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# PLANE WAVE PROPAGATION THROUGH A COMPRESSIBLE INHOMOGENEOUS PLASMA

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## ABSTRACT

The set of hydrodynamic equations together with Maxwell's equations which describe a nonuniform plasma as formulated by Cohen [1962] have been applied to a two dimensional problem where the magnetic field has only one component which is transverse to the direction of propagation and the profile is of the form $e^{-Bx}$. These assumptions lead to a set of second order coupled electromagnetic and hydrodynamic differential equations which describe the region under consideration. Solutions to this coupled set have been obtained by the application of the method of Frobenius. Considerable simplification resulted in the solutions when the assumption that the rms velocity of the electrons is much smaller than the speed of light was made. The resulting set of approximate solutions have been applied to the problem of obtaining reflection and transmission coefficients for the case of a plane wave incident from free space upon a layer of compressible inhomogeneous plasma.

## LIST OF CONTRIBUTORS

The authors are grateful to Dr. David Lomen, Department of Mathematics, the University of Arizona, for several discussions and valuable comments.

## RELATED CONTRACTS AND PUBLICATIONS

The present report is a continuation of research related to the plasma sheath effects on antennas. The previous publications are:

1. "An Experimental Study of Plasma Sheath Effects on Antennas" by G. Tyras, P. C. Bargeliotes, J. M. Hamm, and R. R. Schell, Scientific Report No. 1 AFCRL-65-53, Contract No. AF 19(628)-3834, December, 1964.

2. "An Experimental Study of Plasma Sheath Effects on Antennas" by G. Tyras, P. C. Bargeliotes, J. M. Hamm, and R. R. Schell, Radio Science, J. Res. NBS/VSNC-URSI, Vol. 69D, No. 6, June, 1965, pp. 839-850.

3. "Further Experimental Study of Plasma Sheath Effects on Antennas" by J. M. Hamm and G. Tyras, Scientific Report No. 2, AFCRL-65-608, Contract No. AF 19(628)-3834, July, 1965.

4. "Field of an Axially Slotted Circular Cylinder Clad with an Inhomogeneous Dielectric" by G. Tyras, Scientific Report No. 3, AF 19(628)-3834, February, 1966.

# 1. INTRODUCTION

While problems dealing with compressible homogeneous plasmas and incompressible homogeneous and inhomogeneous plasmas have received considerable attention in the past, little has appeared on the problem of propagation through a plasma that is both compressible and inhomogeneous. Furthermore, one finds that frequent use is made of the boundary condition that the normal component of the electron velocity shall vanish at the free space-plasma interface. While the application of this boundary condition may yield useable answers, it is doubtful that rigid interfaces can be realized in situations dealing with reentry communications. The boundary condition in question can be avoided by suitably choosing a profile which is continuous at the air plasma interface. The electron density will consequently be zero there and the vanishing of the normal component of the electron velocity will be satisfied automatically. Additional boundary conditions for the pressure and its first derivative, however, must be introduced in order to specify the problem uniquely.

In the following, the equations describing a compressible inhomogeneous plasma will be derived first. Then a special case of the latter will be considered and the corresponding solutions will be found. Lastly, these will be applied to the determination of the reflection and transmission coefficients for the case of a plane wave incident upon a layer of compressible inhomogeneous plasma.

## 2. STATEMENT OF THE PROBLEM

Let us focus our attention on the problem of determining the transmission and reflection coefficients for the case of a plane wave incident obliquely upon a plasma layer which is both inhomogeneous and compressible. Let the layer be bounded by free space for the region $x<0$ on one side, and on the other, for $x>t$, by a compressible homogeneous plasma. The profile describing the inhomogeneity of the layer will be of the form $\exp(-Bx)$. The relative dielectric constant of free space will be taken as unity and that of the compressible homogeneous region as $\exp(-Bt)$. From this it follows that the relative dielectric constant is then continuous everywhere. Furthermore, it will be assumed that the magnetic field has only one component $H_z$ and that the latter is transverse to the direction of propagation which has been chosen to be x. Finally, the problem is transformed into a two dimensional one by requiring that all variations along the z-axis to be zero. The reader is referred to Fig. 1, p.22.

Since the relevant equations for the regions $x<0$ and $x>t$ are special cases of the equations describing the compressible inhomogeneous plasma, the latter region will be considered first.

## 3. DERIVATION OF THE EQUATIONS DESCRIBING AN INHOMOGENEOUS COMPRESSIBLE PLASMA

The linearized equations describing a warm, non-uniform, sourceless plasma as formulated by Cohen [1962] are

$$\nabla \times \vec{E} = i\omega\mu_o \vec{H} \tag{1}$$

$$\nabla \times \vec{H} = -i\omega\varepsilon_o \vec{E} - eN_o \vec{v} \tag{2}$$

$$\nabla\cdot\vec{H} = 0 \tag{3}$$

$$\nabla\cdot\vec{E} = \frac{-e}{\varepsilon_o}N_1 \tag{4}$$

$$\nabla\cdot(N_o\vec{v}) = i\omega N_1 \tag{5}$$

$$-i\omega m N_o\vec{v} + N_o e\vec{E} = -\nabla P \tag{6}$$

$$P = \frac{N_o m v_o^2}{3} + N_1 m v_o^2 \tag{7}$$

where $N_o$ and $v_o$ are the average density and rms velocity of the electrons; m and e are the mass and charge of the electrons; $\vec{E}$ and $\vec{H}$ are the electromagnetic field vectors; and $\vec{v}$ and $N_1$ are the velocity vector and density of the electrons imparted by the fields. In addition a time dependence of the form $e^{-i\omega t}$ has been assumed and suppressed throughout.

The velocity vector appearing in (2) may be eliminated by use of (6) yielding

$$\frac{\nabla x\vec{H}}{\varepsilon} = -i\omega\varepsilon_o\vec{E} + \frac{ie}{\omega m}\frac{\nabla P}{\varepsilon} \tag{8}$$

where

$$\varepsilon = 1 - \frac{e^2 N_o(\vec{r})}{\varepsilon_o\omega^2 m} . \tag{9}$$

Multiplying (8) by $\nabla x$, using (1) and (3) and expanding produces

$$\nabla^2\vec{H} + k_o^2\varepsilon\vec{H} + \frac{1}{\varepsilon}\nabla\varepsilon x\nabla x\vec{H} = \frac{ie}{\omega m}\frac{1}{\varepsilon}\nabla\varepsilon x\nabla P \tag{10}$$

where

$$k_o^2 = \omega^2\mu_o\varepsilon_o .$$

Similarly the electric field vector appearing in (2) may be eliminated by the use of (6). Multiplication of the resultant equation by $\nabla\cdot$ produces

$$i\omega m\nabla\cdot(N_o\vec{v}) = \nabla\cdot(\frac{\nabla P}{\varepsilon}) + \frac{ie}{\omega m}\nabla\cdot(\frac{N_o}{\varepsilon}\nabla x\vec{H}). \tag{11}$$

An expression for $\nabla\cdot(N_o\bar{v})$ may be obtained by solving (7) for $N_1$ and substituting the resultant expression into (5) producing

$$\nabla\cdot(N_o\vec{v}) = i\omega(\frac{P}{mv_o{}^2} - \frac{N_o}{3}). \tag{12}$$

The last term appearing in (12) is a static term and is henceforth dropped. Upon substituting of the time varying portion of (12) into (11) and expanding one obtains

$$\nabla^2 P + k_p{}^2\varepsilon P - \frac{1}{\varepsilon}\nabla\varepsilon\cdot\nabla P = \frac{i\omega m}{e}\frac{1}{\varepsilon}\nabla\varepsilon\cdot\nabla xH \tag{13}$$

where

$$k_p^2 = \frac{\omega^2}{v_o{}^2}\ . \tag{14}$$

An expression for the electric field may be obtained by solving (8) for $\vec{E}$. Then

$$\vec{E} = \frac{i}{\omega\varepsilon_o\varepsilon}\left(\nabla x\vec{H} - \frac{ie}{\omega m}\nabla P\right) \tag{15}$$

Similarly, an expression for $\vec{v}$ may be derived by elminating $\vec{E}$ from (2) and (6). Then

$$\vec{v} = \frac{-ie^2}{\omega^3\varepsilon_o m^2\varepsilon(\varepsilon-1)}\left[\nabla P + \frac{i\omega m}{e}(1-\varepsilon)\nabla x\vec{H}\right]. \tag{16}$$

Hence the plasma is completely specified once $\vec{H}$ and P defined by (10) and (13) are known.

## 4. SOLUTIONS OF THE COUPLED SYSTEM

Let us expand (10) and (13) in the cartesian coordinate system for the case where

$$\varepsilon = \varepsilon(x), \ \frac{\partial}{\partial z} = 0, \text{ and } \vec{H} = \vec{\alpha}_z H_z \ .$$

Then

$$\frac{\partial^2 H_z}{\partial x^2} + \frac{\partial^2 H_z}{\partial y^2} - \frac{1}{\varepsilon}\frac{\partial \varepsilon}{\partial x}\frac{\partial H_z}{\partial x} + k_o^{\,2}\varepsilon(x)H_z = \frac{ie}{\omega m}\frac{1}{\varepsilon}\frac{\partial \varepsilon}{\partial x}\frac{\partial P}{\partial y} \qquad (17)$$

and

$$\frac{\partial^2 P}{\partial x^2} + \frac{\partial^2 P}{\partial y^2} - \frac{1}{\varepsilon}\frac{\partial \varepsilon}{\partial x}\frac{\partial P}{\partial x} + k_p^{\,2}\varepsilon(x)P = i\,\frac{\omega m}{e}\frac{1}{\varepsilon}\frac{\partial \varepsilon}{\partial x}\frac{\partial H_z}{\partial y} \ . \qquad (18)$$

In general the y-variation can be removed by a Fourier transform. However, since we are dealing with the propagation of plane waves through the plasma we can simply assume a y-variation of the form $\exp(i\alpha y)$ where $\alpha = k_o \sin \Theta_o$ and $\Theta_o$ is the angle of incidence measured from the normal [Hessel et al 1962]. Equations (17) and (18) become

$$\left(\frac{d^2}{dx^2} - \frac{1}{\varepsilon}\frac{d\varepsilon}{dx}\frac{d}{dx} + k_o^{\,2}\varepsilon(x) - \alpha^2\right)H_z = \frac{-e\alpha}{\omega m}\frac{1}{\varepsilon}\frac{d\varepsilon}{dx}P$$

or in an abbreviated form

$$L_1 H_z = \frac{-e\alpha}{\omega m}\frac{1}{\varepsilon}\frac{d\varepsilon}{dx}P \qquad (19)$$

$$\left\{\frac{d^2}{dx^2} - \frac{1}{\varepsilon}\frac{d\varepsilon}{dx}\frac{d}{dx} + k_p{}^2\varepsilon(x) - \alpha^2\right\}P = \frac{-\omega m}{e}\,\alpha\,\frac{1}{\varepsilon}\frac{d\varepsilon}{dx}H_z$$

$$L_2P = \frac{-\omega m}{e}\,\alpha\,\frac{1}{\varepsilon}\frac{d\varepsilon}{dx}H_z \tag{20}$$

where $L_1$ and $L_2$ correspond to the differential operators in brackets. Equations (19) and (20) are more readily solved if they can be decoupled. Multiplying (19) by $L_2$ produces

$$L_2(L_1H_z) = \frac{-e\alpha}{\omega m}L_2\left(\frac{1}{\varepsilon}\frac{d\varepsilon}{dx}\right)P \tag{21}$$

It is evident that P may be eliminated from (21) by means of (20) if $L_2$ and $\frac{1}{\varepsilon}\frac{d\varepsilon}{dx}$ commute. To meet this end, let us choose

$$\varepsilon(x) = \exp(-Bx). \tag{22}$$

Then

$$L_2L_1H_z = \alpha^2B^2H_z \tag{23}$$

and similarly

$$L_1L_2P = \alpha^2B^2P\ . \tag{24}$$

Before proceeding with the solution of (23) and (24) let us return once more to (19) and (20) to see if an additional simplification can be effected. For the exponential profile (22), (19) and (20) become

$$\frac{d^2H_z}{dx^2} + B\frac{dH_z}{dx} + (k_o{}^2e^{-Bx} - \alpha^2)H_z = \alpha B\overline{P} \tag{25}$$

$$\frac{d^2\overline{P}}{dx^2} + B\frac{d\overline{P}}{dx} + (k_p^{\ 2}e^{-Bx} - \alpha^2)\overline{P} = \alpha BH_z \tag{26}$$

where $\overline{P} = \frac{e}{\omega m} P$ .

An equivalent system may be generated by the addition and subtraction of (25) and (26) which has the form

$$\frac{d^2W}{dx^2} + B\frac{dW}{dx} + \left(\frac{k_p^2 + k_o^2}{2}e^{-Bx} - \alpha B - \alpha^2\right)W = \frac{k_p^2 - k_o^2}{2}e^{-Bx}R \tag{27}$$

$$\frac{d^2R}{dx^2} + B\frac{dR}{dx} + \left(\frac{k_p^2 + k_o^2}{2}e^{-Bx} + \alpha B - \alpha^2\right)R = \frac{k_p^2 - k_o^2}{2}e^{-Bx}W \tag{28}$$

where

$$W = H_z + \overline{P}$$

and

$$R = H_z - \overline{P} \tag{29}$$

Unlike Eqs. (25) and (26), (27) and (28) permit a comparison between the relative magnitudes of $k_o^{\ 2}$ and $k_p^{\ 2}$ . Let us consider the ratio

$$\frac{k_o^{\ 2}}{k_p^{\ 2}} = \frac{v_p^{\ 2}}{c^2} \tag{30}$$

where $v_p$ is the r.m.s. velocity of electrons and c the velocity of light. Normally this ratio will be considerably smaller than unity, then to first order, any contribution due to $k_o^{\ 2}$ may be neglected. The effect of neglecting $k_o^{\ 2}$ in (27) and (28) is equivalent to dropping the term involving $k_o^{\ 2}$ in (25). This result may easily be verified by the addition and subtraction of (27) and (28) and the subsequent use of (29) thus recovering our original system (25) and (26) with the term involving $k_o^2$ absent. Thus we are faced with the task of solving

$$\left(\frac{d^2}{dx^2} + B\frac{d}{dx} - \alpha^2\right)H_z = \frac{e}{\omega m}\alpha BP \tag{31}$$

and

$$\left(\frac{d^2}{dx^2} + B\frac{d}{dx} + k_p^2 e^{-Bx} - \alpha^2\right)P = \frac{\omega m}{e}\alpha BH_z \tag{32}$$

By means of the transformation

$$P = e^{-\frac{Bx}{2}} F(x) \tag{33}$$

$$H_z = e^{-\frac{Bx}{2}} G(x) \tag{34}$$

(31) and (32) become

$$\frac{d^2G}{dx^2} - (\alpha^2 + \frac{B^2}{4})G = \frac{e}{\omega m}\alpha BF \tag{35}$$

and

$$\frac{d^2F}{dx^2} + (k_p^2 e^{-Bx} - \alpha^2 - \frac{B^2}{4})F = \frac{\omega m}{e}\alpha BG \ . \tag{36}$$

Equations (35) and (36) may be decoupled according to the method indicated by (23) and (24). Then

$$\frac{d^4F}{dx^4} + \left(k_p^2 e^{-Bx} - 2(\alpha^2 + \frac{B^2}{4})\right)\frac{d^2F}{dx^2} - 2B\,k_p^2 e^{-Bx}\frac{dF}{dx}$$
$$+ \left((\alpha^2 - \frac{B^2}{4})^2 + k_p^2(\frac{3B^2}{4} - \alpha^2)e^{-Bx}\right)F = 0 \tag{37}$$

$$\frac{d^4G}{dx^4} + \left\{k_p^2e^{-Bx} - 2(\alpha^2 + \frac{B^2}{4})\right\}\frac{d^2G}{dx^2}$$

$$+ \left\{(\alpha^2 - \frac{B^2}{4})^2 - k_p^2(\alpha^2 + \frac{B^2}{4})e^{-Bx}\right\}G = 0 \ . \qquad (38)$$

Before applying the method of Frobenius to each of the decoupled equations it is convenient to remove the factor $e^{-Bx}$. To this end let

$$F(x) = f(z)$$

$$G(x) = g(z) \qquad (39)$$

where $z = \exp(\frac{-Bx}{2})$.

Then

$$z^4 \frac{d^4f}{dz^4} + 6z^3 \frac{d^3f}{dz^3} + z^2\left\{Kz^2 + 7 - 2\nu^2\right\}\frac{d^2f}{dz^2}$$

$$+ z\left\{5Kz^2 + 1 - 2\nu^2\right\}\frac{df}{dz} + \left\{K(4 - \nu^2)z^2 + \nu^4 - \gamma^2\right\}f = 0$$

$$(40)$$

and

$$z^4 \frac{d^4g}{dz^4} + 6z^3 \frac{d^3g}{dz^3} + z^2\left\{Kz^2 + 7 - 2\nu^2\right\}\frac{d^2g}{dz^2}$$

$$+ z\left\{Kz^2 + 1 - 2\nu^2\right\}\frac{dg}{dz} + \left\{\nu^4 - \gamma^2 - K\nu^2z^2\right\}g = 0 \qquad (41)$$

where

$$K = \frac{4k_p^2}{B^2}, \ \nu^2 = \frac{4\alpha^2}{B^2} + 1, \text{ and } \gamma^2 = \frac{16\alpha^2}{B^2} \ .$$

Assuming power series solutions of the form

$$f(z) = \sum_{n=o}^{\infty} a_n z^{n+s}$$

$$g(z) = \sum_{n=o}^{\infty} b_n z^{n+s} \tag{42}$$

one obtains upon substitution into (40) and (41) the following recursion relationships

$$a_n = -\frac{4k_p^2}{B^2} \frac{[(n+s)^2 - v^2]}{[(n+s)^2 - v^2]^2 - \gamma^2} a_{n-2}$$

$$b_n = -\frac{4k_p^2}{B^2} \frac{[(n-2+s)^2 - v^2]}{[(n+s)^2 - v^2]^2 - \gamma^2} b_{n-2} \tag{43}$$

where s is determined from the indicial equation

$$(s^2 - v^2)^2 - \gamma^2 = 0 . \tag{44}$$

Since two of the roots of the indicial equation are also zeros of the denominators of the recursion relations for the case n = 2, both solutions of the first and second kind must be constructed. As this procedure is rather well-known, only the final results will be presented. Thus the solutions generated by the method of Forbenius are

$$P_A(x) = e^{-x(\alpha+B)} \left(1 + \sum_{m=1}^{\infty} (-1)^m \frac{\Delta(1)\Delta(2)\ldots\Delta(m)}{D(1)D(2)\ldots D(m)} \left(\frac{k_p^2}{B^2} e^{-Bx}\right)^m\right) \tag{45a}$$

$$H_A(x) = e^{-x(\alpha+B)}\left(1 + \sum_{m=1}^{\infty} (-1)^m \frac{\Delta(0)\Delta(1)\ldots\Delta(m-1)}{D(1)D(2)\ldots D(m)}\left(\frac{k_p^{\,2}}{B^2}\, e^{-Bx}\right)^m\right) \tag{45b}$$

$$P_B(x) = P_A(x,-\alpha) \tag{45c}$$

$$H_B(x) = H_A(x,-\alpha) \tag{45d}$$

$$P_C(x) = e^{-\alpha x}\left(\frac{-Bx}{2}\right) \frac{1}{\frac{\alpha}{B}(1+\frac{2\alpha}{B})} \sum_{m=1}^{\infty} (-1)^m \frac{\Delta(0)\Delta(1)\ldots\Delta(m-1)}{D(1)D(2)\ldots D(m-1)}\left(\frac{k_p^{\,2}}{B^2}\, e^{-Bx}\right)^m$$

$$+ e^{-\alpha x}\left(1 + \frac{1}{\frac{\alpha}{B}(1+\frac{2\alpha}{B})} \sum_{m=1}^{\infty} (-1)^m \frac{\Delta(0)\Delta(1)\ldots\Delta(m-1)}{D(1)D(2)\ldots D(m-1)}\left(\frac{B^2+2\alpha B+4\alpha^2}{4\alpha(B+2\alpha)}\right.\right.$$

$$\left.\left. + \sum_{k=2}^{m}\left(\frac{\Delta'(k)}{2\Delta(k-1)} - \frac{\Delta'(k)\Delta(k-1)}{D(k-1)}\right)\right)\left(\frac{k_p^{\,2}}{B^2}\, e^{-Bx}\right)^m\right) \tag{45e}$$

$$H_C(x) = e^{-\alpha x}\left(\frac{-Bx}{2}\right) \frac{1}{\frac{\alpha}{B}(1+\frac{2\alpha}{B})} \sum_{m=1}^{\infty} (-1)^m \frac{\Delta(-1)\Delta(0)\ldots\Delta(m-2)}{D(1)D(2)\ldots D(m-1)}\left(\frac{k_p^{\,2}}{B^2}\, e^{-Bx}\right)^m$$

$$+ e^{-\alpha x}\left(1 + \frac{1}{\frac{\alpha}{B}(1+\frac{2\alpha}{B})} \sum_{m=1}^{\infty} (-1)^m \frac{\Delta(-1)\Delta(0)\ldots\Delta(m-2)}{D(1)\ D(2)\ldots D(m-1)}\left(\frac{B^2-6\alpha B-12\alpha^2}{4\alpha(B+2\alpha)}\right.\right.$$

$$\left.\left. + \sum_{k=2}^{m}\left(\frac{\Delta'(k-1)}{2\Delta(k-2)} - \frac{\Delta'(k)\Delta(k-1)'}{D(k-1)}\right)\right)\left(\frac{k_p^{\,2}}{B^2}\, e^{-Bx}\right)^m\right) \tag{45f}$$

$$P_D(x) = P_c(x, -\alpha) \tag{45g}$$

$$H_D(x) = H_c(x, -\alpha) \tag{45h}$$

where

$$\Delta(m) = m(m+1) + \frac{\alpha}{B}(2m + 1)$$

$$D(m) = m(m+1)\left(m + \frac{2\alpha}{B}\right)\left(m + 1 + \frac{2\alpha}{B}\right) \tag{45i}$$

$$\Delta'(m) = \left(2m + \frac{2\alpha}{B} - 1\right) .$$

The above solutions hold for the case where $\frac{2\alpha}{B}$ is not equal to an integer or zero.

Substituting the generated series into (31) and (32) reveals only $P_A$, $\frac{e}{\omega m} H_A$ and $P_B$, $\frac{-e}{\omega m} H_B$ to be solutions. Additional solutions, however, may be found by forming linear superpositions of the various series generated. Thus, if

$$H_3(x) = \frac{e}{\omega m} \gamma_A H_A(x) + \frac{e}{\omega m} \gamma_B H_c(x) \tag{46}$$

$$P_3(x) = \gamma_C P_A(x) + \gamma_D P_c(x) \tag{47}$$

then a substitution of (46) and (47) into (31) and (32) reveals that

$$\gamma_C = \gamma_A + \frac{k_P^2(B - 2\alpha)}{2\alpha B(B + 2\alpha)} \gamma_B \tag{48}$$

$$\gamma_D = - \gamma_B . \tag{49}$$

Hence, the various $\gamma$'s cannot be determined uniquely. Substitution of (48) and (49) into (46) and (47) produces

$$P_3 = \gamma_B \left( \frac{k_P^2(B-2\alpha)}{2\alpha B(B+2\alpha)} P_A - P_C \right) + \gamma_A P_A \tag{50}$$

$$H_3 = \frac{e}{\omega m} \gamma_B H_c + \frac{e}{\omega m} \gamma_A H_A \tag{51}$$

or alternately

$$P_3 = - \gamma_B P_c + \gamma_c P_A \tag{52}$$

$$H_3 = \frac{e}{\omega m} \gamma_B \left( H_c - \frac{k_p^{\ 2}(B-2\alpha)}{2\alpha B(B+2\alpha)} H_A \right) + \frac{e}{\omega m} \gamma_c H_A \tag{53}$$

The last terms in (50) to (53) may be dropped, since being already solutions of the system, they add nothing new. That the alternate forms are equivalent and that either representation may be used for the problem at hand will be demonstrated shortly. Similarly, if

$$n_4(x) = - \frac{e}{\omega m} \gamma_G H_B(x) - \frac{e}{\omega m} \gamma_E h_D(x) \tag{54}$$

$$P_4(x) = \gamma_E P_B + \gamma_F P_D \tag{55}$$

then

$$\gamma_E = \gamma_G + \frac{k_p^{\ 2}(B+2\alpha)}{-2\alpha B(B-2\alpha)} \gamma_H \tag{56}$$

$$\gamma_H = - \gamma_F \tag{57}$$

and

$$P_4 = \gamma_H \left( \frac{k_p^{\ 2}(B + 2\alpha)}{-2\alpha B(B-2\alpha)} P_B - P_D \right) + \gamma_G P_B \tag{58}$$

$$H_4 = - \gamma_H \frac{e}{\omega m} H_D - \frac{e}{\omega m} \gamma_B H_B \tag{59}$$

or

$$P_4 = - \gamma_H P_D + \gamma_E P_B \tag{60}$$

$$H_4 = \frac{e}{\omega m}\left(\frac{k_p^{\,2}(B + 2\alpha)}{-2\alpha B(B-2\alpha)} H_B - H_D\right) - \gamma_E \frac{e}{\omega m} H_B \qquad (61$$

Again the last terms appearing in (59) through (61 may be dropped since they are already solutions of the coupled system.

The sets of solutions satisfying (31) and (32) are summarized below

$$P_1(x) = P_A(x)$$

$$H_1(x) = \frac{e}{\omega m} H_A(x)$$

$$P_2(x) = P_B(x) \qquad (62)$$

$$H_2(x) = -\frac{e}{\omega m} H_B(x)$$

$$P_3(x) = \frac{k_p^{\,2}(B-2\alpha)}{2\alpha B(B+2\alpha)} P_A(x) - P_c(x)$$

$$H_3(x) = \frac{e}{\omega m} H_c(x)$$

$$P_4(x) = \frac{k_p^{\,2}(B+2\alpha)}{-2\alpha B(B-2\alpha)} P_B(x) - P_D(x)$$

$$H_4(x) = -\frac{e}{\omega m} H_D(x) \quad .$$

The complete solution describing the pressure and magnetic field for the region $0 \leq x \leq t$ is obtained by forming a linear superposition of all the solutions tabulated above. Thus

$$H(x) = C_1H_1(x) + C_2H_2(x) + C_3H_3(x) + C_4H_4(x) \qquad (63)$$

and

$$P(x) = C_1P_1(x) + C_1P_1(x) + C_1P_1(x) + C_1P_1(x)$$

Recalling that the C's are yet unspecified the substitution

$$C_1 = C_1' - \frac{k_p^{\,2}(B-2\alpha)}{2\alpha B(B+2\alpha)} \tag{65}$$

$$C_2 = C_2' - \frac{k_p^{\,2}(B+2\alpha)}{-2\alpha B(B-2\alpha)} \tag{66}$$

transforms (65) and (66) into the form which would have resulted if the alternate forms for $P_3$, $P_4$, $H_3$ and $H_4$ had been used. Thus it can be concluded that the various forms are equivalent.

## 5. THE BOUNDARY CONDITIONS

Having obtained solutions to the coupled set of electromagnetic and hydrodynamic equations a consideration of the boundary conditions at the interfaces located at $x = 0$ and $x = t$ is in order. For the electromagnetic fields we shall require that the tangential components of the electric and magnetic fields are continuous at the interfaces. In order to determine the behavior of P at the free space-plasma interface Eq. (6) is expanded in component form. Thus

$$-\frac{\partial P}{\partial y} = N_o e\, E_y - i\omega m\, N_o v_y \tag{67}$$

and

$$-\frac{\partial P}{\partial x} = N_o e\, E_x - i\omega m\, N_o v_x \tag{68}$$

where $N_o = \dfrac{\varepsilon_o m\omega^2}{e^2}\,(1 - e^{-Bx})$ .

Since the electron density $N_o$ is equal to zero for x equal to zero and the components of the electric field and ion velocity are expected to be bounded it follows that

$$\left.\frac{\partial P}{\partial x}\right|_{x=o+} = 0 \tag{69}$$

$$\left. P \right|_{x=o+} = 0 \tag{70}$$

In addition we shall require that P be continuous at x = t.

## 6. THE DETERMINATION OF THE REFLECTION AND TRANSMISSION COEFFICIENTS

The partial differential equation describing the magnetic field in free space is obtained from (15) for the case $\varepsilon = 1$. The solutions are well-known and the expression for the magnetic field may be written as

$$H_{z1} = (e^{ik_o\cos\theta_o x} + Re^{-ik_o\cos\theta_o x})\, e^{ik_o\sin\theta_o y} \tag{71}$$

where R is the reflection coefficient sought and $\theta_o$ the angle of incidence measured from the normal. Using (15) it follows that the expression for the tangential component of the electric field is

$$E_{y1} = \frac{k_o\cos\theta_o}{\omega\varepsilon_o}(e^{ik_o\cos\theta_o x} - Re^{-ik_o\cos\theta_o x})\, e^{ik_o\sin\theta_o y} . \tag{72}$$

For the region occupied by the compressible inhomogeneous plasma we have

$$H_{z2} = \left(C_1H_1(x) + C_2H_2(x) + C_3H_3(x) + C_4H_4(x)\right) e^{ik_o\sin\theta_o y} \tag{73}$$

$$P = \left(C_1P_1(x) + C_2P_2(x) + C_3P_3(x) + C_4P_4(x)\right) e^{ik_o\sin\theta_o y} . \tag{74}$$

The tangential component of the electric field $E_{y2}$ may be calculated from (15). By making use of (73) and (74) this may be expressed as

$$E_{y2} = \frac{-i}{\omega\varepsilon_o\varepsilon(x)}\Big(C_1(H_1' - \frac{e\alpha}{\omega m} P_1) + C_2(H_2' - \frac{e\alpha}{\omega m} P_2) + C_3(H_3' - \frac{e\alpha}{\omega m} P_3)$$

$$+ C_4(H_4' - \frac{e\alpha}{\omega m} P_4)\Big) \tag{75}$$

where the prime denotes differentiation with respect to x and $\alpha = k_o \sin\theta_o$.

The equations describing the pressure and the magnetic field for the region occupied by the homogeneous compressible plasma may be obtained from (17) and (18) setting $\varepsilon = \varepsilon_3 = \exp(-Bt) =$ constant. Then

$$\frac{d^2H_{z3}}{dx^2} + (k_o^2\,\varepsilon_3 - \alpha^2)H_{z3} = 0 \tag{76}$$

$$\frac{d^2P_3}{dx^2} + (k_p^{\ 2}\,\varepsilon_3 - \alpha^2)P_3 = 0\ . \tag{77}$$

Since the region for x > t is semi-infinite only outgoing waves will be present. Thus solutions for this region are of the form

$$H_{z3} = Te^{+i\sqrt{k_o^2\varepsilon_3 - \alpha^2}\ (x-t)}\ e^{ik_o\sin\theta_o y} \tag{78}$$

$$P_3 = Qe^{+i\sqrt{k_p^{\ 2}\varepsilon_3 - \alpha^2}\ (x-t)}\ e^{ik_o\sin\theta_o y}\ . \tag{79}$$

As before the tangential component of the electric field $E_{y3}$ is obtained from (15). Making use of (78) and (79)

$$E_{y3} = \Bigg(\frac{\sqrt{k_o^2\varepsilon_3 - \alpha^2}}{\omega\varepsilon_o\varepsilon_3}\ Te^{i\sqrt{k_o^2\varepsilon_3 - \alpha^2}\ (x-t)}$$

$$+ \frac{ie\alpha}{\omega^2 m\varepsilon_o\varepsilon_3}\ Qe^{i\sqrt{k_p^{\ 2}\varepsilon_3 - \alpha^2}\ (x-t)}\Bigg)e^{ik_o\sin\theta_o y}\ . \tag{80}$$

$$\begin{bmatrix} H_1(0) & H_2(0) & H_3(0) & H_4(0) & -1 & 0 & 0 \\ H_1'(0) & H_2'(0) & H_3'(0) & H_4'(0) & ik_o\cos\theta_o & 0 & 0 \\ P_1(0) & P_2(0) & P_3(0) & P_4(0) & 0 & 0 & 0 \\ P_1'(0) & P_2'(0) & P_3'(0) & P_4'(0) & 0 & 0 & 0 \\ H_1(t) & H_2(t) & H_3(t) & H_4(t) & 0 & -1 & 0 \\ P_1(t) & P_2(t) & P_3(t) & P_4(t) & 0 & 0 & -1 \\ H_1'(t) - \frac{e\alpha}{\omega m} P_1(t) & H_2'(t) - \frac{e\alpha}{\omega m} P_2(t) & H_3'(t) - \frac{e\alpha}{\omega m} P_3(t) & H_4'(t) - \frac{e\alpha}{\omega m} P_4(t) & 0 & -i\sqrt{k_o^2\epsilon_3 - \alpha^2} & \frac{e\alpha}{\omega m} \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \\ R \\ T \\ Q \end{bmatrix} = \begin{bmatrix} 1 \\ ik_o\cos\theta_o \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad (81)$$

Application of the indicated boundary conditions yields (81) where the desired coefficients may be determined by applying Cramers rule.

## 7. NUMERICAL RESULTS

For purposes of numerical evaluation the reflection and transmission coefficients may be expressed as

$$R = -\frac{W_1 - k_o\cos\Theta_o\sqrt{k_o^2\varepsilon_3 - \alpha^2}\;W_4 - i(k_o\cos\Theta_o W_3 + \sqrt{k_o^2\varepsilon_3 - \alpha^2}\;W_2)}{W_1 + k_o\cos\Theta_o\sqrt{k_o^2\varepsilon_3 - \alpha^2}\;W_4 + i(k_o\cos\Theta_o W_3 - \sqrt{k_o^2\varepsilon_3 - \alpha^2}\;W_2)} \tag{82}$$

$$T = \frac{i2k_o\cos\Theta_o W_5}{W_1 + k_o\cos{}_o\sqrt{k_o^2\varepsilon_3 - \alpha^2}\;W_4 + i(k_o\cos\Theta_o W_3 - \sqrt{k_o^2\varepsilon_3 - \alpha^2}\;W_2)} \tag{83}$$

where

$$W_1 = \begin{vmatrix} H_1'(o) & H_2'(o) & H_2'(o) & H_4'(o) & 0 \\ P_1(o) & P_2(o) & P_3(o) & P_4(o) & 0 \\ P_1'(o) & P_2'(o) & P_3'(o) & P_4'(o) & 0 \\ P_1(t) & P_2(t) & P_3(o) & P_4(t) & -1 \\ H_1'(t) - \frac{e\alpha}{\omega m}P_1(t) & H_2'(t) - \frac{e\alpha}{\omega m}P_2(t) & H_3'(t) - \frac{e\alpha}{\omega m}P_3(t) & H_4'(t) - \frac{e\alpha}{\omega m}P_4(t) & \frac{e\alpha}{\omega m} \end{vmatrix} \tag{84}$$

$$W_2 = \begin{vmatrix} H_1'(o) & H_2'(o) & H_3'(o) & H_4'(o) \\ P_1(o) & P_2(o) & P_3(o) & P_4(o) \\ P_1'(o) & P_2'(o) & P_3'(o) & P_4'(o) \\ H_1(t) & H_2(t) & H_3(t) & H_4(t) \end{vmatrix} \tag{85}$$

$$W_3 = \begin{vmatrix} H_1(o) & H_2(o) & H_3(o) & H_4(o) & 0 \\ P_1(o) & P_2(o) & P_3(o) & P_4(o) & 0 \\ P_1'(o) & P_2'(o) & P_3'(o) & P_4'(o) & 0 \\ P_1(t) & P_2(t) & P_3(t) & P_4(t) & -1 \\ H_1'(t) - \frac{e\alpha}{\omega m}P_1(t) & H_2'(t) - \frac{e\alpha}{\omega m}P_2(t) & H_3'(t) - \frac{e\alpha}{\omega m}P_3(t) & H_4'(t) - \frac{e\alpha}{\omega m}P_4(t) & \frac{e\alpha}{\omega m} \end{vmatrix} \tag{86}$$

$$W_4 = \begin{vmatrix} H_1(o) & H_2(o) & H_3(o) & H_4(o) \\ P_1(o) & P_2(o) & P_3(o) & P_4(o) \\ P_1'(o) & P_2'(o) & P_3'(o) & P_4'(o) \\ H_1(t) & H_2(t) & H_3(t) & H_4(t) \end{vmatrix} \tag{87}$$

$$W_5 = \begin{vmatrix} P_1(o) & P_2(o) & P_3(o) & P_4(o) & 0 \\ P_1'(o) & P_2(o) & P_3'(o) & P_4'(o) & 0 \\ H_1(t) & H_2(t) & H_3(t) & H_4(t) & 0 \\ P_1(t) & P_2(t) & P_3(t) & P_4(t) & -1 \\ H_1'(t) - \frac{e\alpha}{\omega m}P_1(t) & H_2'(t) - \frac{e\alpha}{\omega m}P_2(t) & H_3'(t) - \frac{e\alpha}{\omega m}P_3(t) & H_4'(t) - \frac{e\alpha}{\omega m}P_4(t) & \frac{e\alpha}{\omega m} \end{vmatrix} \tag{88}$$

It should be noted that all of the entries in the above determinants are real. Furthermore, expressions for the reflection and transmission coefficients for the case where $k_o^2 \epsilon_3 - \alpha^2 < o$ may be obtained by replacing the factor $(k_o^2 \epsilon_3 - \alpha^2)^{1/2}$ appearing in (82) and (83) by $+i(\alpha^2 - k_o\epsilon_3)^{1/2}$

While all of the series solutions converge, their rate of convergence depends critically upon the magnitude of the factor $(k_p/B)^2$. Using (14) and the expression for the acoustic velocity in an electron gas at a temperature T

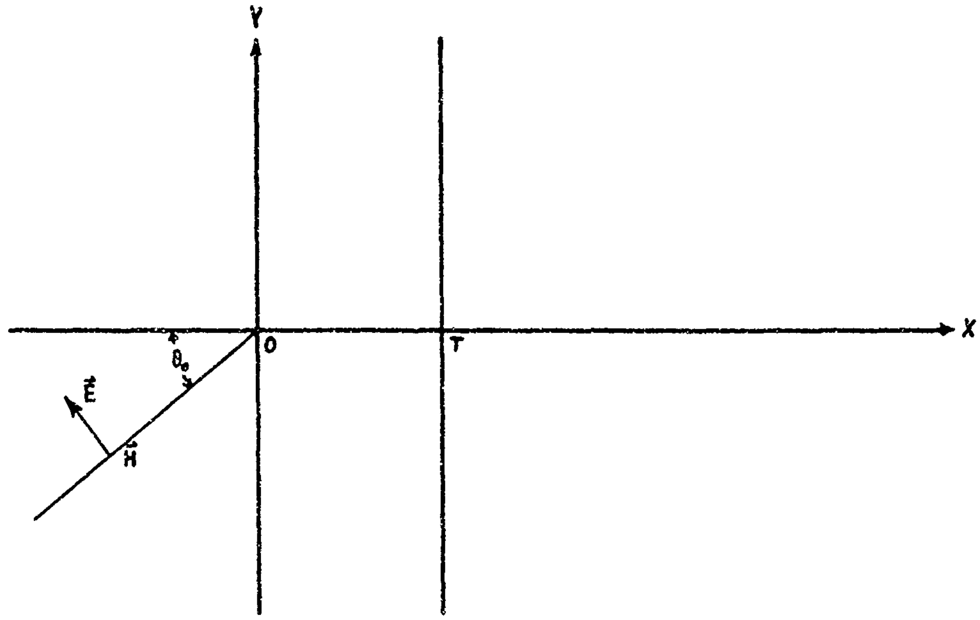
$$v_o^2 = \frac{3kT}{m} \tag{89}$$

it follows that

$$\left(\frac{k_p}{B}\right)^2 = \frac{4\pi^2 m}{3k} \frac{f^2}{TB^2} \tag{90}$$

where m is the electronic mass, k the Boltzmann constant, and f the frequency of the incident electromagnetic radiation. For a frequency of 1GHz, a temperature of 2000°K, and B equal to unity, the value of $(k_p/B)^2$ will be of the order $10^8$. Consequently, all of the series will converge rather slowly and require the calculation of many terms. Furthermore, these terms rapidly reach magnitudes which exceed the capabilities of most computers. While it is believed that this problem can be overcome by special programs, a lack of time prevented any efforts in this direction.

In order to obtain some idea of the behavior of the functions involved, $(k_p/B)^2$ was set equal to unity. Then for a frequency of 1GHz, B=50, and the thickness of the compressible inhomogeneous plasma layer equal to .014 meters, a reflection coefficient of about unity and a transmission coefficient effectively zero were observed for all angles of incidence between 5 and 85 degrees. It must be pointed out that the assumed numerical values actually describe a plasma at a temperature so high that it is no longer consistent with the governing equations and the assumptions behind them.

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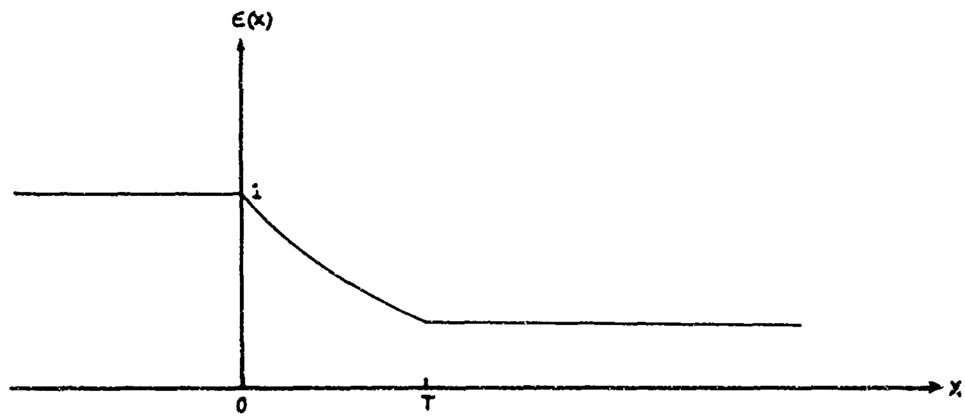


(ii)

Figure 1. (i) Plane wave incident obliquely upon a layer of compressible inhomogeneous plasma occupying the region $o \leq x \leq T$. (ii) The profile for all three regions.

## REFERENCES

Cohen, M. H. (April 1962), Radiation in a plasma II, Equivalent sources, Phys. Rev., 126, 389-397.

Hessel, A., N. Marcuvitz, and J. Shmoys (January 1962), Scattering and guided waves at an interface between air and a compressible plasma, IRE Trans. Antennas and Propagation, AP-10, 48-54.

Rabenstein, A. L. (1966), Introduction to ordinary differential equations, (Academic Press, New York)

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13 ABSTRACT

The set of hydrodynamic equations together with Maxwell's equation which describe a nonuniform plasma as formulated by Cohen (1962) have been applied to a two dimensional problem where the magnetic field has only one component which is transverse to the direction of propagation and the profile is of the form $e^{-Bx}$. These assumptions lead to a set of second order coupled electromagnetic and hydrodynamic differential equations which describe the region under consideration. Solutions to this coupled set have been obtained by the application of the method of Frobenius. Considerable simplification resulted in the solutions when the assumption that the rms velocity of the electrons is much smaller than the speed of light was made. The resulting set of approximate solutions have been applied to the problem of obtaining reflection and transmission coefficients for the case of a plane wave incident from free space upon a layer of compressible inhomogeneous plasma.

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