle function, (4.4) becomes

$$\int_{-\infty}^{-x_0} (x+x_0)h(x)dx + \int_{x_1}^{\infty} (x-x_1)h(x)dx$$

$$(4.5) \qquad \geq \begin{cases} \max\left(0, \int_{-\infty}^{\infty} xh(x)dx - x_1\right) & \text{for } \int_{-\infty}^{+\infty} xh(x)dx \geq 0 \\ \min\left(0, \int_{-\infty}^{\infty} xh(x)dx + x_0\right) & \text{for } \int_{-\infty}^{\infty} xh(x)dx < 0 \end{cases} .$$

Let $p(y) = \sum_1^n a_i \frac{n!}{(n-i)!(i-1)!} y^i(1-y)^{n-i}$ and substituting $y = G(x)$ in (4.5) we obtain

$$\int_0^{1-y_0} (G^{-1}(y) - G^{-1}(1-y_0))p(y)dy + \int_{y_1}^{1} (G^{-1}(y) - G^{-1}(y_1))p(y)dy$$

$$(4.6) \qquad \geq \begin{cases} \max\left(0, \int_0^1 G^{-1}(y)p(y)dy - G^{-1}(y_1)\right) & \text{for } \int_0^1 G^{-1}(y)p(y)dy \geq 0 \\ \min\left(0, \int_0^1 G^{-1}(y)p(y)dy - G^{-1}(1-y_0)\right) & \text{for } \int_0^1 G^{-1}(y)p(y)dy < 0 \end{cases} ,$$

and (4.1) is true if and only if (4.6) is true for all $1 > y_1 \geq y_0 \geq \frac{1}{2}$ and all G^{-1} strictly increasing on $(0,1)$ and antisymmetric about $\frac{1}{2}$.

We shall now show that (4.2) and (4.3) together imply (4.6).

From the well-known equality

$$\frac{n!}{(n-i)!(i-1)!} \int_\eta^1 y^{i-1}(1-y)^{n-i}dy = \sum_{j=0}^{i-1} \binom{n}{j} \eta^j (1-\eta)^{n-j}$$

we see that (4.2) is equivalent to

$$\text{(4.7)} \qquad 0 \le -\int_0^{1-\eta} p(y)\,dy + \int_\eta^1 p(y)\,dy \le 1 \qquad \tfrac{1}{2} \le \eta \le 1$$

and (4.3) is equivalent to

$$\text{(4.8)} \qquad -\int_0^{1-\eta} p(y)\,dy \ge 0 \qquad \tfrac{1}{2} \le \eta \le 1$$

Let $G^{-1}(x)$ be a "double step function"; that is for $\tfrac{1}{2} \le \eta \le 1$

$$G^{-1}(x) = \begin{cases} -1 & 0 \le x \le 1-\eta \\ 0 & 1-\eta < x < \eta \\ 1 & \eta \le x \le 1 \end{cases} .$$

If (4.7) and (4.8) are true we can see by substitution that

$\int_0^1 G^{-1}(y)p(y)\,dy \ge 0$ and

$$\int_0^{1-y_0} (G^{-1}(y)-G^{-1}(1-y_0))p(y)\,dy + \int_{y_1}^1 (G^{-1}(y)-G^{-1}(y_1))p(y)\,dy$$

$$\ge \max\left(0, \int_0^1 G^{-1}(y)p(y)\,dy - G^{-1}(y_1)\right) .$$

Since any strictly increasing function defined on $(0,1)$ and antisymmetric about $\tfrac{1}{2}$ may be approximated arbitrarily closely from below (above) for $x \ge \tfrac{1}{2}$ $(x < \tfrac{1}{2})$ by a positive multiple of double step functions, and since for $\alpha_i \geq 0$,

$$\sum \alpha_i \max\left(0, \int_0^1 G_i^{-1}(y)p(y)dy - G_i^{-1}(y_0)\right)$$

$$\geq \max\left(0, \sum \alpha_i \left\{\int_0^1 G_i^{-1}(y)p(y)dy - G_i^{-1}(y_0)\right\}\right) ,$$

the theorem follows by the Lebesgue monotone convergence theorem. If $1 - F(x) \leq F(-x)$ the proof follows in a similar way. ||

Similarly to Theorem 4.3 we obtain

<u>Corollary 4.5</u>

If $F \underset{s}{<} G$, G symmetric, $F(0) = G(0) = \frac{1}{2}$, $1 - F(x) \overset{(<)}{\geq} F(-x)$, $x \geq 0$, $EY_{i,n}$ exists then

$$F\left(\sum_1^n a_i EX_{i,n}\right) \geq G\left(\sum_1^n a_i EY_{i,n}\right)$$

if

$$-1 \leq \sum_{j=0}^{n-1} \binom{n}{j} \bar{A}_{j+1}\left[y^j(1-y)^{n-j} + y^{n-j}(1-y)^j\right] - \bar{A}_1 \leq 0$$

and

$$\sum_{j=0}^{n-1} \binom{n}{j} \bar{A}_{j+1}\, y^j(1-y)^{n-j} \overset{(<)}{\geq} 0$$

for all $\frac{1}{2} \leq y \leq 1$.

If F is symmetric we can deduce the following corollary from the proof of Theorem 4.3.

Corollary 4.6

If $F \underset{s}{<} G$, F, G symmetric about the origin and $EY_{i,n}$ exists then

$$F\left(\sum_1^n a_i EX_{i,n}\right) \ \overset{(\geq)}{\leq}\ G\left(\sum_1^n a_i EY_{i,n}\right)$$

if

$$0 \ \overset{(\geq)}{\leq} \ \sum_{j=0}^{n-1} \binom{n}{j} \bar{A}_{j+1}\left[y^j(1-y)^{n-j} + y^{n-j}(1-y)^j\right] - \bar{A}_1 \ \overset{(\geq\ -1)}{\leq\ 1}$$

for all $\frac{1}{2} \leq y \leq 1$.

Van Zwet (1964) proves that if $F \underset{s}{<} G$ and F , G symmetric about the origin, then

$$\frac{\left[E|X|^b\right]^a}{\left[E|X|^a\right]^b} \leq \frac{\left[E|Y|^b\right]^a}{\left[E|Y|^a\right]^b} \qquad \text{for } 0 \leq a \leq b \tag{4.9}$$

for those values of a such that $E|Y|^a$ exists and

$$\frac{EX^{2k}}{(EX^2)^k} \leq \frac{EY^{2k}}{(EY^2)^k} \qquad \text{for } k = 1,2, \ldots \tag{4.10}$$

for those values of k for which EY^{2k} exists. We will prove a stronger result; namely that given $F \underset{r}{<} G$ then the inequalities (4.9) and (4.10) hold stochastically for the usual estimates of the expectations, and hence by the strong law of large numbers, for the expectations themselves.

We need to introduce the concept of majorization, and one of the theorems applying this concept. For a fuller treatment see Hardy Littlewood and Pólya (1959) and Ostrowski (1952).

<u>Definition</u>

(i) A sequence $\underline{a} = (a_1, \ldots, a_n)$ is said to majorize a sequence $\underline{b} = (b_1, \ldots, b_n)$ (written $\underline{a} \succ \underline{b}$) if $a_1 \geq \ldots \geq a_n$, $b_1 \geq \ldots \geq b_n$, and $\sum_1^r a_i \geq \sum_1^r b_i$ for $r = 1, \ldots, n-1$, while $\sum_1^n a_i = \sum_1^n b_i$.

<u>Theorem 4.7</u> (Hardy, Littlewood and Pólya)

If ϕ is convex on the interval I and $\underline{x} \succ \underline{y}$ where $x_1, \ldots, x_n$; $y_1, \ldots, y_n$ belong to I , then

$$\sum_1^n \phi(x_i) \geq \sum_1^n \phi(y_i) \quad .$$

<u>Theorem 4.8</u>

If $F \underset{r}{<} G$, F and G symmetric about the origin, then

(i) $$\frac{\left[\sum_1^n |X_i|^b\right]^a}{\left[\sum_1^n |X_i|^a\right]^b} \overset{st}{\leq} \frac{\left[\sum_1^n |Y_i|^b\right]^a}{\left[\sum_1^n |Y_i|^a\right]^b} \qquad 0 \leq a \leq b$$

and if $E|Y|^b$ exists then

(ii) $$\frac{(E|X|^b)^a}{(E|X|^a)^b} \leq \frac{(E|Y|^b)^a}{(E|Y|^a)^b} \quad .$$

<u>Proof</u>

Raise to the a^{th} power the absolute value of the observations from F , and order so that

$$|X|_1^a \geq |X|_2^a \geq \ldots \geq |X|_n^a \quad .$$

Now if $F_a(x) = P_F(|X|^a \le x)$, $a \ge 0$ we see that $G_a^{-1}F_a(x^a) = [G^{-1}F(x)]^a$ and since $\frac{G^{-1}F(x)}{x} \uparrow x \ge 0$ we have

$$\frac{G_a^{-1}F_a(x)}{x} \uparrow x \ge 0 \quad .$$

If follows from a theorem in Marshall, Olkin and Proschan (1966) [cf. Barlow and Proschan (1966), Theorem 3.12] that

$$\sum_{i=1}^{k} \left(\frac{|X|_i^a}{\sum_1^n |X|_i^a} \right) \overset{st}{\le} \sum_{i=1}^{k} \left(\frac{|Y|_i^a}{\sum_{i=1}^n |Y|_i^a} \right)$$

for $k = 1,2, \ldots, n$.

Now from Theorem 4.7, by considering the convex function $\phi(x) = x^c$, $x \ge 0$, $c \ge 1$ we obtain the stochastic inequality

$$\frac{\sum_1^n |X_i|^{ac}}{\left(\sum_1^n |X_i|^a\right)^c} \overset{st}{\le} \frac{\sum_1^n |Y_i|^{ac}}{\left(\sum_1^n |Y_i|^a\right)^c} \quad .$$

Letting $ac = b$ we obtain

$$\frac{\left(\sum_1^n |X_i|^b\right)^a}{\left(\sum_1^n |X_i|^a\right)^b} \overset{st}{\le} \frac{\left(\sum_1^n |Y_i|^b\right)^a}{\left(\sum_1^n |Y_i|^a\right)^b} \quad .$$

Now if $E|Y|^b$ exists, then $E|Y|^a$ exists and by a limiting argument we can see that $E|X|^b$ exists. (ii) is then true by the strong law of large numbers. ||

Corollary 4.9

If $F \underset{r}{\leq} G$, F and G symmetric about the origin, then

(i) $$\frac{\sum_1^n X_i^{2rk}}{\left(\sum_1^n X_i^{2r}\right)^k} \overset{st}{\leq} \frac{\sum_1^n Y_i^{2rk}}{\left(\sum_1^n Y_i^{2r}\right)^k} \qquad \begin{array}{l} r = 1,2, \ldots \\ k = 1,2, \ldots \end{array}$$

(ii) If EY^{2rk} exists then

$$\frac{EX^{2rk}}{(EX^{2r})^k} \leq \frac{EY^{2rk}}{(EY^{2r})^k}$$

Corollary 4.10

If $F \underset{r}{\leq} G$, F and G symmetric about the origin and EY_i^{2rk} exists, then

$$\frac{EX_i^{2rk}}{(EX_i^{2r})^k} \leq \frac{EY_i^{2rk}}{(EY_i^{2r})^k} \qquad \text{for } k = 1,2, \ldots$$

Proof

The proof follows in the same way as for Corollary 4.9 and by the observation that $G_{(i)}^{-1}F_{(i)}(x) = G^{-1}F(x)$, where $F_{(i)}(x) = P(X_i \leq x)$.

REFERENCES

[1] Barlow, R. E., Marshall, A. W. and Proschan, F. (to appear). Some inequalities for star-shaped and convex functions.

[2] Barlow, R. E. and Marshall, A. W. (1964). Bounds for distributions with monotone hazard rate, I. *Ann. Math. Statist.* **35** 1234-1257.

[3] Barlow, R. E. and Proschan, F. (1965). *Mathematical Theory of Reliability*, John Wiley & Sons, New York.

[4] Barlow, R. E. and Proschan, F. (1966a). Inequalities for linear combinations of order statistics from restricted families. *Ann. Math. Statist.* **37** 1574-1592.

[5] Barlow, R. E. and Proschan, F. (1966b). Tolerance and confidence limits for classes of distributions based on failure rate. *Ann. Math. Statist.* **37** 1592-1601.

[6] Gantmacher, F. R. (1959). *Applications of the Theory of Matrices*, Interscience Publishers.

[7] Hardy, G. H., Littlewood, J. E. and Pólya, G. (1959). *Inequalities*, Cambridge University Press.

[8] Hardy, G. H., Littlewood, J. E. and Pólya, G. (1959). Some simple inequalities satisfied by convex functions. *The Messenger of Mathematics.* **58** 145-152.

[9] Karlin, S. (1964). Total positivity; absorbtion probabilities and applications. *Trans. Amer. Math. Soc.* **3** 37, 107.

[10] Karlin, S. and Novikoff, A. (1963). Generalized convex inequalities. *Pacific J. Math.* **13** 1251-1279.

[11] Marshall, A. W., Olkin, I. and Proschan, F. (1965). Monotonicity of ratios of means and other applications of majorization. Boeing Document D1-82-0490.

[12] Ostrowski, A. (1952). Sur quelques applications des fonctions convexes et concaves au sens de i. shur. *J. Math. Pure Applications.* **31** 253-292.

[13] Owen, D. B. (1962). *Handbook of Statistical Tables*, Addison-Wesley Publishing Company, Inc.

[14] van Zwet, W. R. (1964). Convex transformations of random variables. *Math. Centre*, Amsterdam.

[15] van Zwet, W. R. (1966). An inequality for expected values of sample quantiles. Report S-369. *Math. Centre*, Amsterdam.

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Define $F \underset{s}{<} G$ $(F \underset{r}{<} G)$ if F and G have the same median, say the origin and $G^{-1}F(x)$ is concave-convex about the origin $(G^{-1}F(x)/x$ is increasing (decreasing) in x positive (negative)). Conservative tolerance limits are derived for distributions which are s-ordered with respect to the Laplace distribution. These are especially reasonable for mensuration data. In addition, many inequalities concerning combinations of order statistics are obtained. These results are useful in robustness studies of tolerance limits, estimates and statistical tests derived for specified distributions such as the normal distribution. Some examples are given.

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